

1 Closed conical subsets

1.1 Transversality

Definition 1.1.1. Let X be a scheme and E be a vector bundle on X . We say that a closed subset C of X is conical if it is stable under the action of the multiplicative group \mathbf{G}_m . The intersection of a closed conical subset C with the 0-section is called the base of C .

Let k be a field and X be a smooth scheme over k . The covariant vector bundle T^*X associated to the locally free \mathcal{O}_X -module is called the cotangent bundle of X . For a smooth subscheme Z of X , the conormal bundle T_Z^*X is a sub vector bundle of the restriction $T_X^* \times_X Z$ associated to the conormal sheaf $N_{Z/X}$. In particular for $Z = X$, the 0-section of T^*X is denoted by T_X^*X .

Definition 1.1.2. Let $h: W \rightarrow X$ be a morphism of smooth schemes over a field k and C be a closed conical subset of the cotangent bundle T^*X .

1. Let w be a point of W . We say that h is C -transversal at w , if the intersection of the pull-back $C \times_X w$ with the kernel $\text{Ker}(T^*X \times_X w \rightarrow T^*W \times_W w)$ inside $T^*X \times_X Tx$ is a subset $\{0\}$.
2. We say that h is C -transversal if h is C -transversal at every point of W .

The condition that h is C -transversal means that the intersection of the pull-back $h^*C = W \times_X C$ with the kernel $\text{Ker}(W \times_X T^*X \rightarrow T^*W)$ inside $W \times_X T^*X$ is a subset of the 0-section.

Lemma 1.1.3. Let $h: W \rightarrow X$ be a morphism of smooth schemes over a field k and C be a closed conical subset of the cotangent bundle T^*X .

1. h is smooth if and only if h is T^*X -transversal.
2. If C is a subset of the 0-section, then h is C -transversal.
3. If C' is a closed conical subset of T^*X containing C as a subset and if h is C' -transversal, then h is C -transversal. Consequently, if h is smooth, then h is C -transversal.
4. The subset $\{w \in W \mid h \text{ is } C\text{-transversal at } w\}$ is an open subset of W .

Proof. 1. h is smooth if and only if $W \times_X T^*X \rightarrow T^*W$ is an injection.

2. and 3. Clear from the definition and 1.

4. The subset is the complement of the image of $\mathbf{P}(W \times_X C \cap \text{Ker})$ by the projection. \square

Lemma 1.1.4. Assume that $h: W \rightarrow X$ is C -transversal. Then, $W \times_X T^*X \rightarrow T^*W$ is finite on h^*C .

Lemma 1.1.5. $\dim h^*C \geq \dim C + \dim W - \dim X$.

Lemma 1.1.6. Assume that $h: W \rightarrow X$ is C -transversal. For a morphism $g: V \rightarrow W$ of smooth schemes over k , the following conditions are equivalent:

- (1) g is h^*C -transversal.
- (2) $h \circ g$ is C -transversal.

1.2 Acyclicity

Definition 1.2.1. Let $h: W \rightarrow X$ and $f: W \rightarrow Y$ be morphisms of smooth schemes over k and let C be a closed conical subset of the cotangent bundle T^*X . We identify $T^*(X \times Y)$ with $T^*X \times T^*Y$ and regard $C \times T^*Y$ as a closed conical subset of $T^*(X \times Y)$.

We say that (h, f) is C -acyclic if $(h, f): W \rightarrow X \times Y$ is $C \times T^*Y$ -transversal. Further if $h = 1_X$ and $(1_X, f)$ is C -acyclic, we say that f is C -acyclic.

Lemma 1.2.2. Let $f: X \rightarrow Y$ be a morphism of smooth schemes over k and C be a closed conical subset of T^*X .

1. The following conditions are equivalent:

(1) $f: X \rightarrow Y$ is C -acyclic.

(2) The inverse image of C by $X \times_Y T^*Y \rightarrow T^*X$ is a subset of the 0-section.

2. If $f: X \rightarrow Y$ is C -acyclic, then f is smooth on a neighborhood of the base of C .

3. If C equals the 0-section T_X^*X , then $f: X \rightarrow Y$ is C -acyclic if and only if f is smooth.

Lemma 1.2.3. Let $h: W \rightarrow X$ and $f: X \rightarrow Y$ be morphisms of smooth schemes over k and C be a closed conical subset of T^*X .

1. The following conditions are equivalent:

(1) (h, f) is C -acyclic.

(2) $h: W \rightarrow X$ is C -transversal and $f: W \rightarrow Y$ is $h^\circ C$ -acyclic.

2. The following conditions are equivalent:

(1) (h, f) is T_X^*X -acyclic.

(2) $f: W \rightarrow Y$ is smooth.

3. The following conditions are equivalent:

(1) (h, f) is T^*X -acyclic.

(2) $(h, f): W \rightarrow X \times Y$ is smooth.

Proof. 1.

2. The condition (1) is equivalent to the condition that $f: W \rightarrow Y$ is T_W^*W -acyclic by 1, Lemma 1.1.3.2 and $h^\circ T_X^*X = T_W^*W$. Further by Lemma 1.2.2.3, this means that $f: W \rightarrow Y$ is smooth.

3. This follows from Lemma 1.1.3.1. □

Lemma 1.2.4. Let $g: X' \rightarrow X$ a smooth morphism of smooth schemes over k and $C' \subset T^*X'$ be a closed conical subset. Assume that g is C -acyclic.

(1) Let

$$\begin{array}{ccc} X' & \xleftarrow{h'} & W' \\ g \downarrow & & \downarrow g' \\ X & \xleftarrow{h} & W \end{array}$$

be a cartesian diagram of morphisms of smooth schemes over k . Then, g' is $h'^\circ C'$ -acyclic.

(2) Let $f: X \rightarrow Y$ be a smooth morphism of smooth schemes over k . Then, $f \circ g$ is C' -acyclic.

Definition 1.2.5. Let $C \subset T^*X$ be a closed conical subset and $f: X \rightarrow Y$ be a morphism of smooth schemes over k . Assume that f is proper on the base of C . Then, we define a closed conical subset $f_\circ C \subset T^*Y$ by the algebraic correspondence $T^*X \leftarrow X \times_Y T^*Y \rightarrow T^*Y$.

Proposition 1.2.6. *Let $g: X' \rightarrow X$ be a morphism of smooth schemes over k and let $C' \subset T^*X'$ be a closed conical subset. Assume that g is proper on the basis B' of C' and define $C = g_\circ C' \subset T^*X$. Let $h: W \rightarrow X$ be a morphism of smooth schemes over k and*

$$\begin{array}{ccc} X' & \xleftarrow{h'} & W' \\ g \downarrow & & \downarrow g' \\ X & \xleftarrow{h} & W \end{array}$$

be a cartesian diagram and assume that h is C -transversal.

