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開多様体の Lefschetz 跡公式。

抽象跡公式.

U を F 上の d 次元スムーズスキームとする．X をコンパクト化とする．$\gamma \in H^{2d}(X \times X, j_{1!}Rj_{2*}\mathbb{Q}_\ell(d))$ に対し，自己準同型 $\gamma^* : H^q_c(U, \mathbb{Q}_\ell) \to H^q_c(U, \mathbb{Q}_\ell)$ を，合成

$$H^q_c(U,\mathbb{Q}_\ell) \xrightarrow{pr_2^*} H^{2d}(X \times X, j_{2!}\mathbb{Q}_\ell) \xrightarrow{\gamma\cup} H^{2d+q}(X \times X, (j \times j)_!\mathbb{Q}_\ell(d)) = H^{2d+q}_c(U \times U, \mathbb{Q}_\ell(d)) \xrightarrow{pr_{1*}} H^q_c(U, \mathbb{Q}_\ell)$$

と定義する．このとき

$$\sum_{q=0}^{2d} (-1)^q \mathrm{Tr}(\gamma^* : H^q_c(X, \mathbb{Q}_\ell)) = \mathrm{Tr}(\Delta_X^*(\gamma)).$$

$\Gamma \subset U \times U$　次元 d の閉部分多様体. $p_i : \Gamma \to U$: the composition with the projections $pr_i : U \times U \to U$.

$[\Gamma] \in H^{2d}(X \times X, j_{1!}Rj_{2*}\mathbb{Z}_\ell(d))$ は無条件には定義されない.

仮定：$D \times X \cap \bar{\Gamma} \subset X \times D \cap \bar{\Gamma}$.

この条件は次のように言い換えられる.

$p_2 : \Gamma \to U$ が固有.

$\bar{\Gamma} \cap X \times U = \Gamma$.

このとき，$\Gamma \subset X \times U$ は閉部分多様体であり，

$$\begin{aligned} [\Gamma] \quad &\in \quad H^{2d}_\Gamma(X \times U, \mathbb{Z}_\ell(d)) = H^{2d}_\Gamma(X \times U, j_{1!}\mathbb{Z}_\ell(d)) \\ &\to \quad H^{2d}(X \times U, j_{1!}\mathbb{Z}_\ell(d)) = H^{2d}(X \times X, j_{1!}Rj_{2*}\mathbb{Q}_\ell(d)) \end{aligned}$$

が定義される.

Write $\mathrm{Tr}(\Gamma^* : H^*_c(U, \mathbb{Q}_\ell)) = \sum_{q=0}^{2d} (-1)^q \mathrm{Tr}(\Gamma^* : H^q_c(U, \mathbb{Q}_\ell))$.

Lemma 1 *p_2 is proper if and only if*

$$\tilde{\Gamma} \cap (D \times X) \subset \tilde{\Gamma} \cap (X \times D). \tag{1}$$

To have a nice formula, we need more assumption. Assume $D = D_1 \cup \cdots \cup D_m$ is a divisor with simple normal crossings and define

$p : (X \times X)' \to X \times X$: the blow-up at $D_1 \times D_1, \ldots, D_m \times D_m$

$\Delta'_X = X \to (X \times X)'$: the log diagonal.

Theorem 2 *Let $\tilde{\Gamma}'$ be the closure of Γ in $(X \times X)'$ and assume*

$$\tilde{\Gamma}' \cap (D \times X)' \subset \tilde{\Gamma}' \cap (X \times D)' \tag{2}$$

where $(D \times X)'$ and $(X \times D)'$ are the proper transforms of $D \times X$ and $X \times D$. Then, $p_2 : \Gamma \to U$ is proper and we have

$$\mathrm{Tr}(\Gamma^* : H^*_c(U, \mathbb{Q}_\ell)) = \deg(\tilde{\Gamma}', \Delta'_X)_{(X \times X)'}.$$

The assumption is satisfied in our case: $\overline{W \times_U W} \cap (D \times Y)' = \overline{W \times_U W} \cap (Y \times D)'$.

Can not replace (2) $\tilde{\Gamma}' \cap D^{(1)\prime} \subset \tilde{\Gamma}' \cap D^{(2)\prime}$ by (1) $\tilde{\Gamma} \cap D^{(1)} \subset \tilde{\Gamma} \cap D^{(2)}$.

