

Conformal Field Theory and Operator Algebras

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Abstract

We review recent progress in operator algebraic approach to conformal quantum field theory. Our emphasis is on use of representation theory in classification theory. This is based on a series of joint works with R. Longo.

1 Introduction

A mathematically rigorous approach to quantum field theory based on operator algebras is called an *algebraic quantum field theory*. It has a long history since pioneering works of Araki, Haag, Kastker. (See [22] for a general treatment of algebraic quantum field theory.) This theory works on Minkowski spaces on any spacetime dimension, and there have been some recent results on curved spacetimes or even noncommutative spacetimes. In the case of 1+1-dimensional Minkowski space with higher spacetime symmetry, *conformal symmetry*, we have *conformal field theory* and there we have seen many new developments in the recent years, so we survey such results here. Our emphasis is on representation theoretic aspects of the theory and we make various comparison with another mathematically rigorous and more recent approach to conformal field theory, that is, theory of vertex operator algebras.

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Roughly speaking, a mathematical study of quantum field theory is a study of Wightman fields, which are certain type of operator-valued distributions on a spacetime with covariance with respect to a given spacetime symmetry group. We have mathematically rigorous axioms for such Wightman fields, but they involve distributions and unbounded operators, so these cause various kinds of technical difficulty. In contrast, in the algebraic quantum field theory, our fundamental object is a *net of von Neumann algebras* of bounded linear operators on a Hilbert space. (See [46] for general theory of von Neumann algebras.) Technical problems on definition domains of unbounded operators do not arise in this approach.

A basic idea is as follows. Suppose we have a Wightman field Φ on a spacetime. Fix a bounded region \mathcal{O} in the space time and consider a test function φ with support contained in \mathcal{O} . Then the pairing $\langle \Phi, \varphi \rangle$ produces an (unbounded) operator. We have many Φ and φ for a fixed \mathcal{O} and obtain many unbounded operators from such pairing. Then we consider a von Neumann algebra of *bounded* linear operators on this Hilbert space generated by these unbounded operators. (For example, if we have a self-adjoint unbounded operators, we consider its spectral projections which are obviously all bounded. In this way, we deal with only bounded operators.) This is regarded as a von Neumann algebra generated by *observables* in the spacetime region \mathcal{O} . A von Neumann algebra is an algebra of bounded linear operators which is closed under the adjoint operation and the strong operator topology. In this way, we have a family $\{\mathcal{A}(\mathcal{O})\}$ of von Neumann algebras on the same Hilbert space parameterized by spacetime regions. Since the spacetime regions make a net with respect to the inclusion order, we call such a family a net of von Neumann algebras. Now we forget Wightman fields and consider only a net of von Neumann algebras. We have some expected properties for such nets of von Neumann algebras from a physical consideration, and now we use these properties as *axioms*. So our mathematical object is a net of von Neumann algebras subject to certain set of axioms. Our mathematical aim is to study such nets of von Neumann algebras.

2 Conformal Quantum Field Theory

We first explain formulation of full conformal quantum field theory on the 1 + 1-dimensional Minkowski space in algebraic quantum field theory. As a spacetime region \mathcal{O} above, it is enough to consider only open rectangles \mathcal{O}

with edges parallel to $t = \pm x$ in $(1 + 1)$ -dim Minkowski space. In this way, we get a family $\{\mathcal{A}(\mathcal{O})\}$ of operator algebras parameterized by spacetime regions \mathcal{O} (rectangles). In order to realize conformal symmetry, we have to make a partial compactification of the $1 + 1$ -dimensional Minkowski space. If two rectangles are spacelike separated, then we have no interactions between them even at the speed of light, so our axiom requires that the corresponding two von Neumann algebras commute with each other. This is the *locality axiom*. Since this is not our main object in this paper, we omit details of the other axioms. See [29] for full details.

