

# Topological Quantum Field Theories and Operator Algebras

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## 1 Introduction

We have seen much fruitful interactions between 3-dimensional topology and operator algebras since the stunning discovery of the Jones polynomial for links [19] arising from his theory of subfactors [18] in theory of operator algebras. In this paper, we review the current status of theory of “quantum” topological invariants of 3-manifolds arising from operator algebras. The original discovery of topological invariants arising from operator algebras was for knots and links, as above, rather than 3-manifolds, but here we concentrate on invariants for 3-manifolds. On the way of studying such topological invariants, we naturally go through topological invariants of knots and links. From operator algebraic data, we construct not only topological invariants of 3-manifolds, but also topological quantum field theories of dimension 3, in the sense of Atiyah [2], as the title of this paper shows, but for simplicity of expositions, we consider mainly complex number-valued topological invariants of oriented compact manifolds of dimension 3 without boundary.

All the constructions of such topological invariants we discuss here are given in the following steps.

1. Obtain combinatorial data arising from representation theory of an operator algebraic system.
2. Realize a manifold concretely using basic building blocks.
3. Multiply or add the complex numbers appearing in the data in Step 1, in a way specified by how the basic building blocks are composed in Step 2, and compute the resulting complex number.

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Table 1: Topological invariants arising from operator algebras

Operator Algebras	Representation Theory	Combinatorial Construction
Subfactors	Quantum $6j$ -symbols	TVO invariants
Nets of factors on $S^1$	Braided tensor categories	RT invariants

4. Prove that the complex number in Step 3 is independent of how the basic building blocks are composed, as long as the homeomorphism class of the resulting manifold is fixed.

In Step 1, the prototype of the representation theory for operator algebras is the one for finite groups. That is, for a finite group  $G$ , we consider representatives of unitary equivalence classes of irreducible unitary representations. This finite set has an algebraic structure arising from the tensor product operation of representations, and it produces combinatorial data such as fusion rules and  $6j$ -symbols. In our setting, we work on some form of representation theory of operator algebraic systems analogous to this classical representation theory of finite groups.

Steps 2 and 3 already appear in the original definition of the Jones polynomial [19], where each link is represented as a closure of a braid, the Jones polynomial is defined from such a braid through certain representation theory, and then it is proved that this polynomial is independent of a choice of a braid for a fixed link.

This strategy should work, in principle, in any dimension, but so far, most of the interesting constructions arising from operator algebras are for dimension 3, so we concentrate in this case in this survey.

There have been many constructions of such topological invariants for 3-dimensional manifolds and two of them are particularly related to operator algebras. One is a construction of Turaev-Viro [36] in a generalized form due to Ocneanu, and the other is the one by Reshetikhin-Turaev [33]. For these two, the triple of operator algebraic systems, representation theoretic data, and the topological invariants in each case is listed as in Table 1.

Since both operator algebras and (topological) quantum field theory are of infinite dimensional nature, one expects a direct and purely infinite dimensional construction of the latter from the former, but such a construction has not been known yet. All the constructions below go through representation theoretic combinatorial data who “live in” finite dimensional spaces, so one could eliminate the initial infinite dimensionality entirely, if one is interested in only new constructions and computations of topological invariants of 3-dimensional manifolds. Still, the infinite dimensional framework of operator algebras is useful, as we see below, even in such a case, because it gives a conceptually convenient working place for various constructions and computations.

We also mention one reason we operator algebraists are interested in this type of theory, even purely from a viewpoint of operator algebras. Classification theory is a central topic in theory of operator algebras, and representation theory gives a very important invariant for classification. Since a series of great works of A. Connes

in 1970's, it is believed that under some nice analytic condition, generally called “amenability”, a certain representation theory should give a complete invariant of operator algebraic systems, such as operator algebras themselves, group actions on them, or certain families of operator algebras. For this reason, studies of representation theories in operator algebraic theory are quite important since old days of theory of operator algebras. What is new after the emergence of the Jones theory is that the representation theory now has a “quantum” nature, whatever it means.

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## 2 Turaev-Viro-Ocneanu invariants

Here we review the Turaev-Viro-Ocneanu invariants of 3-dimensional manifolds. The book [11] is a basic reference.

