Macroscopic index theory and geometric quantum Hall effect

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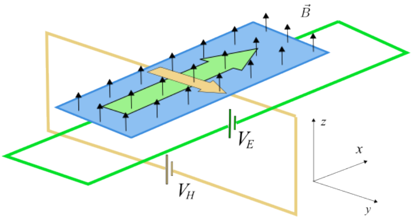
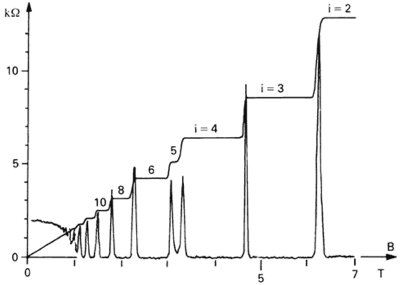
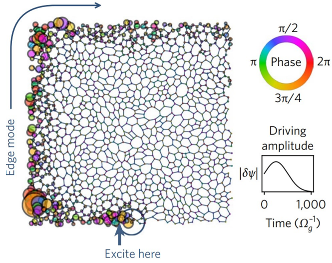
[1] Joint with M. Ludewig, Y. Kubota, J. Xia.
CMP'21,'22; arXiv:2308.02819,2401.07449

- 1980 **IQHE**: Classical/macroscopic $\mathbb{R}$-valued Hall conductance measurement, whose range is $\frac{e^2}{h}\mathbb{Z}$, with $\sim 10^{-10}$ error.
- Most theory $\sim$ perfect Euclidean plane sample.
- Experimental samples are bumpy and finite sized.
- Quantized Hall conductance persists, ignores fine geometric details. This is why it is useful, e.g. redefinition of Kilogram.
- **Summary:** Macroscopic index theory explains how macroscopic quantization emerges from microscopic QM.







N. Mitchell et al, Nature Phys. (2018)

Are 2D Interfaces Really Flat?

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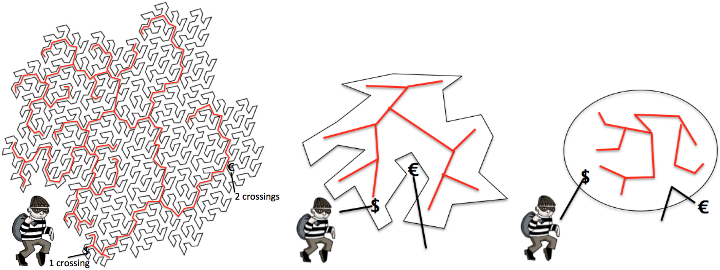
to examine the cross-sectional structure of various 2D interfaces on the length scale of an array of electronic devices (~12.5 μm in total). Contrary to the conventional assumption that 2D interfaces are always flat, we find that these interfaces can be quite intricate and complex. Correlating the interface deformation with the corresponding device performance, we

One calculation to rule them all

- "Topological invariance": deform complicated calculations to special simple ones.
- Non-trivial step is to *justify* deformation invariance of relevant calculation.



- For IQHE, geometric invariances and basic calculation seem to be missing.

- I. Differential geometry of Landau quantization
    - Riemannian, spin, gauge.
- II. Functional analysis and traces of commutators
    - Cancelling infinities in QM.
- III. Coarse geometry and index theory
    - Macroscopic quantization of Hall conductance.

Focus is on Geometry and Analysis, not on Topology and algebra.

No specialized condensed matter ideas, models, or background assumed.

- Usually, Schrödinger operator on Riemannian M is

$$H = -\nabla^2 + V,$$

  where V is real-valued potential.
- H, and its spectral projections P, will commute with complex conjugation.
- Such "real" Hamiltonians cannot describe QHE.
- Must replace ∇ with $\nabla - iA$, where A is connection 1-form for $\mathrm{U}(1)$ line bundle.
- This is a fundamental feature of quantum mechanics!

Quantum phenomena do not occur in a Hilbert space, they occur in a laboratory. — A. Peres.

- A natural Hilbert space to represent position operators on M is $L^2(M)$.
- Is quantum state a L^2 "wave*function*" $\psi : M \to \mathbb{C}$?
- No! Even in *spinless* case, ψ is not a $\mathbb{C}$-scalar field, but a section of a $\mathrm{U}(1)$ line bundle[2] $\mathcal{L} \to M$.
- Need *connection*, to compare copies of $\mathbb{C}$ living at different points of M.
- On contractible M with no magnetic field, this may be forgotten without consequence. (But recall AB-effect!)

[2] Consider what Galilean invariant "momentum operator" means. . .

- Generally, we have a Hermitian vector bundle $\mathcal{V} \to M$.
- A (local) "gauge" is an orthonormal frame, making the bundle look (locally) like $M \times \mathbb{C}^N$. Gauge group unitarily represented in $\mathbb{C}^N$.
- Relative to a gauge choice, a section of $\mathcal{V}$ is described as a $\mathbb{C}^N$-valued function.
- To differentiate ψ "gauge-covariantly", we use a "connection", or "parallel transport", defined by properties

$$\nabla_{u+f\cdot v}\psi = \nabla_u\psi + f \cdot \nabla_v\psi,$$
$$\nabla_v(h \cdot \psi) = v(h) \cdot \psi + h \cdot \nabla_v\psi,$$

u, v vector fields, f, h smooth functions, ψ section.

For simplicity, work with $\mathrm{U}(1)$ line bundles, $\mathcal{L} \to M$.

- After choosing a (local) gauge and coordinates, ψ is a $\mathbb{C}$-valued function of $x^1, \ldots, x^d$.
- Correspondingly, ∇ becomes a covariant derivative,

  $$\nabla_j \psi = (\partial_j - i\mathcal{A}_j)\psi,$$

  where $\mathcal{A} = \sum_j \mathcal{A}_j \, dx^j$ is a $\mathbb{R} = -i\mathfrak{u}(1)$-valued 1-form, called the connection 1-form.
- "Constant function" with respect to gauge choice generally differs from "∇-constant". Mismatch is encoded by $\mathcal{A}$.