Next we briefly explain that boundary conformal field theory can be handled within the same framework. Now we consider the half-space $\{(x, t) \mid x > 0\}$ in the $1 + 1$ -dimensional Minkowski space and only rectangles \mathcal{O} contained in this half-space. In this way, we have a similar net of von Neumann algebras $\{\mathcal{A}(\mathcal{O})\}$ parameterized with rectangles in the half-space. See [38] for full details of the axioms.

If we have a net of von Neumann algebras over the $1 + 1$ -dimensional Minkowski space, we can *restrict* the net of von Neumann algebras to two *chiral* conformal field theories on the light cones $\{x = \pm t\}$. In this way, we have two nets of von Neumann algebras on the compactified S^1 as description of two *chiral* conformal field theories. Since this net is our main mathematical object in this article, we give a full set of axioms. (See [29] for details of this “restriction” procedure.)

Now our “spacetime” is S^1 and a “spacetime region” is an interval I , which means a non-empty, non-dense open connected subset of S^1 . We have a family $\{\mathcal{A}(I)\}$ of von Neumann algebras on a fixed Hilbert space H . These von Neumann algebras are simple and such von Neumann algebras are called *factors*, so the family $\{\mathcal{A}(I)\}$ satisfying the axioms below is called a *net of factors* (or an *irreducible local conformal net of factors*, strictly speaking). Actually, the set of intervals on S^1 is not directed with respect to inclusions, so the terminology *net* is not mathematically appropriate, but is widely used.

1. (isotony) For intervals $I_1 \subset I_2$, we have $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.
2. (locality) For intervals I_1, I_2 with $I_1 \cap I_2 = \emptyset$, we have $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = 0$
3. (Möbius covariance) There exists a strongly continuous unitary representation U of $PSL(2, \mathbb{R})$ on H satisfying $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$ for any $g \in PSL(2, \mathbb{R})$ and any interval I .

4. (positivity of energy) The generator of the one-parameter rotation subgroup of U , called the *conformal Hamiltonian*, is positive.
5. (existence of the vacuum) There exists a unit U -invariant vector Ω in H , called the *vacuum vector*, and the von Neumann algebra $\bigvee_{I \in S^1} \mathcal{A}(I)$ generated by all $\mathcal{A}(I)$'s is $B(H)$.
6. (conformal covariance) There exists a projective unitary representation U of $\text{Diff}(S^1)$ on H extending the unitary representation of $PSL(2, \mathbb{R})$ such that for all intervals I , we have

$$\begin{aligned} U(g)\mathcal{A}(I)U(g)^* &= \mathcal{A}(gI), \quad g \in \text{Diff}(S^1), \\ U(g)AU(g)^* &= A, \quad A \in \mathcal{A}(I), \quad g \in \text{Diff}(I'), \end{aligned}$$

where $\text{Diff}(S^1)$ is the group of orientation-preserving diffeomorphisms of S^1 and $\text{Diff}(I')$ is the group of diffeomorphisms g of S^1 with $g(t) = t$ for all $t \in I$.

The isotony axiom is natural because we have more test functions (or more observables) for a larger interval. The locality axiom takes this simple form on S^1 . The choice of the spacetime symmetry is not unique, and we can use the Poincaré symmetry on the Minkowski space or the Möbius covariance on S^1 , for example, but in the *conformal* field theory, we use *conformal* symmetry, which means diffeomorphism covariance as above. This set of axioms imply various nice conditions such as the Reeh-Schlieder property, the Bisognano-Wichmann property and the Haag duality. See [28] and references there for details.

In the usual situation, all the von Neumann algebras $\mathcal{A}(I)$ are isomorphic to the so-called Araki-Woods type III_1 factor for all nets \mathcal{A} and all intervals I . So each von Neumann algebra does not contain any information about the conformal field theory, but it is the relative position of the von Neumann algebras in the family that encodes the physical information of the theory. (It is similar to subfactor theory of Jones where we study a relative position of one factor in another.)

