An analogue of a solvable lattice model in classification of subfactors
— Ocneanu's theory and orbifold models —

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Abstract

We explain a strong similarity between Ocneanu's paragroups in classification of subfactors and solvable lattice models. Especially we show that an idea of orbifold models in the theory of solvable lattice models proves an announcement of Ocneanu for the first time that each of the Dynkin diagrams D_{2n} has a unique corresponding subfactor but D_{2n+1} have none.

§0 Introduction

In 1983, V. F. R. Jones introduced a new notion index for subfactors in [J1]. Since then, more and more deep connections among subfactors, knot invariants, conformal field theory, quantum groups, etc., have appeared and a great deal of research has been done in this area. Among them, A. Ocneanu introduced a new notion *paragroup* for classification of subfactors in [O1]. In a sense, this is a kind of *quantized* Galois groups, and at the same time, this can be regarded as a *discrete* version of a compact manifold and as a solvable lattice model without a spectral parameter. We work on this paragroup with emphasis on its similarity to solvable lattice models.

We work on pairs of factor-subfactor. In a sense, a *factor* is a simple $*$-algebra of bounded linear operators on a Hilbert space which is closed under a certain topology. It can be regarded as an infinite dimensional analogue of matrix algebras $M_n(\mathbf{C})$. For a factor M, we can consider a *subfactor* N, which is a simple $*$-subalgebra of M. Roughly speaking, the Jones index measures a relative size of the bigger algebra M with respect to the smaller algebra N. Because we are interested in the case M and N are infinite dimensional, this need a rigorous definition, but it can be done by a coupling constant of von Neumann as in [J1]. The Jones index, denoted by $[M : N]$, is a positive real number bigger than or equal to 1 *a priori*, but Jones showed the following surprising result in [J1].

Theorem (Jones). *The index value* $[M : N]$ *is contained in the set*

$$\{4\cos^2\frac{\pi}{n} \mid n = 3, 4, 5, \dots\} \cup [4, \infty],$$

and all the values in this set can be realized.

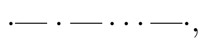
Since this work, classification of subfactors has become one of the central problems in the theory of operator algebras. This problem is related to the following topics.

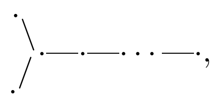
(1) Classification of group actions on factors
(2) Fundamental groups for factors
(3) link invariants
(4) quantum groups
(5) solvable lattice models

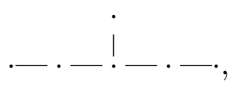
This paper deals with connections to the last topic mainly.

A viewpoint emphasized by A. Ocneanu for study of subfactors is an analogue of the Galois theory. That is, instead of a field and a subfield in the Galois theory, we work on a factor and a subfactor. From this viewpoint, the Jones index can be regarded as the order of this "quantized Galois group". Ocneanu calls this "quantized Galois group" a *paragroup*. In this picture, the underlying set of a group is replaced by a graph, and the order of the group corresponds to the square of the Perron-Frobenius eigenvalue of the incidence matrix of the graph. For example, a finite graph of order n is represented by a graph with a central vertex which has n edges from it. The Perron-Frobenius eigenvalue for this graph is n, and the eigenvector is realized as $(\sqrt{n}, 1, 1, \dots, 1)$. (Here the first entry corresponds to the central vertex and the others to the other n vertices.)

Since the first work of Jones [J1], the index value 4 has had a special meaning as shown in the above theorem. In a paragroup setting, the index less than 4 means that the Perron-Frobenius eigenvalue of the graph is between 1 and 2. It is known since Kronecker that such a graph must be one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8. (See [GHJ] for more on this.) For one of these graphs, the Perron-Frobenius eigenvalue is given by $2\cos\dfrac{\pi}{N}$, where N is the Coxeter number of the graph. These are represented by the following pictures.

A_n :  $N = n+1$,

D_n :  $N = 2n-2$,

E_6 :  $N = 12$,

$$E_7 : \quad \cdot\!-\!\cdot\!-\!\overset{\overset{\cdot}{|}}{\cdot}\!-\!\cdot\!-\!\cdot\!-\!\cdot, \qquad N = 18,$$

$$E_8 : \quad \cdot\!-\!\cdot\!-\!\overset{\overset{\cdot}{|}}{\cdot}\!-\!\cdot\!-\!\cdot\!-\!\cdot\!-\!\cdot, \qquad N = 30,$$

We will work on these graphs. A. Ocneanu announced a complete classification of subfactors with index less than 4 in [O1], but his proof has not been presented. His outline is divided into the following three steps. Let $N \subset M$ be a subfactor with index less than 4 of the AFD type II_1 factor. (See [GHJ, O1, P1] for related definitions.)