1. *There exists an open neighborhood U' of the inverse image $B'_{W'} = h'^{-1}(B') \subset W'$ smooth over W .*

2. *Let $f: W \rightarrow Y$ be a morphism of smooth schemes over k and $U' \subset W'$ be an open subscheme as in 1. Then, the following conditions are equivalent:*

- (1) *(h, f) is C -acyclic.*
- (2) *$(h'|_{U'}, f \circ g'|_{U'})$ is C' -acyclic.*

Lemma 1.2.7. *Let $h: W \rightarrow X$ and $f: W \rightarrow Y$ be smooth morphisms of smooth schemes over k . Assume that $(h, f): W \rightarrow X \times Y$ is an immersion of codimension 1 and that $f: W \rightarrow Y$ is proper. Let $C \subset T^*X$ be a closed conical subset such that h is C -transversal.*

*Define $C' \subset T^*_W(X \times Y)$ to be the inverse image of C by the composition $T^*_W(X \times Y) \rightarrow (T^*X \times T^*Y) \times_{X \times Y} W \rightarrow T^*X$ and let $E \subset W$ be the projectivization $\mathbf{P}(C') \subset \mathbf{P}(T^*_W(X \times Y)) = W$. Then, we have $f_\circ h^\circ C \cup T^*_Y Y = f_\circ(h^\circ T^*X \times_W E) \cup T^*_Y Y$.*

Proof. The pull-back $h^\circ C \subset T^*W$ is the image of $(C \times T^*_Y Y) \times_{X \times Y} W$ by the surjection $(T^*X \times T^*Y) \times_{X \times Y} W \rightarrow T^*W$. Since the kernel of the surjection $(T^*X \times T^*Y) \times_{X \times Y} W \rightarrow T^*W$ is the line bundle $T^*_W(X \times Y)$, the inverse image of $h^\circ C \subset T^*W$ by $T^*Y \times_Y W \rightarrow T^*W$ equals the image of C' by the composition $T^*_W(X \times Y) \rightarrow (T^*X \times T^*Y) \times_{X \times Y} W \rightarrow T^*Y \times_Y W$. Since C' equals $T^*_W(X \times Y) \times_W E$ outside of the 0-section, the assertion follows. \square

1.3 Legendre transform

Let \mathbf{P} be a projective space. The dual projective space \mathbf{P}^\vee parametrizes hyperplanes in \mathbf{P} and the universal hyperplane $Q \subset \mathbf{P} \times \mathbf{P}^\vee$ may be considered as $\{(x, H) \in \mathbf{P} \times \mathbf{P}^\vee \mid x \in H\}$. Let $p: Q \rightarrow \mathbf{P}$ and $p^\vee: Q \rightarrow \mathbf{P}^\vee$ denote the projections. The canonical injection of the conormal bundle $T^*_Q(\mathbf{P} \times \mathbf{P}^\vee) \rightarrow (T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P} \times \mathbf{P}^\vee} Q$ induces an isomorphism

$$T^*_Q(\mathbf{P} \times \mathbf{P}^\vee) \rightarrow \text{Ker}((T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P} \times \mathbf{P}^\vee} Q \rightarrow T^*Q).$$

Since a hyperplane of \mathbf{P} containing a point of \mathbf{P} defines a line of the fiber of $T^*\mathbf{P}$ at the point and vice versa, the universal hyperplane Q is identified with the projective space bundle $\mathbf{P}(T^*\mathbf{P})$ associated to the vector bundle $T^*\mathbf{P}$. The image of the conormal bundle $T^*_Q(\mathbf{P} \times \mathbf{P}^\vee) \subset (T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P} \times \mathbf{P}^\vee} Q$ to the first factor $T^*\mathbf{P} \times_{\mathbf{P}} Q$ is the tautological line bundle. Symmetrically, Q is identified with $\mathbf{P}(T^*\mathbf{P}^\vee)$.

Definition 1.3.1. *Let $C \subset T^*\mathbf{P}$ be a closed conical subset. Define the Legendre transform $C^\vee = LC$ by*

$$C^\vee = p^\vee_\circ p^\circ C.$$

We consider the projectivization

$$E = \mathbf{P}(C) \subset \mathbf{P}(T^*\mathbf{P}) = Q$$

as a closed subset of Q .

Proposition 1.3.2. *Let $C \subset T^*\mathbf{P}$ be a closed conical subset and $B \subset \mathbf{P}$ be the base of C . Let $C' \subset T_Q^*(\mathbf{P} \times \mathbf{P}^\vee)$ be the inverse image of $C \times T^*\mathbf{P}^\vee$ by the injection with $T_Q^*(\mathbf{P} \times \mathbf{P}^\vee) \rightarrow (T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P} \times \mathbf{P}^\vee} Q$ and $C^\vee \subset T^*\mathbf{P}^\vee$ be the Legendre transform.*

1. C' equals the union of the restriction $T_Q^*(\mathbf{P} \times \mathbf{P}^\vee) \times_Q E$ on $E = \mathbf{P}(C) \subset Q$ with the 0-section on $p^{-1}B$.

2. C is equal to the image of C' by the composition $(T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P} \times \mathbf{P}^\vee} Q \rightarrow T^*\mathbf{P} \times_{\mathbf{P}} Q \rightarrow T^*\mathbf{P}$.

3. As a subset of $Q = \mathbf{P}(T^*\mathbf{P}) = \mathbf{P}(T^*\mathbf{P}^\vee)$, we have $E = \mathbf{P}(C) = \mathbf{P}(C^\vee)$.

Proof. 1. Since $T_Q^*(\mathbf{P} \times \mathbf{P}^\vee)$ is a line bundle over Q , the fiber of C' at a point of Q is either a line, a point or empty. The base of C' equals $p^{-1}B$. Since the image of $T_Q^*(\mathbf{P} \times \mathbf{P}^\vee) \subset (T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P} \times \mathbf{P}^\vee} Q$ in $T^*\mathbf{P} \times_{\mathbf{P}} Q$ is the tautological line bundle, the fiber is a line if and only if the point is contained in $E = \mathbf{P}(C) \subset Q$.

2. Since $C \subset T^*\mathbf{P}$ is a conical subset, it is the union of the intersections with lines in fibers. Since Q parametrizes the lines in fibers and the image of $T_Q^*(\mathbf{P} \times \mathbf{P}^\vee)$ is the tautological line in $T^*\mathbf{P} \times_{\mathbf{P}} Q$, the assertion follows.

3. The subset $E = \mathbf{P}(C) \subset Q = \mathbf{P}(T^*\mathbf{P})$ consists of the pairs of points of \mathbf{P} and the lines in the fibers of $T^*\mathbf{P}$ at the points contained in C .

The Legendre transform $C^\vee = p^\vee p^\circ C$ is defined as the image of the intersection of $(C \times T^*\mathbf{P}^\vee) \cap T_Q^*(\mathbf{P} \times \mathbf{P}^\vee)$ by the composition $(T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P} \times \mathbf{P}^\vee} Q \rightarrow T^*\mathbf{P}^\vee \times_{\mathbf{P}^\vee} Q \rightarrow T^*\mathbf{P}^\vee$. Hence $\mathbf{P}(C^\vee) \subset Q$ consists of the points of Q such that the fiber of $T_Q^*(\mathbf{P} \times \mathbf{P}^\vee)$ is contained in $C \times T^*\mathbf{P}^\vee$. Since the image of the conormal bundle $T_Q^*(\mathbf{P} \times \mathbf{P}^\vee) \subset (T^*\mathbf{P} \times T^*\mathbf{P}^\vee) \times_{\mathbf{P} \times \mathbf{P}^\vee} Q$ in $T^*\mathbf{P} \times_{\mathbf{P}} Q$ by the first projection is the tautological line bundle, the subset $\mathbf{P}(C^\vee) \subset Q$ equals $\mathbf{P}(C)$. \square

Corollary 1.3.3. *Let C be a closed conical subset of $T^*\mathbf{P}$ and let $E = \mathbf{P}(C) \subset Q = \mathbf{P}(T^*\mathbf{P})$ be the projectivization. Then, the complement $Q - E$ is the largest open subset of Q where p^\vee is $p^\circ C$ -acyclic.*

Proof. The pair (p, p^\vee) is C -acyclic precisely outside the projectivization $E = \mathbf{P}(C) \subset Q = \mathbf{P}(T_Q^*(\mathbf{P} \times \mathbf{P}^\vee))$. Since p is smooth, the condition that (p, p^\vee) is C -acyclic is equivalent to the condition that p^\vee is $p^\circ C$ -acyclic. \square

Proposition 1.3.4. *We consider a cartesian diagram*

$$\begin{array}{ccccc} \mathbf{P}^\vee & \xleftarrow{p^\vee} & Q & \xleftarrow{h'} & Q_W \\ & & p \downarrow & \square & \downarrow p' \\ & & \mathbf{P} & \xleftarrow{h} & W \xrightarrow{f} Y \end{array}$$

of smooth schemes over k . For a closed conical subset $C \subset T^*\mathbf{P}$, its Legendre transform $C^\vee \subset T^*\mathbf{P}^\vee$ and the union $C^+ = C \cup T_{\mathbf{P}}^*\mathbf{P}$ with the 0-section, the following conditions are equivalent:

(1) (h, f) is C^+ -acyclic.