At the end of this section, we compare our formulation of conformal quantum field theory with another mathematically rigorous approach, the theory of vertex operator algebras. A vertex operator algebra is an algebraic axiomatization of Wightman fields on S^1 , called vertex operators. If we have an operator valued distribution on S^1 , its Fourier expansion should give

countably many (possibly unbounded) operators as the Fourier coefficients. Under the so-called state-field correspondence, any vector in the space of “states” should give an operator-valued distribution, a quantum “field”, and its Fourier expansion gives countably many operators. In this way, one vector should give countably many operators on the space of these vectors. In other words, for two vectors v, w we have countably many binary operations $v_{(n)}w$, $n \in \mathbb{Z}$, the action of the n -th operator given by v on w . An axiomatization of this idea gives a notion of *vertex operator algebra*. (See [16] for a precise definition. There is a slightly weaker notion of a *vertex algebra*. See [27] for its precise definition and related results.) In theory of vertex operator algebra, one considers a vector space of states without an inner product and even when we have a positive definite inner product, one considers this vector space without completion. Here in comparison to nets of factors, we are interested in the case where we have positive definite inner products on the spaces of states. We say that such a vertex operator algebra is *unitary*.

Both of one (unitary) vertex operator algebra and one net of factors should describe one chiral conformal field theory. So unitary vertex operator algebras and nets of factors should be in a bijective correspondence, at least under some “nice” additional conditions, but no general theorems have been known for such a correspondence, though there is a recent progress due to S. Carpi and M. Weiner. However, if we have one construction or an idea on one side, we can often “translate” it to the other side, though it can be highly non-trivial from a technical viewpoint. Fundamental sources of constructions for vertex operator algebras are affine Kac-Moody algebras and integral lattices. The corresponding constructions for nets of factors have been done by A. Wassermann [47] and his students, and Dong-Xu [12], respectively, after the initial construction of Buchholz-Mack-Todorov [5]. If we have examples with some nice properties, we can often construct new examples from them, and as such methods of constructions of vertex operator algebras, we have simple current extensions, the coset construction, and the orbifold construction. The simple current extensions for nets of factors are simply crossed products by DHR-automorphisms and easy to realize. (See the next section for a notion of DHR-endomorphisms.) The coset and orbifold constructions for nets of factors have been studied in detail by F. Xu [50, 51, 52].

For nets of factors, we have introduced a new construction of examples in [28] based on Longo’s notion of Q -systems [36]. Further examples have been constructed by Xu [55] with this method. This can be translated to the setting of vertex operator algebras, as we will see in this article later.

3 Representation Theory

An important tool to study nets of factors is a representation theory. For a net of factors $\{\mathcal{A}(I)\}$, all the algebras $\mathcal{A}(I)$ act on the initial Hilbert space H from the beginning, but we also consider their representations on another Hilbert space, that is, a family $\{\pi_I\}$ of representations $\pi_I : \mathcal{A}(I) \rightarrow B(K)$, where K is another Hilbert space, common for all I . For $I_1 \subset I_2$, we must have that the restriction of π_{I_2} on $\mathcal{A}(I_1)$ is equal to π_{I_1} . The representation on the initial Hilbert space is called the *vacuum representation* and plays a role of a trivial representation. We also have to take care of the spacetime symmetry group when we consider a representation, but this part is often automatic (see [20]), so we now ignore it for simplicity. See [20] for a more detailed treatment. Note that a representation of a net of factors is a counterpart of a module over a vertex operator algebra.