- Connection 1-form $\mathcal{A}$ is gauge-dependent description of ∇.
- Rotate gauge choice by applying

$$g = e^{i\Lambda} : M \to \mathrm{U}(1),$$
$$\mathcal{A} \rightsquigarrow \mathcal{A} + d\Lambda.$$

- If $\mathcal{A}$ happens to be exact, $\mathcal{A} = d\Lambda$, we can apply $g = e^{-i\Lambda}$ to find a "better" gauge in which the connection looks "trivial",

$$\nabla_j = \partial_j.$$

- Since $d^2 = 0$, an obstruction to $\mathcal{A}$ being exact is

$$d\mathcal{A} = \sum_{i,j} (\partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i)\, dx^i \wedge dx^j \equiv \mathcal{F}^{\nabla}.$$

- $\mathcal{F}^\nabla$ is called the *curvature* 2-form of ∇. It is *gauge-independent*.
- *Minimal coupling* of charge-q particle to E&M Faraday 2-form $\mathcal{F}$ means: ∇ is locally

$$\nabla_j = \partial_i - i\frac{q}{\hbar}\mathcal{A}_j, \qquad d\mathcal{A} = \mathcal{F}^\nabla = \mathcal{F}.$$

- "Free Hamiltonian" is

$$H_{\text{free}} = \frac{\hbar^2}{2m}\nabla^*\nabla.$$

- On $M = \mathbb{R}^2$ and uniformly magnetic field $\mathcal{F} = b \cdot dx \wedge dy$, this is called the *Landau Hamiltonian*. But it is defined for much more realistic geometries.

- Lab is a Riemannian 3-manifold $(\widetilde{M}, g)$, say Euclidean $\mathbb{R}^3$.
- Charge q is confined to 2D orientable submanifold $\iota : M \hookrightarrow \widetilde{M}$ (the sample).
- Magnetic field $\widetilde{\mathcal{F}} \in \Omega^2(\widetilde{M})$ is set up.
- Since velocity vector $v \in TM$, only restricted field

$$\mathcal{F} := \iota^* \widetilde{\mathcal{F}}$$

  affects tangential motion (Lorentz force).
- Pick an orientation on M, thus vol_M. Then

$$\mathcal{F} = B \cdot \mathrm{vol}_M$$

  for *magnetic field strength* $B \in C^\infty(M)$.

- If M is contractible, $B \cdot \mathrm{vol}_M$ is the curvature of some connection ∇, unique up to gauge equivalence. Then we have the *Landau operator*

$$H_B = \nabla^* \nabla.$$

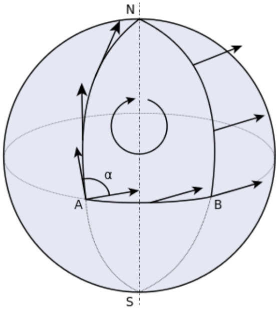
- Simplest example,

$$M = \mathbb{R}^2, \qquad B \equiv b \in \mathbb{R} \setminus \{0\}$$

has spectrum (Landau '30),

$$\mathrm{Spec}(H_b) = (2\mathbb{N} + 1)|b|.$$

- I will explain the differential geometry behind "Landau quantization".



On a Riemannian manifold, Levi–Civita connection parallel transports tangent vectors.

Spin connection ∇^{Spin} on spinor fields (fermions) gives Dirac operator,

$$D = \sum_{i=1}^{d} c(e_i) \cdot \nabla^{\mathrm{Spin}}_{e_i} \qquad (\mathrm{physics:}\ -i\gamma^\mu \nabla_\mu, \quad \gamma^\mu = \mathbf{e}^\mu_i e^i)$$

where $\{e_i\}_{i=1,\dots,d}$ is orthonormal frame and $\{c(e_i), c(e_j)\} = -2\delta_{ij}$.

- On 2D spin Riemannian manifold M, spinor bundle is,

$$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$$

where $\mathcal{S}^\pm \to M$ is a line bundle with connection $\nabla^{\mathrm{Spin},\pm}$.

- $\pm$ corresponds to

$$\begin{aligned} \mathrm{U}(1) \cong \mathrm{Spin}(2) &\xrightarrow{2:1} \mathrm{SO}(2) \\ e^{i\theta} &\mapsto e^{\pm i\theta/2}. \end{aligned}$$

- Curvature of $\nabla^{\mathrm{Spin},\pm}$ is

$$\mp \frac{R}{4} \mathrm{vol}_M$$

where $R \in C^\infty(M)$ is Riemannian scalar curvature function.

Landau quantization and Dirac index

- Dirac operator is an odd operator on $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$,

$$D_{\mathcal{S}} = \begin{pmatrix} 0 & D_{-+} \\ D_{+-} & 0 \end{pmatrix}.$$

- Couple charged fermion to E&M, twist by $\mathcal{L}_B \to M$ with curvature $B \cdot \mathrm{vol}_M$.
  Twisted spin connection on $\mathcal{S}^\pm \otimes \mathcal{L}_B$ has curvature

$$\mathcal{F}^{\nabla_B^{\mathrm{Spin},\pm}} = (B \mp \tfrac{R}{4}) \cdot \mathrm{vol}_M.$$

- Two possible Laplacians on $\mathcal{S} \otimes \mathcal{L}_B$.
  - Dirac^2.
  - ∇_B^{Spin} Laplacian.

