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Superconformal Field Theory and Operator Algebras

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Abstract.

We present an operator algebraic approach to superconformal field theory and a classification result in this framework. This is based on a joint work with S. Carpi and R. Longo

§1. Introduction

This is a review on operator algebraic approach to superconformal field theory based on a joint work [6] with S. Carpi and R. Longo.

In general, we study *quantum fields* in quantum field theory. From a mathematical viewpoint, they are certain operator-valued distributions on a spacetime and often called *Wightman fields*. We also need to fix a certain type of a spacetime symmetry group on the spacetime.

An operator algebraic approach to quantum field theory is called an *algebraic quantum field theory* [19] and has been studied for more than 40 years. When we deal with operator-valued distributions, they cause technical difficulties since they are distributions, rather than functions, and they usually produce unbounded operators. In algebraic quantum field theory, we deal with a family of algebras of *bounded* linear operators instead, and their algebraic operations are much easier to handle.

A basic idea in algebraic quantum field theory is as follows. In one quantum field theory on one spacetime, we assign to each (bounded) region in the spacetime an algebra of bounded linear operators generated by *observables* on the region. Recall that observables are represented by (generally unbounded) self-adjoint operators in quantum mechanics. Our operator algebras are assumed to be closed under the adjoint operation and the weak operator topology, and such an operator algebra is called a *von Neumann algebra*. On the Hilbert space on which these

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operators act, we also assume to have a projective unitary representation of the spacetime symmetry group. Thus one quantum field theory is described with a Hilbert space, a family of von Neumann algebras parameterized by spacetime regions and a projective unitary representation of the spacetime symmetry group, subject to certain set of physically natural axioms. We also have a distinguished vector representing a *vacuum* on this Hilbert space.

We review *conformal (quantum) field theory* before dealing with *superconformal* quantum field theory. In conformal field theory, we work on a $(1+1)$-dimensional Minkowski space with *conformal symmetry*, which is explained below. Then we can restrict the theory onto the two light rays $\{(x,t) \mid x = \pm t\}$ and their compactifications, where x, t are the space and time coordinates of the $(1+1)$-dimensional Minkowski space. In this way, we have two restricted theories and each of such two is called a *chiral* conformal field theory. An operator algebraic formulation of a chiral conformal field theory is given as follows. (See [24] for a more precise formulation. Also see [25] for a formulation on the $(1+1)$-dimensional Minkowski space and the exact meaning of the "restriction" procedure. A *boundary* conformal field theory can be studied in a similar framework [34, 29]. See [9] for other aspects of conformal field theory including more physical discussions.)

Now the space and time are mixed and compactified into a one-dimensional circle S^1. A spacetime region is an *interval* I which is a non-empty, non-dense, connected open subset of S^1. We have a corresponding von Neumann algebra $A(I)$ for each such interval I, all acting on the same Hilbert space.

We explain basic axioms as follows.

If we have a larger spacetime region, we expect to have more observables, hence a larger operator algebra. That is, for intervals $I_1 \subset I_2$, we assume to have $A(I_1) \subset A(I_2)$. This axiom is called *isotony*.

On the $(1+1)$-dimensional Minkowski space, if we have two spacelike separated regions, we have no interactions between them, so two observables on two respective regions commute. In a chiral conformal field theory after restriction on a compactified light ray, this condition takes an even simpler form. That is, when we have two disjoint intervals I_1, I_2, our axiom requires $[A(I_1), A(I_2)] = 0$, where the bracket means the commutator. This axiom is called a *locality* axiom.

Now our "spacetime symmetry" group is the conformal group $\mathrm{Diff}(S^1)$, that is, the group of orientation group preserving diffeomorphisms on S^1. We assume to have a projective unitary representation U of this group on the Hilbert space satisfying $U(g)A(I)U^*(g) = A(gI)$, where gI is the image of the interval I under the diffeomorphism g. We further assume to have $U(g)xU^*(g) = x$ for $x \in A(I)$ if g acts as the identity on I. This axiom is called *conformal covariance*.