Our operator algebra here is a so-called *von Neumann algebra*, which is an algebra of bounded linear operators on a certain Hilbert space that is closed under the  $*$ -operation and the strong operator topology. (Here we consider only infinite dimensional separable Hilbert spaces, though a general theory exists for other Hilbert spaces.) Requiring closedness under the weak operator topology, we obtain the same class of operator algebras. If we use a norm topology, we have a wider class of operator algebras called  $C^*$ -algebras. Although von Neumann algebras give a subclass of  $C^*$ -algebras, it is not very useful, except for some elementary aspects of the theory, to regard a von Neumann algebra as a  $C^*$ -algebra, because a von Neumann algebra is far from being a “typical”  $C^*$ -algebra. For example, most of natural  $C^*$ -algebras are separable, as Banach spaces, but von Neumann algebras are never separable, unless they are finite dimensional. We assume, as usual, that a von Neumann algebra contains the identity operator, which is the unit of the algebra. A commutative  $C^*$ -algebra having a unit is the algebra of all the continuous functions on a compact Hausdorff space, and a commutative von Neumann algebra is the algebra of  $L^\infty$ -functions on a measure space. This gives a reason for a basic idea that a general  $C^*$ -algebra is a “noncommutative topological space” and a general von Neumann algebra is a “noncommutative measure space”. A finite dimensional  $C^*$ - or von Neumann algebra is isomorphic to a finite direct sum of full matrix algebras  $M_n(\mathbf{C})$ . In this paper, we deal with only *simple* von Neumann algebras in the sense that they have only trivial two-sided closed ideals in the strong or weak operator topology. This simplicity is equivalent to triviality of the center of the algebra, and we call such a von Neumann algebra a *factor*, rather than a simple von Neumann algebra.

In the Murray-von Neumann classification, factors are classified into type I, type  $\text{II}_1$ , type  $\text{II}_\infty$ , and type III. Factors of type I are simply all the bounded linear operators on some Hilbert space, and they are not interesting for the purpose of this survey. We are interested in factors of type  $\text{II}_1$  in the following two sections and those of type III in the last section. Although technical details on these factors are not necessary for conceptual understanding of the theory, we give brief explanations on

how to construct such factors.

We start with a countable group  $G$ . The (left) regular representation gives a unitary representation of  $G$  on the Hilbert space  $\ell^2(G)$ . We consider the von Neumann algebra generated by its image. If the group  $G$  is commutative, the resulting von Neumann algebra is isomorphic to  $L^\infty(\hat{G})$ . If the group  $G$  is “reasonably noncommutative” in an appropriate sense, the resulting von Neumann algebra is a factor of type  $\text{II}_1$ . One example of such a group is that of all permutations of a countable set that fix all but finite elements.

Another construction of a factor arises from an infinite tensor product of the  $n \times n$ -matrix algebra  $M_n(\mathbf{C})$ . We can define such an infinite tensor product in an appropriate sense, and then this infinite dimensional algebra has a natural representation on a separable Hilbert space. The von Neumann algebra generated by its image is a type  $\text{II}_1$  factor and these are all isomorphic, regardless  $n$ . This infinite tensor product also has many other representations on Hilbert spaces and “most” of them generate factors of type  $\text{III}$ .

The most natural starting point of a representation theory for factors is certainly a study of all representations of a fixed factor on Hilbert spaces. (A factor is an algebra of operators on a certain Hilbert space by definition, but we consider representations on other Hilbert spaces. In our setting, it is enough to consider only representations on infinite dimensional separable Hilbert spaces.) We certainly have a natural notion of unitary equivalence for representations of factors of type  $\text{II}_1$  or  $\text{III}$ , but this notion is not particularly interesting, as follows. Such representations are never irreducible, and for a fixed type  $\text{II}_1$  factor, we can classify representations completely, up to unitary equivalence, with a single invariant, called a coupling constant, due to Murray and von Neumann, having values in  $(0, \infty]$ . (This invariant produces the Jones index as below, and produces something deep in this sense, but the classification of representations themselves is rather simple and classical.) For factors of type  $\text{III}$ , the situations are even simpler; they are all unitarily equivalent for a fixed type  $\text{III}$  factor.