- The Laplacian for ∇_B^{Spin},

$$(\nabla_B^{\mathrm{Spin}})^*\nabla_B^{\mathrm{Spin}} \cong \begin{pmatrix} H_{B-R/4} & 0 \\ 0 & H_{B+R/4}, \end{pmatrix}$$

  is direct sum of two Landau operators.
- Related to Dirac^2 by Schrödinger–Lichnerowicz identity:

$$D^2_{\mathcal{S}\otimes\mathcal{L}_B} = (\nabla_B^{\mathrm{Spin}})^*\nabla_B^{\mathrm{Spin}} + \frac{R}{4} + \begin{pmatrix} -B & 0 \\ 0 & B \end{pmatrix}.$$

  Equivalently,

$$D^2_{\mathcal{S}\otimes\mathcal{L}_{B+\frac{R}{4}}} = \begin{pmatrix} H_B - B & 0 \\ 0 & H_{B+\frac{R}{2}} + \frac{R}{2} + B \end{pmatrix}.$$

- For large uniform magnetic field strength, $B \equiv b > -\frac{R}{2}$,

$$D^2_{\mathcal{S}\otimes\mathcal{L}_{b+\frac{R}{4}}} = \begin{pmatrix} H_b - b & 0 \\ 0 & H_{b+\frac{R}{2}} + \underbrace{\frac{R}{2} + b}_{>0} \end{pmatrix} \geq 0.$$

  Only $H_b - b$ can have zero eigenvalues.

- "Lowest Landau level" is Dirac kernel,

$$\begin{aligned} P_{\mathrm{LLL}} \equiv \ker(H_b - b) &= \ker D_{+-}, \\ 0 &= \ker D_{-+}, \qquad b > 0. \end{aligned}$$

  Breaks SUSY…but by "how much"?

- Original Landau quantization is $R \equiv 0$ case,

$$H_b \ \text{Spectrum}: \qquad \bullet_{b} \qquad \bullet_{3b} \qquad \bullet_{5b} \qquad \bullet_{7b} \qquad \cdots$$

- For almost constant $B \approx b$, small R and potentials, get "Landau band".

$$H_B \ \text{Spectrum}$$

- We need completeness for essential self-adjointness. Very different story when M has a boundary!
- How does Landau quantization of spectrum relate to quantized Hall conductance?!

Landau quantization and Dirac index

- LLL is infinitely-degenerate when M is non-compact.
- Its "size" is the K-theoretic "Dirac index",

$$[P_{\mathrm{LLL}}] = \mathrm{Index}(D) \in K_0(C^*(M)),$$

  which is a *coarse geometric* invariant of M.
- Higher-dimensional Landau quantization exists (L+T'24).
- K-theory index is an *abstract* abelian group element.
- Numerical "macroscopic trace" homomorphism $K_0(C^*(M)) \to \mathbb{R}$, which is "Kubo formula" for Hall conductance from physics.

Values are quantized!

- By abstract invariances (later), it suffices to deform to basic Euclidean plane case, $R \equiv 0$, and $B = b$.
- In complex coordinates,

$$D_{\mathcal{S}\otimes\mathcal{L}_b} = -2i \begin{pmatrix} 0 & \partial - \frac{b}{4}\bar{z} \\ \bar{\partial} + \frac{b}{4}z & 0 \end{pmatrix},$$

and lowest Landau level is

$$\begin{aligned} \ker(\bar{\partial} + \frac{b}{4}z) &= \overline{\mathrm{span}_{\mathbb{C}}}\{z^m e^{-b|z|^2/4} \; : \; m \geq 0\} \\ &= \mathrm{Bargmann} - \mathrm{Fock\ space}. \end{aligned}$$

We will calculate the Hall conductance of this space.

End of Lecture 1

Last time/today

- Last time, geometric mechanism for Landau level phenomenon was given, and LLL was identified with twisted Dirac kernel.
- They have abstract K-theoretic indices, which I will turn into quantized numbers today.
- These are "renormalized" numbers hidden in intricate traces-of-commutators (e.g. Kubo formula).
- Explicit calculation for Euclidean Landau levels.
- New technique of Carey–Pincus–Helton–Howe theory will be introduced.

$$[\mathrm{Position}, \mathrm{Momentum}] = [X, P] = i\hbar.$$

- Can you measure $\hbar$ this way?

$$\langle\psi| - i[X, P]|\psi\rangle = \hbar \ \ ??$$

- **Textbook QM**: Density matrix ρ, bounded observable $A = A^*$,

$$\langle A\rangle_\rho = \mathrm{Tr}(\rho A) \in \mathbb{R}.$$

  Need ρ to be "trace-class" here.

- Local observables in Dirac sea state?

- For operators on $\mathbb{C}^N$,

$$\mathrm{Tr} = \begin{cases} \sum \text{e-values} \\ \sum \text{diagonal} \end{cases}$$

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \Rightarrow \mathrm{Tr}[A, B] = 0.$$

- What about

$$\mathrm{Tr}[X, P] = \mathrm{Tr}(i\hbar) = \begin{cases} 0? \\ \infty? \end{cases}$$

- Correct answer is

$$\mathrm{Tr}[X_{\mathrm{reg}}, P_{\mathrm{reg}}] = \frac{i}{2\pi}.$$

- On a Hilbert space $\mathscr{H}$, a bounded operator A is *trace class* if

$$\sum_k \langle e_k || A || e_k \rangle < \infty, \qquad \{e_k\}_{k\in\mathbb{N}} \text{ some O.N.B.}$$

  where $|A| = \sqrt{A^* A}$. Then

$$\mathrm{Tr}(A) := \sum_k \langle e_k | A | e_k \rangle \in \mathbb{C}, \qquad \{e_k\}_{k\in\mathbb{N}} \text{ any O.N.B.}$$

- Set of trace class ops. is denoted $\mathcal{L}_1(\mathscr{H})$.
  Finite-rank $\subset$ Trace class $\subset$ Compact operators.

