One-parameter automorphism groups of the injective II_1 factor arising from the irrational rotation C^*-algebra

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Abstract. We show a certain one-parameter automorphism group of the injective II_1 factor $\mathcal{R}$ arising from the irrational rotation C^*-algebra A_θ is cocycle conjugate to an infinite tensor product type action, hence, unique up to cocycle conjugacy. An $SL(2,\mathbf{Z})$-action on A_θ, a Rieffel projection in A_θ, and central sequence technique in $\mathcal{R}$ are used.

§0 Introduction

In the irrational rotation C^*-algebra A_θ with $uv = e^{2\pi i\theta} vu$, consider the following one-parameter automorphism group α_t: $\alpha_t(u) = e^{i\lambda t}u, \alpha_t(v) = e^{i\mu t}v$. Here λ and μ are non-zero real numbers with $\lambda/\mu \notin \mathbf{Q}$. We extend this one-parameter automorphism group to the weak closure $\mathcal{R}$ of A_θ with respect to the trace τ, which is the AFD (approximately finite dimensional) II_1 factor. We will show this one-parameter automorphism group is cocycle conjugate to an infinite tensor product type one-parameter automorphism group with full Connes spectrum $\mathbf{R}$ if and only if λ/μ is not in the $GL(2,\mathbf{Q})$ orbit of θ. Then such a one-parameter automorphism group is unique up to cocycle conjugacy by our previous result [12].

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Now we explain the motivation of investigating this one-parameter automorphism group. After Connes' seminal work on the classification of single automorphisms of AFD II_1 factor [7], there have been remarkable progress in the cocycle conjugacy classification of discrete amenable group actions on AFD factors. But these developments are restricted to the *discrete* case. Without question, the cocycle conjugacy problem of continuous group actions on AFD factors is one of the major problems in the theory of von Neumann algebras. In fact, the completion of the cocycle conjugacy classification of one-parameter automorphism groups on the AFD II_∞ factor would give more insight into the structure of the AFD III_1 factor, whose uniqueness was recently established by Connes [8] and Haagerup [9] — a deep result of the subject whose proof is still considered difficult and mysterious beyond the validity of the result. Note that Haagerup's result can be formulated as follows: One-parameter automorphism group α of the AFD type II_∞ factor is unique up to conjugacy if it satisfies $\mathrm{tr} \cdot \alpha_t = e^{-t}\mathrm{tr}$. In the previous papers [11],[12], we started to challenge the problem, and obtained positive partial results, completion of the classification in the cases that the Connes spectrum $\Gamma(\alpha)$ is not equal to $\mathbf{R}$, and that the action fixes a Cartan subalgebra of $\mathcal{R}$ elementwise. In the latter cases, the condition $\Gamma(\alpha) = \mathbf{R}$ implies uniqueness of α up to cocycle conjugacy. Note that these cases include infinite tensor product type actions. In this paper, we consider the above one-parameter automorphism group α of the AFD II_1 factor $\mathcal{R}$ which is far from the infinite tensor product type, i.e., our actions are ergodic and almost periodic.

The results in [11],[12] are analogous to the classification of the AFD type III factors. Thus one might expect that $\Gamma(\alpha) = \mathbf{R}$ would imply the uniqueness of α up to cocycle conjugacy, as an analogue of the uniqueness of the AFD type III_1 factor, but this is not the case. For the type III factors, the condition $S(\mathcal{M}) = [0, \infty)$ implies $T(\mathcal{M}) = \{0\}$. (See Connes [6].) But now $\Gamma(\alpha) = \mathbf{R}$ for a one-parameter automorphism group α of the AFD II_1 factor $\mathcal{R}$ does not imply $\{t \in \mathbf{R} \,|\, \alpha_t \in \mathrm{Int}(\mathcal{R})\} = \{0\}$. Indeed, for the above one-parameter automorphism group α of $\mathcal{R}$, it is easy to see that we have $\Gamma(\alpha) = \mathbf{R}$ because of $\lambda/\mu \notin \mathbf{Q}$, but we have $\{t \in \mathbf{R} \,|\, \alpha_t \in \mathrm{Int}(\mathcal{R})\} = \{0\}$ if and only if λ/μ is not in $GL(2, \mathbf{Q})$-orbit of θ. (Here $GL(2, \mathbf{Q})$-action is given by a fractional transformation.)