(1) Prove that for a certain choice of tunnel $N \supset N_1 \supset N_2 \supset \cdots$, the pairs of the relative commutants $N_k' \cap N \subset N_k' \cap M$ generate the pair $N \subset M$.
(2) Characterize the pairs $N_k' \cap N \subset N_k' \cap M$ combinatorially.
(3) Determine all the possible pairs $N_k' \cap N \subset N_k' \cap M$ using (2).

In [O1], A. Ocneanu claimed that (1) is valid if the subfactor has finite index, finite depth, and the trivial relative commutants. (Subfactors with index less than 4 automatically have the trivial relative commutants and the finite depth condition.) He called this Spanning Theorem, but his proof has not been presented. S. Popa then gave a proof for this statement in [P1] without an assumption of the trivial relative commutants, and further announced necessary and sufficient conditions for this spanning condition in [P2].

In this paper we explain on the above (2) and work on (3).

§1 String algebras and connections

We fix a graph and choose a vertex of it, which will be denoted by $*$. Then consider all the paths on the graph starting from $*$ with length k. A *string* is a pair of such paths (ξ, η) with equal endpoints. That is, we require $s(\xi) = s(\eta) = *$, $r(\xi) = r(\eta)$, $|\xi| = |\eta|$, where $s(\xi), r(\xi), |\xi|$ denote the starting point, the end point, and the length of ξ. We construct a string algebra Str_*^k as follows. The $\mathbf{C}$-algebra Str_*^k is spanned by pairs (ξ, η) as above as a linear space. We define a multiplication and a $*$-structure by

$$(\xi, \eta) \cdot (\xi', \eta') = \delta_{\eta, \xi'} (\xi, \eta'),$$
$$(\xi, \eta)^* = (\eta, \xi).$$

We also define an embedding $\mathrm{Str}_*^k \subset \mathrm{Str}_*^{k+1}$ by

$$(\xi, \eta) \mapsto \sum_{\alpha, |\alpha|=1} (\xi \cdot \alpha, \eta \cdot \alpha).$$

This gives an increasing sequence of finite dimensional $*$-algebras $\mathrm{Str}_*^1 \subset \mathrm{Str}_*^2 \subset \mathrm{Str}_*^3 \subset \cdots$. Taking a certain closure of $\bigcup_k \mathrm{Str}_*^k$, we get a factor, which is called the *AFD II_1 factor*. It is known since Murray and von Neumann that any graph produces a unique factor.

As an easy example, consider a graph having two vertices and double edges connecting these two. Applying the above construction to this graph, we get a sequence

$$M_2(\mathbf{C}) \subset M_4(\mathbf{C}) \subset M_8(\mathbf{C}) \subset \cdots,$$

and the inclusion is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix},$$

for example.

We would like to construct not only a factor but also a subfactor. For this purpose, we need another construction.

Label the sequence $\mathrm{Str}_*^1 \subset \mathrm{Str}_*^2 \subset \mathrm{Str}_*^3 \subset \cdots$ as $A_{0,0} \subset A_{0,1} \subset A_{0,2} \subset \cdots$, and also construct $A_{0,0} \subset A_{1,0} \subset A_{1,1} \subset A_{1,2} \subset \cdots$, $A_{0,0} \subset A_{1,0} \subset A_{2,0} \subset A_{2,1} \subset \cdots$, and so on in the same method. By this way, we get

$$\begin{array}{ccccccc} A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \cdots & \to & A_{0,\infty} \\ \cap & & & & & & & \\ A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \cdots & \to & A_{1,\infty} \\ \cap & & & & & & & \\ A_{2,0} & \subset & A_{2,1} & \subset & A_{2,2} & \cdots & \to & A_{2,\infty} \\ \cap & & & & & & & \\ \vdots & & & & & & & \end{array}$$