- A deep result:

$$\mathrm{Tr}(A) = \sum \text{e-values}(A)$$

Theorem: If AB and BA are trace class, then

$$\mathrm{Tr}[A, B] = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = 0.$$

- Interesting situation is "almost commuting pair",

$$[A, B] \in \mathcal{L}_1 \quad \text{but} \quad \mathrm{Tr}[A, B] \neq 0.$$

In such a situation,

$$0 \neq \mathrm{Tr}[A, B] = \text{``}\mathrm{Tr}(AB) - \mathrm{Tr}(BA)\text{''} = \text{``}\infty - \infty\text{''}.$$

Consider $f : \mathbb{R} \to [-\frac{1}{2}, \frac{1}{2}]$, $t \mapsto \frac{t}{2\sqrt{1+t^2}}$. The regularized

$$X_{\mathrm{reg}} = f(X), \qquad P_{\mathrm{reg}} = f(P),$$

are almost-commuting, and it can be calculated that

$$\mathrm{Tr}[X_{\mathrm{reg}}, P_{\mathrm{reg}}] = \frac{i}{2\pi}.$$

After the lectures, think about *why* this is "robust"!

$X_{\mathrm{reg}}, P_{\mathrm{reg}}$ is called an almost-commuting self-adjoint pair.

Carey–Pincus–Helton–Howe '70s

- Suppose $[A, B] \in \mathcal{L}_1$. Then polynomials $p_i(A, B)$ also almost-commute, e.g.,

$$[A, AB] = A[A, B], \qquad [A, BA] = [A, B]A \qquad \in \mathcal{L}_1.$$

- No ordering ambiguity in $p(A, B)$ under the trace,

$$\mathrm{Tr}[A, AB] = \mathrm{Tr}(A[A, B]) \overset{\text{Lidskii}}{=} \mathrm{Tr}([A, B]A) = \mathrm{Tr}[A, BA].$$

- Conclude that $\exists$ bilinear, antisymmetric map

$$(p_1, p_2) \mapsto \mathrm{Tr}[p_1(A, B), p_2(A, B)].$$

Theorem

Let A, B be self-adjoint a.c. pair. Then there exists principal function $G_{A,B}$, such that

- $$\int_{\mathbb{C}} \{p_1, p_2\}\, G_{A,B} = \mathrm{Tr}[p_1(A,B), p_2(A,B)]$$

  holds for any polynomials $p_i \equiv p_i(x,y) \equiv p_i(x+iy)$.
- *Whenever λ is not an essential spectral value of $A+iB$,*

  $$2\pi i \cdot G_{A,B}(\lambda) = -\mathrm{Index}(A+iB-\lambda) \in \mathbb{Z}.$$

Cf. "quantization" of Poisson brackets,

- Each a.c. pair A, B gives an "exact quantization" of phase space weighted by principal function $G_{A,B}$.
- $G_{A,B}$ is basically unknown on essential spectrum, so trace formula had little practical utility.
- New interesting examples come from Kubo formula for Landau levels!

Reminder: A bounded operator S is Fredholm if

$$\mathrm{Index}(S) := \dim \ker S - \dim \operatorname{coker} S < \infty.$$

Away from essential spectrum, $S - \lambda$ is Fredholm.

- Kubo formula for Hall conductance has the form

$$\sigma_{\text{Hall}} = i\mathrm{Tr}[A, B].$$

- Furthermore, $A + iB$ has essential spectrum being the unit square $\square$.
- Away from unit square,

$$\mathrm{Index}(A + iB - \lambda) = \begin{cases} -1 & \lambda \text{ inside square}, \\ 0, & \lambda \text{ outside square}. \end{cases}$$

$$\Rightarrow G_{A,B} = \frac{1}{2\pi i}\chi_{\blacksquare}$$

- Thus $\sigma_{\text{Hall}} = i\mathrm{Tr}[A, B]$ is quantized.

Write $P : L^2(\mathbb{C}) \to \mathscr{H}_{LLL}$ for the orthogonal projection onto LLL/Fock space,

$$\mathscr{H}_{\mathrm{LLL}} := \overline{\mathrm{span}}_{\mathbb{C}}\{z^m e^{-|z|^2/2} \ : \ m \in \mathbb{N}\} \subset L^2(\mathbb{C}).$$

$f \in L^\infty(\mathbb{C})$ acts on $L^2(\mathbb{C})$ by multiplication operators. Compress it to the Landau level,

$$P_f := P\, f\, P,$$

to get *Toeplitz operator* on $\mathscr{H}_{\mathrm{LLL}}$ with symbol f.

- Define *switch functions*,

$$\eta : \mathbb{R} \to [0,1], \qquad \eta(t) = \begin{cases} 0, & t \leq -a, \\ 1, & t \geq a, \end{cases}$$

for some interpolation interval $[-a, a]$. Also

$$\begin{array}{ll} & \mathbb{C} \to [0,1] \\ x\text{-switch} & (x+iy) \mapsto \eta_1(x) \\ y\text{-switch} & (x+iy) \mapsto \eta_2(y) \end{array}$$

- These are "half-plane position observable".

Chiral asymmetry of Landau levels

- Unlike $L^2(\mathbb{C})$, subspace $\mathcal{H}_{\mathrm{LLL}}$ prefers anticlockwise.
- Let f_1, f_2 by x-switch and y-switch respectively. When projected to $\mathcal{H}_{\mathrm{LLL}}$,

$$\underbrace{P_{f_1} P_{f_2}}_{\text{clockwise}} - \underbrace{P_{f_2} P_{f_1}}_{\text{anticlockwise}} \neq 0.$$

- Individually, $P_{f_1} P_{f_2}$ and $P_{f_2} P_{f_1}$ are not trace class.

