One-parameter automorphism groups of the injective factor of type II_1 with Connes spectrum zero

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Abstract. We construct a one-parameter automorphism group of the injective type II_1 factor with Connes spectrum $\{0\}$ which is not stably conjugate to an infinite tensor product action. We construct a countable family of one-parameter automorphism groups of the injective type II_1 factor such that all are stably conjugate but no two are cocycle conjugate.

§0 Introduction

We exhibit a one-parameter automorphism group of the approximately finite dimensional (AFD) factor of type II_1 which has Connes spectrum $\{0\}$ and is not stably conjugate to an infinite tensor product action. We also construct a countable family of one-parameter automorphism groups of the AFD factor of type II_1, all of which are stably conjugate but no two of which are cocycle conjugate. This shows the difference between the two notions, cocycle conjugacy and stable conjugacy.

At a certain stage of development, the existence of a non-ITPFI AFD type III_0 factor was a focal point of the structure analysis of factors. In the first section, we prove a corresponding result for one-parameter automorphism groups.

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In our previous work [10], we considered a one-parameter automorphism group which fixes a Cartan subalgebra of the AFD type II_1 factor $\mathcal{R}$ elementwise, and showed that if it has full spectrum, i.e., spectrum equal to $\mathbf{R}$, then it is cocycle conjugate to an infinite tensor product one-parameter automorphism group (Theorem 1.6 of [10]). Now, we consider the problem of whether all one-parameter automorphism groups of the AFD type II_1 factor $\mathcal{R}$ are cocycle (or stably) conjugate to an infinite tensor product one-parameter automorphism group. The answer to this problem would be expected to be negative, in analogy with the existence of an AFD type III_0 factor which is not ITPFI. (See Araki-Woods [1] and Connes-Woods [4] for related definitions and results.) In fact, we shall construct an example of a one-parameter automorphism group α of the AFD type II_1 factor $\mathcal{R}$, with $\Gamma(\alpha) = \{0\}$, which is not stably conjugate to an infinite tensor product one-parameter automorphism group. The main technical tool is taken from Connes-Woods [4].

In §2, we exhibit a countable family of one-parameter automorphism groups of the AFD type II_1 factor $\mathcal{R}$, with Connes spectra $\{0\}$, all of which are all stably conjugate, but no two of which are cocycle conjugate.

In our earlier work [9] on one-parameter automorphism groups of $\mathcal{R}$, we used stable conjugacy for classification when the Connes spectrum is $\{0\}$, and obtained two complete invariants: the type of the crossed product algebra and the flow given by the dual action on the center of the crossed product algebra. For actions fixing a Cartan subalgebra and certain actions arising from the irrational rotation algebra, we also showed the uniqueness, up to cocycle conjugacy, of one-parameter automorphism groups with full Connes spectrum in [10,11]. One is naturally led to ask whether cocycle conjugacy and stable conjugacy coincide for one-parameter automorphism groups or not. Cocycle conjugacy for general actions trivially implies stable conjugacy, but the converse is true in some cases, and false in others. For example, the two notions coincide for discrete amenable groups and do not coincide for $\mathbf{T}^d$, $d > 1$. (See the first paragraph of §2.) The problem for $\mathbf{T}$ or $\mathbf{R}$ is more subtle than for discrete groups or $\mathbf{T}^d$, $d > 1$. We will show in §2 that stable conjugacy does not imply cocycle conjugacy for either $\mathbf{T}$ or $\mathbf{R}$. The main tools are the basic construction for subalgebras and the method developed by Christensen in [2].

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§1 One-parameter automorphism groups of non-product type

We construct an example of a one-parameter automorphism group α of the AFD type II_1 factor $\mathcal{R}$ with $\Gamma(\alpha) = \{0\}$ which is not stably conjugate to an infinite tensor product one-parameter automorphism group in this section, using a technique from Connes-Woods [4]. We consider the following property first. This is an analogue of the condition in Lemma 2.1 of Connes-Woods [4].