(2) $f: W \rightarrow Y$ is smooth and $(p^\vee \circ h', f \circ p'): Q_W \rightarrow \mathbf{P}^\vee \times Y$ is $T^*\mathbf{P}^\vee$ -acyclic on a neighborhood of the inverse image $E_W = E \times_Q Q_W$.

These equivalent conditions imply that $(p^\vee \circ h', f \circ p'): Q_W \rightarrow \mathbf{P}^\vee \times Y$ is C^\vee -acyclic outside $E_W = E \times_Q Q_W$.

The second condition in (2) means that $(p^\vee \circ h', f \circ p'): Q_W \rightarrow \mathbf{P}^\vee \times Y$ is smooth on a neighborhood of the inverse image $E_W = E \times_Q Q_W$.

Proof. Since $\mathbf{P}(C^\vee) = \mathbf{P}(C) \subset Q$ by Proposition 1.3.2.3, this equals the subset E in Lemma 1.2.7. Hence $C^+ = p_\circ p^{\vee\circ} C^\vee \cup T_{\mathbf{P}}^* \mathbf{P}$ equals $p_\circ(p^{\vee\circ} T^* \mathbf{P}^\vee \times_Q E) \cup T_{\mathbf{P}}^* \mathbf{P}$ by Lemma 1.2.7. Thus the condition (1) is equivalent to the combination of the following conditions:

(1') (h, f) is $T_{\mathbf{P}}^* \mathbf{P}$ -acyclic.

(1'') (h, f) is $p_\circ(p^{\vee\circ} T^* \mathbf{P}^\vee \times_Q E)$ -acyclic.

By Lemma 1.2.3.2, the condition (1') means that $f: W \rightarrow Y$ is smooth.

Since p is proper and smooth, by Proposition 1.2.6.2, the condition (1'') is equivalent to the condition that $(h', f \circ p')$ is $p^{\vee\circ} T^* \mathbf{P}^\vee \times_Q E$ -acyclic. Since the transversality is an open condition by Lemma 1.1.3.4, this is equivalent to that $(h', f \circ p')$ is $p^{\vee\circ} T^* \mathbf{P}^\vee$ -acyclic on a neighborhood U of $E_W \subset Q_W$. By Lemma 1.1.6, this is further equivalent to that $(p^\vee \circ h', f \circ p')$ is $T^* \mathbf{P}^\vee$ -acyclic on U . By Lemma 1.1.3.3, this means that $U \rightarrow \mathbf{P}^\vee \times Y$ is smooth.

Since $p: Q \rightarrow \mathbf{P}$ is $p^{\vee\circ} C$ -acyclic outside E by Corollary 1.3.3, $p': Q_W \rightarrow W$ is $(p^\vee \circ h')^\circ C$ -acyclic outside E_W by Lemma 1.2.4.1. Since $f: W \rightarrow Y$ is smooth, the composition $f p': Q_W \rightarrow W \rightarrow Y$ is also $(p^\vee \circ h')^\circ C$ -acyclic outside E_W by Lemma 1.2.4.2. \square

Let $h: W \rightarrow \mathbf{P}$ be an immersion and $f: W \rightarrow Y$ be a smooth morphism. Define sub vector bundles $C_W \subset C_f \subset T^* \mathbf{P} \times_{\mathbf{P}} W$ by $C_W = T_W^* \mathbf{P}$ and C_f as the inverse image of $W \times_Y T^* Y \subset T^* W$ by the surjection $T^* \mathbf{P} \times_{\mathbf{P}} W \rightarrow T^* W$.

Lemma 1.3.5. *Let $C^\vee \subset T^* \mathbf{P}^\vee$ be a closed conical subset and let $C = L^\vee C^\vee \subset T^* \mathbf{P}$ be the inverse Legendre transform.*

1. *The following conditions are equivalent:*

(1) h is C -transversal.

(2) *The intersection of $\mathbf{P}(C) \subset \mathbf{P}(T^* \mathbf{P}) = Q$ and $\mathbf{P}(C_W) \subset \mathbf{P}(T^* \mathbf{P} \times_{\mathbf{P}} W) = Q \times_{\mathbf{P}} W \subset Q$ is empty.*

2. *Assume that $h: W \rightarrow \mathbf{P}$ is C -transversal. Then $Q \times_{\mathbf{P}} W \rightarrow \mathbf{P}^\vee$ is C^\vee -transversal. The complement $Q \times_{\mathbf{P}} W - \mathbf{P}(C \cap C_f)$ equals the largest open subset $U \subset Q \times_{\mathbf{P}} W$ where $(p^\vee: Q \times_{\mathbf{P}} W \rightarrow \mathbf{P}^\vee, f p: Q \times_{\mathbf{P}} W \rightarrow W \rightarrow Y)$ is C^\vee -acyclic. Further $\mathbf{P}(C \cap C_f)$ is a subset of the inverse image of the complement of the largest open subset where f is $h^\circ C$ -transversal.*

3. *Further if $\dim Y = 1$, the closed subset $\mathbf{P}(C \cap C_f) \subset Q \times_{\mathbf{P}} W$ is finite over W .*

Proof. 1. (1) means $C \cap C_W$ is a closed subset of the zero-section and is equivalent to (2).

2. By Proposition 1.2.6, the C -transversality of $h: W \rightarrow \mathbf{P}$ implies the C^\vee -transversality of $Q \times_{\mathbf{P}} W \rightarrow Q$. Since $p^\vee: Q \rightarrow \mathbf{P}^\vee$ is smooth, the first assertion follows.

The largest open subset $U \subset Q \times_{\mathbf{P}} W$ is the same as that where (p^\vee, p) is $C^\vee \times C_f$ -transversal. Hence, it equals the complement of $\mathbf{P}(C^\vee) \cap \mathbf{P}(C_f) = \mathbf{P}(C) \cap \mathbf{P}(C_f) = \mathbf{P}(C \cap C_f)$.

If f is $h^\circ C$ -acyclic, then $(p^\vee, f p)$ is C^\vee -acyclic and the last assertion follows.

3. Since $\dim Y = 1$, the subvector bundle $C_W \subset C_f$ is of codimension 1 and the complement $\mathbf{P}(C_f) - \mathbf{P}(C_W)$ is a vector bundle over W . Since $\mathbf{P}(C \cap C_W)$ is empty by 1, the intersection $\mathbf{P}(C \cap C_f)$ is a closed subset of $\mathbf{P}(C_f - C_W)$. Hence its closed subset $\mathbf{P}(C \cap C_f)$ proper over W is finite over W .

2 Singular support

2.1 Local acyclicity

Let $f: X \rightarrow S$ be a morphism of schemes. Let $x \rightarrow X$ and $t \rightarrow S$ be geometric points and let $S_{(s)}$ be the strict localization at the image $s = f(x) \rightarrow S$ of x . Then a specialization $x \leftarrow t$ is a lifting of $t \rightarrow S$ to $t \rightarrow S_{(s)}$.