Thus when we try to prove the uniqueness for the case $\Gamma(\alpha) = \mathbf{R}$ in more general situations than in Kawahigashi [12], we have to use the condition $\{t \in \mathbf{R} \,|\, \alpha_t \in \mathrm{Int}(\mathcal{R})\} = \{0\}$ in an essential way. (Note that $\{t \in \mathbf{R} \,|\, \alpha_t \in \mathrm{Int}(\mathcal{R})\} = \{0\}$ does not imply $\Gamma(\alpha) = \mathbf{R}$, either.) But at this point, we do not know the method of making use of this condition in general situations. Thus we are led to investigate the above action α in detail as the next step of [11],[12]. Because infinite tensor product type one-parameter automorphism groups with the full Connes spectrum $\mathbf{R}$ are unique up to cocycle conjugacy by Kawahigashi [12], we can consider an action of this type as a model action, and we compare it with our action α. Our one-parameter automorphism group α has a delicate and interesting property, because when we change parameters λ, μ by a very small number, we get a periodic action or an action with $\Gamma(\alpha) = \mathbf{R}$ and $\{t \in \mathbf{R} \,|\, \alpha_t \in \mathrm{Int}(\mathcal{R})\} \neq \{0\}$. Another interesting property is that it is an ergodic action. The key to the uniqueness in our previous result [12] was the existence of a Cartan subalgebra in the fixed point subalgebra. That is, it is a good condition that a fixed point algebra is large, from this viewpoint, and the ergodic actions are clearly the most difficult ones. (The point is our group

$\mathbf{R}$ is, of course, non-compact.) Another important point of this action is that it is almost periodic. That is, we can extend this action to a $\mathbf{T}^2$ action. Because compact abelian group actions have been classified in Jones-Takesaki [10], it would be natural to try to use this extension to our problem. But Proposition 4.7 in Olesen-Pedersen-Takesaki [14] says that we cannot get a non-ergodic action from an ergodic action by a cocycle perturbation on $\mathbf{T}^2$. What we would like to get is now a model, an infinite tensor product type action, which has a large fixed point algebra. This means that we have to get out of the "compact world" to obtain a large fixed point algebra, though the action is extended to a compact group. Indeed, the hardest step in our proof is fixing countably many projections by successive cocycle perturbations. (Fixing a projection by cocycle perturbation was stated as a problem in the introduction of Takesaki [16]. We solve this problem for our ergodic actions.)

The contents of the sections are as follows. We show existence of a solution of a certain system of inequalities for Diophantine approximation in order to get a desired automorphism of A_θ coming from an $SL(2, \mathbf{Z})$ action. We also need a well-behaved Rieffel projection and a well-behaved unitary in A_θ, and the choice is made in §1. By these, we will make a central sequence of almost 2×2 matrix units which are well-behaved with respect to the derivation in A_θ. (Note that we cannot make a matrix unit in A_θ because the range of the trace of the projections does not contain any rational number.) In §2, we prepare several lemmas for norm estimates of a derivation in holomorphic functional calculus. We have to change projections and unitaries from given ones to better ones while keeping estimates of a derivation. In §3, we will show the splitting of a model action. That is, our action α is cocycle conjugate to the tensor product of α and a model. This is done by central sequence technique. The key point is making almost matrix units commute with each other while they only almost commute at first. We also have to make a central sequence of true matrix units in $\mathcal{R}$ from almost matrix units in A_θ. In §4, the main theorem, Theorem 16, and a corollary is given by showing that a model action can absorb our action α as a factor of tensor product: a model action is cocycle conjugate to the tensor product of a model action and α. The almost periodicity is used for this statement to reduce the case to our previous result [12]. (The key to our result in [12] for almost periodic action with the irreducible fixed point algebra was Ocneanu's theorem [13].)

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§1 An $SL(2,\mathbf{Z})$-action on A_θ

We will have to solve a certain system of inequalities for Diophantine approximation for constructing well-behaved elements in a C^*-algebra A_θ. We show in this section that it is possible to solve the system. This enables us to find a desirable automorphism of A_θ arising from $SL(2,\mathbf{Z})$-action.