Definition 1.1. Let $\mathcal{M}$ be the AFD type II_1 factor $\mathcal{R}$ or the AFD type II_∞ factor $\mathcal{R}_{0,1}$. For a one-parameter automorphism group α of $\mathcal{M}$, consider the Sakai flip $\sigma : x \otimes y \mapsto y \otimes x$ on $\mathcal{M} \otimes \mathcal{M}$. Because σ commutes with $\alpha \otimes \alpha$ on $\mathcal{M} \otimes \mathcal{M}$, we can extend σ to σ_α on $(\mathcal{M} \otimes \mathcal{M}) \rtimes_{\alpha\otimes\alpha} \mathbf{R}$. We consider the following property:

$$(*) \qquad \sigma_\alpha \textit{ is trivial on } \mathcal{Z}((\mathcal{M} \otimes \mathcal{M}) \rtimes_{\alpha\otimes\alpha} \mathbf{R}).$$

Proposition 1.2. *The property* $(*)$ *is invariant under stable conjugacy.*

Proof. First consider replacing α_t by β_t which is cocycle conjugate to α_t. Then σ_α on $(\mathcal{M} \otimes \mathcal{M}) \rtimes_{\alpha\otimes\alpha} \mathbf{R}$ is conjugate to σ_β on $(\mathcal{M} \otimes \mathcal{M}) \rtimes_{\beta\otimes\beta} \mathbf{R}$. Next replace α_t by $\alpha_t \otimes i_t$, where i_t is the trivial action of $\mathbf{R}$ on $\mathcal{L}(\mathcal{H})$. Then σ_α on $\mathcal{Z}((\mathcal{M}\otimes\mathcal{M}) \rtimes_{\alpha\otimes\alpha} \mathbf{R})$ is conjugate to $\sigma_{\alpha\otimes i}$ on $\mathcal{Z}((\mathcal{M} \otimes \mathcal{L}(\mathcal{H}) \otimes \mathcal{M} \otimes \mathcal{L}(\mathcal{H})) \rtimes_{\alpha\otimes i\otimes\alpha\otimes i} \mathbf{R})$. Thus we get the conclusion. Q.E.D.

The following result corresponds to Lemma 2.1 in Connes-Woods [4].

Proposition 1.3. *If a one-parameter automorphism group* α *of the AFD type* II_1 *factor* $\mathcal{R}$ *is of infinite tensor product type, then it has the property* $(*)$.

Proof. Let α_t be of infinite tensor product type with respect to the decomposition $\mathcal{R} = \bigotimes_{n=1}^\infty M_n$, where M_n is a matrix algebra, and consider the infinite tensor product with respect to the trace. Let σ_m be the Sakai flip on $(\bigotimes_{n=1}^m M_n) \otimes (\bigotimes_{n=1}^m M_n)$. Then $\sigma = \lim_{m\to\infty} \sigma_m \otimes 1$, and $\sigma_\alpha = \lim_{m\to\infty} (\sigma_m \otimes 1)_\alpha$. Thus it is enough to show that $(\sigma_m \otimes 1)_\alpha$ is trivial on $\mathcal{Z}((\mathcal{R} \otimes \mathcal{R}) \rtimes_{\alpha\otimes\alpha} \mathbf{R})$. But for this statement, we may assume that α_t is trivial for the first m components by perturbing α_t by a unitary cocycle. Then the above assertion is trivial, and we are done. Q.E.D.

We now have the following theorem. The argument is parallel to the proof of Theorem 2.3 in Connes-Woods [4].

Theorem 1.4. *Let a one-parameter automorphism group α of the AFD type II_∞ factor $\mathcal{R}_{0,1}$ be defined as follows. For an ergodic and infinite-measure preserving transformation T on X such that $T \times T^{-1}$ is dissipative, construct the crossed product algebra $L^\infty(X) \rtimes_T \mathbf{Z} \cong \mathcal{R}_{0,1}$. Define α by $\alpha_t(x) = x$ for $x \in L^\infty(X)$ and $\alpha_t(u) = e^{it}u$ for the implementing unitary u. Then the action α does not have the property $(*)$ of Definition 1.1, and hence it is not stably conjugate to an infinite tensor product one-parameter automorphism group.*

Proof. The flow given by $(\alpha \otimes \alpha)\hat{}$ on $\mathcal{Z}((\mathcal{R}_{0,1} \otimes \mathcal{R}_{0,1}) \rtimes_{\alpha\otimes\alpha} \mathbf{R})$ is a Poincaré flow, given by the 1-cocycle obtained by the equivalence relation $(Tx, y, t{+}1) \sim (x, y, t) \sim (x, Ty, t+1)$ on $X \times X \times \mathbf{R}$. (See Proposition 1.3 in [10].) Let E be any $T \times T^{-1}$ invariant set in $X \times X$, σ_X the flip on $X \times X$. Let $\tilde{E}$ be the subset of $X \times X \times \mathbf{R}$ generated by $E \times [0,1[$ and the above equivalence relation. Suppose that α has the property $(*)$. Because σ_α acts on $\mathcal{Z}((\mathcal{R}_{0,1} \otimes \mathcal{R}_{0,1}) \rtimes_{\alpha\otimes\alpha} \mathbf{R})$ by σ_X, we get that the set $\tilde{E}$ is invariant under $\sigma_X \otimes id$. This implies that σ_X preserves E. Now by Lemma 2.2 in Connes-Woods [4], we get $\sigma_X \in [T \times T^{-1}]$. Then for almost all $(x, y) \in X \times X$, we have an integer $n(x, y)$ such that

$$\sigma_X(x, y) = (y, x) = (T^{n(x,y)}x, T^{-n(x,y)}y),$$

so that $y \in T$-orbit of x, but this is impossible because the orbit is countable.