Notions of irreducibility and a direct sum for such representations are easy to formulate. Non-trivial notions are dimensions and tensor products. Each representation $\{\pi_I\}$ is in a bijective correspondence to a certain *endomorphism* λ of an infinite dimensional operator algebra, called a Doplicher-Haag-Roberts (DHR) endomorphism [13, 15], and we can restrict λ to a single factor $\mathcal{A}(I)$ for an arbitrarily but fixed interval I . Then $\lambda(\mathcal{A}(I)) \subset \mathcal{A}(I)$ is a subfactor and we have its Jones index [26]. (See [14, 41, 43] for general theory of subfactors.) The square root of this Jones index plays the role of the *dimension* of the representation [35]. In algebraic quantum field theory, such a dimension was called a *statistical dimension*, and it is analogous to a quantum dimension in the theory of quantum groups. It is a positive real numbers in the interval $[1, \infty]$. We can also compose endomorphisms and this composition gives the correct notion of *tensor products*. We then get a *braided* tensor category as in [15].

In representation theory of a vertex operator algebra (and also a quantum group), it sometimes happens that we have only finitely many irreducible representations. Such finiteness is often called *rationality*, possibly with some extra assumptions on some finite dimensionality. This also plays an important role in theory of quantum invariants in low dimensional topology. In [32], we have introduced an operator algebraic condition for such rationality for nets of factors as follows and we called it *complete rationality*. We split the circle into four intervals I_1, I_2, I_3, I_4 in this order, say, counterclockwise. Then complete rationality is given by the finiteness of the Jones index for a subfactor $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$ where $'$ means the commutant, together

with the *split property*. The split property is known to hold if the *vacuum character*, $\sum_{n=0}^{\infty} (\dim H_n) q^n$, is convergent for $|q| < 1$ by [9], so it usually holds and is easy to verify. (Here $H = \bigoplus_{n=0}^{\infty} H_n$ is the eigenspace decomposition of the original Hilbert space for the positive generator of the rotation group. So this convergence property can be verified simply by looking at the Hilbert space, not the von Neumann algebras.) In the original definition of complete rationality in [32], we required another condition called strong additivity, but it was proved to be redundant by Longo-Xu [39]. We have proved in [32] that this complete rationality implies that we have a *modular tensor category* as a representation category of $\{\mathcal{A}(I)\}$. A modular tensor category produces a 3-dimensional topological quantum field theory. (See [45] for general theory of topological quantum field theory.) The $SU(N)_k$ -net of Wassermann has been shown to be completely rational by [49].

We now introduce an important notion of α -*induction*. For an inclusion of nets of factors, $\mathcal{A}(I) \subset \mathcal{B}(I)$, we have an induction procedure analogous to the group representation. So from a representation of the smaller net \mathcal{A} , we would like to construct a representation of the larger net \mathcal{B} , but what we actually obtain is not a genuine representation of the larger net \mathcal{B} in general, and is something weaker called *solitonic*. This induction procedure is called the α -induction and depends a choice of braiding, so we write α^+ and α^- . This was first defined in Longo-Rehren [37] and studied in detail in Xu [48]. Then Böckenhauer-Evans [1] made a further study, and [2, 3] unified this study with Ocneanu's graphical method [42]. The intersection of the irreducible endomorphisms appearing in the images of α^+ -induction and α^- -induction gives the true representation category of $\{\mathcal{B}(I)\}$ if \mathcal{A} is completely rational by [2, 32].

This α -induction opens an important and new connection with theory of *modular invariants*. A modular tensor category produces a unitary representation π of $SL(2, \mathbb{Z})$ through its braiding as in [44], and its dimension is the number of irreducible objects. So a completely rational net of factors produces such a unitary representation. (Note that our representation of $SL(2, \mathbb{Z})$ comes from the braiding structure, not from the action of this group on the characters through change of variables $\tau \mapsto \frac{a\tau + b}{c\tau + d}$, though in all the "nice" known examples, these two representations coincide. See [30] for a discussion on this matter.)