We fix some notations. Let A_θ be the C^*-algebra generated by two unitaries u, v with the relation $uv = e^{2\pi i\theta}vu$, $\theta \in [0,1] \setminus \mathbf{Q}$. Note that if an element x is in A_θ, then it can be expressed as an ℓ^2-sum $x = \sum_{n,m\in\mathbf{Z}} a_{n,m}u^n v^m$, $a_{n,m} \in \mathbf{C}$. We define a subalgebra A_θ^∞ of smooth elements by

$$A_\theta^\infty = \{ \sum_{n,m\in\mathbf{Z}} a_{n,m}u^n v^m \in A_\theta \,|\, ((|n|^k + |m|^k)a_{n,m}) \in \ell^2(\mathbf{Z}^2) \text{ for any integer } k \geq 0\}.$$

We write τ for the unique normalized trace on A_θ. We consider the derivation δ of A_θ defined by $\delta(u) = i\lambda u$, $\delta(v) = i\mu v$, where λ and μ are non-zero real numbers. (This is the generator of a one-parameter automorphism group of A_θ.) We assume $\lambda/\mu \notin \mathbf{Q}$, and λ/μ is not in the $GL(2,\mathbf{Q})$ orbit of θ. (A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on θ by $A\theta = (a\theta + b)/(c\theta + d)$.) We identify the one-dimensional torus $\mathbf{T}$ with $\mathbf{R}/\mathbf{Z}$ and $[0,1[$, and for $x \in \mathbf{R}$ we use the notation $\{x\} = x \bmod \mathbf{Z} \in \mathbf{T}$, and $\|x\|$ for the distance between x and the nearest integer. The following Lemmas 1 and 2 are preliminaries for Lemma 3, which is the key lemma in this paper.

Lemma 1. *Let the real numbers θ, λ, μ be as above. Let $\varepsilon > 0$, and I, J be open intervals in $\mathbf{T}$ with the width 2ε. Then there exist integers a, b such that $\{a\theta\} \in I$, $\{b\theta\} \in J$ and $|a\lambda + b\mu| < \varepsilon$.*

Proof. Set

$$A = \{(a\theta, b\theta, a\lambda + b\mu) \mid a, b \in \mathbf{Z}\} \subset \mathbf{T}^2 \times \mathbf{R}.$$

It is enough to show $\bar{A} = \mathbf{T}^2 \times \mathbf{R}$. Suppose A is not dense. Then there exists $n, m \in \mathbf{Z}$ and $\nu \in \mathbf{R}$ such that $(n, m, \nu) \neq (0,0,0)$ and

$$\exp 2\pi i(na\theta + mb\theta + \nu(a\lambda + b\mu)) = 1, \qquad \text{for } a, b \in \mathbf{Z}.$$

Taking $a = 0, b = 1$ and $a = 1, b = 0$, we get $m\theta + \mu\nu = m'$ and $n\theta + \lambda\nu = n'$ for some $m', n' \in \mathbf{Z}$ respectively. Because $\theta \notin \mathbf{Q}$, we get $\nu \neq 0$, and $\lambda/\mu = (n' - n\theta)/(m' - m\theta)$, which is a contradiction. Thus we are done. Q.E.D.

In the next step, we take the above a, b so that they are relatively prime.

Lemma 2. *Let the real numbers θ, λ, μ be as above. Let $\varepsilon > 0$, and I, J be open intervals in $\mathbf{T}$ with the width 2ε. Then there exist integers a, b such that $(a, b) = 1$, $\{a\theta\} \in I$, $\{b\theta\} \in J$ and $|a\lambda + b\mu| < \varepsilon$.*

Proof. We may assume the transfomation S on $\mathbf{T}^2$ defined by a translation by $(\varepsilon, \varepsilon/\sqrt{2})$ is ergodic by replacing ε by a smaller number, if necessary. (Choose ε so that $\varepsilon \notin \{\sqrt{2}k/(\sqrt{2}n + m) \,|\, k, n, m \in \mathbf{Z}, (n, m) \neq (0, 0)\}$.)