Theorem (T; T+Xia '24)

For any x-switch f_1 and y-switch f_2 (at any angle $0 < \theta < \pi$),

$$\mathrm{Tr}[P_{f_1}, P_{f_2}] = \frac{1}{2\pi i}. \qquad (\star)$$

Proof of $\mathrm{Tr}[P_{f_1}, P_{f_2}] = \frac{1}{2\pi i}$

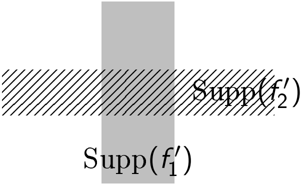
- Rewrite in "Kubo form",

$$[P_{f_1}, P_{f_2}] = P\big[[f_1, P], [f_2, P]\big]$$

- Integral kernel of P is smooth and rapidly decaying,

$$p(z, w) = \frac{1}{\pi} e^{-\frac{|z|^2+|w|^2}{2}} e^{z\bar{w}} = \frac{1}{\pi} e^{-\frac{1}{2}|z-w|^2} e^{iw\wedge z}.$$

  So P is
  - Approx. finite propagation,
  - Locally trace class.
- Product $[f_1, P][f_2, P]$ is supported near origin,



Proof of $\mathrm{Tr}[P_{f_1}, P_{f_2}] = \frac{1}{2\pi i}$

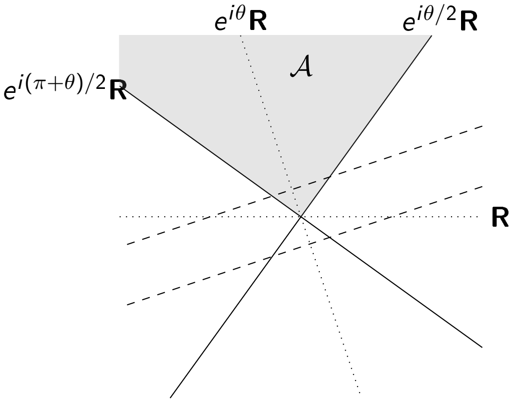
- So $P[f_1, P][f_2, P]$ and $P[f_1, P][f_2, P]$ are trace class, and we can integrate their kernels along diagonal.
- Magnetic translational symmetry + geometry of switches + residue theorem gives exact integral as $\frac{1}{2\pi i}$. (arXiv:2401.06660, cf. Avron–Seiler–Simon '90s) $\square$

Relation to CPHH:

- Write $\pi : \mathcal{B} \to \mathcal{B}/\mathcal{K}$.
- P_{f_1}, P_{f_2} are self-adjoint a.c., so $\pi(P_{f_1} + iP_{f_2})$ is normal in $\mathcal{B}/\mathcal{K}$.
- $\pi(P_{f_1})$ and $\pi(P_{f_2})$ have spectrum inside $[0, 1]$, so

$$\text{ess-Spec}(P_{f_1+if_2}) = \mathrm{Spec}(\pi(P_{f_1+if_2})) \subset \blacksquare.$$

In T+Xia, we constructed Fredholm inverses for $P_{f_1+if_2} - \lambda$ when λ is in interior of square.



This shows $\text{ess-Spec}(P_{f_1+if_2}) = \quad \square$.

- As discussed earlier, principal function is therefore

$$G_{P_{f_1},P_{f_2}} = \frac{in}{2\pi}\chi_{\blacksquare}, \qquad n = \mathrm{Index}(P_{f_1+if_2} - \lambda),\ \lambda \in \mathring{\blacksquare}.$$

- CPHH base formula is

$$2\pi\sigma_{\mathrm{Hall}}(P) = \underbrace{-2\pi i \mathrm{Tr}[P_{f_1}, P_{f_2}]}_{-1} = -2\pi i \int_{\blacksquare} \frac{in}{2\pi} = n,$$

with $n = -1$ due to calculation ($\star$).

- For polynomials $p_i \in \mathbb{Z}[x, y]$, get rational quantization,

$$2\pi i \mathrm{Tr}[p_1(P_{f_1}, P_{f_2}), p_2(P_{f_1}, P_{f_2})] \in \mathbb{Q}.$$

Roughly: contribution along indirect paths, e.g., $1212 - 2121$.

Summary of LLL integer conductance

- LLL is a classical function space, whose chiral asymmetry is detected by pairing its P with switch functions f_1, f_2.
- By abstract functional analytic reasons + plane geometry, this pairing is index-theoretic and must be integral.
- Explicitly computable in LLL case. Works for higher LL (T+Xia)!
- This gives (first?) explicit calculation that every LL has one unit of Hall conductance.
- It seems that there are no "finite-propagation" projections which can exhibit non-zero Hall conductance. The analysis is mandatory!

- Rigorous derivation of Kubo formula for σ_{Hall}?
  e.g. De Roeck–Elgart–Fraas, Inventiones '23.
- Such σ_{Hall} describes a "thought experiment" on infinite-sized sample, where we "measure" transverse current induced by slow introduction of voltage.
- Actual QHE experiments measure an aggregate *resistivity* on finite-sized sample with boundary.
- Meaning of spectral gap, mobility gap, etc., is not sharp.
- Basic mystery: why quantization of 1/resistivity is almost exact in sufficiently large finite-sized sample.

- The phrase "topological phase" is arguably a misnomer.
- Microscopically, dynamics $\leftrightarrow$ geometry.
- Analysis is crucial — estimating complicated system (finite/with boundary) against clean limiting model (infinite).
- Algebraic topology, roughly **Top** $\rightarrow$ **Algebra** has great functorial properties, but its locality principle (Mayer–Vietoris) is much too flabby.
- Rigidity via imposition of symmetry "by hand" defeats the purpose of genericity.

- "Index" or "zero modes" is what's left behind in the low energy/large-scale limit.
- For bounded M, large-scale limit is a point. Only topological data remains, and determine the index $= \#$ zero-modes.
- For unbounded M, the idea is to label different types of ∞-degenerate zero mode spaces by its homotopy class in a suitable operator algebra.
- Roe achieved such an index theory of Dirac operators, at the level of K-theory of Roe-algebras.

End of Lecture 2.

- Kubo formula has $\mathrm{Tr}[P_{f_1}.P_{f_2}]$ form, makes sense for general spectral projections P, and manifolds M. Trace converges under mild conditions given later.
- Quantized Kubo formula is explicitly computed in basic LLL case, and proved in limited settings using "topology".
- Real life quantization persists far beyond "topology" setting.
- Quantization has functional analytic origin, valid once very mild geometric conditions are satisfied (given later). Can deform from LLL case to these.
- Finite, but macroscopic, aspect becomes apparent.