It has been proved in [2] that the matrix $(Z_{\lambda,\mu})$ defined by

$$Z_{\lambda,\mu} = \dim \text{Hom}(\alpha_\lambda^+, \alpha_\mu^-)$$

is in the commutant of the representation π , using Ocneanu's graphical calculus [42]. Such a matrix Z is called a *modular invariant*, and we have only finitely many such Z for a given π . For any completely rational net $\{\mathcal{A}(I)\}$, any extension $\{\mathcal{B}(I) \supset \mathcal{A}(I)\}$ produces such Z . Matrices Z are certainly much easier to classify than extensions and this is a source of classification theory in the next section.

4 Classification Theory

For a net of factors, we can naturally define a *central charge* and it is well-known to take discrete values $1 - 6/m(m+1)$, $m = 3, 4, 5, \dots$, below 1 and all values in $[1, \infty)$ by [17, 18]. We have the *Virasoro net* $\{\text{Vir}_c(I)\}$ for each such c and it is the operator algebraic counterpart of the Virasoro vertex operator algebra with the same c . Any net of factors $\{\mathcal{A}(I)\}$ with central charge c is an extension of the Virasoro net with the same central charge and it is automatically completely rational if $c < 1$, as shown in [28]. So we can apply the above theory and we get the following complete classification list for the case $c < 1$ as in [28].

1. The Virasoro nets $\{\text{Vir}_c(I)\}$ with $c < 1$.
2. The simple current extensions of the Virasoro nets with index 2.
3. Four exceptionals at $c = 21/22, 25/26, 144/145, 154/155$.

The unitary representations of $SL(2, \mathbb{Z})$ for the Virasoro nets are the well-known ones, and all the modular invariants for these have been classified by [6]. Our result shows that each of the so-called type I modular invariants in the classification list of [6] corresponds to a net of factors uniquely. They are labeled with pairs of A - D_{2n} - $E_{6,8}$ Dynkin diagrams with Coxeter numbers differing by 1. Three in (3) of the above list have been identified with coset models, but the remaining one does not seem to be related to any other known constructions. This is constructed with "extension by Q-system". Xu [55] recently applied this construction to many other coset models and

obtained infinitely many new examples based on [54], called *mirror extensions*. Classification for the case $c = 1$ has been also done under some extra assumption [7, 53].

This classification theorem also implies a classification of certain types of vertex operator algebras as follows.

Let V be a (rational) vertex operator algebra and W_i be its irreducible modules. We would like to classify all vertex operator algebras arising from putting a vertex operator algebra structure on $\bigoplus_i n_i W_i$ and using the same Virasoro element as V , where n_i is multiplicity and $W_0 = V$, $n_0 = 1$. From a viewpoint of tensor category, this classification problem of extensions of a vertex operator algebras is the “same” as the classification problem of extensions of a completely rational net of factors, as shown in [24].

So the above classification theorem of local conformal nets implies a classification theorem of extensions of the Virasoro vertex operator algebras with $c < 1$ as above, and we obtain the same classification list. That is, besides the Virasoro vertex operator algebras themselves, we have their simple current extensions, and four exceptionals at $c = 21/22, 25/26, 144/145, 154/155$. With the usual notation of $L(c, h)$ for a module with central charge c and conformal weight h of the Virasoro vertex operator algebras with $c < 1$, the four exceptionals are listed as follows.

1. $L(21/22, 0) \oplus L(21/22, 8)$. It has 15 irreducible representations and has two coset realizations, from $SU(9)_2 \subset (E_8)_2$ and $(E_8)_3 \subset (E_8)_2 \otimes (E_8)_1$.
2. $L(25/26, 0) \oplus L(25/26, 10)$. It has 18 irreducible representations and has a coset realization from $SU(2)_{11} \subset SO(5)_1 \otimes SU(2)_1$.
3. $L(144/145, 0) \oplus L(144/145, 24) \oplus L(144/145, 78) \oplus L(144/145, 189)$. It has 28 irreducible representations and no coset realization has been known.
4. $L(154/155, 0) \oplus L(154/155, 26) \oplus L(154/155, 84) \oplus L(154/155, 203)$. It has 30 irreducible representations and has a coset realization from $SU(2)_{29} \subset (G_2)_1 \otimes SU(2)_1$.