Q.E.D.

An example of a transformation T as in Theorem 1.4 is given in Harris-Robins[5]. (See also §3 of Connes-Woods [4].) If we form α^e for the above one-parameter automorphism group α and an invariant projection $e \in L^\infty(X) \subset \mathcal{R}_{0,1}$

with finite trace, we get an example of a one-parameter automorphism group β of the AFD type II_1 factor $\mathcal{R}$, with $\Gamma(\beta) = \{0\}$, which is not stably conjugate to an infinite tensor product one-parameter automorphism group.

Remark 1.5. There exists another type of one-parameter automorphism group α of the AFD type II_1 factor $\mathcal{R}$, with $\Gamma(\alpha) = \mathbf{R}$, which is not cocycle conjugate to an infinite tensor product one-parameter automorphism group. This type of one-parameter automorphism group α does not satisfy $\alpha_t \in \mathrm{Out}(\mathcal{R}),\ t \neq 0$, while $\Gamma(\alpha) = \mathbf{R}$, and therefore it cannot be cocycle conjugate to an action of product type. (See Introduction of [11] for a more detailed explanation.) This does not have an analogue in the case of AFD type III factors.

§2 Cocycle conjugacy and stable conjugacy

Cocycle conjugacy for general actions trivially implies stable conjugacy, but the converse is true in some cases, and false in other cases, as shown below. (Recall that α and β are said to be stably conjugate if $\alpha \otimes id_\infty$ and $\beta \otimes id_\infty$ are cocycle conjugate, where id_∞ is the trivial action on the separable type I_∞ factor.) Stable conjugacy implies cocycle conjugacy for discrete amenable group actions on $\mathcal{R}$ because the characteristic invariant is a stable conjugacy invariant. (See Theorem 2.7 in Ocneanu [12].) For the tori $\mathbf{T}^d$, $d \geq 2$, it is not difficult to construct two actions on $\mathcal{R}$ which are stably conjugate, but not cocycle conjugate: Take an ergodic action α on $\mathcal{R}$ and set $\beta = \alpha \otimes id_2$, where id_2 is the trivial action on $M_2(\mathbf{C})$. (See Olesen-Pedersen-Takesaki [13] for the construction of ergodic actions.) Proposition 4.7 in [13] asserts that these are not cocycle conjugate. But this construction does not work for $\mathbf{T}$: The one-dimensional torus $\mathbf{T}$ does not have an ergodic action on $\mathcal{R}$

and if an action of $\mathbf{T}$ on $\mathcal{R}$ has a factor as its fixed point algebra, then it is unique up to conjugacy. (See Corollary 4.7 in Paschke [14], our Theorem 2.2 [9], and a remark on p. 185 of Jones [7].) Thus, the problem for $\mathbf{T}$ or $\mathbf{R}$ is more subtle. We will show that stable conjugacy does not imply cocycle conjugacy for either $\mathbf{T}$ or $\mathbf{R}$.

Define two actions α and β by

$$\alpha_t = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\exp 2\pi i t \begin{pmatrix} 3^n/2 & 0 \\ 0 & -3^n/2 \end{pmatrix}\right),$$

$$\beta_t = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\exp 2\pi i t \begin{pmatrix} 3^{n+2}/2 & 0 \\ 0 & -3^{n+2}/2 \end{pmatrix}\right).$$

Here we identify the AFD type II_1 factor $\mathcal{R}$ with the infinite tensor product $\bigotimes_{n=1}^{\infty} M_2(\mathbf{C})$ with respect to the trace τ. We denote by $e_{jk}(n)$, $1 \leq j,k \leq 2$, the matrix units in the n-th factor $M_2(\mathbf{C})$. We also denote by D_2 the set of diagonal matrices in $M_2(\mathbf{C})$. Because $\bigotimes_{n=1}^{\infty} D_2$ is in the fixed point algebra of α, we can show that $\Gamma(\alpha) \subseteq 3^n\mathbf{Z}$ for each n, whence $\Gamma(\alpha) = \{0\}$. Note that

$$\alpha_t = \mathrm{Ad}\left(\exp 2\pi i t \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}\right) \otimes \beta_t,$$