A representation of a factor can be regarded as a (left) module over a factor, trivially. It was Connes who realized first that the right setting for studying representation theory of factors is to study *bimodules*, rather than modules. That is, we consider two factors  $M$  and  $N$ , which could be equal, and study a Hilbert space  $H$  which is a left  $M$ -module and a right  $N$ -module with the two actions commuting. We call such  $H$  an  $M$ - $N$  bimodule and write  ${}_M H_N$ . The situation where both  $M$  and  $N$  are of type  $\text{II}_1$  is technically simpler. We have natural notions of irreducible decomposition, dimensions having values in  $(0, \infty]$  which are defined in terms of the coupling constants, contragredient bimodules, and relative tensor products. For example, for two bimodules  ${}_M H_N$  and  ${}_N K_P$ , we can define an  $M$ - $P$  bimodule  ${}_M H \otimes_N K_P$  and the dimension is multiplicative. For a factor  $M$ , the algebra  $M$  itself trivially has the left and right actions of  $M$ , so it has a bimodule structure, but this  $M$  is not a Hilbert space. We have a natural method to put an inner product on  $M$  and complete it, and in this way, we obtain an  $M$ - $M$  bimodule. By an abuse of notation, we often

write  ${}_M M_M$  for this bimodule, by ignoring the completion. This bimodule has dimension one, and plays a role of a trivial representation. In this way, our representation theory is quite analogous to that of a compact group. Connes used a terminology *correspondences* rather than bimodules. See [30] for a general theory on bimodules.

Jones initiated studies of inclusions of factors  $N \subset M$  in [18]. Such  $N$  is called a *subfactor* of  $M$ . By an abuse of terminology, the inclusion  $N \subset M$  is often called a subfactor. Technically simpler situations are that both  $M$  and  $N$  are of type  $\text{II}_1$ . Then we have an  $M$ - $M$  bimodule  ${}_M M_M$  as above, and we restrict the left action to the subalgebra  $N$  to obtain  ${}_N M_M$ . The dimension of this bimodule is called the *Jones index* of the subfactor  $N \subset M$  and written as  $[M : N]$ . (This terminology and notation come from an analogy to a notion of an index of a subgroup.) Jones proved in [18] an astonishing statement that this index takes values in  $\{4 \cos^2(\pi/n) \mid n = 3, 4, 5, \dots\} \cup [4, \infty]$  and all the values in this set are realized. This is in a sharp contrast to the fact that the coupling constant of a type  $\text{II}_1$  factor  $M$  can take all values in  $(0, \infty]$ . Jones introduced the *basic construction* whose successive uses produce an increasing sequence  $N \subset M \subset M_1 \subset M_2 \subset \dots$  and using this, he introduced the *higher relative commutants* and the *principal graph* for subfactors. Although we do not give their definitions here, we only mention that if the subfactor has index less than 4, then the principal graph is one of the *A-D-E* Dynkin diagrams, as noted by Jones. (See [11, Chapter 9] for precise definitions.)

It was Ocneanu [27] who realized that these invariants and further finer structures related to them can be captured by theory of bimodules and that they can be characterized by a set of combinatorial axioms. We explain his theory here. See [11, Chapter 9] for more details. We start with a type  $\text{II}_1$  subfactor  $N \subset M$  with finite Jones index. (If we have a finite index and one of  $N$  and  $M$  is of type  $\text{II}_1$ , then the other is also of type  $\text{II}_1$  automatically.) Ocneanu's idea was to develop a *representation theory for a pair  $N \subset M$* . We start with  ${}_N M_M$  and this plays a role of the fundamental representation. We also have  ${}_M M_N$  and make relative tensor products such as  ${}_N M \otimes_M M \otimes_N M_M$ . They are not irreducible in general, so we make irreducible decompositions. We look at all unitary equivalence classes of  $N$ - $N$  bimodules arising in this way. In general, we expect to have infinitely many equivalence classes, but it sometimes happens that we have only finitely many equivalence classes. This is the situation we are interested in, and in such a case, we say that the subfactor  $N \subset M$  has a finite depth. (The terminology "depth" comes from the way of Jones to write higher relative commutants.) This finite depth condition is similar to rationality condition in conformal field theory and quantum group theory. If we have a finite depth, we also have only finitely many equivalence classes of irreducible  $M$ - $M$  bimodules arising in the above way. Note that a compact group has only finitely many equivalence classes of irreducible unitary representations if and only if the group is finite. We assume the finite depth condition and fix a finite set of representatives of equivalence classes of irreducible  $N$ - $N$  bimodules arising as above from  $N \subset M$ . Note that it contains a trivial bimodule, that for each bimodule in the set, its contragredient bimodule is equivalent to one in the set, and that a relative tensor