We further assume that U restricts to a unitary representation of the Möbius group $\mathsf{M\ddot{o}b}$, identified with $PSL(2,\mathbb{R})$, and this restriction has a unit invariant vector Ω, unique up to phase. This vector Ω is called a *vacuum vector*. The restriction of U to the rotation group gives a one-parameter unitary group. We assume its generator is positive. This axiom is called *positivity of the energy*.

The above net is sometimes called *bosonic*. We now modify the above definition and present a *fermionic* counterpart to deal with a *supersymmetric* theory.

First we denote the n-cover of $\mathsf{M\ddot{o}b}$ by $\mathsf{M\ddot{o}b}^{(n)}$ for $n = 1, 2, 3, \ldots, \infty$. Note that $\mathsf{M\ddot{o}b}^{(2)}$ and $\mathsf{M\ddot{o}b}^{(\infty)}$ are naturally identified with $SL(2,\mathbb{R})$ and the universal cover of $\mathsf{M\ddot{o}b}$, respectively.

We now assume to have a $\mathbb{Z}_2$-grading Γ on the Hilbert space satisfying $\Gamma\Omega = \Omega$, $\Gamma^2 = \mathrm{Id}$, and $\Gamma A(I)\Gamma = A(I)$ for all intervals I. We write γ for $\mathrm{Ad}(\Gamma)$ and for each element x in some $A(I)$, we say that x is bosonic [fermionic] if $\gamma(x) = x$ [$\gamma(x) = -x$], respectively. We can naturally define the graded commutator and still use the same symbol $[x, y]$ for it.

Now the locality axiom takes the form $[x, y] = 0$ for $x \in A(I_1)$, $y \in A(I_2)$ with $I_1 \cap I_2 = \varnothing$, which is the formula as before, but now we mean the graded commutator by $[x, y]$. This is called *graded locality*.

Note that for a graded local net A, the fixed point subnet A^γ, called the Bose subnet of A, satisfies the usual locality.

Let $\mathrm{Diff}^{(2)}(S^1)$ and $\mathrm{Diff}_I^{(2)}(S^1)$ be the 2-cover of $\mathrm{Diff}(S^1)$ and the connected component of the identity of the preimage of $\mathrm{Diff}_I(S^1)$ in $\mathrm{Diff}^{(2)}(S^1)$, respectively. (Here the group $\mathrm{Diff}_I(S^1)$ consists of orientation preserving diffeomorphisms acting trivially on the complement of I.) Then the *conformal covariance* for a graded local net means the following.

We have a projective unitary representation U extending the unitary representation of $\mathsf{M\ddot{o}b}^{(2)}$ satisfying $U(g)A(I)U^*(g) = A(\dot{g}I)$ for $g \in \mathrm{Diff}^{(2)}(S^1)$ and $U(g)xU^*(g) = x$ for $x \in A(I')$ and $g \in \mathrm{Diff}_I^{(2)}(S^1)$. Here $\dot{g}$ represents the image of g under the natural quotient map onto $\mathrm{Diff}(S^1)$, and I' is the interior of the complement of I.

A graded local net with conformal covariance is called a *Fermi conformal net*. This is our mathematical object to study.

Now at the end of this section, we briefly mention a theory of *vertex operator algebra*, which is another mathematical framework to study a chiral conformal field theory. A *vertex operator* is the name for a certain operator-valued distribution, and this notion gives a direct algebraic axiomatization of Wightman fields on the circle S^1. See [14] for a detailed treatment. A discovery of mysterious relations between sporadic finite simple groups and elliptic modular functions predates this theory. After the initial discovery due to McKay, a general conjecture called the *Moonshine conjecture* was established by Conway and Norton [8]. The theory of vertex operator algebras gives a realization of a new predicted algebraic structure and the Moonshine conjecture has been solved by Borcherds [2].