Note that it is not obvious that the representation category of the Virasoro net Vir_c and the representation category of the Virasoro vertex operator algebra $L(c, 0)$ are isomorphic, but as long as the two are braided tensor category and have the same S - and T -matrices, the arguments in [28] work, so we obtain the above classification result for vertex operator algebras.

Using the above results and more techniques, we can also completely classify full conformal field theories within the framework algebraic quantum field theory for the case $c < 1$. Full conformal field theories are given as certain nets of factors on $1 + 1$ -dimensional Minkowski space. Under natural symmetry and maximality conditions, those with $c < 1$ are completely labeled with the pairs of A - D - E Dynkin diagrams with the difference of their Coxeter numbers equal to 1, as shown in [29]. We now naturally have D_{2n+1} , E_7 as labels, unlike in the chiral case. The main difficulty in this work lies in proving uniqueness of the structure for each modular invariant in the Cappelli-Itzykson-Zuber list [6]. This is done through 2-cohomology vanishing for certain tensor categories. in the spirit of [25].

Furthermore, using the above results and more techniques we can also completely classify boundary conformal field theories for the case $c < 1$. Boundary conformal field theories are given as certain nets of factors on a $1 + 1$ -dimensional Minkowski half-space. Under a natural maximality condition, these with $c < 1$ are now completely labeled with the pairs of A - D - E Dynkin diagrams with distinguished vertices having the difference of their Coxeter numbers equal to 1, as shown in [33] based on a general theory in [38]. The “chiral fields” in a boundary conformal field theory should produce a net of factors on the boundary (which is compactified to S^1) as in the operator algebraic approach. Then a general boundary conformal field theory restricts to this boundary to produce a *non-local* extension of this *chiral* conformal field theory on the boundary.

5 Moonshine Conjecture

The Moonshine conjecture, formulated by Conway-Norton [8], is about mysterious relations between finite simple groups and modular functions, since an observation due to McKay.

Today the classification of all finite simple groups is complete and the classification list contains 26 sporadic groups in addition to several infinite series. The largest group among the 26 sporadic groups is called the *Monster* group and its order is about 8×10^{53}

One the one hand, the non-trivial irreducible representation of the *Monster* having the smallest dimension is 196883 dimensional. On the other hand, the following function, called *j-function*, has been classically studied

in algebra.

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

For $q = \exp(2\pi i\tau)$, $\text{Im } \tau > 0$, we have modular invariance property, $j(\tau) = j\left(\frac{a\tau + b}{c\tau + d}\right)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, and this is the only function, up to the constant term, satisfying this property and starting with q^{-1} ,

McKay noticed $196884 = 196883 + 1$, and similar simple relations for other coefficients of the j -function and dimensions of irreducible representations of the Monster group turned out to be true. Then Conway-Norton [8] formulated the Moonshine conjecture roughly as follows, which has been now proved by Borcherds [4] in 1992.

1. We have a “natural” infinite dimensional graded vector space $V = \bigoplus_{n=0}^{\infty} V_n$ with some algebraic structure having a Monster action preserving the grading and each V_n is finite dimensional.
2. For any element g in the Monster, the power series $\sum_{n=0}^{\infty} (\text{Tr } g|_{V_n}) q^{n-1}$ is a special function called a *Hauptmodul* for some discrete subgroup of $SL(2, \mathbb{R})$. When g is the identity element, the series is the j -function minus constant term 744.

For the part (1) of this conjecture, Frenkel-Lepowsky-Meurman [16] gave a precise definition of “some algebraic structure” as a *vertex operator algebra* and constructed a particular example V , which is now called the *Moonshine vertex operator algebras* and denoted by V^{\natural} .

