Conformal quantum field theory and subfactors

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Abstract

We survey a recent progress on algebraic quantum field theory in connection to subfactor theory. We mainly concentrate on one-dimensional conformal quantum field theory.

1 Introduction

Algebraic quantum field theory is an operator algebraic approach to quantum field theory. Here we review methods of Haag-Kastler nets of operator algebras on a spacetime with emphasis on recent progresses in low dimensions in connection to subfactor theory and modular invariants.

In algebraic quantum field theory, we have a family of operator algebras parameterized by regions in a certain spacetime. Each algebra represents a system of physical quantities observable in the corresponding region. Representation theory of such a family of operator algebras has turned out to be quite interesting mathematically. (See [24] for a general theory of algebraic quantum field theory.) A natural "spacetime" for such a formulation is a 4-dimensional Minkowski space, but in this article, we will concentrate on one-dimensional compactified "spacetime", S^1. (One way to get this situation naturally is making a tensor product decomposition of a theory of 2-dimensional spacetime. Such a one-dimensional theory is often called a *chiral* theory.) A one-dimensional theory has caught much attention recently and provides a rich source of mathematical problems and insight.

2 Conformal nets and representation theory

Now our "spacetime" is one-dimensional circle S^1 and a region in this spacetime is an *interval* which means a non-empty, non-dense, open, and connected set in S^1. We study

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a family of von Neumann algebras $\mathcal{A}(I)$ on a fixed Hilbert space H parameterized by intervals I under the following set of axioms.

Axiom 2.1. *For intervals $I \subset J$, we have $\mathcal{A}(I) \subset \mathcal{A}(J)$.*

This axiom is called *isotony* and means that we have more observable for a larger region. Each algebra $\mathcal{A}(I)$ is called a *local algebra*.

Axiom 2.2. *If two intervals I, J have no intersection, then we have $xy = yx$ for all operators $x \in \mathcal{A}(I)$, $y \in \mathcal{A}(J)$.*

This axiom is called *locality*. In a 4-dimensional Minkowski space, the locality axiom means that two space-like separated regions have no interactions since we cannot reach one region from the other even with speed of light, and hence operators in the corresponding two algebras commute with each other. In this one-dimensional setting, the natural assumption for "space-like disjointness" is simply disjointness.

We also need to encode a role of spacetime symmetries. The next axiom is called *Möbius covariance* or *conformal covariance*.

Axiom 2.3. *We have a unitary representation U_g of $PSL(2,\mathbb{R})$ on H with $U_g\mathcal{A}(I)U_g^* = \mathcal{A}(gI)$ for $g \in PSL(2,\mathbb{R})$ and each interval I, where $PSL(2,\mathbb{R})$ acts on S^1 by the Möbius transformation.*

We next have a positive energy condition. Note that the above action of $PSL(2,\mathbb{R})$ on S^1 contains the rotation as a subgroup.

Axiom 2.4. *The generator of the one-parameter automorphism subgroup of U_g given by rotation is positive.*

We further assume existence of a special vector called a *vacuum vector* Ω.

Axiom 2.5. *We have a U-invariant unit vector $\Omega \in H$.*

The final axiom here is called irreducibility.

Axiom 2.6. *The von Neumann algebra $\bigvee_I \mathcal{A}(I)$ generated by all $\mathcal{A}(I)$'s is $B(H)$.*

Such a family of von Neumann algebras satisfying the above set of axioms is simply called a *net of von Neumann algebras* and denoted by $\mathcal{A}$. (The inclusion order on the set of intervals is not directed, so the terminology *net* is not appropriate, strictly speaking, but this terminology has been often used in literature.) The *Haag duality*, $\mathcal{A}(I') = \mathcal{A}(I)'$ now follows from these axioms, where I is an interval, I' is the interior of its complement, and $\mathcal{A}(I)'$ is the *commutant* of $\mathcal{A}(I)$, that is, $\{x \in B(H) \mid xy = yx, \quad \forall y \in \mathcal{A}(I)\}$. We also have that the U-invariant vector is unique up to scalar. Each algebra $\mathcal{A}(I)$ is a type III_1 factor, except for the trivial case $\mathcal{A}(I) = \mathbb{C}$ for all I. So, a net of von Neumann algebra in the above sense is also called a net of factors. See [7, 9, 18, 22, 23] for more explanations on the axioms and proofs of these statements. Also, if one does not like compactification

S^1, one can work on $\mathbb{R}$ instead. See the appendix of [32] for a relation between the two formulations.