Definition 2.1.1. *Let $f: X \rightarrow S$ be a morphism of schemes and \mathcal{F} be a complex of torsion sheaves on X . We say that f is locally acyclic relatively to \mathcal{F} or \mathcal{F} -acyclic for short if for each geometric points $x \rightarrow X$ and $t \rightarrow S$ and each specialization $x \leftarrow t$, the canonical morphism $\mathcal{F}_x \rightarrow R(X_{(x)} \times_{S_{(s)}} t, \mathcal{F})$ is an isomorphism.*

We say that f is universally locally acyclic relatively to \mathcal{F} , if for every morphism $S' \rightarrow S$, the base change of f is locally acyclic relatively to the pull-back of \mathcal{F} .

For geometric points s, t of S and a specialization $t \rightarrow S_{(s)}$, let $i: X_s \rightarrow X \times_S S_{(s)}$ and $j: X_t \rightarrow X \times_S S_{(s)}$ denote the canonical morphisms. Then, the local acyclicity is equivalent to that the canonical morphism $i^* \mathcal{F} \rightarrow i^* Rj_* \mathcal{F}$ is an isomorphism for each s, t and $s \leftarrow t$.

If \mathcal{F} is a constructible sheaf on X , \mathcal{F} is locally constant if and only if 1_X is locally acyclic relatively to \mathcal{F} .

The local acyclicity is preserved by quasi-finite base change $S' \rightarrow S$. Hence for constructible \mathcal{F} , the universal local acyclicity is reduced to smooth base change.

Lemma 2.1.2. 1. *The following conditions are equivalent:*

- (1) *The identity $1_X: X \rightarrow X$ is \mathcal{F} -acyclic.*
- (2) *\mathcal{F} is locally constant.*

2. *The following conditions are equivalent:*

- (1) *The constant morphism $0: X \rightarrow \mathbf{A}^1$ is \mathcal{F} -acyclic.*
- (2) *$\mathcal{F} = 0$.*

Theorem 2.1.3. 1. (local acyclicity of smooth morphism) *Assume that $f: X \rightarrow S$ is smooth and that \mathcal{F} is locally constant killed by an integer invertible on S . Then f is ula relatively to \mathcal{F} .*

2. (generic local acyclicity) *Assume that $f: X \rightarrow S$ is of finite type and that \mathcal{F} is constructible. Then, there exists a dense open subscheme $U \subset S$ such that the base change of f to U is ula relatively to the restriction of \mathcal{F} .*

Corollary 2.1.4. *Assume that $g: Y \rightarrow S$ is smooth, that $f: X \rightarrow Y$ is la relatively to \mathcal{F} and \mathcal{F} is killed by an integer invertible on S . Then, gf is locally acyclic relatively to \mathcal{F} .*

Lemma 2.1.5. *Let $f: X \rightarrow Y$ be a proper morphism of schemes over S and assume that $X \rightarrow S$ is locally acyclic relatively to \mathcal{F} . Then $Y \rightarrow S$ is locally acyclic relatively to $Rf_* \mathcal{F}$.*

Proof. Proper base change theorem.

2.2 Micro support

Definition 2.2.1. Let \mathcal{F} be a constructible complex on X and $C \subset T^*X$ be a closed conical subset. We say that \mathcal{F} is micro supported on C , if for every C -acyclic pair (h, f) of morphisms $h: W \rightarrow X$ and $f: W \rightarrow Y$ of smooth schemes over k , the morphism f is (universally) locally acyclic relatively to $h^*\mathcal{F}$.

If \mathcal{F} is micro supported on $C \subset C'$, then \mathcal{F} is micro supported on C' .

Lemma 2.2.2. 1. The following conditions are equivalent:

- (1) \mathcal{F} is micro supported on \emptyset .
 - (2) $\mathcal{F} = 0$.
2. The following conditions are equivalent:
- (1) \mathcal{F} is micro supported on the 0-section T_X^*X .
 - (2) \mathcal{F} is locally constant.

Proof. 1. Any pair (h, f) is \emptyset -acyclic.

(1) \Rightarrow (2): Since the pair $(1_X, 0)$ of the identity $1_X: X \rightarrow X$ and the constant morphism $0: X \rightarrow \mathbf{A}^1$ is \emptyset -acyclic, the condition (1) implies that the morphism $0: X \rightarrow \mathbf{A}^1$ is \mathcal{F} -acyclic. This means $\mathcal{F} = 0$.

(2) \Rightarrow (1): Any morphism $f: W \rightarrow Y$ is h^*0 -acyclic.

2. By Lemma 1.2.3.2, (h, f) is T_X^*X -acyclic if and only if f is smooth.

(1) \Rightarrow (2): Since the pair $(1_X, 1_X)$ is T_X^*X -acyclic, the condition (1) implies that the identity 1_X is \mathcal{F} -acyclic. By Lemma 2.1.2, this means that \mathcal{F} is locally constant.

(2) \Rightarrow (1): If f is smooth and \mathcal{F} is locally constant, then f is $h^*\mathcal{F}$ -acyclic by the local acyclicity. \square

Lemma 2.2.3. Any constructible \mathcal{F} is micro supported on T^*X .

Proof. Suppose (h, f) is T^*X -acyclic. Then $W \rightarrow X \times Y$ is smooth by Lemma 1.2.3.3. Locally, $W \rightarrow Y$ is the composition of an étale morphism $W \rightarrow X \times \mathbf{A}^n \times Y$ with the projection $X \times \mathbf{A}^n \times Y \rightarrow Y$. By the generic local acyclicity, the projection $X \times \mathbf{A}^n \times Y \rightarrow \mathbf{A}^n \times Y$ is $\text{pr}_1^*\mathcal{F}$ -acyclic. Since $W \rightarrow X \times \mathbf{A}^n \times Y$ is étale and the projection $\mathbf{A}^n \times Y \rightarrow Y$ is smooth, f is $h^*\mathcal{F}$ -acyclic by Corollary 2.1.4. \square

Lemma 2.2.4. Assume that \mathcal{F} is micro supported on C . Let $U \subset X$ be an open subscheme and A be the complement. If the restriction $\mathcal{F}|_U$ is micro supported on C'_U , then \mathcal{F} is micro supported on the union of $C|_A = C \times_X A$ and the closure C' of C'_U . In particular, if $\mathcal{F}|_U = 0$, then \mathcal{F} is micro supported on $C|_A$.

Lemma 2.2.5. 1. Let $X = \bigcup_i U_i$ be an open covering. For a closed conical subset C , the following conditions are equivalent:

- (1) \mathcal{F} is micro supported on C .
- (2) $\mathcal{F}|_{U_i}$ is micro supported on C_{U_i} for every i .

2. Let $\rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow$ be a distinguished triangle and suppose that \mathcal{F}' and \mathcal{F}'' are micro supported on C' and on C'' respectively. Then \mathcal{F} is micro supported on $C = C' \cup C''$.

Lemma 2.2.6. Assume that \mathcal{F} is micro supported on C .

1. If $h: W \rightarrow X$ is C -transversal, then $h^*\mathcal{F}$ is micro supported on $h^\circ C$.
2. The support of \mathcal{F} is a subset of the base B of C .
3. If $f: X \rightarrow Y$ is proper on the base of C , then $Rf_*\mathcal{F}$ is micro supported on $f_\circ C$.

Proof. 1. Suppose $g: V \rightarrow W, f: V \rightarrow Y$ is $h^\circ C$ -transversal. Then, (hg, f) is C -transversal and f is locally acyclic relatively to $(hg)^* \mathcal{F}$.