One local conformal net and one vertex operator algebra are both supposed to describe one conformal field theory, so we should have a mathematical theorem on a bijective correspondence between local conformal nets and vertex operator algebras, at least under natural extra assumptions on some kind of finiteness. The Hilbert space for a local conformal net should be a completion of the underlying space of a vertex operator algebra, and the von Neumann algebras should be generated by smeared vertex operators. However, no such theorems have been known so far, unfortunately. Still, if one has an example, a construction or a technique for one of them, it is often possible to "translate" it to the other side. (Note that unitarity, existence of a positive definite inner product on the underlying space, is an essential part of the operator algebraic approach and we cannot drop this assumption, while vertex operator algebras without unitarity have been often studied. So the "translation" is actually for local conformal nets and unitary vertex operator algebras.) See [27, 23] for more on relations between the two approaches. For example, we have a construction of an operator algebraic counterpart of the Moonshine vertex operator algebra [27] based on [10].

There is also a *super* version of vertex operator algebras. See [20] for a recent progress in this approach.

§2. Representation theory of Fermi nets

Representation theory is a very useful tool to study local/Fermi conformal nets. This is one of the main advantages of the operator algebraic approach, while the counterpart for vertex operator algebras, theory of modules, has a more complicated general theory. For a Fermi conformal net A, a slight adaptation of the classical Doplicher-Haag-Roberts theory [11] gives a framework to study representations as follows.

A *DHR representation* of a net A is a pair of a family of representations λ_I of $A(I)$ on the same Hilbert space with $\lambda_{I_2}|_{A(I_1)} = \lambda_{I_1}$ for $I_1 \subset I_2$ and a projective unitary representation U_λ of the universal cover

$\mathrm{Diff}^{(\infty)}(S^1)$ of $\mathrm{Diff}(S^1)$ on the same Hilbert space with

$$\lambda_{\dot{g}I}(U(g)xU^*(g)) = U_\lambda(g)\lambda_I(x)U_\lambda^*(g),$$

for $x \in A(I)$ and $g \in \mathrm{Diff}^{(\infty)}(S^1)$.

For the net A, we can define a certain universal C^*-algebra $C^*(A)$ generated by $A(I)$'s with $I \subset S^1$. Then any DHR representation λ is given, up to unitary equivalence, by a certain type of endomorphism, called a *localized endomorphism*, of $C^*(A)$, as long as the Hilbert space involved is separable. Then we can compose such endomorphisms and this composition gives a right notion of a *tensor product* for representations. (Note that there are no obvious notions of a tensor product for representations of a family of algebras.) If we have a DHR representation of a Fermi conformal net A, then it gives a DHR representation of its Bose subnet A^γ.

Now we recall some results on representation theory for local conformal nets A. The tensor product operation of representations makes the representation category a *tensor category*. It is known that the tensor category is actually *braided* [13]. Fix an interval I. Then a localized endomorphism λ actually gives an endomorphism of $A(I)$ (after a possible change of representative within the unitary equivalence class). Then for the inclusion $\lambda(A(I)) \subset A(I)$, we have a notion of the *Jones index* [22], which measures the relative size of $A(I)$ with respect to the subalgebra $\lambda(A(I)$. The index $[A(I) : \lambda(A(I))]$ takes a real value in the interval $[1,\infty]$ and this number is independent of I. (Actually, the von Neumann algebra $A(I)$ is a so-called *type III factor*, and we need Kosaki's version of the Jones index.) A von Neumann algebra is called a *factor* when its center is trivial and now each $A(I)$ is automatically a factor. The subalgebra $\lambda(A(I))$ is automatically isomorphic to $A(I)$, so in particular it is also a factor, and called a *subfactor*. Jones [22] initiated a systematic study of theory of subfactors. Longo [30, 31] has shown that the square root of the Jones index $[A(I) : \lambda(A(I))]$ is equal to the statistical dimension of λ, which plays the role of a dimension of the representation in the Doplicher-Haag-Roberts theory. See [12] for a general theory of subfactors and its connection to various topics such as quantum invariants in 3-dimensional topology.