We next consider representations of such a family of von Neumann algebras together with a "compatible" unitary representation of the Möbius group on different Hilbert spaces. A representation π of a net $\mathcal{A}$ means we have a family of representations π_I on a Hilbert space K parameterized by intervals I on S^1 such that $\pi_J\mid_{\mathcal{A}(I)}=\pi_I$ for $I\subset J$. Here we deal with only the case where K is separable and then each π_I is automatically normal and unitarily equivalent to the identity representation of $\mathcal{A}(I)$ on the original Hilbert space H. This property of unitary equivalence of π_I is called *localizability* of π. As in [16, II, Section 5], we can construct the universal C^*-algebra $C^*(\mathcal{A})$ from the net $\mathcal{A}$. (Roughly speaking, this is something like a union of $\mathcal{A}(I)$'s, but the set of intervals on S^1 is not directed, so we cannot take an inductive limit simply, and we need to be more careful.) Since we have a canonical embedding of $\mathcal{A}(I)$ into $C^*(\mathcal{A})$, we regard $\mathcal{A}(I)$ as a subalgebra of $C^*(\mathcal{A})$. One can show that we have a bijective correspondence between representations of the net $\mathcal{A}$ and those of the C^*-algebra $C^*(\mathcal{A})$. By the Haag duality, each representation of the net $\mathcal{A}$ is unitarily equivalent to $\sigma_0\cdot\rho$, where ρ is an endomorphism of $C^*(\mathcal{A})$ and σ_0 is the representation of $C^*(\mathcal{A})$ corresponding to the identity representation of the net $\mathcal{A}$ on the original Hilbert space H. See the appendix of [32] for handling of representations of a net on $\mathbb{R}$ along the line of the DHR analysis [13]. In the following, we will often consider DHR endomorphisms rather than representations of a net. Note that a net of von Neumann algebras on $\mathbb{R}$ is indeed a net in the usual sense and thus we can make an inductive limit C^*-algebra of local algebras. Then each representation of a net is realized as a special endomorphism, called a DHR endomorphism, of the inductive limit C^*-algebra.

We would like to pursue an analogy between the representation theory of a net of von Neumann algebras as above and that of a compact group. It turns out that we can define a notion of (*statistical*) *dimension* of a representation which takes a value in $[1,\infty]$, possibly a non-integer, and also a notion of *tensor product* through a composition of endomorphisms. (Note that a tensor product of representations does not make sense literally, so we *define* a tensor product of representations as composition of DHR endomorphisms.) In the following, we consider only representations with finite statistical dimensions. One way to see this analogy more concretely is to fix an interval I and realize representations as endomorphisms of a factor $\mathcal{A}(I)$. Then the statistical dimension of an endomorphism ρ is simply the square root of the Jones-Kosaki index $[\mathcal{A}(I):\rho(\mathcal{A}(I))]$ [27, 34]. The tensor product operation is given by composition of endomorphisms on a single factor $\mathcal{A}(I)$. We have a notion of *conjugate endomorphisms* as in [36] which correspond to that of contragredient representations. We also have notions such as direct sums and irreducible decompositions. Irreducibility is defined as $\rho(\mathcal{A}(I))'\cap\mathcal{A}(I)=\mathbb{C}$, for example. A unitary equivalence class of a representation is called a *superselection sector* and a category of representations has a strong formal similarity to that of unitary representations of a compact group. Actually, if a spacetime dimension is four, the category of representations of a net is equivalent to that of representations of a compact group, and we can recover the compact group through abstract duality [14]. In this case, the statistical dimensions are integers, in particular. In our current one-dimensional spacetime S^1, however, the statistical dimensions are not necessarily integers, and we have some category not arising from a compact group, in general.

One property of a category of unitary representations of a compact group is that for two representations π_1 and π_2, two tensor products $\pi_1 \otimes \pi_2$ and $\pi_2 \otimes \pi_1$ are trivially unitarily equivalent. The corresponding commutative property in a category of endomorphisms is that we have a unitary $u \in \mathcal{A}(I)$ with $\mathrm{Ad}(u) \cdot \rho_1 \cdot \rho_2 = \rho_2 \cdot \rho_1$. Since ρ_1, ρ_2 are endomorphisms of an infinite algebra, we have no reason to expect that the compositions commute, even up to unitary equivalence, but it turns out that they do have this commutativity up to unitary equivalence, due to locality. In the case of higher spacetime dimensions, this commutativity holds in a rather simple way and this unitary u corresponds to a permutation of two objects ρ_1, ρ_2, but for a net on S^1, we have more non-trivial commutativity giving a braid relation for the two objects ρ_1, ρ_2. In this way, the category of representations of a net of von Neumann algebras becomes a *braided tensor category* as in [16]. See [45] for a precise definition and related properties of braided tensor categories. (Also see [42] for a definition of a braiding in the setting of endomorphisms of a von Neumann algebra.)