and so α_t and β_t are stably conjugate. In the following, we will prove that these two actions are not cocycle conjugate. To obtain a contradiction, suppose that they are cocycle conjugate: there is an automorphism θ of $\mathcal{R}$ and an α-unitary cocycle u_t such that $\theta \cdot \beta_t \cdot \theta^{-1} = \mathrm{Ad}(u_t) \cdot \alpha_t$, for all $t \in \mathbf{R}$. We will reach a contradiction at the end of this section. The basic idea of the proof is as follows: For large N, the

u_t's are almost contained in the first N factors, and thus $\beta_t \cdot \theta^{-1}$ is almost equal to $\theta^{-1} \cdot \alpha_t$ at the $(N+1)$-st factor and later on. Then θ^{-1} should be like a backward shift by 2 there, but such an automorphism does not exist.

Because both the groups α_t and β_t have period 1, u_1 is a scalar; thus we may assume u_t also has also period 1, without loss of generality. Let $\varepsilon < 1/51200$, and choose $N \geq 2$ such that for any $t \in \mathbf{R}$, there exists a such that

$$(1) \qquad a \in \underbrace{M_2(\mathbf{C}) \otimes \cdots \otimes M_2(\mathbf{C})}_{N \text{ times}} \otimes \mathbf{C} \otimes \mathbf{C} \otimes \cdots, \qquad \|u_t - a\|_2 \leq \varepsilon.$$

Assertion 2.1. *In the context above, for any*

$$x \in \underbrace{M_2(\mathbf{C}) \otimes \cdots \otimes M_2(\mathbf{C})}_{N \text{ times}} \otimes \mathbf{C} \otimes \mathbf{C} \otimes \cdots, \qquad \|x\|_\infty \leq 1,$$

there exists y such that

$$(2) \qquad \begin{cases} y \in \underbrace{M_2(\mathbf{C}) \otimes \cdots \otimes M_2(\mathbf{C})}_{N-2 \text{ times}} \otimes D_2 \otimes D_2 \otimes \cdots, \\ \|\theta^{-1}(x) - y\|_2 \leq 10\sqrt{\varepsilon}. \end{cases}$$

For the proof, we introduce the following action ρ_g of $\prod_{n=1}^\infty \mathbf{Z}_2$ on $\mathcal{R}$:

$$\rho_g = id_{2^{N-2}} \otimes \bigotimes_{n=1}^{\infty} \operatorname{Ad} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{g_n} \end{pmatrix},$$

where $id_{2^{N-2}}$ means the trivial action on $M_{2^{N-2}}(\mathbf{C})$, and $g = (g_n)$, $g_n = 0$ or 1. Fix $g = (g_n) \in \bigoplus_{n=1}^\infty \mathbf{Z}_2 \subset \prod_{n=1}^\infty \mathbf{Z}_2$. Suppose that

$$g_n = \begin{cases} 1, & \text{for } n = k_1, \ldots, k_m,\ 1 \leq k_1 \leq \cdots \leq k_m, \\ 0, & \text{otherwise.} \end{cases}$$

We use the notation $J = (j_1, \ldots, j_m) \in \prod_{n=1}^{m}\{1,2\}$ and define projections $p_J, \bar{p}_J$ by

$$p_J = e_{j_1 j_1}(k_1 + N - 2) \cdots e_{j_m j_m}(k_m + N - 2),$$

$$\bar{p}_J = e_{3-j_1,3-j_1}(k_1 + N - 2) \cdots e_{3-j_m,3-j_m}(k_m + N - 2).$$

We also set

$$e_J = e_{j_1 j_1}(k_1 + N) \cdots e_{j_m j_m}(k_m + N),$$

$$w_J = e_{3-j_1,j_1}(k_1 + N) \cdots e_{3-j_m,j_m}(k_m + N),$$

$$\sigma(J) = (2j_1 - 3) \cdots (2j_m - 3) \in \{1, -1\},$$

$$\lambda_J = (2j_1 - 3)3^{k_1+N} + \cdots + (2j_m - 3)3^{k_m+N}.$$

Note that $e_J = w_J^* w_J$ and $\alpha_t(w_J) = \exp(2\pi i \lambda_J t) w_J$. We define P_J to be the projection onto the β_t-eigenspace for the eigenvalue $2\pi\lambda_J$. The range of this projection is generated by $\bar{p}_J \mathcal{R} p_J$.