We would like to get vertical inclusions, too, and we use a *connection* for this purpose. Take $A_{2,2}$ for example. To get a string in this algebra, we use vertical paths of length 2 composed with horizontal paths of length 2. But we would like to use other type of paths, too. For example, we could use horizontal path of length 2 composed with vertical paths of length 2. In general, we get several mutually isomorphic finite dimensional string algebras corresponding to choices of types of paths. (There are ${}_{k+l}C_k$ types of paths for $A_{k,l}$.) We give explicit isomorphisms between these algebras. When we regard paths as a basis of a finite dimensional Hilbert space, strings are rank one operators given by them. So it is enough to give unitaries identifying other types of paths. For this purpose, it is enough to give a unitary identification of the following type.

$$\begin{array}{ccc} a & & \\ \downarrow & & \\ b & \longrightarrow & d \end{array} = \sum_c W\begin{pmatrix} a & \longrightarrow & c \\ \downarrow & & \downarrow \\ b & \longrightarrow & d \end{pmatrix} \begin{array}{ccc} a & \longrightarrow & c \\ & & \downarrow \\ & & d \end{array},$$

where the left hand side is a vector in a finite dimensional Hilbert space and the right hand side is a linear combination of vectors with complex scalar coefficients $W\begin{pmatrix} a & \longrightarrow & c \\ \downarrow & & \downarrow \\ b & \longrightarrow & d \end{pmatrix}$ defined for each 1×1-square whose vertices and edges come from the original graph. This W is called a connection, and we often use a notation $\begin{array}{ccc} a & \longrightarrow & c \\ \downarrow & & \downarrow \\ b & \longrightarrow & d \end{array}$ for $W\begin{pmatrix} a & \longrightarrow & c \\ \downarrow & & \downarrow \\ b & \longrightarrow & d \end{pmatrix}$. This is an analogue of a Boltzmann weight in the theory of solvable lattice models and our unitarity requirement corresponds to the first inversion relations in solvable lattice model theory. (See [B], [DJMO] for general background in solvable lattice model theory.) With the connection, we can get the following double sequence of string algebras.

$$\begin{array}{ccccccccc} A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \cdots & \to & A_{0,\infty} \\ \cap & & \cap & & \cap & & & \cap \\ A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \cdots & \to & A_{1,\infty} \\ \cap & & \cap & & \cap & & & \cap \\ A_{2,0} & \subset & A_{2,1} & \subset & A_{2,2} & \cdots & \to & A_{2,\infty} \\ \cap & & \cap & & \cap & & & \cap \\ \vdots & & \vdots & & \vdots & & & \vdots \end{array}$$

Note that for identifications in general, we use the following method. We make all the possible fillings of squares for a large diagram, whose four edges are given, as follows.

$$
\begin{array}{ccccccc}
\cdot & \longrightarrow & \cdot & \longrightarrow \cdots & \cdot & \longrightarrow & \cdot \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\cdot & \longrightarrow & \cdot & \longrightarrow \cdots & \cdot & \longrightarrow & \cdot \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\cdot & \longrightarrow & \cdot & \longrightarrow \cdots & \cdot & \longrightarrow & \cdot \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\cdot & \longrightarrow & \cdot & \longrightarrow \cdots & \cdot & \longrightarrow & \cdot
\end{array}
$$

Such a choice is called a configuration. We multiply the connection values of all the small squares in a configuration and sum them over all the configurations. This is the value assigned to the above large diagram, and we mean this value by the diagram. This is an analogue of a partition function in solvable lattice model theory. Our main goal is computing certain partition functions explicitly which have operator algebraic meanings.

But in order to get an interesting information on the subfactor, we need another condition as follows.

$$
\begin{array}{ccc}
a & \longrightarrow & b \\
\Big\downarrow & & \Big\downarrow \\
c & \longrightarrow & d
\end{array}
= \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}}\;
\overline{\begin{array}{ccc}
b & \longrightarrow & a \\
\Big\downarrow & & \Big\downarrow \\
d & \longrightarrow & c
\end{array}}
$$

$$
= \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}}\;
\overline{\begin{array}{ccc}
c & \longrightarrow & d \\
\Big\downarrow & & \Big\downarrow \\
a & \longrightarrow & b
\end{array}},
$$

where $\mu(\cdot)$ denote the entries of the Perron-Frobenius eigenvector of the incidence matrix of the graph. This condition corresponds to the famous *commuting square* condition of Jones. If we make a connection arising from a subfactor by Ocneanu's Galois functor, this condition is always satisfied. This ia an analogue of the crossing symmetry condition in solvable lattice model theory. An example of $\mu(\cdot)$ is given as follows. If we take the Dynkin diagram A_n and label its vertices $1, 2, \dots, n$, then the entry $\mu(k)$ is given by $\sin \frac{k\pi}{n+1}$. Then the Perron-Frobenius eigenvalue arises as follows.

$$
\sin \frac{(k-1)\pi}{n+1} + \sin \frac{(k+1)\pi}{n+1} = 2\cos \frac{\pi}{n+1} \sin \frac{k\pi}{n+1}.
$$

There are also explicit formulas for μ for the other (extended) Dynkin diagrams. See [GHJ] for example.