Choose a positive integer N such that for every $(x, y) \in \mathbf{T}^2$ there exists j with $0 \leq j \leq N$, $\|x - j\varepsilon\| < \varepsilon/2$ and $\|y - j\varepsilon/\sqrt{2}\| < \varepsilon/2\sqrt{2}$. This is possible because of the ergodicity of S. Choose integers a' and b' by Lemma 1 so that

$$\varepsilon - \varepsilon/2N < \{a'\theta\} < \varepsilon,$$

$$(\varepsilon - \varepsilon/2N)/\sqrt{2} < \{b'\theta\} < \varepsilon/\sqrt{2},$$

$$0 < |a'\lambda + b'\mu| < \varepsilon/2N.$$

Let $k = (a', b')$. Because $k \neq 0$ and $(a'/k, b'/k) = 1$, there exist integers c, d such that $a'd/k - b'c/k = 1$. We may assume $|c\lambda + d\mu| < \varepsilon/2N$ by replacing c, d by $c + la', d + lb'$ respectively for an appropriate l, if necessary. Let I' and J' be the open intervals in $\mathbf{T}$ which have the same centers as I and J respectively, and have the width ε. Then by the definition of N, there exists a positive integer $j \leq N$ such that $\{c\theta + j\varepsilon\} \in I'$ and $\{d\theta + j\varepsilon/\sqrt{2}\} \in J'$. By the choice of a' and b', this implies $\{c\theta + ja'\theta\} \in I$ and $\{d\theta + jb'\theta\} \in J$. It suffices to set $a = c + ja'$ and $b = d + jb'$ because we have $(a, b) = 1$ by $(a'/k)b - (b'/k)a = 1$, and $|a\lambda + b\mu| < \varepsilon/2N + \varepsilon/2 \leq \varepsilon$.

Q.E.D.

Now we can prove the key lemma.

Lemma 3. *Let the real numbers θ, λ, μ be as above. For any $\varepsilon > 0$ and $\nu \in \mathbf{R}$, there exist integers a, b, c, d such that*

$$\|a\theta\|, \|b\theta\|, \|c\theta\|, \|d\theta\| < \varepsilon,$$

$$ad - bc = 1,$$

$$|a\lambda + b\mu|, |c\lambda + d\mu - \nu| < \varepsilon.$$

Proof. The proof is very similar to that of Lemma 2. We may assume again that the transfomation S on $\mathbf{T}^2$ defined by a translation by $(\varepsilon, \varepsilon/\sqrt{2})$ is ergodic. Choose a positive integer N such that for every $(x, y) \in \mathbf{T}^2$ there exists j with $0 \leq j \leq N$, $\|x - j\varepsilon\| < \varepsilon/2$ and $\|y - j\varepsilon/\sqrt{2}\| < \varepsilon/2\sqrt{2}$. Choose integers a and b by Lemma 2 so that

$$(a, b) = 1,$$

$$\varepsilon - \varepsilon/2N < \{a\theta\} < \varepsilon,$$

$$(\varepsilon - \varepsilon/2N)/\sqrt{2} < \{b\theta\} < \varepsilon/\sqrt{2},$$

$$0 < |a\lambda + b\mu| < \varepsilon/2N.$$

There exist integers c', d' such that $ad' - bc' = 1$. We may assume $|c'\lambda + d'\mu - \nu| < \varepsilon/2N$ by replacing c', d' by $c' + la, d' + lb$ respectively for an appropriate l, if necessary. Then by the definition of N, there exists a positive integer $j \leq N$ such that $\|c'\theta + j\varepsilon\| < \varepsilon/2$ and $\|d'\theta + j\varepsilon/\sqrt{2}\| < \varepsilon/2$. By the choice of a and b, this implies $\|c'\theta + ja\theta\| < \varepsilon$ and $\|d'\theta + jb\theta\| < \varepsilon$. It suffices to set $c = c' + ja$ and $d = d' + jb$ because we then have $ad - bc = 1$ and $|c\lambda + d\mu - \nu| < \varepsilon/2N + \varepsilon/2 \leq \varepsilon$.

Q.E.D.

We introduce a new definition here.

Definition 4. Let A be a unital C^*-algebra with a normalized trace τ. For a positive number ε, a pair of a projection e and a unitary w in A is called an ε-pair if e and wew^* are orthogonal, and $\tau(e + wew^*) \geq 1 - \varepsilon$.

Note that this gives us an "almost" 2×2 matrix unit in the sense that e, ew^*, we, wew^* make a 2×2 matrix unit of $(A_\theta)_{e+wew^*}$ and $\tau(e + wew^*) \geq 1 - \varepsilon$.