Lemma 2.2. *In the context above, we have*

$$\|(I - P_J)(\theta^{-1}(w_J))\|_2^2 = \frac{1}{2}\int_0^1 \|[u_t, w_J]\|_2^2 \, dt.$$

Proof. Let $\theta^{-1}(w_J) = \sum_\lambda b_\lambda$ be the decomposition of $\theta^{-1}(w_J)$ into β_t-eigenspaces with eigenvalues $2\pi\lambda$. Because all the λ's are integers and every λ can be expressed as a sum of finitely many $\pm 3^n$'s in a unique way, we get

$$\begin{aligned}
&2\|(I-P_J)(\theta^{-1}(w_J))\|_2^2 \\
=&2\sum_{\lambda\neq\lambda_J}\|b_\lambda\|_2^2 \\
=&\int_0^1 \|\sum_\lambda(\exp(2\pi i\lambda t)b_\lambda - \exp(2\pi i\lambda_J t)b_\lambda)\|_2^2\,dt \\
=&\int_0^1 \|\beta_t(\theta^{-1}(w_J)) - \exp(2\pi i\lambda_J t)\theta^{-1}(w_J)\|_2^2\,dt \\
=&\int_0^1 \|u_t\cdot\exp(2\pi i\lambda_J t)w_J\cdot u_t^* - \exp(2\pi i\lambda_J t)w_J\|_2^2\,dt \\
=&\int_0^1 \|[u_t, w_J]\|_2^2\,dt.
\end{aligned}$$

Q.E.D.

Now we set

$$f_J = \theta^{-1}(w_J^* w_J) = \theta^{-1}(e_J),$$

$$a_J = P_J(\theta^{-1}(w_J))^* P_J(\theta^{-1}(w_J)).$$

We have $p_J a_J p_J = a_J$. Note that by Lemma 2.2,

$$\begin{aligned}
&\|f_J - a_J\|_2 \\
\leq&\|\theta^{-1}(w_J^*)(\theta^{-1}(w_J) - P_J(\theta^{-1}(w_J)))\|_2 \\
&\quad + \|(\theta^{-1}(w_J^*) - P_J(\theta^{-1}(w_J))^*)P_J(\theta^{-1}(w_J))\|_2 \\
\leq&\sqrt{2}\left(\int_0^1 \|[u_t, w_J]\|_2^2\,dt\right)^{1/2}.
\end{aligned} \tag{3}$$

Lemma 2.3. *In the context above, we get*

$$\sum_{\sigma(J)=-1} \|[a, w_J]\|_2^2 \leq 4\|a\|_2^2, \qquad a \in \mathcal{R}.$$

Proof. We have

$$\begin{aligned} &\sum_{\sigma(J)=-1} \|[a, w_J]\|_2^2 \\ =&\sum_{\sigma(J)=-1} (\|aw_J - w_J a\|_2^2 + \|aw_J + w_J a\|_2^2) \\ =&2\tau(\sum_{\sigma(J)=-1} w_J w_J^* a^* a) + 2\tau(\sum_{\sigma(J)=-1} w_J^* w_J a a^*) \\ \leq&4\|a\|_2^2. \end{aligned}$$

Q.E.D.

Lemma 2.4. *In the context above, we get*

$$\|\sum_{\sigma(J)=-1} (f_J - a_J)\|_2^2 \leq 8\varepsilon^2.$$

Proof. We have

$$
\begin{aligned}
&\|\sum_{\sigma(J)=-1}(f_J-a_J)\|_2^2\\
=&\tau(\sum_{\sigma(J),\sigma(K)=-1} f_Jf_K+a_Ja_K-f_Ja_K-a_Jf_K)\\
=&\tau(\sum_{\sigma(J)=-1} f_J)+\tau(\sum_{\sigma(J)=-1} a_J^2)-2\tau(\sum_{\sigma(J),\sigma(K)=-1} f_Ka_Jf_K)\\
\le&\tau(\sum_{\sigma(J)=-1} f_J)+\tau(\sum_{\sigma(J)=-1} a_J^2)-2\tau(\sum_{\sigma(J),\sigma(K)=-1} f_Ka_Jf_K)\\
&+2\tau(\sum_{\sigma(J),\sigma(K)=-1,J\neq K} f_Ka_Jf_K)\\
=&\tau(\sum_{\sigma(J)=-1}(f_J+a_J^2-2f_Ja_Jf_J))\\
=&\sum_{\sigma(J)=-1}\|f_J-a_J\|_2^2\\
\le&2\sum_{\sigma(J)=-1}\int_0^1\|[u_t,w_J]\|_2^2\,dt,
\end{aligned}
$$

by (3). By (1), there exists a_t such that

$$
a_t\in\underbrace{M_2(\mathbf{C})\otimes\cdots\otimes M_2(\mathbf{C})}_{N\text{ times}}\otimes\mathbf{C}\otimes\mathbf{C}\otimes\cdots,\quad \|u_t-a_t\|_2\le\varepsilon,
$$

for each t. By Lemma 2.3, we get

$$
\begin{aligned}
&\|\sum_{\sigma(J)=-1}(f_J-a_J)\|_2^2\\
\le&2\sum_{\sigma(J)=-1}\int_0^1\|[u_t-a_t,w_J]\|_2^2\,dt\\
\le&8\varepsilon^2.
\end{aligned}
$$

Q.E.D.