In some very nice situation, we have only finitely many unitary equivalence classes of DHR representations of a local conformal net A and all have finite statistical dimensions. A similar situation has been well studied in theory of quantum groups and the terminology *"rational"* has been used to express this situation. In [28], we have introduced a notion called *complete rationality* as an operator algebraic counterpart of this rationality. Then we have proved the following theorem [28, Theorem 33].

Theorem 1. *Let A be a local conformal net on the circle satisfying the following conditions.*

(1) *It has a split property.*

(2) *The Jones index for a subfactor $A(I_1)\vee A(I_3) \subset (A(I_2)\vee A(I_4))'$ is finite, where we split the circle into four intervals I_1, I_2, I_3, I_4 in this order, say, counterclockwise.*

(3) *It is strongly additive.*

Then the number of irreducible DHR representations of A, up to unitary equivalence, is finite and all have finite dimensions. Furthermore, the braiding on these representations is nondegenerate.

When we have the above three conditions for A, we say that A is completely rational. The third assumption on strong additivity has been later shown to be redundant in [35]. The first condition, the split property, is known to hold if the vacuum character $\mathrm{Tr}(\exp -tL_0)$ is convergent for all $t > 0$, where L_0 is the conformal Hamiltonian, so the main condition of complete rationality is the finiteness of the Jones index. The nondegeneracy of the braiding of the representation category is called *modular* in the sense of Turaev [37]. This notion plays an important role in theory of quantum invariants in 3-dimensional topology [37]. Rehren [36] showed that we have a unitary representation of $SL(2,\mathbb{Z})$ for such a modular tensor category where the dimension of the representation is the number of unitary equivalence classes of irreducible representations. That is, this representation arises from the structure of braiding. In all the known concrete examples, this representation coincides with the one arising from linear fractional transformations on the characters of the representations, which has a well-established general theory for vertex operator algebras as in [21, 45]. In [26], we have called a local conformal net with such coincidence *modular*, but we do not have a nice characterization of such local conformal nets. The action of $SL(2,\mathbb{Z})$ is very important in many aspects of conformal field theory, and its role in classification theory, as we will see below, is one of such examples of this importance.

We have a natural notion of an extension $A(I) \subset B(I)$ for local conformal nets. Such a situation was first systematically studied in [33] under the name of *nets of subfactors*. An extension with $(\bigvee_I A(I))' \cap B(I_0) = \mathbb{C}$, for some, hence all I_0, is said to be *irreducible*. In a usual representation theory for groups, we have a notion of an *induced representation* for a subgroup $H \subset G$, which produces a representation of the larger group from one for the smaller group. For a net of subfactors, we have a similar notion called an *α-induction*. For a DHR representation λ of A, we have an α-induction α_λ of λ, but this induction procedure depends on a choice of braiding (of λ and the so-called dual canonical endomorphism of the extension), so we use a symbol $\alpha^\pm_\lambda$ to denote the choice of over/under crossing. Furthermore, the induced "representation" is not a genuine DHR representation in general, but a so-called *soliton* representation. (Actually, this α-induction is more similar to *restriction* rather than to induction in the classical situation, but we use the name α-induction.) This α-induction was first defined in [33], and many interesting properties and examples were given in [39]. We have unified this theory of α-induction in [1] with Ocneanu's graphical calculus. In particular, we have shown the following theorem [1, Theorem 5.7]. Actually, this holds for a more general braided tensor category, as explained in [1].

Theorem 2. *Let $Z_{\lambda,\mu} = \dim\ \mathrm{Hom}(\alpha^+_\lambda, \alpha^-_\mu)$. Then the matrix Z is a modular invariant, which means that the matrix Z is in the commutant of the image of the unitary representation of $SL(2,\mathbb{Z})$ arising from the braiding of the DHR representations of the local conformal net, each $Z_{\lambda,\mu}$ is a nonnegative integer, and Z_{00} is 1, where 0 denotes the vacuum representation.*

If a completely rational local conformal net A is given, its any extension B produces a modular invariant matrix Z through the above procedure. The number of possible matrices Z is always finite and often very small for a given modular tensor category. Together with Longo's notion of Q-system [32, 33], we can, in principle, classify all (irreducible) extensions B of A.