Lemma 5. *For any positive ε, there exists an ε-pair (e, w) in A_θ^∞ in the form*

$$
\begin{aligned}
e &= f(u) + g(u)v + v^* g(u), \\
w &= v^m,
\end{aligned}
$$

where f and g are C^∞-functions on $\mathbf{T}$. Moreover, for a given positive integer m_0, m can be chosen so that $m > m_0$.

Proof. We identify the unit circle in the complex plane, $\mathbf{R}/\mathbf{Z}$ and the unit interval $[0, 1]$. (The points 0 and 1 are identified.) By the functional calculus, we identify u and the function $e^{2\pi it}$ of t on the unit interval. Then for a continuous function $f(t)$

on the unit interval, we get the relation $vfv^*(t) = f(t-\theta)$. Take a positive integer n such that $(1-\varepsilon)/2 < \{n\theta\} < 1/2$, and take $\varepsilon' > 0$ such that $\{n\theta\} + \varepsilon' < 1/2$.

First we assume $n = 1$ for simplicity. Choose a Rieffel projection e as follows. (See Theorem 1.1 in Rieffel [15].) Set $e = f(t) + g(t)v + v^*g(t)$, where $0 \leq f(t) \leq 1$ on the interval $[0, \varepsilon']$, $f(t) = 1 - f(t-\theta)$ on $[\theta, \theta + \varepsilon']$, $f(t) = 1$ on $[\varepsilon', \theta]$, and $f(t) = 0$ elsewhere, and $g(t) = ((1-f(t))f(t))^{1/2}$ on the interval $[\theta, \theta+\varepsilon']$, and $g(t) = 0$ elsewhere. We can take f and g among C^∞-functions here. We choose a positive integer $m > m_0$ such that $1/2 < \{m\theta\} < 1 - \theta - \varepsilon'$. Set $w = v^m$. First we show $wew^*e = 0$. Because $wew^* = f(t-m\theta) + g(t-m\theta)v + v^*g(t-m\theta)$, we get, by a direct computation,

$$
\begin{aligned}
wew^*e =& f(t-m\theta)f(t) + f(t-m\theta)g(t)v + v^*f(t-\theta-m\theta)g(t) \\
&+ g(t-m\theta)f(t-\theta)v + g(t-m\theta)g(t-\theta)v^2 + g(t-m\theta)g(t) \\
&+ v^*g(t-m\theta)f(t) + g(t-m\theta+\theta)g(t+\theta) + v^{*2}g(t-m\theta-\theta)g(t).
\end{aligned}
$$

Then all the nine terms on the right hand side turn out to be zero. We also have

$$
\tau(e + wew^*) = 2\tau(e) = 2\theta > 1 - \varepsilon.
$$

In general cases $n > 1$, we can apply the same technique as in the proof of Theorem 1.1 in Rieffel [15]. Q.E.D.

Next we consider an action of $SL(2, \mathbf{Z})$ on A_θ. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$. We define the automorphism σ_g of A_θ by $\sigma_g(u) = u^a v^b$ and $\sigma_g(v) = u^c v^d$. (This action was considered in Brenken [5] and Watatani [17].) Note that this actually defines an automorphism because we have $(u^a v^b)(u^c v^d) = e^{2\pi i\theta}(u^c v^d)(u^a v^b)$ by $ad - bc = 1$.

We need easy lemmas for norm estimates for the $SL(2, \mathbf{Z})$-action. Let δ_1, δ_2 be the canonical derivations of A_θ defined by $\delta_1(u) = iu, \delta_1(v) = 0$ and $\delta_2(u) = 0, \delta_2(v) = iv$. For $x = \sum_{n,m\in\mathbf{Z}} a_{n,m} u^n v^m \in A_\theta^\infty$, we set $\|x\|_1 = \sum_{n,m\in\mathbf{Z}} |a_{n,m}|$. Note that this is bounded because $x \in A_\theta^\infty$ and also note that $\|x\| \leq \|x\|_1$.