product of two in the set decomposes into a finite direct sum of irreducible bimodules each of which is equivalent to one in the set. We say such a finite set of bimodules is a finite system of bimodules. Choose three, not necessarily distinct, irreducible  $N$ - $N$  bimodules  $A, B, C$  in the system. Then we can decompose  $A \otimes_N B \otimes_N C$  in two ways. That is, we first decompose  $A \otimes_N B$  in one, and we first decompose  $B \otimes_N C$  in the other. In this way, we obtain the “quantum” version of the classical  $6j$ -symbols which produce a complex number from six bimodules and four intertwiners. Such quantum  $6j$ -symbols were known in the quantum group theory, and Ocneanu found that a general system of bimodules produce similar  $6j$ -symbols and that classical properties such as the Frobenius reciprocity also holds in this setting. Associativity of the relative tensor product gives a so-called pentagonal relation as in the classical setting. This finite system of bimodules and quantum  $6j$ -symbols are the combinatorial data arising from a representation theory of a subfactor  $N \subset M$ .

Turaev and Viro [36] constructed topological invariants of 3-dimensional manifolds using the quantum  $6j$ -symbols for the quantum group  $U_q(sl_2)$  at roots of unity, and Ocneanu realized that a generalized version of this construction works for general quantum  $6j$ -symbols arising from a subfactor of finite Jones index and finite depth as above. The construction goes as follows for a fixed finite system of bimodules. (See [11, Chapter 12] for more details.)

We first make a triangulation of a manifold. That is, we regard a manifold made of gluing faces of finitely many tetrahedra so that we have an empty boundary and compatible orientation. Then we label each of the six edges with bimodules in the system and each of the four faces, triangles, with (co-)isometric intertwiners. When all the tetrahedra are labeled in this way, the quantum  $6j$ -symbol produce a complex number for each labeled tetrahedron. This number is simply a composition of the four intertwiners, up to normalization arising from dimensions of the four bimodules. (The composed intertwiners give a complex number because of irreducibility of the bimodules.) The well-definedness of this number comes from the so-called tetrahedral symmetry of quantum  $6j$ -symbols. Then we multiply all these numbers over all the tetrahedra in the triangulation, and add these products over all isometric intertwiners in an orthonormal basis for each face and over all labeling of edges with bimodules. With an appropriate normalization arising from dimensions of the bimodules, the resulting number is a topological invariant of the original 3-dimensional manifold. In order to prove this topological invariance, one has to prove that the complex number is independent of triangulations of a manifold. The relations of two triangulation of a manifold have been known by Alexander. That is, one triangulation is obtained from the other by successive applications of finitely many local changes of triangulations, called Alexander moves. (This result of Alexander holds in any dimension.) Pachner has proved that a different set of local moves also gives a similar theorem, and this set is more convenient for our purpose. That is, it is enough for us to prove that the above complex number is invariant under each of the Pachner moves. This invariance follows from properties of the quantum  $6j$ -symbols, such as the pentagon relation. So we conclude that the above complex number gives a well-defined topological in-

variant of 3-dimensional closed oriented manifolds. If we reverse the orientation, the topological invariant becomes the complex conjugate of the original value. In the original setting of Turaev-Viro [36] based on the quantum  $6j$ -symbols of  $U_q(\mathfrak{sl}_2)$ , the resulting invariants are real, so they do not detect orientations, but there is an example of a subfactor which produces a non-real invariant for some manifold and thus can detect orientations. (Actually, the original construction of Turaev-Viro [36] works without orientability.) Also note that in our setting, each intertwiner space has a Hilbert space structure and each dimension of a bimodule, which is sometimes called a *quantum dimension*, is positive. Such a feature is called *unitarity* of quantum  $6j$ -symbols, and this unitarity does not necessarily hold in a purely algebraic setting of quantum  $6j$ -symbols for quantum groups. We can apply the same construction by using the system of the  $M$ - $M$  bimodules instead of that of the  $N$ - $N$  bimodules, but the resulting invariant is the same.