3 Complete rationality

As we have seen above, we have a braided tensor category arising from representations of a net of von Neumann algebras. In connection to quantum groups and 3-dimensional topological quantum field theory, a braided tensor category with finitely many irreducible objects has caught much attention. Such a category is called *rational*. Furthermore, a braiding on a rational tensor category produces two finite-dimensional scalar-valued matrices, S- and T-matrices. (See [42] for their operator algebraic definition in the setting of endomorphisms.) The T-matrix is always unitary, but S-matrix can be non-invertible in general. Its invertibility is an important property, particularly in connection to 3-dimensional topological quantum field theory, and it is often very difficult to prove this invertibility for a concrete category arising from, say, quantum groups or vertex operator algebras. When we have this invertibility, in addition to rationality, we say that the tensor category is *modular*, since we then have a unitary representation of a modular group $SL(2, \mathbb{Z})$. (See [1, 45] for this invertibility and related results.)

So in an operator algebraic approach to study of modular tensor categories, it is important to know when the tensor category of representations of a net becomes modular. In [32], we have proposed one set of conditions and proved that it indeed implies modularity of the tensor categories. We now give the set of axioms and explanations.

We have three more axioms in addition to those in the previous section. The first one below is called *strong additivity*.

Axiom 3.1. *Let I be an interval and p a point on it. Let I_1, I_2 be two connected components of $I \setminus \{p\}$. Then we have $\mathcal{A}(I) = \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$.*

The next one is called a *split property*. It is known that this holds if $\mathrm{Tr}(e^{-\beta L_0}) < \infty$ for all $\beta > 0$, where L_0 is the conformal Hamiltonian. (See [8, 11].)

Axiom 3.2. *Let I, J be two intervals with two disjoint closures. Then $\mathcal{A}(I) \vee \mathcal{A}(J)$ is naturally isomorphic to $\mathcal{A}(I) \otimes \mathcal{A}(J)$.*

The next one involves a notion of *μ-index* as follows.

Definition 3.3. *Split the circle to four intervals I_1, I_2, I_3, I_4 in the counterclockwise order. The μ-index of the net $\mathcal{A}$, $\mu_{\mathcal{A}}$, is defined to be the Jones-Kosaki index of the subfactor $\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$.*

It turns out that this μ-index is independent of the choice of the four intervals. (See the book [15] for general theory on subfactors.) Then the final axiom for complete rationality is the following.

Axiom 3.4. *The μ-index of the net $\mathcal{A}$ is finite.*

The main result in [32] is that under the set of these axioms, we have modularity of the tensor category of representations of the net $\mathcal{A}$. In particular, this tensor category is rational and this is why we use the terminology "complete rationality". It has been also proved in [32] that the subfactor in Definition 3.3 is the Longo-Rehren subfactor as in [39, Proposition 4.10] arising from the system of irreducible DHR endomorphisms of the net $\mathcal{A}$ and the index of this subfactor measures the size of this tensor category. (See [25, 40] for more on the Longo-Rehren subfactors.) The $SU(N)_k$ net of factors on S^1 constructed in [46] is completely rational by the results of [46, 49]. (The structure of the tensor categories of the DHR endomorphisms of these nets is the same as those arising from the WZW $SU(N)_k$ models. See the book [12] on WZW models.) Coset nets and orbifold nets have been studied in the context of completely rational nets in [50, 51, 52, 53] and several interesting results including invariants of 3-manifolds have been obtained. Longo [38] has proved that if we have a net of subfactors $\{\mathcal{A}(I) \subset \mathcal{B}(I)\}$ with finite index and one of the two is completely rational, so is the other. (See the next section for more on nets of subfactors.)

4 α-induction and modular invariants

In the usual representation theory, we have a machinery of induction and restriction for a group G and its subgroup H. For nets of subfactors $\{\mathcal{A}(I) \subset \mathcal{B}(I)\}$ on the circle, we have a similar machinery. A general theory of nets of subfactors was started in [39] and a method of induction and restriction was also proposed there in Proposition 3.9 based on an old suggestion of Roberts [44]. This machinery was extensively studied by Xu [47, 48] in the setting of conformal inclusions and several general useful properties and interesting examples have been obtained. This has been further studied in [2, 3, 4, 5, 6] under the name of α-induction.