§3. Classification results for superconformal nets with $c < 3/2$

First we review our previous classification result for local conformal nets with small central charge [24].

When we have conformal covariance for a local net A, the Hilbert space also has a unitary representation of the Virasoro algebra, which is an infinite dimensional Lie algebra generated by L_n, $n \in \mathbb{Z}$, and one central element c with the relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

It has been known [15] that the central element c is mapped to a positive scalar for an irreducible representation and the value is of the form $1 - 6/m(m+1)$, $m = 3, 4, 5, \ldots$, if it is less than 1. The unitary representation of the Virasoro algebra arising from conformal covariance is not irreducible in general, but still one has that the central element c is mapped to a scalar. In this way, we obtain a numerical invariant, called the *central charge*, of a local conformal net A. This number is also denoted by c. The representation of the Virasoro algebra produces a local conformal subnet of A, called the *Virasoro net*, and if $c < 1$, then one can show that A is an irreducible extension of the Virasoro net. In this way, a classification of A with $c < 1$ reduces to a classification of irreducible extensions of the Virasoro nets with $c < 1$.

The Virasoro nets can be also realized with the coset construction [18]. In the operator algebraic framework, the coset construction has been studied well by Xu [41, 42, 43]. The case of the Virasoro net with $c < 1$ uses Wassermann's construction of the local conformal nets corresponding to the Wess-Zumino-Witten models $SU(2)_k$ [38]. By using another paper of Xu [40] together, one can show that the Virasoro nets with $c < 1$ are completely rational in the sense of [28]. The modular invariant matrices for the representation categories of these Virasoro nets have been classified by [5], and we have shown in [24] that the so-called type I modular invariant matrices in the classification list of [5] are in a bijective correspondence to the local conformal nets with $c < 1$. (For uniqueness of the local conformal net corresponding to each modular invariant matrix, also see [25].) In this way, we have obtained a first classification result in algebraic quantum field theory as follows [24, Theorem 5.1].

Theorem 3. *The following is a complete list of the local conformal nets on the circle with central charge less than 1.*

(1) *The Virasoro nets with* $c = 1 - 6/m(m-1)$.
(2) *The index 2 extensions of the Virasoro nets with* $c = 1 - 6/m(m-1)$, *where* $m \equiv 1, 2 \bmod 4$.
(3) *The four exceptionals at* $c = 1 - 6/m(m-1)$, *where* $m = 11, 12, 29, 30$.

The four exceptionals in the list arise from the modular invariants labeled with pairs of the Dynkin diagrams (A_{10}, E_6), (E_6, A_{12}), (A_{28}, E_8), (E_8, A_{30}). Three of them with $m = 11, 12, 30$, can be constructed with another known construction, the coset construction, but the other one with $m = 29$ does not seem to arise from any other known constructions. This new construction has been generalized as a mirror extension [44].

The vertex operator algebras corresponding to the Virasoro nets with $c < 1$ and their extensions of index 2 are well-known. A result of Huang, Kirillov and Lepowsky on extensions of vertex operator algebras ensures that we do have vertex operator algebras corresponding to the four exceptionals, including the one at $m = 29$, but we do not have a good understanding on what it really is.