A large class of subfactors are constructed with methods related to classical theory of groups and Hopf algebras, and their “quantum” counterparts, that is, quantum group theory and conformal field theory such as the Wess-Zumino-Witten models. For such subfactors, we have various interesting studies from an operator algebraic viewpoint, but if we are interested only in resulting topological invariants through the above machinery, they do not produce really new invariants. It is, however, expected that we have much wider varieties of subfactors in general. One “evidence” for such expectation is study of Haagerup [15]. By purely combinatorial arguments, he found a list of candidates of subfactors of finite depth in the index range  $(4, 3 + \sqrt{2})$ , and it seems that most of these are indeed realized. None of them seem to be related to conformal field theory or today’s theory of quantum groups so far. Haagerup himself proved that the first one in the list is indeed realized, and Asaeda and he further proved that another in the list is also realized in [1]. The nature of topological invariants arising from these two subfactors is not understood yet, but we expect that they contain some interesting information. Since the list of Haagerup is only for a small range of the index values, we expect that we would have by far more examples of “exotic” subfactors as mentioned above, but an explicit construction of even a single example is highly difficult. We know almost nothing about topological meaning of invariants arising from such subfactors. Izumi [17] has some more examples of such interesting subfactors.

### 3 Reshetikhin-Turaev invariants

Another construction of topological invariants due to Reshetikhin-Turaev [36] requires a “higher symmetry” for combinatorial data arising from a representation theory. This higher symmetry is called a *modularity* of a tensor category. It is also called a *nondegenerate braiding*.

Wenzl has a series of work [39, 37, 38, 40], partly with V. G. Turaev, on related constructions, but here we concentrate on two methods producing a modular tensor category from a general operator algebraic representation theory. One is within sub-

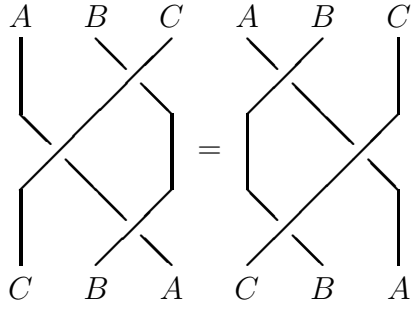


Figure 1: Yang-Baxter equation

factor theory, due to Ocneanu, and presented in this section, and the other is due to Longo, Müger and the author [22], explained in the next section.

We first give a brief explanation on braiding. In a representation theory of a group, two tensor products  $\pi \otimes \sigma$  and  $\sigma \otimes \pi$  are obviously unitarily equivalent for two representations  $\pi$  and  $\sigma$ , but for two  $N$ - $N$  bimodules  $A$ ,  $B$ , we have no reason to expect that  $A \otimes_N B$  and  $B \otimes_N A$  are equivalent, and they are indeed not equivalent in general. It is, however, possible that for all  $A$  and  $B$  in a finite system, we have equivalence of  $A \otimes_N B$  and  $B \otimes_N A$ . If we can choose isomorphisms of these two bimodules in a certain compatible way simultaneously for all bimodules in the system, we say that the system has a *braiding*. See [32] for more details, where an equivalent, but slightly different formulation using endomorphisms, rather than bimodules, is presented.

The isomorphism between  $A \otimes_N B$  and  $B \otimes_N A$  can be graphically represented as an overcrossing of two wires labeled with  $A$  and  $B$ , respectively. Then the assumption on “compatibility” implies, for example, the Yang-Baxter equation, which represents the Reidemeister move of type III as in Fig. 1, where each crossing represents an isomorphism and each hand side is a composition of three such isomorphisms.

In representation theory of groups, the tensor product operation is trivially commutative in the above sense. This is “too commutative” in the sense that we have no distinction between an overcrossing and an undercrossing in the above graphical representation, and this is not very useful for construction of topological invariants, obviously. So, in order to obtain an interesting topological invariant, an overcrossing and an undercrossing must be “sufficiently different”. Such a condition is called non-degeneracy of the braiding. This condition can be also formulated in the language of tensor categories, and then it is called a modularity of the tensor category. A non-degenerate braiding, or a modular tensor category, produces a unitary representation of the modular group  $SL(2, \mathbf{Z})$ .