The construction roughly goes as follows. In dimension 24, we have an exceptional lattice Λ called the *Leech lattice*. Then there is a general construction of a vertex operator algebra from a certain lattice, and the one for the Leech lattice gives something very close to our final object V^{\natural} . Then we take a fixed point algebra under a natural action of $\mathbb{Z}/2\mathbb{Z}$ arising from the lattice symmetry, and then make a simple current extension of order 2. The resulting vertex operator algebra is the Moonshine vertex operator algebra V^{\natural} . (The final step is called a twisted orbifold construction). The series $\sum_{n=0}^{\infty} (\dim V_n^{\natural}) q^{n-1}$ is indeed the j -function minus constant term 744.

Miyamoto [40] has a new realization of V^{\natural} as an extension of a tensor power of the Virasoro vertex operator algebra with $c = 1/2$, $L(1/2, 0)^{\otimes 48}$

(based on Dong-Mason-Zhu [11]). This kind of extension of a Virasoro tensor power is called a *framed vertex operator algebra* as in [10].

We have given an operator algebraic counterpart of such a construction in [31].

We realize a Leech lattice net of factors on S^1 as an extension of $\text{Vir}_{1/2}^{\otimes 48}$ using certain \mathbb{Z}_4 -code. Then we can perform the twisted orbifold construction in the operator algebraic sense to obtain a net of factors, the *Moonshine net* \mathcal{A}^\natural . Theory of α -induction is used for obtaining various decompositions. We then get a Miyamoto-type description of this construction, as an operator algebraic counterpart of the framed vertex operator algebras. We then have the following properties.

1. $c = 24$.
2. The representation theory is trivial.
3. The automorphism group is the Monster.
4. The Hauptmodul property (as above).

Outline of the proof of these four properties is as follows.

It is immediate to get $c = 24$. We can show complete rationality passes to an extension (and an orbifold) in general with control over the size of the representation category, using the Jones index. With this, we obtain (2) very easily. Such a net is called *holomorphic*. Property (3) is the most difficult part. For the Virasoro VOA $L(1/2, 0)$, the vertex operator is indeed a well-behaved Wightman field and smeared fields produce the Virasoro net $\text{Vir}_{1/2}$. Using this property and the fact that $\bigcup_g g(L(1/2, 0)^{\otimes 48})$ for all $g \in \text{Aut}(V^\natural)$ generate the entire Moonshine VOA V^\natural , we can prove that the automorphism group as a vertex operator algebra and the automorphism group as a net of factors are indeed the same. Then (4) is now a trivial corollary of the Borchers theorem [4].

We note that the Baby Monster, the second largest among the 26 sporadic finite simple groups, can be treated similarly with Höhn's construction of the shorter Moonshine super vertex operator algebra.

Still, these examples are treated with various tricks case by case. We expect a bijective correspondence between vertex operator algebras and nets of factors on S^1 under some nice conditions. On the side of vertex operator

algebras, the most natural candidate for such a “nice” condition is the C_2 -finiteness condition of Zhu [56] (with unitarity). On the operator algebraic side, our complete rationality in [32] seems to be such a “nice” condition, but the actual relations between the two notions are not clear at this moment. The essential condition for complete rationality is the finiteness of the Jones index arising from four intervals on the circle, and this finiteness somehow has formal similarity to the finiteness appearing in the definition of the C_2 -finiteness.

At the end, we list some open problems. The operator algebraic approach has an advantage in control of representation theory, but is behind of theory of vertex operator algebras in the theory of characters.

For a net of factors, we can naturally define a notion of a character for each representation. But even convergence of these characters have not been proved in general, and the modular invariance property, the counterpart of Zhu’s result [56], is unknown, though we certainly expect it to be true. We also expect the Verlinde identity holds, which has been proved in the context of vertex operator algebras recently by Huang [23]. We would need an S -matrix version of the spin-statistics theorem [21] for nets of factors.

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