The unitarity and the crossing symmetry are strong enough to determine connection on the Dynkin diagrams. Indeed, there is a unique connection for each of A_n and D_n, and there are two (and only two) connections for each of E_6, E_7, E_8, up to certain equivalence relation explained in [O1]. These can be expressed by the single following formula as in [O1, O2].

$$\begin{array}{ccc} a & \longrightarrow & b \\ \Big\downarrow & & \Big\downarrow \\ c & \longrightarrow & d \end{array} = \delta_{bc}\varepsilon + \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}}\delta_{ad}\bar{\varepsilon},$$

where $\varepsilon = \sqrt{-1}\exp\dfrac{\pi\sqrt{-1}}{2N}$ and N is the Coxeter number of the diagram.

In general cases arising from subfactors, we have two graphs as in [O1, O2], but in the case index less than or equal to 4, the two graphs are the same and they are (extended) Dynkin diagrams, so our above explanation is enough for this case.

§2 Flatness for connections

Connections arising from subfactors satisfy another condition which is called *flatness*. Flat connections are complete invariants for subfactors with finite depth and finite index of the AFD type II_1 factor. (See [GHJ], [O1], [P1] for related definitions.) From Popa's viewpoint, subfactors are classified by the towers of relative commutants, and in the finite depth case, the canonical commuting square in the sense of Popa contains complete information. Ocneanu's paragroup gives a combinatorial characterization of the canonical commuting square.

In the double sequence of the string algebras, flatness means that the vertical strings and the horizontal strings commute. That is, for $x \in A_{k,0}$ and $y \in A_{0,l}$, we get $xy = yx$. (This equality holds in a bigger string algebra to which we embed $A_{k,0}$ and $A_{0,l}$ The algebra $A_{k,l}$ is an example of such an algebra.) This condition depends on the choice of the starting point $*$ of the graph. (If we have two graphs, we also require this condition for the case two graphs are interchanged.) M. Izumi has a remark in [I] that the starting vertex $*$ must have the smallest entry of the Perron-Frobenius eigenvector. The name "flat connection" comes from its analogy to flat connections in differential geometry. (See [K, Theorem 2.1]. Flatness can be stated in the form that a loop does not change its form under parallel transport from the origin $*$ to the origin.)

Operator algebraic meaning of flatness is as follows. Crossing symmetry implies that each line of the double sequence of the string algebras gives an algebra of the Jones tower. That is, $A_{0,\infty} \subset A_{1,\infty} \subset A_{2,\infty} \subset \cdots$ is the Jones tower for a factor-subfactor pair $A_{0,\infty} \subset A_{1,\infty}$. Ocneanu's compactness argument in [O2] shows that $A'_{0,\infty} \cap A_{k,\infty} \subset A_{k,0}$. Flatness of the connection means that we have equalities $A'_{0,\infty} \cap A_{k,\infty} = A_{k,0}$ for all k, thus the tower of relative commutants is given by the first vertical line of the double sequence. Because we have another flatness for the other graph, we get that the canonical commuting squares in the sense of Popa [P1] are given by the two vertical lines of the double sequence of the string algebras. In this way, we get one-to-one correspondence between the paragroups and the towers of the relative commutants of subfactors with finite index and finite depth.

Crossing symmetry implies that the subalgebra of $A_{k,0}$ generated by the vertical Jones projections commute with the horizontal string algebra from $*$. In [O2], this fact is referred as flatness of the Jones projections. For the Dynkin diagrams A_n, the string algebra is generated by the Jones projections as in [J1], so we get flatness for A_n. These A_n's have fundamental importance in subfactor theory and other theories. These subfactors correspond to the famous link invariant, the Jones polynomial, [J2], in knot theory, to $U_q(sl_2)$ in quantum group theory, and to Andrews-Baxter-Forrester model [ABF] in solvable lattice model theory. In solvable lattice model theory, we have the Yang-Baxter equation. Because the Yang-Baxter equation implies flatness of the face operators as in [R] and the face operators in the A_n cases are linear combinations of the identity and the Jones projection, we may also say that the flatness for A_n follows from the Yang-Baxter equation.