We first explain how to obtain such a nondegenerate braiding in subfactor theory. We start with a subfactor  $N \subset M$  with finite Jones index and finite depth. Then



Ocneanu has found a construction of a new subfactor from this subfactor, which is called the asymptotic inclusion [27]. He realized that the system of bimodules for this new subfactor has a nondegenerate braiding and it can be regarded as the “quantum double” of the original system of  $N$ - $N$  (or  $M$ - $M$ ) bimodules arising from the subfactor  $N \subset M$ . Note that the original system of  $N$ - $N$  bimodules and that of  $M$ - $M$  bimodules are not isomorphic in general, but they have the same “quantum double” system of bimodules. Popa has a more general construction of this type, called the symmetric enveloping algebra [31]. Longo-Rehren [25] has found the essentially same construction as the asymptotic inclusion in the setting of algebraic quantum field theory. See [11, Chapter 12] for more details on the asymptotic inclusion and [16, 17] for detailed analysis based on the Longo-Rehren approach.

Suppose we have a nondegenerate braiding. It is also known that such a braiding can arise from quantum groups or conformal field theory. Reshetikhin-Turaev [33] has constructed a topological invariant of 3-dimensional manifold from such a system. First we draw a picture of a link on a plane. This has various overcrossings and undercrossings. We label each connected component with an irreducible bimodule in the system, then each crossing gives an isomorphism arising from the braiding. Then this labeled picture produces a complex number as a composition of these isomorphisms. This is an invariant of “colored links”, where coloring means labeling of each component with an irreducible bimodule. Actually, this number is not invariant under the Reidemeister move of type I, and it is invariant under only the Reidemeister moves of type II and type III, so this is not a topological invariant of colored links, but it gives a “regular isotopy” invariant of colored links, for which invariance under the Reidemeister moves of type II and type III is sufficient. Then we sum these complex numbers over all possible colorings, with appropriate normalizing weights arising from dimensions of the bimodules. In this way, we obtain a complex number from a planar picture of a link. There is a method to construct a 3-dimensional oriented closed manifold from such a planar picture of a link, called the *Dehn surgery*. Roughly speaking, we embed a link in the 3-sphere, and remove a tubular neighbourhood, consisting of a disjoint union of solid tori, from the 3-sphere, and then put back the solid tori in a different way. Different links can produce the same 3-dimensional manifolds, but again, it is known that in such a case, the two links can be transformed from one to the other with successive applications of local moves. Such moves are called Kirby moves. Reshetikhin and Turaev have proved that nondegeneracy of the braiding implies invariance of the above complex number, the weighted sum of colored link invariants, under Kirby moves, thus we obtain a topological invariant of 3-dimensional manifolds in this way. Reshetikhin and Turaev considered an example arising from the quantum groups  $U_q(sl_2)$  at roots of unity, but the general machinery applies to any nondegenerate braiding. See the book [35] for more details on this construction.

So, starting with a subfactor with finite Jones index and finite depth, we have two topological invariants of 3-dimensional manifolds. One is the Turaev-Viro-Ocneanu invariant arising from the system of  $N$ - $N$  bimodules. The other is the Reshetikhin-

Turaev invariant of the “quantum double” system of the original system of  $N$ - $N$  bimodules. It is quite natural to investigate the relation between these two invariants. Sato, Wakui and the author proved in [23] that these two invariants coincide. Ocneanu [29] has also announced such coincidence and it seems to us that his method is different from ours. Sato and Wakui [34] also made explicit computations of this invariant for various concrete examples of subfactors and manifolds, based on Izumi’s explicit computations of the representations of the modular group arising from some subfactors, including the “exotic” one due to Haagerup, in [17].

Another computation of topological invariants arising from subfactors is based on  $\alpha$ -induction [25, 41, 3, 4, 5]. This method, in particular, produces subfactors with principal graphs  $D_{2n}$ ,  $E_6$ , and  $E_8$ , and the corresponding Turaev-Viro-Ocneanu invariants can be computed once we have a description of the “quantum doubles” by [23], and these quantum doubles were computed in [6]. (Also see [29].) This  $\alpha$ -induction is also related to theory of modular invariants [7]. See [3, 4, 5, 20, 21] for more on this topic.