2. Let $j: U = X - B \rightarrow X$ be the open immersion. Then, since j is C -transversal and $j^\circ C = \emptyset$, the restriction $\mathcal{F}|_U$ is micro supported on \emptyset . Hence we have $\mathcal{F}|_U = 0$ by Lemma 2.2.2.1.

3. Suppose $h: W \rightarrow Y, g: W \rightarrow Z$ is $f_\circ C$ -transversal. Then, $h_X: W \times_Y X \rightarrow X, g \circ f_W: W \times_Y X \rightarrow W \rightarrow Z$ is C -transversal and $h_X^* \mathcal{F}$ is locally acyclic relatively to $g \circ f_W$. Hence $h^* Rf_* \mathcal{F} = Rf_{W*} h_X^* \mathcal{F}$ is locally acyclic relatively to g .

2.3 Singular support

Definition 2.3.1. We say that $C \subset T^*X$ is the singular support of \mathcal{F} if for $C' \subset T^*X$, the inclusion $C \subset C'$ is equivalent to the condition that \mathcal{F} is micro supported on C .

Lemma 2.3.2. Let \mathcal{F} be a constructible sheaf on X .

1. Let $U \subset X$ be an open subscheme. Assume that $C \subset T^*X$ is the singular support of \mathcal{F} . Then, $C|_U$ is the singular support of $\mathcal{F}|_U$.

2. Let (U_i) be an open covering of X and C_i be the singular support of $\mathcal{F}|_{U_i}$. Then, $C = \bigcup_i C_i$ is the singular support of \mathcal{F} .

Proof. 1.

2. By 1, for every i, j , the restrictions $C_i|_{U_i \cap U_j}$ and $C_j|_{U_i \cap U_j}$ are the singular support of $\mathcal{F}|_{U_i \cap U_j}$ and are the same. Hence the union $C = \bigcup_i C_i$ is a closed conical subset of T^*X . By Lemma 2.2.5.1, \mathcal{F} is micro supported on C . We show that C is the smallest. Let C' be a closed conical subset of T^*X on which \mathcal{F} is micro supported. Then, for each i , we have $C_i \subset C'|_{U_i}$. Hence we have $C \subset C'$. \square

Lemma 2.3.3. Let $i: X \rightarrow P$ be a closed immersion. and let \mathcal{F} be a sheaf on X .

1. Assume that \mathcal{F} is micro supported on C . Then, $i_* \mathcal{F}$ is micro supported on $i_\circ C$.

2. Let $s: T^*X \rightarrow T^*P|_X$ be a section of the surjection $T^*P|_X \rightarrow T^*X$. Assume that $i_* \mathcal{F}$ is micro supported on C_P and let $B \subset X$ be the support of \mathcal{F} . Then, we have $T_X^* P|_B \subset C_P$ and \mathcal{F} is micro supported on $C = s^{-1}(C_P|_X)$.

Proof. 1. Lemma 2.2.6.3.

2. Let $h: W \rightarrow X$ be a C -transversal morphism. By replacing P by a smooth scheme over P and X by the inverse image, we may assume that h is an immersion. We extend the immersion $h: W \rightarrow X$ to an immersion $V \rightarrow P$ transversal with the immersion $X \rightarrow P$ such that $T_V^* P$ is a sub vector bundle of the image of the section $T^*X \rightarrow T^*P$. Since $T_V^* P \rightarrow T_W^* X$ is an isomorphism, the C -transversality of $h: W \rightarrow X$ implies the C_P -transversality of $V \rightarrow P$. Since $i_* \mathcal{F}$ is micro supported on C_P , this implies that $V \rightarrow P$ is $i_* \mathcal{F}$ -transversal. Since the intersection $W = V \cap X$ is transversal, this implies that $h: W \rightarrow X$ is \mathcal{F} -transversal. Hence \mathcal{F} is micro supported on C . \square

Proposition 2.3.4. Let $i: X \rightarrow P$ be a closed immersion. Assume that $C_P \subset T^*P$ is the singular support of $i_* \mathcal{F}$.

1. C_P is a subset of $T^*P|_X$.

2. Let $C \subset T^*X$ be its image of C_P by the surjection $T^*P|_X \rightarrow T^*X$. Then, we have $C_P = i_\circ C$.

3. C in 2. is the singular support of \mathcal{F} .

Proof. 1. Since \mathcal{F} is micro supported on $C_P|_X = C_P \cap T^*P|_X \subset C_P$ by Lemma 2.2.4, the inclusion is an equality $C_P|_X = C_P$.

2. Since the assertion is local on X , we may assume that there exists a section $s: T^*X \rightarrow T^*P|_X$. Then, by Lemma 2.3.3.2, \mathcal{F} is micro supported on $C_s = s^{-1}(C_P)$. Further by Lemma 2.3.3.1, $i_*\mathcal{F}$ is micro supported on i_oC_s . Hence we have $C_P \subset i_oC_s$. For any other section s' , this implies $C_{s'} \subset C_s$ and hence $C_{s'} = C_s$. This means that $C_s = C$ and $C_P = i_oC$.

3. Assume that \mathcal{F} is micro supported on C' . Then, since $i_*\mathcal{F}$ is micro supported on i_oC' by Lemma 2.3.3.1, we have $C_P = i_oC \subset i_oC'$. This implies $C \subset C'$. Since \mathcal{F} is micro supported on $C = C_s$, we have $C = SS\mathcal{F}$. \square

Theorem 2.3.5. (Beilinson) $SS\mathcal{F}$ exists.

Proof will be given at the end of next section.

Theorem 2.3.6. (Beilinson) 1. $\dim E \leq \dim \mathbf{P} - 1$.

2. Every irreducible component of E has $\dim \mathbf{P} - 1$.

2.4 Radon transform

We define the naive Radon transform $R\mathcal{F}$ to be $Rp_*^\vee p^*\mathcal{F}$ and the naive inverse Radon transform $R^\vee\mathcal{G}$ to be $Rp_*p^\vee\mathcal{G}$.

Proposition 2.4.1. There exists a distinguished triangle

$$\rightarrow \bigoplus_{q=0}^{n-2} R\Gamma(\mathbf{P}_{\bar{k}}, \mathcal{F})(q)[2q] \rightarrow R^\vee R\mathcal{F} \rightarrow \mathcal{F}(n-1)[2(n-1)] \rightarrow .$$

Proof. By the cartesian diagram

$$\begin{array}{ccccc} \mathbf{P} & \xleftarrow{p} & Q & \xleftarrow{pr_1} & Q \times_{\mathbf{P}^\vee} Q \\ & & p^\vee \downarrow & & \downarrow pr_2 \\ & & \mathbf{P}^\vee & \xleftarrow{p^\vee} & Q \\ & & & & \downarrow p \\ & & & & \mathbf{P} \end{array}$$

and the proper base change theorem, we have a canonical isomorphism

$$R^\vee R\mathcal{F} \rightarrow Rpr_{2*}(pr_1^*\mathcal{F} \otimes R(p \times p)_*\Lambda_{Q \times_{\mathbf{P}^\vee} Q})$$

for $p \times p: Q \times_{\mathbf{P}^\vee} Q \rightarrow \mathbf{P} \times \mathbf{P}$.