Lemma 6. *Let the real numbers θ, λ, μ and the derivation δ be as above. For $x \in A_\theta^\infty$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$, we get*

$$\|\sigma_g^{-1} \cdot \delta \cdot \sigma_g(x)\|_1 \leq \max(|a\lambda + b\mu|, |c\lambda + d\mu|)(\|\delta_1(x)\|_1 + \|\delta_2(x)\|_1).$$

Proof. Let $x = \sum_{n,m\in\mathbf{Z}} a_{n,m} u^n v^m \in A_\theta^\infty$, $a_{n,m} \in \mathbf{C}$. Because

$$\sigma_g^{-1} \cdot \delta \cdot \sigma_g(x) = \sum_{n,m\in\mathbf{Z}} a_{n,m}(n(a\lambda + b\mu) + m(c\lambda + d\mu))u^n v^m,$$

we get

$$\begin{aligned}\|\sigma_g^{-1} \cdot \delta \cdot \sigma_g(x)\|_1 &\leq \sum_{n,m\in\mathbf{Z}} |a_{n,m}|(|n||a\lambda + b\mu| + |m||c\lambda + d\mu|) \\ &\leq \max(|a\lambda + b\mu|, |c\lambda + d\mu|)(\|\delta_1(x)\|_1 + \|\delta_2(x)\|_1).\end{aligned}$$

Q.E.D.

Lemma 7. *Let the real numbers θ, λ, μ and the derivation δ be as above. For $x, y \in A_\theta^\infty$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$, we get*

$$\begin{aligned} &\|[\sigma_g(x), y]\|_1 \\ &\le 2\pi \max(\|a\theta\|, \|b\theta\|, \|c\theta\|, \|d\theta\|)(\|\delta_1(x)\|_1 + \|\delta_2(x)\|_1)(\|\delta_1(y)\|_1 + \|\delta_2(y)\|_1). \end{aligned}$$

Proof. Let $x = \sum_{n,m \in \mathbf{Z}} a_{n,m} u^n v^m \in A_\theta^\infty$, $y = \sum_{n,m \in \mathbf{Z}} b_{n,m} u^n v^m \in A_\theta^\infty$, $a_{n,m}, b_{n,m} \in \mathbf{C}$. By direct computation, we get

$$\begin{aligned} &\|[\sigma_g(x), y]\|_1 \\ &\le \sum_{n,m,n',m' \in \mathbf{Z}} |a_{n,m}||b_{n',m'}|\|[(u^a v^b)^n (u^c v^d)^m, u^{n'} v^{m'}]\|_1 \\ &\le \sum_{n,m,n',m' \in \mathbf{Z}} |a_{n,m}||b_{n',m'}||1 - \exp(2\pi i \theta(-n'md - n'mb + m'mc + m'na))| \\ &\le \sum_{n,m,n',m' \in \mathbf{Z}} |a_{n,m}||b_{n',m'}|2\pi\|(-n'md - n'mb + m'mc + m'na)\theta\| \\ &\le \sum_{n,m,n',m' \in \mathbf{Z}} |a_{n,m}||b_{n',m'}|2\pi(|n'||m|\|d\theta\| + |n'||m|\|b\theta\| + |m'||m|\|c\theta\| + |m'||n|\|a\theta\|) \\ &\le 2\pi \max(\|a\theta\|, \|b\theta\|, \|c\theta\|, \|d\theta\|) \sum_{n,m,n',m' \in \mathbf{Z}} |a_{n,m}||b_{n',m'}|(|n| + |m|)(|n'| + |m'|) \\ &\le 2\pi \max(\|a\theta\|, \|b\theta\|, \|c\theta\|, \|d\theta\|)(\|\delta_1(x)\|_1 + \|\delta_2(x)\|_1)(\|\delta_1(y)\|_1 + \|\delta_2(y)\|_1). \end{aligned}$$

Q.E.D.

The next lemma will give us an almost matrix unit which is well-behaved with respect to the derivation and almost commutativity. This can be regarded as a variant of the non-commutative Rohlin Theorem in the sense that it produces a piece of a model action. (See §6.1 in Ocneanu [13].) This is also related to property L'_λ. (See Araki [1].)