Before going into this theory, we make one remark. In the usual subfactor theory on $N \subset M$ as in [15], the roles of N and M are symmetric, since we can perform the Jones basic construction [27], but in the theory of nets of subfactors, the roles of the two nets are not symmetric. For $\{\mathcal{A}(I) \subset \mathcal{B}(I)\}$, fix one interval I. Longo's dual canonical endomorphism [36] for the subfactor $\mathcal{A}(I) \subset \mathcal{B}(I)$ gives a DHR endomorphism of the net $\mathcal{A}$, but the canonical endomorphism of this subfactor is not a DHR endomorphism of the net $\mathcal{B}$. We cannot make a basic construction for nets of subfactors.

For a DHR endomorphism λ of a net $\mathcal{A}$ and a fixed interval I, we may assume that λ is localized on I, that is, λ gives an endomorphism $\mathcal{A}(I)$. Then using the formula

$$\alpha_\lambda^\pm = \gamma^{-1} \cdot \mathrm{Ad}(\varepsilon^\pm(\lambda, \gamma|_{\mathcal{A}(I)})) \cdot \lambda \cdot \gamma$$

in [39, Proposition 3.9], where γ is a canonical endomorphism [36] of the subfactor $\mathcal{A}(I) \subset \mathcal{B}(I)$ and $\varepsilon(\lambda, \gamma|_{\mathcal{A}(I)})$ is the braiding on the tensor category of DHR endomorphisms of the net $\mathcal{A}$, we have an endomorphism $\alpha_\lambda^\pm$ of $\mathcal{B}(I)$. These endomorphisms are not DHR endomorphisms of the net $\mathcal{B}$ in general, but it turns out that the intersection of the irreducible endomorphisms appearing in the decompositions of α_λ^+'s and those of α_μ^-'s is exactly the system of irreducible DHR endomorphisms of the net $\mathcal{B}$. Note that the system of the irreducible DHR endomorphisms of the net $\mathcal{B}$ is smaller than that of the net $\mathcal{A}$ and the ratio of the size is given by the square of the index $[\mathcal{B} : \mathcal{A}]$ by [32, Proposition 24].

For irreducible DHR endomorphisms λ, μ of the net $\mathcal{A}$, we set $Z_{\lambda\mu} = \dim \mathrm{Hom}(\alpha_\lambda^+, \alpha_\mu^-)$. Then it has been proved in [4, Theorem 5.7] that this matrix Z commutes with the unitary representation of the $SL(2, Z)$ arising from the braiding on the system of irreducible DHR endomorphisms of the net $\mathcal{A}$. Thus, this matrix is a *modular invariant* in the sense that $Z_{\lambda\mu} \in \mathbb{N}$, $ZS = SZ$, $ZT = TZ$, and $Z_{00} = Z_{00}$, where the index 0 means the vacuum representation. (Actually, the results in [4] hold in a much more general situation where we have just an abstract braiding in the sense of [42].) Several results have been obtained about the categorical structures of systems of endomorphisms arising from α-induction in [4, 5, 6].

Around the same time as Longo-Rehren [39], Ocneanu [41, Part 5] introduced a graphical method to study Goodman-de la Harpe-Jones subfactors [21, Section 4.5] arising from A-D-E Dynkin diagrams. It has been shown in [4, Theorem 5.3] that this method is essentially the same as α-induction. The results on Goodman-de la Harpe-Jones subfactors in [29] has given a prototype for such studies.

For a given braiding, it is easy to see that the number of possible modular invariant matrices Z is finite. In a natural concrete example, this finite number is often very small such as 1, 2, and 3. In the case of the $SU(2)_k$ WZW-models, all the modular invariant matrices have been classified in [10, 28] and they are labeled with A-D-E Dynkin diagrams. (See [19] and references there for recent results on classification of modular invariants.) It has been shown in [2, 4, 5] that all of them arise from subfactors with α-induction in the above way.

5 Central charge and classification

In this last section, we replace the Möbius group with the orientation preserving diffeomorphism group $\mathrm{Diff}(S^1)$, an infinite dimensional Lie group, as the symmetry group of the "spacetime". We then need some modifications of the axioms as follows. We now assume that we have a *projective* unitary representation of $\mathrm{Diff}(S^1)$ and the covariance axiom holds with respect to this representation, but we assume that invariance of the vacuum vector only for the Möbius group, a subgroup of $\mathrm{Diff}(S^1)$. (It is impossible that the vacuum vector is invariant under the whole $\mathrm{Diff}(S^1)$.) Furthermore, we assume that if $g \in \mathrm{Diff}(S^1)$ acts trivially on an interval I, then $U(g)$ implements the identity automorphism of $\mathcal{A}(I)$.