Now we work on *superconformal* nets. As a "super" version of the Virasoro algebra, we have two super Virasoro algebras and they are the super Lie algebras generated by even elements L_n, $n \in \mathbb{Z}$, odd elements G_r, and a central even element c, satisfying the relations,

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0},\\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r},\\
[G_r, G_s] &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0},
\end{aligned}
$$

where we have $r \in \mathbb{Z} + 1/2$ in the *Neveu-Schwarz* case and $r \in \mathbb{Z}$ in the *Ramond* case. The corresponding super Lie algebras are called the Neveu-Schwarz and Ramond algebras. We can define the central charge again for representations, and now the discrete part of the possible values is up to the value 3/2. Again by using a (different) coset construction involving the $SU(2)_k$-models, one can construct the super Virasoro nets with $c < 3/2$, and again, their Boson parts are completely rational, so we can apply the above procedure to classify irreducible extensions. We now *define* such irreducible extensions to be *superconformal* nets with $c < 3/2$. (The discrete part of the local conformal nets is given by the condition $c < 1$ as above. Now the discrete part of the superconformal nets is given by the condition $c < 3/2$, since we have contribution 1 from a boson and 1/2 from a fermion.)

We first need a classification of modular invariant matrices for the modular tensor categories arising from the representations of the Boson parts of the super Virasoro nets with $c < 3/2$. Such a classification list was proposed by Cappelli [4], and certain completeness of the classification list has been shown in [16, 17] for the case *without* so-called fixed point resolution. The modular tensor categories for the case having the fixed point resolution have been studied in [42], so with this result, one can extend the method of [16, 17] to show completeness of the classification list of the modular invariant matrices. In this way, we reach a complete classification of superconformal nets with $c < 3/2$ as follows [6, Theorem 36]. (We first classify extensions of the Boson parts of the super Virasoro nets and deal with Fermionic extensions after that.)

Theorem 4. *The following gives a complete list of superconformal nets with $c < 3/2$, together with the labels for the modular invariants.*

(1) *The super Virasoro net with* $c = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right)$, *labeled with* (A_{m-1}, A_{m+1}).

(2) *Index 2 extensions of the above (1), labeled with* $(A_{4m'-1}, D_{2m'+2})$, $(D_{2m'+2}, A_{4m'+3})$.

(3) *Six exceptionals labeled with* (A_9, E_6), (E_6, A_{13}), (A_{27}, E_8), (E_8, A_{31}), (D_6, E_6), (E_6, D_8).

Again some of the exceptionals can be realized with the coset construction, and others with the mirror extension of [44].

At the end, we mention that some connections to noncommutative geometry of Connes [7] have been expected and one possible direction is given as follows [26, Theorem 30].

Theorem 5. *Let* A *be a Fermi conformal net and* λ *a supersymmetric irreducible representation of* A. *Then*

$$\mathrm{ind}(Q_{\lambda+}) = \frac{d(\rho)}{\sqrt{\mu_A}} \sum_{\nu\in\mathfrak{R}} \Phi_\nu(\varepsilon(\rho,\nu)^*\varepsilon(\nu,\rho)^*) d(\nu)\mathrm{null}(\nu, \mathrm{c}/24),$$

where ρ *is one of the two irreducible components of* λ_b.

Here $d(\rho)$ is the dimension of ρ, μ_A is the μ-index of A, which is the square sum of the dimensions of all irreducible representations of the net A. A *supersymmetric representation* λ means that we have

$$H_\lambda - \frac{c}{24} = Q_\lambda^2,$$

where H_λ is the conformal Hamiltonian and Q_λ is some odd selfadjoint operator called the *supercharge* and $\mathrm{ind}(Q_{l+})$ is the Fredholm index of the upper off diagonal part of Q_λ. The symbol $\mathrm{null}(\nu, \mathrm{h})$ denotes the dimension of the kernel of $H_\nu - h$. The set $\mathfrak{R}$ is the set of σ-Fermi irreducible sectors of Boson part A_b. A sector ν of A_b is said to be σ-Fermi if the monodromy of ν and σ is trivial, where σ is the sector of A_b dual to the grading of A. The symbol Φ_ν denotes the left inverse of ν and ε means the braiding. Note that $\Phi_\nu(\varepsilon(\rho,\nu)^*\varepsilon(\nu,\rho)^*)$ is a complex number.

In this theorem, the Jones index and the Fredholm index are related, while these have been unrelated despite their common name.

Also see [3] for a recent, but different treatment of supersymmetry within algebraic quantum field theory.

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