We compute $R(p \times p)_*\Lambda_{Q \times_{\mathbf{P}^\vee} Q}$. The closed scheme $Q \times_{\mathbf{P}^\vee} Q \subset \mathbf{P} \times \mathbf{P} \times \mathbf{P}^\vee$ is the \mathbf{P}^{n-1} -bundle Q on the diagonal $\mathbf{P} \subset \mathbf{P} \times \mathbf{P}$. On the complement $\mathbf{P} \times \mathbf{P} - \mathbf{P}$, it is a sub \mathbf{P}^{n-2} -bundle. Hence, we have a distinguished triangle

$$\rightarrow \tau_{\leq 2(n-2)} R\Gamma(\mathbf{P}_{\bar{k}}^\vee, \Lambda) \otimes \Lambda_{\mathbf{P} \times \mathbf{P}} \rightarrow R(p \times p)_*\Lambda_{Q \times_{\mathbf{P}^\vee} Q} \rightarrow \Lambda_{\mathbf{P}}(n-1)[2(n-1)] \rightarrow .$$

Proposition 2.4.2. For \mathcal{G} on \mathbf{P}^\vee and $C^\vee \subset T^*\mathbf{P}^\vee$, we have implications (1) \Rightarrow (2) \Rightarrow (3).

- (1) \mathcal{G} is micro supported on C^\vee .
- (2) p is universally $p^{\vee*}\mathcal{G}$ -acyclic outside $E = \mathbf{P}(C^\vee)$.
- (3) $R^\vee\mathcal{G}$ is micro supported on C^+ .

Proof. (1) \Rightarrow (2): By Corollary 1.3.3, the pair (p^\vee, p) of $p^\vee: Q \rightarrow \mathbf{P}^\vee$ and $p: Q \rightarrow \mathbf{P}$ is C^\vee -acyclic outside $E = \mathbf{P}(C^\vee)$. Hence (1) implies that p is universally $p^{\vee*}\mathcal{G}$ -acyclic outside E .

(2) \Rightarrow (3): Assume that a pair of morphisms $h: W \rightarrow \mathbf{P}, f: W \rightarrow Y$ is C^+ -acyclic. We consider the cartesian diagram

$$\begin{array}{ccccc} \mathbf{P}^\vee & \xleftarrow{p^\vee} & Q & \xleftarrow{h'} & Q_W \\ & & p \downarrow & \square & \downarrow p' \\ & & \mathbf{P} & \xleftarrow{h} & W & \xrightarrow{f} & Y \end{array}$$

and set $\mathcal{G}_{Q_W} = h'^*p^{\vee*}\mathcal{G}$. Since $h^*R^\vee\mathcal{G} = Rp'_*\mathcal{G}_{Q_W}$, it suffices to show that (2) implies that $f: W \rightarrow Y$ is $Rp'_*\mathcal{G}_{Q_W}$ -acyclic. Since p' is proper, by Lemma 2.1.5, further it suffices to show that (2) implies that $fp': Q_W \rightarrow Y$ is \mathcal{G}_{Q_W} -acyclic.

By (2), $p': Q_W \rightarrow W$ is \mathcal{G}_{Q_W} -acyclic outside the inverse image $E_W \subset Q_W$ of E . By Proposition 1.3.4, the C^+ -acyclicity of (h, f) means that $f: W \rightarrow Y$ is smooth and $Q_W \rightarrow \mathbf{P}^\vee \times Y$ is $T^*\mathbf{P}^\vee$ -acyclic on a neighborhood U of $E_W \subset Q_W$. Hence by Corollary 2.1.4, $fp': Q_W \rightarrow Y$ is \mathcal{G}_{Q_W} -acyclic outside E_W . Since \mathcal{G} is micro supported on $T^*\mathbf{P}^\vee$ by Lemma 2.2.3, the restriction $fp'|_U: U \rightarrow Y$ is \mathcal{G}_{Q_W} -acyclic. Thus $fp': Q_W \rightarrow Y$ is \mathcal{G}_{Q_W} -acyclic as required. \square

We prove Theorem 2.3.5 for $X = \mathbf{P}$.

Corollary 2.4.3. Let \mathcal{F} on \mathbf{P} and $\mathcal{G} = R\mathcal{F}$ on \mathbf{P}^\vee be the Radon transform. Let $E \subset Q$ be the smallest closed subset such that $p: Q \rightarrow \mathbf{P}$ is universally $p^{\vee*}\mathcal{G}$ -acyclic on the complement $\mathbf{P} - E$ and define a closed conical subset $C \subset T^*\mathbf{P}$ by $C = p_\circ(p^{\vee\circ}T^*\mathbf{P}^\vee \times_Q E)$. Then the union of C and the restriction of the 0-section $T^*\mathbf{P} \times_{\mathbf{P}} B$ is the singular support of \mathcal{F} .

Proof. By Proposition 2.4.2 (2) \Rightarrow (3), $R^\vee\mathcal{G} = R^\vee R\mathcal{F}$ is micro supported on C^+ . Hence by Proposition 2.4.1, Lemma 2.2.5.2 and Lemma 2.2.2.2, \mathcal{F} is also micro supported on C^+ . By Lemma 2.2.4, \mathcal{F} is micro supported on $C^+|_B$ and we have $C^+|_B = C \cup (T^*\mathbf{P} \times_{\mathbf{P}} B)$.

We show that $C^+|_B$ is the smallest. Suppose \mathcal{F} is micro supported on $C' \subset T^*\mathbf{P}$. Then by Proposition 2.4.2 (1) \Rightarrow (3), $\mathcal{G} = R\mathcal{F}$ is micro supported on $C'^{\vee+}$. Hence by Proposition 2.4.2 (1) \Rightarrow (2), $p: Q \rightarrow \mathbf{P}$ is universally $p^{\vee*}\mathcal{G}$ -acyclic outside $E' = \mathbf{P}(C'^\vee) = \mathbf{P}(C')$. Since E is the smallest, we have $E \subset E'$ and hence $C \subset C'$. By Lemma 2.2.6.2, we have $T^*\mathbf{P} \times_{\mathbf{P}} B \subset C'$. Hence we have $C^+|_B = C \cup (T^*\mathbf{P} \times_{\mathbf{P}} B) \subset C'$. \square

Proof of Theorem 2.3.5. Since the assertion is local by Lemma 2.3.2, it is reduced to the case where X is affine. By Proposition 2.3.4, it is further reduced to the case where X is an affine space. Further by Lemma 2.3.2, we may assume that X is a projective space. In this case, the assertion follows from Corollary 2.4.3. \square

3 Characteristic cycles

3.1 Characteristic cycles

Theorem 3.1.1. *There exists a unique way to attach a \mathbf{Z} -linear combination $CC\mathcal{F} = \sum_a m_a C_a$ of irreducible components $SS\mathcal{F} = \bigcup_a C_a$ for each constructible complex \mathcal{F} of Λ -modules on a smooth scheme X over k , satisfying the following axioms:*

(1) (normalization) For $X = \text{Spec } k$ and $\mathcal{F} = \Lambda$, we have

$$(3.1) \quad CC\Lambda = T_X^*X.$$

(2) (additivity) For distinguished triangle $\rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow$, we have

$$(3.2) \quad CC\mathcal{F} = CC\mathcal{F}' + CC\mathcal{F}''.$$

(3) (pull-back) For $SS\mathcal{F}$ -transversal morphism $h: W \rightarrow X$ of smooth schemes over k , we have

$$(3.3) \quad CCh^*\mathcal{F} = h^!CC\mathcal{F}.$$

(4) (closed immersion) For closed immersion $i: X \rightarrow P$ of smooth schemes over k , we have

$$(3.4) \quad CCi_*\mathcal{F} = i_!CC\mathcal{F}.$$

(5) (Radon transform) For $X = \mathbf{P}^n$ and for the Radon transform, we have

$$(3.5) \quad CCR\mathcal{F} = LCC\mathcal{F}.$$

Corollary 3.1.2. (index formula) *Assume that X is projective and smooth. Then, we have*

$$(3.6) \quad \chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T_X^*X).$$

Proof. By (1), (2) and (3), if \mathcal{F} is locally constant, we have

$$(3.7) \quad CC\mathcal{F} = (-1)^n \text{rank } \mathcal{F} \cdot T_X^*X.$$

By (4), we may assume that $X = \mathbf{P}^n$ and $n \geq 2$. Then, we have

$$(3.8) \quad CCR^\vee R\mathcal{F} = CC\mathcal{F} + (n-1) \cdot \chi(\mathbf{P}_{\bar{k}}^n, \mathcal{F})[T_{\mathbf{P}^n}^*\mathbf{P}^n].$$

By (5) and (2), we have $CC(R^\vee R\mathcal{F}) - CC\mathcal{F} = L^\vee LCC\mathcal{F} - CC\mathcal{F}$. Hence, we have $(n-1)\chi(\mathbf{P}_{\bar{k}}^n, \mathcal{F}) = (n-1)(CC\mathcal{F}, T_{\mathbf{P}^n}^*\mathbf{P}^n)$ and (3.6).