Lemma 8. *Let the real numbers θ, λ, μ and the derivation δ of A_θ be as above. Let $F \subset A_\theta^\infty$ be a finite set, $\nu \in \mathbf{R}$ and $\varepsilon > 0$. Then there exists an ε-pair (e, w) such that*

$$\|[e,y]\|, \|[w,y]\| \le \varepsilon, \qquad \textit{for every } y \in F,$$

$$\|\delta(e)\|, \|\delta(w) - i\nu w\| \le \varepsilon.$$

Proof. Choose a Rieffel projection e as in Lemma 5. Set

$$\varepsilon_1 = \varepsilon / \max(\|\delta_1(e)\|_1 + \|\delta_2(e)\|_1, 1).$$

We choose $w = v^m$ by Lemma 5 so that $m > (|\nu| + \varepsilon_1)/\varepsilon_1$. We also set

$$K = \max_{y \in F}(\|\delta_1(y)\|_1 + \|\delta_2(y)\|_1, 1),$$

and

$$\varepsilon_2 = \frac{\varepsilon}{2\pi K \max(\|\delta_1(e)\|_1 + \|\delta_2(e)\|_1, \|\delta_1(w)\|_1 + \|\delta_2(w)\|_1, 1)}.$$

Choose integers a, b, c, d by Lemma 3 so that

$$\|a\theta\|, \|b\theta\|, \|c\theta\|, \|d\theta\| < \varepsilon_2,$$

$$ad - bc = 1,$$

$$|a\lambda + b\mu|, |c\lambda + d\mu - \nu/m| < \varepsilon_1/m,$$

and let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$. Note that $|a\lambda + b\mu| \leq \varepsilon_1$ and $|c\lambda + d\mu| < (\varepsilon_1 + |\nu|)/m \leq \varepsilon_1$. We will show $\sigma_g(e)$ and $\sigma_g(w)$ satisfy the desired properties. By Lemma 6, we have

$$\|\delta(\sigma_g(e))\| \leq \varepsilon_1(\|\delta_1(e)\|_1 + \|\delta_2(e)\|_1) \leq \varepsilon.$$

We also have

$$\|\delta(\sigma_g(w)) - i\nu w\| = |m(c\lambda + d\mu) - \nu| \leq \varepsilon_1 \leq \varepsilon.$$

For commutators, we have, by Lemma 7,

$$\|[\sigma_g(e), y]\| \leq 2\pi\varepsilon_2(\|\delta_1(e)\|_1 + \|\delta_2(e)\|_1)(\|\delta_1(y)\|_1 + \|\delta_2(y)\|_1) \leq \varepsilon,$$

and similarly $\|[\sigma_g(w), y]\| \leq \varepsilon$ for every $y \in F$. Thus we can replace e, w by $\sigma_g(e)$,$\sigma_g(w)$ respectively. Q.E.D.

§2 Holomorphic functional calculus and norm estimates of a derivation

We need some preliminaries for norm estimates of a derivation for holomorphic functional calculus and inner perturbation of a derivation. These will be used for getting a unitary or a projection which commutes with given elements when we have a unitary or projection which almost commutes with them. We have to keep the estimate of a derivation for this change.

The first lemma is for inner perturbation of a derivation.

Lemma 9. *Let A be a C^*-algebra, and δ be the generator of a one-parameter automorphism group on A. Let A^∞ be a subalgebra of C^∞-elements with respect to δ. If an ε-pair (e, w) in A^∞ satisfies $\|\delta(e)\| \leq \varepsilon'$, $\|\delta(w) - i\nu w\| \leq \varepsilon'$ for $\varepsilon' > 0$ and $\nu \in \mathbf{R}$, and $e, w \in B'$, the commutant of B, for a subset $B \subset A^\infty$ with $\delta = 0$ on B, then there exists a self-adjoint element $h \in A^\infty \cap B'$ such that $\|h\| \leq 100\varepsilon'$, $(\delta + ad(ih))(e) = 0$ and $(\delta + ad(ih))(we) = i\nu we$.*

Proof. Set $e_{11} = e$, $e_{12} = ew^*$, $e_{21} = we$, and $e_{22} = wew^*$. Note that these make a matrix unit of $A_{e_{11}+e_{22}}$. Set $ih_0 = (i\nu e_{11} - i\nu e_{22})/2$, and $ih_1 = [e_{11} + e_{22}, \delta(e_{11} + e_{22})]$. Then it is well known and easy to see that $(\delta + \mathrm{ad}(ih_0 + ih_1))(e_{11} + e_{22}) = 0$. Now we set

$$ih_2 = e_{11}(\delta + \mathrm{ad}(ih_0 + ih_1))(e_{11}) + e_{21}(\delta + \mathrm{ad}(ih_0 + ih_1))(e_{12}).$$