The corresponding infinite dimensional Lie algebra to $\mathrm{Diff}(S^1)$ is the celebrated Virasoro algebra with relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n},$$

where $m, n \in \mathbb{Z}$, and and $[L_n, c] = 0$. The number $c \in \mathbb{C}$ is called a *central charge*. If $c < 1$, then the value of c must belong to the set

$$\{1 - 6/m(m+1) \mid m = 2, 3, 4, \dots\}$$

and all these values are realized by [17, 20]. For each admissible value of c, we have a unique irreducible, projective unitary representation U, with positive energy, of $\mathrm{Diff}(S^1)$ such that the lowest eigenvalue of the conformal Hamiltonian L_0 is 0. This is called the vacuum representation with central charge c. Then we define the Virasoro net $\mathrm{Vir}_c(I) = U(\mathrm{Diff}(I))''$, where $\mathrm{Diff}(I)$ is the group of diffeomorphisms S^1 which fix the points outside of I. We call this $\mathrm{Vir}_c(I)$ the *Virasoro net* with central charge c. From a viewpoint of the coset construction of unitary representations of the Virasoro algebras with central charge less than 1 by Goddard-Kent-Olive [20], it is natural to expect that the Virasoro net with central charge $c = 1 - 6/m(m+1)$ coincides with the coset model arising from the diagonal embedding $SU(2)_{m-1} \subset SU(2)_{m-2} \times SU(2)_1$ as in Xu [50]. One can prove that this is indeed the case. (See [31] for more details.) Then Longo's results in particular implies [38] that the Virasoro net Vir_c with $c < 1$ is completely rational. (Strong additivity is a part of the axioms for complete rationality. Strong additivity of Vir_c was claimed in [35], but the proof there contains a serious gap.)

Now we would like to classify diffeomorphism covariant nets on S^1. For such a net $\mathcal{A}$, the projective unitary representation of $\mathrm{Diff}(S^1)$ gives a subnet of $\mathcal{A}$ in the same way as above. Furthermore, we can prove that this subnet is irreducible, that is, we have $\mathrm{Vir}_c(I)' \cap \mathcal{A}(I) = \mathbb{C}$ for an interval I. Because the Virasoro net is completely rational, a result in [26] implies that the inclusion $\mathrm{Vir}_c(I) \subset \mathcal{A}(I)$ has a finite index. In this way, the classification problem of such nets is reduced to the classification problem of irreducible extensions of the Virasoro nets with $c < 1$. We can now apply the method of α-induction and modular invariants in the above section.

The S- and T-matrices arising from the braiding of the category of DHR endomorphisms of the Virasoro net Vir_c with $c < 1$ is explicitly known. (See [12], for example.) The modular invariant matrices for this have been explicitly classified in [10]. They are labeled with pairs of A-D-E Dynkin diagrams with difference of their Coxeter numbers being one. So the operator algebraic problems are existence and uniqueness of nets of factors corresponding to each modular invariant. This problem is reduced to classification problem of Q-systems in the sense of [37] for each modular invariant. As in [31], the problems of existence and uniqueness can be solved affirmatively for each of the so-called type I modular invariants in [10] and we do not have any net corresponding to the type II modular invariants. In this way, diffeomorphism covariants nets of factors on S^1 are in a bijective correspondence to pairs of A-D_{2n}-$E_{6,8}$ Dynkin diagrams with difference of their Coxeter numbers being one.

Note that a general classification problem of conformal nets on S^1 seems very difficult, but a relative version of this classification problem is more tractable. That is, for a given net $\mathcal{A}$, we would like to classify all the irreducible extensions $\mathcal{B}$ of $\mathcal{A}$. For a given completely rational net $\mathcal{A}$, we have only finitely many such $\mathcal{B}$, and a general strategy for classification is just as above; first we classify (type I) modular invariant matrices, and then solve existence and uniqueness problems of Q-systems for each modular invariant. Kirillov-Ostrik [33] considers the same type of classification problems from a different context. The results in [33] can be translated to a classification of irreducible extensions of the $SU(2)_k$ nets. Note that the tensor category of representation of a *larger* net is *smaller*. So considering *extensions* of a given net corresponds to considering *subsystems* of a given category.

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