We will deduce Theorem 3.1.1 from the following variant.

Theorem 3.1.3. *There exists a unique way to attach a \mathbf{Q} -linear combination $CC\mathcal{F} = \sum_a m_a C_a$ of irreducible components $SS\mathcal{F} = \bigcup_a C_a$ for each constructible complex \mathcal{F} of Λ -modules on smooth smooth scheme X over k , satisfying the following axioms:*

(1) (Milnor formula) *Let $f: X \rightarrow Y$ be a proper morphism over k to a smooth curve Y over k and $x \in X$ be a closed point such that f is $SS\mathcal{F}$ -transversal on the complement of x . Then, the coefficient of the fiber T_y^*Y at $y = f(x)$ in $f_*CC\mathcal{F}$ is minus the Artin conductor $-a_x Rf_*\mathcal{F}$.*

(3) *For étale morphism $h: W \rightarrow X$ of smooth schemes over k , we have (3.3).*

(4) *For closed immersion $i: X \rightarrow P$ of smooth schemes over k , we have (3.4).*

Outline and key points of proof of theorems.

Proof of Theorem 3.1.3. We show the uniqueness. By (3), we may assume X is affine. By (4), we may assume $X = \mathbf{A}^n$. By (3), we may assume X is projective. We may take a Lefschetz pencil. Since it suffices to determine the coefficient m_a for each C_a , we may assume that $f: W \rightarrow L$ is C_b -transversal for $C_b \neq C_a$ and C_a -transversal except at x and is not C_a -transversal at x . Then, by (1), we have

$$(3.9) \quad m_a(C_a, df)_x = -a_x$$

and the uniqueness follows.

To show the existence, first we show that the coefficient m_a determined by (3.9) is well-defined. This follows from the (semi-)continuity of Swan conductor and the formalism of vanishing cycles over general base. Then $CC\mathcal{F}$ characterized by (3.9) satisfies the conditions (3) and (4) by standard properties of usual vanishing cycles.

Proof of the uniqueness in Theorem 3.1.1. By Corollary 3.1.2, we have the index formula (3.6) for projective and smooth X . By comparing the index formula (3.6) for proper smooth curve X and the Grothendieck-Ogg-Shafarevich formula and using (3) for étale morphism of smooth curves and (3.7), we obtain (1) in Theorem 3.1.3 for $f = 1_X: X \rightarrow X$.

Similarly as in the proof of Theorem 3.1.3, it is reduced to the case where X is projective and smooth. Then by taking a Lefschetz pencil, it follows from (5), (3) and (1) in Theorem 3.1.3.

Proof of the existence in Theorem 3.1.1. We deduce the existence from Theorem 3.1.3. We show that $CC\mathcal{F}$ satisfying the conditions in Theorem 3.1.3 also satisfies those in Theorem 3.1.1. The conditions (1) and (2) in Theorem 3.1.1 follow from (1) in Theorem 3.1.3. The condition (4) in Theorem 3.1.1 is the same as (4) in Theorem 3.1.3. Hence it remains to show the conditions (3), (5) and the integrality.

The condition (3) for smooth morphism is a consequence of the Thom-Sebastiani formula. The integrality in the case $p \neq 2$ or non-exceptional case in $p = 2$ follows from (1) in Theorem 3.1.3. In the exceptional case, it is reduced to the non-exceptional case using the condition (3) for $X \times \mathbf{A}^1 \rightarrow X$.

To show (3) in the case where h is an immersion, we first consider the case where X is an projective space \mathbf{P}^n .

Lemma 3.1.4. *Let $h: W \rightarrow P = \mathbf{P}^n$ be an immersion and*

$$\begin{array}{ccc} W & \xleftarrow{p_W} & W \times_P Q & \xrightarrow{p_W^\vee} & P^\vee \\ h \downarrow & & \downarrow & & \\ P & \xleftarrow{p} & Q & & \end{array}$$

be the cartesian diagram. Let \mathcal{G} be a constructible complex on P^\vee micro supported on C^\vee and assume that h is properly C -transversal for $C = L^\vee C^\vee$. Then, we have

$$\mathbf{P}(CCRp_{W*}p_W^{\vee*}\mathcal{G}) = \mathbf{P}(p_{W!}p_W^{\vee!}CC\mathcal{G}).$$

Proof. Since the characteristic cycle is characterized by the Milnor formula, it suffices to show that $p_{W!}p_W^{\vee!}CC\mathcal{G}$ satisfies the Milnor formula for $Rp_{W*}p_W^{\vee*}\mathcal{G}$ and for smooth morphisms $f: W \rightarrow Y$ to a curve defined locally on W . Since h is C -transversal, $p_W^{\vee}: Q \times_{\mathbf{P}} W \rightarrow \mathbf{P}^{\vee}$ is C^{\vee} -transversal by Lemma 1.3.5.2 and $p_W^*\mathcal{G}$ is micro supported on $p_W^{\vee}C^{\vee}$. Since $p_W^{\vee}: Q \times_{\mathbf{P}} W \rightarrow \mathbf{P}^{\vee}$ is smooth outside $\mathbf{P}(C_W)$, we have $CCp_W^{\vee*}\mathcal{G} = p_W^{\vee\circ}CC\mathcal{G}$ outside $\mathbf{P}(C_W)$ as (3) is already proved for smooth morphisms.

Assume that f is smooth and has only isolated characteristic point. Then, by Lemma 1.3.5.2, the composition fp_W is $p^{\vee}C$ -transversal outside the inverse images of the characteristic points. Further it is $p^{\vee}C$ -transversal outside of finitely many closed points in the inverse images by Lemma 1.3.5.3 and these points are not contained in $\mathbf{P}(C_W)$ by Lemma 1.3.5.1. Hence the assertion follows.

Lemma 3.1.4 implies also $\mathbf{P}(CC h^*\mathcal{F}) = \mathbf{P}(h^!CC\mathcal{F})$. Since the coefficient of the 0-section is determined by the generic rank as in (3.7), we deduce (3) in the case $X = \mathbf{P}$. In the general case, since the assertion is local, we may assume that there exists an open subscheme $U \subset \mathbf{P}$ and a cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{h} & X \\ j \downarrow & \square & \downarrow i \\ V & \xrightarrow{g} & U \subset P \end{array}$$

where $i: X \rightarrow U$ and $g: V \rightarrow U$ are closed immersions of smooth subschemes meeting transversely. Then, since h is properly C -transversal, g is properly $i_{\circ}C$ -transversal. Hence the case where $X = \mathbf{P}$ implies $CCg^*i_*\mathcal{F} = g^!CCi_*\mathcal{F} = g^!i_!CC\mathcal{F}$. This implies $j_!CC h^*\mathcal{F} = CCj_*h^*\mathcal{F} = j_!h^!CC\mathcal{F}$ and (3.3).