Example 1. $X = \mathbb{P}^1$, $U = \mathbb{A}^1$, $F : U \to U$ Frobenius. $\Gamma = \Gamma_F$

Then, $\mathrm{Tr}(F^* : H_c^*(U, \mathbb{Q}_\ell)) = p$ and $(\Gamma, \Delta)_{(X \times X)'} = p$. On the other hand, $\mathrm{Tr}(F_* : H_c^*(U, \mathbb{Q}_\ell)) = 1$ and $(\Gamma^t, \Delta)_{(X \times X)'} = p$.

Example 2. $X = \mathbb{P}^1$, $U = \mathbb{A}^1$, $\sigma : U \to U$ defined by $x \mapsto x + 1$. $\Gamma = \Gamma_\sigma$.

Then, $\mathrm{Tr}(\sigma^* : H_c^*(U, \mathbb{Q}_\ell)) = 1$ and $(\Gamma, \Delta)_{(X \times X)'} = 1$. There is an intersection at ∞. $t = \frac{1}{x} \mapsto \frac{1}{\frac{1}{t}+1} = \frac{t}{1+t}$. $\frac{s}{1+s} - t = t \cdot \frac{u-(1+ut)}{1+ut}$.

Proof of Theorem 2.

Classical case ($X = U$ is proper).

$$\begin{array}{ccccc}
\Gamma^* \in & \bigoplus \mathrm{End} H^q(X, \mathbb{Q}) & \xrightarrow{\simeq} & H^{2d}(X \times X, \mathbb{Q}_\ell(d)) & \ni [\Gamma] \\
& \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \Delta^*} & \\
\mathrm{Tr}\ \Gamma^* \in & \mathbb{Q}_\ell & \xleftarrow{\mathrm{Tr}} & H^{2d}(X, \mathbb{Q}_\ell(d)) & \ni \Delta^*[\Gamma]
\end{array}$$

The isomorphism in the upper line is given by the Poincaré duality and Künneth formula.

$\Delta^*[\Gamma] = [(\Delta, \Gamma)]$: compatibility of the cup-product with the intersection product.

$\deg = \mathrm{Tr}$.

Our case.

$$\begin{array}{ccccc}
\Gamma^* \in & \bigoplus \mathrm{End} H_c^q(U, \mathbb{Q}) & \xrightarrow{\simeq} & H_{!*}^{2d}(U \times U, \mathbb{Q}_\ell(d)) & \ni [\Gamma] \\
& \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \Delta^*} & \\
\mathrm{Tr}\ \Gamma^* \in & \mathbb{Q}_\ell & \xleftarrow{\mathrm{Tr}} & H_c^{2d}(U, \mathbb{Q}_\ell(d)) & \ni \Delta^*[\Gamma]
\end{array}$$

where $H_{!*}^{2d}(U \times U, \mathbb{Q}_\ell(d)) = H^{2d}(X \times X, j_{1!} R j_{2*} \mathbb{Q}_\ell(d))$, $j_2 : U \times U \to U \times X, j_1 : U \times X \to X \times X$.

Need to relate $\Delta^*[\Gamma]$ with $[(\Delta, \tilde{\Gamma})_{(X \times X)'}]$. We have a commutative diagram

$$\begin{array}{ccccc}
[\Gamma] \in & H_{!*}^{2d}(U \times U, \mathbb{Q}_\ell(d)) & \longrightarrow & H_{!\emptyset*}^{2d}(U \times U, \mathbb{Q}_\ell(d)) & \ni [\tilde{\Gamma}] \\
& \downarrow{\scriptstyle \Delta^*} & & \downarrow{\scriptstyle \Delta^*} & \\
\Delta^*[\Gamma] \in & H_c^{2d}(U, \mathbb{Q}_\ell(d)) & \longrightarrow & H^{2d}(X, \mathbb{Q}_\ell(d)) & \ni \Delta^*[\tilde{\Gamma}]
\end{array}$$

where $H_{!\emptyset*}^{2d}(U \times U, \mathbb{Q}_\ell(d)) = H^{2d}((X \times X)', j_{1!} R j_{2*} \mathbb{Q}_\ell(d))$, $k_2 : (X \times X)' - (D \times X)' \cup (X \times D)') \to (X \times X)' - (D \times X)', k_1 : (X \times X)' - (D \times X)' \to (X \times X)'$. The assumption implies that $[\tilde{\Gamma}] \in H_{!\emptyset*}^{2d}(U \times U, \mathbb{Q}_\ell(d))$ is defined. The key point is that the upper horizontal map sends $[\Gamma] \to [\tilde{\Gamma}]$. This follows from the fact that the map is an isomorphism observed by Faltings and Pink. ■