Lemma 2.5. *In the context of Claim 2.1, we have*

$$\|\rho_g(\theta^{-1}(x)) - \theta^{-1}(x)\|_2 \leq 10\sqrt{\varepsilon}.$$

Proof. First we have

$$\begin{aligned}
&\|\rho_g(\theta^{-1}(x)) - \theta^{-1}(x)\|_2^2 \\
=&4\|[\sum_{\sigma(J)=-1} p_J,\, \theta^{-1}(x)]\|_2^2 \\
=&4\|[\sum_{\sigma(J)=-1} p_J - f_J,\, \theta^{-1}(x)]\|_2^2 \\
=&16\|\sum_{\sigma(J)=-1} p_J - f_J\|_2^2 \\
=&16\tau(\sum_{\sigma(J),\sigma(K)=-1} (p_Jp_K + f_Jf_K - p_Jf_K - f_Jp_K)) \\
=&32\tau(\sum_{\sigma(J)=-1} f_J - \sum_{\sigma(J),\sigma(K)=-1} p_Kf_Jp_K) \\
=&32\tau(\sum_{\sigma(J)=-1} (f_J - a_J) - \sum_{\sigma(J),\sigma(K)=-1} p_K(f_J - a_J)p_K) \\
\leq&32\|\sum_{\sigma(J)=-1} (f_J - a_J) - \sum_{\sigma(J),\sigma(K)=-1} p_K(f_J - a_J)p_K\|_2 \\
\leq&32\|\sum_{\sigma(J)=-1} (f_J - a_J)\|_2.
\end{aligned}$$

Then by Lemma 2.4, we get

$$\|\rho_g(\theta^{-1}(x)) - \theta^{-1}(x)\|_2^2 \leq 64\sqrt{2}\varepsilon,$$

and hence

$$\|\rho_g(\theta^{-1}(x)) - \theta^{-1}(x)\|_2 \leq 10\sqrt{\varepsilon}.$$

Q.E.D.

Now we can prove Assertion 2.1.

Proof of Assertion 2.1. By Lemma 2.5, we have

$$\|\rho_g(\theta^{-1}(x)) - \theta^{-1}(x)\|_2 \leq 10\sqrt{\varepsilon}, \tag{4}$$

for each $g \in \bigoplus_{n=1}^{\infty} \mathbf{Z}_2$. Hence by continuity the estimate (4) is also valid for each $g \in \prod_{n=1}^{\infty} \mathbf{Z}_2$. Then set

$$y = \int_{\prod_{n=1}^{\infty} \mathbf{Z}_2} \rho_g(\theta^{-1}(x))\, dg,$$

where the integral is performed with respect to the Haar measure of $\prod_{n=1}^{\infty} \mathbf{Z}_2$. Now by (4), we get

$$\|y - \theta^{-1}(x)\|_2 \leq 10\sqrt{\varepsilon},$$

which is (2). Q.E.D.

We introduce the following notation for the second step:

$$\mathcal{P} = \underbrace{M_2(\mathbf{C}) \otimes \cdots \otimes M_2(\mathbf{C})}_{N \text{ times}} \otimes \mathbf{C} \otimes \mathbf{C} \otimes \cdots \subset \mathcal{R},$$

$$\mathcal{Q} = \underbrace{M_2(\mathbf{C}) \otimes \cdots \otimes M_2(\mathbf{C})}_{N-2 \text{ times}} \otimes D_2 \otimes D_2 \otimes \cdots \subset \mathcal{R},$$

$$\mathcal{M} = M_2(\mathbf{C}) \otimes \mathcal{R}, \quad \mathcal{N} = \mathbf{C} \otimes \mathcal{Q}, \quad \mathcal{L} = \mathbf{C} \otimes \mathcal{P},$$

$$\mathcal{A} = \mathbf{C} \otimes \underbrace{\mathbf{C} \otimes \cdots \otimes \mathbf{C}}_{N-2 \text{ times}} \otimes D_2 \otimes D_2 \otimes \cdots \subset \mathcal{M}.$$

We also write $L^\infty(X,\mu)$, $\mu(X)=1$, for $\mathcal{A}$. So far, we have proved $\mathcal{P} \overset{10\sqrt{\varepsilon}}{\subset} \mathcal{Q}$ in this notation. We would like to embed $\mathcal{P}$ into $\mathcal{Q}$ by a perturbation to get a contradiction. The main difficulty is that we have no control over the size of N here. But by the technique of Christensen, [2], we can embed $\mathcal{P}$ into $M_2(\mathbf{C}) \otimes \mathcal{Q}$, which is enough for our purpose.