- Let M be a Riemannian manifold, or some "good discretization" of it.
- On Hilbert space $L^2(M)$, a subset $A \subset M$ is regraded as multiplication operator by χ_A. For example,

$$AA^c = A \cap A^c = 0.$$

- "Roe algebra" $\mathcal{B}_{\mathrm{fin}}(M)$ comprises operators L satisfying,
  1. Locally trace class, $KL, LK \in \mathcal{L}_1$ when $K \subset M$ compact.
  2. Finite propagation, $\exists R$, such that $KLK' = 0$ when $d(K, K') > R$.

Periodic table for topological insulators and superconductors

Alexei Kitaev

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31. A. Connes, *Noncommutative geometry*, Academic Press, San Diego, 1994.

Anyons in an exactly solved model and beyond

Alexei Kitaev *

In general, a *quasidiagonal matrix* is a lattice-indexed matrix $A = (A_{jk})$ with sufficiently rapidly decaying off-diagonal elements. Technically, one requires that

$$|A_{jk}| \leqslant c|j-k|^{-\alpha}, \quad \alpha > d,$$

where c and α are some constants, and d is the dimension of the space. Note that "lattice" is simply a way to impose coarse $\mathbb{R}^d$ geometry at large distances. We may think about the problem in these terms: matrices are operators acting in some Hilbert space, and lattice points are basis vectors. But the choice of the basis need not be fixed. One may safely replace the basis vector corresponding to a given lattice point by a linear combination of nearby points. One may also use some kind of coarse-graining, replacing the basis by a decomposition into orthogonal subspaces corresponding to groups of points, or regions in $\mathbb{R}^d$.

Roe algebras

- $\mathcal{B}_{\mathrm{fin}}(M)$ is too naïve — does not even contain the most important example, e.g., LLL projection.
- To fix this, one usually takes the norm completion, $C^*(M)$.
  - Convenient for K-theory calculations.
  - But Kubo formula typically diverges!
- We constructed a Fréchet Roe algebra $\mathcal{B}(M)$:
  - Contains all reasonable low energy spectral projections of sensible gapped Hamiltonians on M;
  - Kubo formula always converges, homotopy invariant, always quantized.

Idea: Non-trivial Roe algebra projections P are necessarily "delocalized[3]".

We detect delocalization by partitioning M and correlations of P along various paths connecting the partitioning sets.

The Kubo formula $\sigma_{\mathrm{Hall}}(P) = -i\mathrm{Tr}[P_{f_1}, P_{f_2}]$ is precisely such a projection-partition pairing.

Precise choice of partition/knowledge of P is unimportant, due to "automatic cancellation of small-scale contributions".

[3] Physicists: No localized Wannier representation

In 2000, Kitaev proposed "real-space Chern number".
Recently, amorphous practitioners use this.

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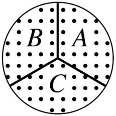
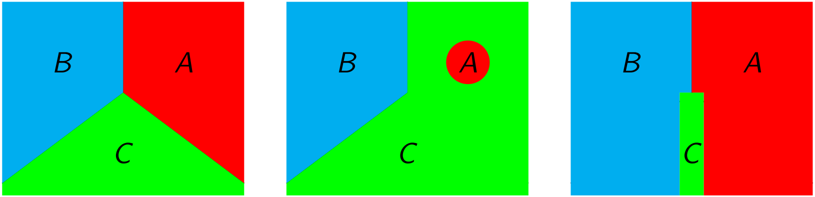


$$\nu(P) = h(A,B,C) \stackrel{\text{def}}{=} \sum_{j\in A}\sum_{k\in B}\sum_{l\in C} h_{jkl}.$$

The idea is to consider hopping amplitudes along triangles, with one vertex in each partitioning set. Sum over all clockwise triangles minus anticlockwise triangles.

Miracle: Something can survive (but only on unbounded sample!?)



Coarse, non-trivial — Coarse but trivial — Not coarse.

Write $B_r(A)$ for set of points within distance r of A.

A partition $M = \bigsqcup_i A_i$ is *coarse* (transverse) if

$$\bigcap_i B_r(A_i) \text{ bounded } \forall r > 0.$$

Coarse partitions pair with Roe algebra operators

- If $L_i \in \mathcal{B}_{\text{fin}}(M)$, then

$$A_0 L_0 A_1 L_1 \dots A_q L_q$$

  is supported within some coarsened intersection $\bigcap_i B_r(A_i)$, which is a compact set K.

- Since L_q locally trace class,

$$A_0 L_0 A_1 L_1 \dots A_q L_q = A_0 L_0 A_1 L_1 \dots A_q \underbrace{L_q K}_{\text{t.c.}} \in \mathcal{L}_1 .$$

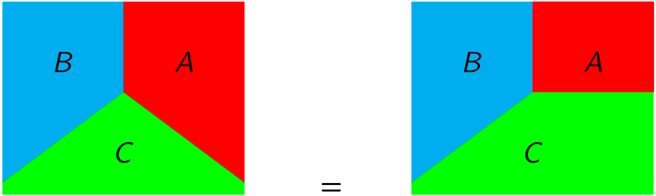
- In particular, for projection $P = P^2 \in \mathcal{B}_{\text{fin}}(M)$, define

$$\mathrm{Tr}[A_0, \dots, A_q]_P := \sum_{\sigma \in S_{q+1}} \mathrm{sgn}(\sigma) \cdot \mathrm{Tr}\left(P A_{\sigma(0)} P A_{\sigma(1)} \dots P A_{\sigma(q)} P\right) .$$

$$\mathrm{Tr}[A_0, \ldots, A_q]_P := \sum_{\sigma \in S_{q+1}} \mathrm{sgn}(\sigma) \cdot \mathrm{Tr}\left(P_{A_{\sigma(0)}} P_{A_{\sigma(1)}} \ldots P_{A_{\sigma(q)}}\right)$$

"Trace of totally antisymmetrized product of Toeplitz operators".