We show (5). The case $W = \mathbf{P}$ in Lemma 3.1.4 means the projectivization

$$(3.10) \quad \mathbf{P}(CCR\mathcal{F}) = \mathbf{P}(LCC\mathcal{F})$$

of (5). Hence it remains to show that the coefficients of the 0-section in $CCR\mathcal{F} = LCC\mathcal{F}$ are the same. Similarly as in the proof of Corollary 3.1.2, this is equivalent to the index formula (3.6) for $X = \mathbf{P}^n$. To prove this, we introduce the characteristic class.

3.2 Characteristic class

We identify the Chow group of the projective completion $\mathbf{P}(T^*X \oplus \mathbf{A}_X^1)$ by the canonical isomorphism

$$(3.11) \quad \mathrm{CH}_{\bullet}(X) = \bigoplus_{i=0}^n \mathrm{CH}_i(X) \rightarrow \mathrm{CH}_n(\mathbf{P}(T^*X \oplus \mathbf{A}_X^1)).$$

For a constructible complex \mathcal{F} on X with the characteristic cycle $CC\mathcal{F} = \sum_a m_a C_a$, we define the characteristic class

$$(3.12) \quad cc_X(\mathcal{F}) \in \mathrm{CH}_{\bullet}(X)$$

to be the class of $\sum_a m_a \bar{C}_a \in \mathrm{CH}_n(\mathbf{P}(T^*X \oplus \mathbf{A}_X^1))$.

Let $K(X, \Lambda)$ denote the Grothendieck group of the category of constructible complexes of Λ -modules on X . By the additivity, we have a morphism

$$(3.13) \quad cc_X: K(X, \Lambda) \rightarrow \mathrm{CH}_{\bullet}(X)$$

sending the class \mathcal{F} to $cc_X \mathcal{F}$. In characteristic 0, we recover the MacPherson Chern class.

The pull-back by the immersion $\mathbf{P}(T^*X) \rightarrow \mathbf{P}(T^*X \oplus \mathbf{A}_X^1)$ and the push-forward by $\mathbf{P}(T^*X \oplus \mathbf{A}_X^1) \rightarrow X$ induce an isomorphism

$$\mathrm{CH}_n(\mathbf{P}(T^*X \oplus \mathbf{A}_X^1)) \rightarrow \mathrm{CH}_{n-1}(\mathbf{P}(T^*X)) \oplus \mathrm{CH}_n(X).$$

For $A = \sum_a m_a C_a$, the images of $\bar{A} = \sum_a m_a \bar{C}_a$ is the pair of $\mathbf{P}(A) = \sum_a m_a \mathbf{P}(C_a)$ and the coefficient of the 0-section.

End of Proof of Theorem 3.1.3. Under (3.10), the equality (3.5) is equivalent to the condition that the diagram

$$(3.14) \quad \begin{array}{ccc} K(\mathbf{P}^n, \Lambda) & \xrightarrow{cc_{\mathbf{P}^n}} & \mathrm{CH}_\bullet(\mathbf{P}^n) \\ R \downarrow & & \downarrow L \\ K(\mathbf{P}^{n\vee}, \Lambda) & \xrightarrow{cc_{\mathbf{P}^{n\vee}}} & \mathrm{CH}_\bullet(\mathbf{P}^{n\vee}) \end{array}$$

gets commutative after composed with the projection $\mathrm{CH}_\bullet(\mathbf{P}^{n\vee}) \rightarrow \mathrm{CH}_n(\mathbf{P}^{n\vee})$ and also to the commutativity of the diagram (3.14) itself.

We prove the commutativity of (3.14) (CD n) and the index formula (3.6) for \mathbf{P}^n (IF n) by a simultaneous induction on n along the diagram; (IF $n-1$) \Rightarrow (CD n) \Rightarrow (IF n). For $n \leq 1$, the commutativity of (3.14) is obvious. For $n=0$, the index formula follows from (3.7). For $n=1$, this is nothing but the Grothendieck-Ogg-Shafarevich formula.

We prove (IF $n-1$) \Rightarrow (CD n). Let $i: H \rightarrow \mathbf{P}^n$ be the immersion of a hyperplane. Then, the right square in

$$(3.15) \quad \begin{array}{ccccc} K(\mathbf{P}^n, \Lambda) & \xrightarrow{cc_{\mathbf{P}^n}} & \mathrm{CH}_\bullet(\mathbf{P}^n) & \xrightarrow{i^!} & \mathrm{CH}_{n-1}(H) \\ R \downarrow & & \downarrow L & & \downarrow \mathrm{deg} \\ K(\mathbf{P}^{n\vee}, \Lambda) & \xrightarrow{cc_{\mathbf{P}^{n\vee}}} & \mathrm{CH}_\bullet(\mathbf{P}^{n\vee}) & \longrightarrow & \mathrm{CH}_n(\mathbf{P}^{n\vee}) = \mathbf{Z} \end{array}$$

is commutative. Hence it suffices to show that the long rectangle is commutative. For \mathcal{F} on \mathbf{P}^n , the generic rank of $R\mathcal{F}$ equals the Euler number $\chi(H_{\bar{k}}, \mathcal{F})$ for a generic H . Hence the composition via lower left sends the class of \mathcal{F} to $\chi(H_{\bar{k}}, \mathcal{F})$. By (3) for the immersion $i: H \rightarrow \mathbf{P}^n$ and (IF $n-1$), we have $\chi(H_{\bar{k}}, \mathcal{F}) = (CCi^* \mathcal{F}, T_H^* H) = \mathrm{deg} i^! cc_{\mathbf{P}^n} \mathcal{F}$ and the long rectangle is commutative.

We prove (CD n) \Rightarrow (IF n). Let $\chi: K(\mathbf{P}^n, \Lambda) \rightarrow \mathbf{Z}$ be the morphism sending the class of \mathcal{F} to the Euler number $\chi(\mathbf{P}_{\bar{k}}^n, \mathcal{F})$. We show that there is a commutative diagram

$$(3.16) \quad \begin{array}{ccc} K(\mathbf{P}^n, \Lambda) & \xrightarrow{cc_{\mathbf{P}^n}} & \mathrm{CH}_\bullet(\mathbf{P}^n) \\ & \searrow \chi & \downarrow \\ & & \mathbf{Z} \end{array}$$

Since $cc_{\mathbf{P}^n}$ is a surjection, it suffices to show that $cc_{\mathbf{P}^n} \mathcal{F} = 0$ implies $\chi(\mathbf{P}_{\bar{k}}^n, \mathcal{F}) = 0$. By (3.8), (CD n) and the assumption $cc_{\mathbf{P}^n} \mathcal{F} = 0$ imply $\chi(\mathbf{P}_{\bar{k}}^n, \mathcal{F}) = 0$ for $n-1 \neq 0$. Thus, there exists a unique morphism $\mathrm{CH}_\bullet(\mathbf{P}^n) \rightarrow \mathbf{Z}$ making the diagram (3.16) commutative. We show that the morphism $\mathrm{CH}_\bullet(\mathbf{P}^n) \rightarrow \mathbf{Z}$ equals the degree mapping. This is reduced to the case where $\mathcal{F} = \Lambda_{\mathbf{P}^i}$, $i = 0, \dots, n$ generating $\mathrm{CH}_\bullet(\mathbf{P}^n) = \mathbf{Z}^{n+1}$.