Assertion 2.6. *There is a non-trivial (non-unital) homomorphism Φ of $\mathcal{P}$ into $\mathcal{N}$.*

We will show Assertion 2.6 by arguments similar to Christensen's in [2]. Note that it follows from $\mathcal{P} \overset{10\sqrt{\varepsilon}}{\subset} \mathcal{Q}$ that $\mathcal{L} \overset{10\sqrt{\varepsilon}}{\subset} \mathcal{N}$ in $\mathcal{M}$. We make the basic construction (see Jones [6], or, for that matter, Christensen [2]) for the pair $\mathcal{N} \subset \mathcal{M}$. Denote by E the conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$, and write e for the projection in $L^2(\mathcal{M})$ arising from E. Then an easy computation shows that the basic construction $\langle \mathcal{M}, e\rangle$ is isomorphic to $M_4(\mathbf{C}) \otimes M_{2^{N-2}}(\mathbf{C}) \otimes L^\infty(X,\mu) \otimes \mathcal{L}(\ell^2(\mathbf{Z}))$ because $L^\infty(X,\mu)$ is a Cartan subalgebra in

$$\mathbf{C} \otimes \underbrace{\mathbf{C} \otimes \cdots \otimes \mathbf{C}}_{N-2 \text{ times}} \otimes M_2(\mathbf{C}) \otimes M_2(\mathbf{C}) \otimes \cdots \cong \mathcal{R}.$$

(Note that the basic construction for $\mathbf{C} \subset M_2(\mathbf{C})$ is $M_4(\mathbf{C})$.) We can define a centre-valued trace T on $\langle \mathcal{M}, e\rangle$ by the formula

$$T(xe) = (\mathrm{tr}_{2^{N-2}} \otimes id_{\mathcal{A}})E(x), \qquad \text{for } x \in \mathcal{M},$$

where $\mathrm{tr}_{2^{N-2}}$ is the normalized trace on $M_{2^{N-2}}(\mathbf{C})$. Under the above isomorphism,

$$\langle \mathcal{M}, e\rangle \cong M_4(\mathbf{C}) \otimes M_{2^{N-2}}(\mathbf{C}) \otimes L^\infty(X,\mu) \otimes \mathcal{L}(\ell^2(\mathbf{Z})),$$

and T corresponds to $\mathrm{Tr}_4 \otimes \mathrm{tr}_{2^{N-2}} \otimes id_{L^\infty(X)} \otimes \mathrm{Tr}$, where Tr_4 is the unnormalized trace on $M_4(\mathbf{C})$, and Tr is the usual trace on $\mathcal{L}(L^2(\mathbf{Z}))$. Then $T(e)$ is the constant function 1 in $L^\infty(X, \mu)$.

Lemma 2.7. *In the context above, there exists a projection $f \in \mathcal{L}' \cap \langle \mathcal{L}, e\rangle$ such that*

$$\int_X T((e-f)^2)\, d\mu \leq \frac{\sqrt{200\varepsilon}}{(1-(200\varepsilon)^{1/4})^2} < \frac{1}{4}.$$

Proof. For any $u \in \mathcal{U}(\mathcal{L})$, we get, by the same argument as after the formula (8) on p. 21 in Christensen [2],

$$T((e-u^*eu)^2) = 2E_{\mathcal{A}}((u-E(u))^*(u-E(u))),$$

where $E_{\mathcal{A}}$ is a conditional expectation of $\mathcal{M}$ onto $\mathcal{A}$. Thus by the argument on p. 22 and Lemma 2.1 of [2], we obtain f as desired. (Christensen's φ corresponds to our $\int_X T(x)\, dx$.) Q.E.D.