- **Cobordism invariance:**



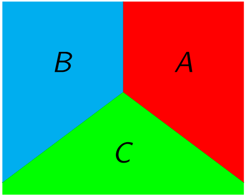
- **Boundary profile invariance:** Can replace $A_i = \chi_{A_i}$ by "partition of unity",

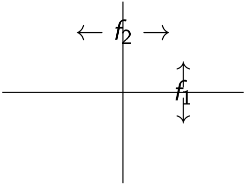
$$\sum_i \tilde{A}_i = 1,$$

with $\mathrm{Supp}(A_i)$ coarsely transverse.

- Using these invariances, we can show that

$$2\mathrm{Tr}[A, B, C]_P = \mathrm{Tr}[P_{f_1}, P_{f_2}].$$





Kubo formula is a coarse projection-partition pairing

- Mathematically, Kubo formula for $\sigma_{\mathrm{Hall}}(P)$ is a pairing between P and the coarse partition $\{A, B, C\}$.
- The pairing has invariances, meaning that it descends to a bilinear pairing

$$\begin{aligned} K_0(\mathcal{B}_{\mathrm{fin}}(M)) \times HX^2(M) &\to \mathbb{C} \\ [P], [\{A, B, C\}] &\mapsto \mathrm{Tr}[A, B, C]_P. \end{aligned}$$

- **Important technical lie**: Any realistic P does not have finite propagation!
- Analysis (trace estimates) needed to make above work, and to find any interesting examples.

- Functional analytic way to see that pairing lands inside $\frac{1}{2\pi i}\mathbb{Z}$.
$\rightarrow$ Universal quantization!
- The range is possibly 0, so we must calculate one non-trivial example.
- For $M = \mathbb{R}^2$, the $K_0(C^*(\mathbb{R}^2)) \cong \mathbb{Z}$, generated by "Dirac index". LLL realizes this concretely.
- Calculation is robust against deformations of M preserving the coarse geometry, perturbations of Hamiltonian.

"topological, but not **Top**"

We outlined a principal function approach in last lecture. Today, I'll proceed more "coarse geometrically".

- Operators A with $A - 1 \in \mathcal{L}_1$ have a *Fredholm determinant*.
- Intuitively, the infinite product of eigenvalues converges, since most of them are 1.
- If $U - 1 \in \mathcal{L}_1$ and $V - 1 \in \mathcal{L}_1$ individually, then

$$\det(UVU^{-1}V^{-1}) = \det(U)\det(V)\det(U)^{-1}\det(V)^{-1} = 1.$$

- Generally, determinant of $UVU^{-1}V^{-1}$ need not vanish: If $[C, D] \in \mathcal{L}_1$, then

$$\det(e^C e^D e^{-C} e^{-D}) = \exp(\mathrm{Tr}[C, D]). \qquad (\mathrm{Pincus} + \mathrm{HH})$$

- Kitaev '00 conjecture: For unitaries U, V,

$$(U-1)(V-1) \in \mathcal{L}_1 \Rightarrow \det(UVU^{-1}V^{-1}) = 1.$$

This was proved by Elgart–Fraas '23.

- The motivation is that if C, D are self-adjoint a.c., and we know that

$$(e^{2\pi i C} - 1)(e^{2\pi i D} - 1) \in \mathcal{L}_1, \tag{1}$$

then

$$\begin{aligned} 1 &= \det\left(e^{2\pi i C} e^{2\pi i D} e^{-2\pi i C} e^{-2\pi i D}\right) \\ &= \exp(\mathrm{Tr}[2\pi i C, 2\pi i D]) \;\Rightarrow\; 2\pi i \mathrm{Tr}[C, D] \in \mathbb{Z}. \end{aligned}$$

- Condition (1) is hard to use in practice.
- We argued that if

$$(C - C^2)(D - D^2) \in \mathcal{L}_1, \tag{2}$$

then Condition (1) holds, so $\mathrm{Tr}[C, D] \in \frac{1}{2\pi i}\mathbb{Z}$.

- **Proof:** The holomorphic function $\varphi : z \mapsto e^{2\pi i z} - 1$ has zeros at $z = 0, 1$, so

$$\varphi(z) = \psi(z)(z - z^2).$$

$$\begin{aligned}(e^{2\pi i C} - 1)(e^{2\pi i D} - 1) &\equiv \varphi(C)\varphi(D) \\ &= \psi(C)\underbrace{(C - C^2)(D - D^2)}_{\text{t.c. by assumption}}\psi(D) \in \mathcal{L}_1.\end{aligned}$$

- Our Condition (2) holds "by physics":
- For (magnetic) Schrödinger operators $H = H^*$ with spectral gap, Fermi projection P lies in $\mathcal{B}(M)$. Furthermore,

$$(P_{f_1} - P_{f_1}^2)(P_{f_2} - P_{f_2}^2) \overset{\text{algebra}}{=} Pf_1 P(1 - f_1) Pf_2 P(1 - f_2) P$$

  is supported near the compact set

$$\mathrm{Supp}(f_1) \cap \mathrm{Supp}(1 - f_1) \cap \mathrm{Supp}(f_2) \cap \mathrm{Supp}(1 - f_2),$$

  so we deduce it's trace class.
- Thus $\sigma_{\mathrm{Hall}}(P) = -i\mathrm{Tr}[P_{f_1}, P_{f_2}] \in \frac{1}{2\pi}\mathbb{Z}$.