## 4 Algebraic quantum field theory

Another occurrence of nondegenerate braiding in theory of operator algebras is in algebraic quantum field theory [14], which has its own long history. This theory is an approach to quantum field theory based on operator algebras. That is, in each bounded region on a spacetime, we assign a von Neumann algebra on a fixed Hilbert space. We think that each such von Neumann algebra is generated by observable physical quantities in the bounded region in the spacetime. In this way, we think that this family of von Neumann algebras parametrized by bounded regions gives a mathematical description of a physical theory. We often restrict bounded regions to those of a special form. We impose a physically natural set of axioms on this family of von Neumann algebras and make a mathematical study of such axiomatized systems. A spacetime of any dimension is allowed in this axiomatized approach, and the four dimensional case was studied originally for an obvious physical reason. These studies of Doplicher-Haag-Roberts [8] and Doplicher-Roberts [9, 10] have been quite successful. Recently, it has been realized that this theory in lower dimensional spacetime has quite interesting mathematical structures. Two-dimensional case has caught much attention in connection to conformal field theory and one-dimensional case also naturally appears in a “chiral” decomposition of a two-dimensional theory. Mathematical structures of one-dimensional theory was studied in [12]. In one-dimensional case, our “spacetime” is simply  $\mathbf{R}$  and a bounded region is simply a bounded interval. It is often convenient to compactify the space  $\mathbf{R}$  to obtain  $S^1$  and consider “intervals” contained in  $S^1$ . In this setting, our mathematical structure is a family of von Neumann algebras on a fixed Hilbert space parameterized by intervals in  $S^1$ . We impose a set of axioms. For example, one axiom requires that we have a larger von Neumann algebra for a larger interval. Another axiom requires “covariance” of the theory with respect to a projective unitary representation of a certain group of the “spacetime

symmetry”. We also have an axiom on “locality” which says if two regions are space-like separated, then the corresponding von Neumann algebras mutually commute. Another requires existence of a “vacuum” vector in the Hilbert space. Positivity of energy in the sense that a certain self-adjoint operator is positive is also assumed. See [13, 22] for a precise description of the set of axioms. (Actually the main results in [22] hold under a weaker set of axioms, but we do not go into details here.) Under the usual set of axioms, each von Neumann algebra for an interval becomes a factor of type III, so we call such a family a *net of factors*. Now the index set of intervals on the circle  $S^1$  is not directed with respect to inclusions, since the entire circle is not allowed as an interval, so it is not appropriate to call such a family a *net*, but this terminology has been commonly used.

This family is our operator algebraic system and we consider a representation of such a family of von Neumann algebras. Such an idea is due to Doplicher-Haag-Roberts [8] and is called the DHR theory. We have a natural notion of irreducibility, dimensions, and tensor products for such representations. Note that we do not have an obvious definition of tensor products for two representations of such a net of factors. The key idea was that the tensor product operation is realized through compositions of endomorphisms. Also the dimension in the usual sense is always infinite. So it was highly nontrivial to obtain sensible definitions of the tensor product and the dimension. This work is much older than the subfactor theory in the previous section, and its similarity to subfactor theory was soon recognized in [24] in a precise form.

In this way, we have a representation theory for a net of factors. A tensor product operation is “too commutative” for higher dimensional spacetime, but in dimensions one and two, it has an appropriate level of commutativity, and naturally produces a braiding. (See [12] for example.) So we have two problems for getting a modular tensor category from such a representation of a net of factors on  $S^1$ . One is whether we have only finitely many equivalence classes of irreducible representations or not. The other is whether the braiding is nondegenerate or not. In [22], Longo, Müger and the author have found a nice operator algebraic condition that implies positive answers to these two problems and we introduced the terminology “complete rationality” for this notion. One of the key conditions for this notion is finiteness of a certain Jones index. Note that in subfactor theory in the previous section, our “family of operator algebras” has only two factors  $N$  and  $M$ , and its representation theory produced a tensor category, without a braiding in general. Now our “family of operator algebras” is a net of factors and has continuously many factors with more structures, and its representation theory produces a braided tensor category.

Xu has proved in [42] that the  $SU(N)_k$ -nets corresponding to the WZW-models  $SU(N)_k$  are completely rational. Xu worked on coset models in the setting of nets of factors on  $S^1$  in [43], and obtained several interesting examples. He then studied in [44] about topological invariants arising from these nets, which seems to be quite interesting topologically. He also worked on orbifold models in this context in [45]. Finally, we also note that complete rationality is also important in classification

theory of nets of factors as in [20, 21].

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