Proof of Assertion 2.6. We first follow the proof of Theorem 4.7 in Christensen [2]. Let e_{jk}, $1 \leq j, k \leq 2$, be the matrix units in

$$M_2(\mathbf{C}) \otimes \mathbf{C} \otimes \mathbf{C} \cdots \subset \mathcal{M} \subset \langle \mathcal{M}, e\rangle.$$

Define p_{jk} to be the range projection of $e_{jk}e$. Then by Christensen's argument, these are mutually orthogonal, and equivalent to e. Setting $p = p_{11} + p_{22} \in \langle \mathcal{M}, e\rangle$, $T(p)$ is the constant 2 in $L^\infty(X, \mu)$. Christensen's argument also shows that

$$\left|\int_X (T(f) - T(e))\, d\mu\right| = \left|\int_X T(f)\, d\mu - 1\right| < \frac{1}{4}. \tag{5}$$

Suppose that $T(f) = 0$ on $Y \subseteq X$. Then by Lemma 7, we get $\mu(Y) = \int_Y T((e-f)^2)\,d\mu < 1/4$. If $T(f) > T(p) = 2$ on $X \setminus Y$, we would have $\int_X T(f)\,d\mu \geq 2(1-\mu(Y)) > 3/2$, contradicting (5). Thus we have a subset Z of X such that $\mu(Z) > 0$ and $0 < T(f) \leq T(p)$ on Z. This implies that $f\chi_Z \prec p\chi_Z$, i.e., there exists a projection $q = q\chi_Z \in \langle \mathcal{M}, e\rangle$ such that $f\chi_Z \sim q\chi_Z \leq p\chi_Z$. Choose $v \in \langle \mathcal{M}, e\rangle$ so that $v^*v = q\chi_Z$ and $vv^* = f\chi_Z$, and define a map

$$x \longmapsto v^*xv \in \langle \mathcal{M}, e\rangle_p \cong \langle \mathcal{M}, e\rangle_e \otimes M_2(\mathbf{C}) \cong \mathcal{N} \otimes M_2(\mathbf{C}),$$

for $x \in \mathcal{L} \cong \mathcal{P}$. (See Proposition 3.1.5 in Jones [6].) The above map defines a non-trivial homomorphism Φ from $\mathcal{P}$ to $\mathcal{N} \otimes M_2(\mathbf{C})$ because $f \in \mathcal{L}$. Q.E.D.

Now finally we obtain a contradiction as follows. Our Φ is a map from $M_{2^N}(\mathbf{C})$ into $M_{2^{N-1}}(\mathbf{C}) \otimes L^\infty(Z)$. Consider the centre-valued trace $T' = \mathrm{tr}_{2^{N-1}} \otimes id_{L^\infty(Z)}$ on $M_{2^{N-1}}(\mathbf{C}) \otimes L^\infty(Z)$. Choose minimal projections $q_1, \ldots, q_{2^N}$ in $M_{2^N}(\mathbf{C})$ with $q_1 + \cdots + q_{2^N} = 1$. Then $T'(\Phi(q_1)) = \cdots = T'(\Phi(q_{2^N}))$ and, hence, $T'(\Phi(q_1)) \leq 1/2^N$ in $L^\infty(Z)$, which is impossible in $T'(M_{2^{N-1}}(\mathbf{C}) \otimes L^\infty(Z))$. Thus we conclude that α_t and $\beta_t = \alpha_{9t}$ are not cocycle conjugate. This also implies α_t and α_{3t} are not cocycle conjugate because otherwise we would have $\alpha_t \simeq \alpha_{3t} \simeq \alpha_{9t}$, a contradiction. (The symbol $\simeq$ means cocycle conjugacy.)

Remark 2.8. The proof of Theorem 4.7 in [2] contains a small mistake. The statement $(L_0 \cup r)'' = (L_0 \cup p)''$ in line 20 on page 25 is invalid. Thus the map constructed in the proof is a homomorphism from M to $(L_0 \cup p)''_r \cong (L_0 \cup p)''_p \otimes M_4(\mathbf{C}) \cong N \otimes M_4(\mathbf{C})$, not $N \otimes M_2(\mathbf{C})$. But if we use $r = r_{11} + r_{22}$ as above instead of $r = r_{11} + r_{12} + r_{21} + r_{22}$ in [2], we still have $\varphi(q) < 2 = \varphi(r)$, and therefore by the same argument, we get a homomorphism from M to $(L_0 \cup p)''_r \cong N \otimes M_2(\mathbf{C})$. Hence the conclusion of Theorem 4.7 is valid, and our proof is not affected.

Theorem 2.9. *There exists a countably infinite family of one-parameter automorphism groups of the AFD type II_1 factor $\mathcal{R}$, all members of which are stably conjugate, but no two members of which are cocycle conjugate.*

Proof. Consider

$$\alpha_t^{(k)} = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\exp 2\pi i t \begin{pmatrix} 3^{n+k}/2 & 0 \\ 0 & -3^{n+k}/2 \end{pmatrix}\right),$$

for each $k \geq 0$. The above argument gives the conclusion. Q.E.D.

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