Exact quantization

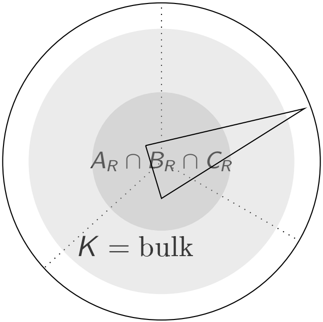
- Hall conductance is quantized to $\frac{1}{2\pi}\mathbb{Z}$, without need for translation symmetry, homogeneity, Euclidean geometry, etc.
- Identification as a coarse geometry pairing shows its invariances against small-scale perturbations/imperfections
  - Flatness of sample
  - Uniformity of magnetic field
  - Holes in sample
  - Geometric assumptions in deriving Kubo formula
- Gap-closing is of course necessary for transitions, but what kind of gap closing?
  - Large-scale geometric changes drive the transitions.

Finite-sized sample and approximate quantization

- In real world, M has finite size, so H has discrete spectrum, and P is finite-rank. Then $[P_{f_1}, P_{f_2}]$ is traceless, so

$$\sigma_{\mathrm{Hall}}(P) = 0 \ldots ??$$

- There is no paradox: non-trivial exact quantization holds in thought experiment with infinite-sized sample.
- Hall conductivity is a *bulk* property.
- Let r be the (approximate) propagation of P. Choose a bulk subset $K \subset M$ which, for some $R \gg r$,
  - Contains the R-thickened intersection of the partition
  - Stays at least distance R from the sample boundary ∂M



- The bulk contributes

$$\sigma_{\mathrm{Hall},K}(P) = -2i\mathrm{Tr}\big(KAKPKBKPKCKPK + \mathrm{antisymm}\big)$$

Dependence on cut-off $R \gg r$ is small, because P has rapid decay.

- Now $\sigma_{\mathrm{Hall},K}(P)$ need not vanish, and is *not exactly quantized*, because KPK is not a projection.
- Sample M is embedded in infinite-sized $\widehat{M}$. True Hamiltonian H on M is a restriction of the fictitious $\widehat{H}$ on $\widehat{M}$, with some boundary conditions imposed.
- For the K-truncated Fermi projections, we have

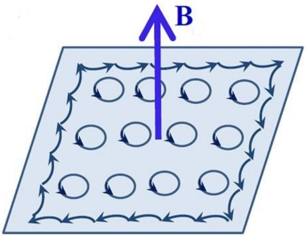
$$KPK \approx K\widehat{P}K.$$

- We get *approximate quantization*,

$$\sigma_{\mathrm{Hall},K}(P) \approx \sigma_{\mathrm{Hall},K}(\widehat{P}) \approx \sigma_{\mathrm{Hall}}(\widehat{P}) \in \frac{1}{2\pi}\mathbb{Z}.$$

Precision of quantization depends on how well $\widehat{P}$ approximates P, decay rate of $\widehat{P}$ (spectral gap size), volume growth rate of $\widehat{M}$... Numerical convergence [Mitchell et al 2018].

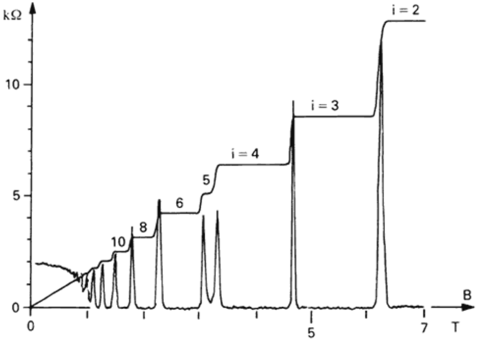
Basic requirement is $R \gg r \sim B^{-1/2}$, i.e. macroscopic sample!



Note: The boundary part, $K^c P K^c$ is *not* approximated by $K^c \widehat{P} K^c$, it must be removed!

Quantum Hall experiment

- Material has electron density μ, non-interacting ($\sim 0K$).
- Each Landau level has $\frac{eb}{h}$ electrons/unit area.
- Filling fraction $\nu = \frac{\mu h}{e}\frac{1}{b}$ is varied by controlling b.



- At large b, it was found that Hall resistivity $\rho_{\mathrm{Hall}} = \sigma_{\mathrm{Hall}}^{-1}$ is

$$\rho_{\mathrm{Hall}}(\nu) = \frac{h}{e^2}\frac{1}{\mathrm{Int}(\nu)}, \qquad \forall \nu \approx \text{integer}.$$

- So $\sigma_{\text{Hall}} \approx \frac{e^2}{h}\mathrm{Int}(\nu)$ when ν is near an integer.
- Geometric imperfections broaden the Landau levels, and allow ν to vary "continuously".

- $\nu \approx$ integer corresponds to gaps between idealized Landau levels; states are localized (maybe Anderson). Varying ν here doesn't change integral conductance.
- $\nu \approx$ half-integer corresponds to core of Landau band; states are "very delocalized". Interpolation of integral conductance occurs here.

Macroscopic Planck constant

- Magnetic length scale is

$$26\mathrm{nm}/\sqrt{\#\mathrm{Tesla}} \sim 10\mathrm{nm}.$$

- Size of $M \gg 10\mathrm{nm}$, so quantization would be very close to exact. Experimentally, 10^{-10} error.
- QHE gives access to $\frac{e^2}{h}$.
- Josephson junction (macroscopic) gives $\frac{h}{2e}$.
- Ironically, macroscopic measurements give best access to h.

$$s(\text{econd}) = 9192631770 \;\text{ ticks of Cs-133 atomic clock}$$

$$m(\text{etre}) = \frac{(\text{speed of light}) \times s}{299792458}$$

- Prior to 2019, **kilogram** was *locally* defined,

$$\text{Old } kg = \text{prototype in Paris.}$$

- For > 100 years, we've known about Planck's constant, and could define "quantum kilogram" by

$$h =: \underbrace{6.626\ldots??}_{\text{fix convention}} \times 10^{-34} (kg_h) \cdot m^2\, s^{-1}.$$

- After QHE,

$$h \overset{\text{new}}{=:} 6.62607015 \times 10^{-34} \cdot (kg_h) \cdot m^2\, s^{-1},$$