§3 Orbifold methods

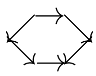
For the Dynkin diagrams D_n, A. Ocneanu announced in [O1] that the connections on D_{2n} are flat and those on D_{2n+1} are not flat, but his proof has been unavailable. Here we show that we can prove this statement with an idea of orbifold models in solvable lattice model theory. (Recently, M. Izumi showed non-flatness of the connections on D_{2n+1} independently in [I] by a different method based on Longo's sector theory [L1, L2].)

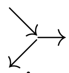
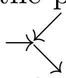
First note that the graph A_{2n-3} has a flip fixing the central vertex. If we construct a string algebra with A_{2n-3} using paths starting from *either* of the two endpoints, the flip can acts on this string algebra as a $*$-algebra automorphism. Then it is easy to see that the fixed point algebra under this automorphism is isomorphic to the string algebra of D_n. In this way, the graph D_n can be regarded as an orbifold of A_{2n-3} by this $\mathbf{Z}_2$-action. (See [C] for more about this in operator algebraic situations.) The point in this construction is that the connection of the A_{2n-3} graph defined in the above section is also invariant under the $\mathbf{Z}_2$-flip. In this case, we can get a connection on the D_n graph in this orbifold procedure. This is an analogue of orbifold models [FG, Ko] in solvable lattice models. Then we get the same connection on D_n as the one defined above. That is, we label A_{2n-3} and D_n as follows.

$$A_n: \qquad 0 — 1 \cdots n-2 — n-1$$

$$D_n: \qquad \begin{matrix} 0 \diagdown \\ \quad 2 — 3 \cdots n-1, \\ 1 \diagup \end{matrix}$$

Then we have the cell system is given as in [R]. Here all the values for small squares are 1, $\frac{1}{\sqrt{2}}$, or $\frac{-1}{\sqrt{2}}$. With this cell system, one can embed the string algebra of D_n into that of A_{2n-3} with the double starting points. Now we would like to embed the string algebra double sequence of D_n into that of A_{2n-3} with the double starting points. For this, we have to check identifications based on connections are compatible with this embedding. It is enough to check the following formula, a kind of the star triangle relation, as follows. Take a hexagon



where the left two downward edges are from the graph D_n, the right two downward edges from A_{2n-3}, and the two horizontal edges connect the two graphs. For each such fixed hexagon, we first consider configurations  for inside of the hexagon. For each configuration, we have a connection of D_n for the left parallelogram and two cell system values for the right parallelograms. We multiply these three numbers and make a sum over all the configurations as in the partition functions. Similarly we make another sum over configurations . We have to show these two sums for all the hexagons, but this can be done by direct computations. (There are 34 cases to be checked.)

With these, we can embed the string algebra double sequence of D_n into that of A_{2n-3} with double starting points, and we can reduce the flatness problem for D_n to a certain problem of computing partition functions for A_{2n-3}. By several combinatorial arguments, it is shown that flatness of D_n

is equivalent to the following equality

$$\mathrm{Re}\ \begin{array}{ccccccccc} 0 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & 2n-5 & \longrightarrow & 2n-4 \\ \downarrow &&&&&&&& \downarrow \\ 1 &&&&&&&& 2n-5 \\ \downarrow &&&&&&&& \downarrow \\ \vdots &&&&&&&& \vdots \\ \downarrow &&&&&&&& \downarrow \\ 2n-5 &&&&&&&& 1 \\ \downarrow &&&&&&&& \downarrow \\ 2n-4 & \longrightarrow & 2n-5 & \longrightarrow & \cdots & \longrightarrow & 1 & \longrightarrow & 0 \end{array} = 1$$

for A_{2n-3}.

Then another combinatorial argument using induction shows that the above value is $(-1)^n$, which proves Ocneanu's announcement.

In short, there arises a $\mathbf{Z}_2$-obstruction for flatness when we make an orbifold from A_{2n-3} and D_{2n+1}'s are killed by this obstruction.

Further details on orbifold subfactors and applications are found in [K] and forthcoming papers of the author with David Evans and Masaki Izumi.

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