It is well known that $(\delta + \mathrm{ad}(ih_0) + \mathrm{ad}(ih_1 + ih_2))(e_{jk}) = 0$ for all $j, k = 1, 2$. (See Remark 1.6.7 in Bratteli [4]. We can apply it because $(\delta + \mathrm{ad}(ih_0 + ih_1))(e_{11} + e_{22}) = 0$ though $e_{11} + e_{22} \neq 1$ in general.) We can set $h = h_1 + h_2$. Because $\|h_1\| \leq 2 \cdot 2 \cdot 4\varepsilon' = 16\varepsilon'$, we get

$$\|h_2\| \leq \varepsilon' + 2\|h_1\| + 2\|h_1\| + \|\delta(e_{12}) + [ih_0, e_{12}]\| \leq 67\varepsilon',$$

and $\|h\| \leq \|h_1\| + \|h_2\| \leq 100\varepsilon'$. It is easy to check $h \in A^\infty \cap B'$, $(\delta + \mathrm{ad}(ih))(e) = 0$ and $(\delta + \mathrm{ad}(ih))(we) = i\nu we$. Q.E.D.

The following three lemmas deal with unitaries and projections. We will change almost unitaries and projections into true unitaries and projections with norm estimates.

Lemma 10. *Let A be a C^*-algebra, and δ be the generator of a one-parameter automorphism group on A. Let A^∞ be a subalgebra of C^∞-elements with respect to δ. Suppose we have a unitary $u \in A$, and a subset $B \subset A^\infty$. If we have $0 < \varepsilon < 1/12$, $\varepsilon' > 0$, $\nu \in \mathbf{R}$, $x \in A^\infty \cap B'$, $\|x - u\| \leq \varepsilon$, and $\|\delta(x) - i\nu x\| \leq \varepsilon'$, then there exists a unitary $v \in A^\infty \cap B'$ given by the polar decomposition of x with $\|v - x\| \leq 5\varepsilon$ and $\|\delta(v) - i\nu v\| \leq 30\varepsilon'$.*

Proof. First note that $\|x\| \leq 1 + \varepsilon$, and

$$\|x^*x - 1\| \leq \|(x^* - u^*)x\| + \|u^*(x - u)\| \leq \varepsilon(1 + \varepsilon) + \varepsilon \leq 3\varepsilon.$$

Thus $1 - 3\varepsilon \leq x^*x \leq 1 + 3\varepsilon$, which implies $1 - 3\varepsilon \leq |x| \leq 1 + 3\varepsilon$ and $1 - 4\varepsilon \leq |x|^{-1} \leq 1 + 4\varepsilon$. Now set $v = x|x|^{-1}$. Clearly this is a unitary in $A^\infty \cap B'$. We have

$$\|v - x\| \leq \|x(1 - |x|^{-1})\| \leq (1 + \varepsilon)4\varepsilon \leq 5\varepsilon.$$

Let C be a circle with the center 1 and the radius 1/2 on the complex plane. By holomorphic functional calculus, we get the equality

$$|x|^{-1} = \frac{1}{2\pi i} \int_C z^{-1/2}(z - x^*x)^{-1}\, dz.$$

Because δ is a closed derivation, by differentiating this under the integral sign, we get

$$
\begin{aligned}
\|\delta(|x|^{-1})\| &\le \frac{1}{2\pi}\|\int_C \delta(z^{-1/2}(z-x^*x)^{-1})\,dz\| \\
&\le \frac{1}{2\pi}\|\int_C z^{-1/2}(z-x^*x)^{-1}\delta(x^*x)(z-x^*x)^{-1}\,dz\| \\
&\le \frac{2\pi(1/2)}{2\pi}\sqrt{2}\cdot 4^2\|\delta(x^*x)\| \\
&\le 8\sqrt{2}\|\delta(x^*)x + x^*\delta(x)\| \\
&= 8\sqrt{2}\|(\delta(x^*)+i\nu x^*)x + x^*(\delta(x)-i\nu x)\| \\
&\le 16\sqrt{2}(1+\varepsilon)\varepsilon' \le 25\varepsilon'.
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
\|\delta(v)-i\nu v\| &\le \|\delta(x)|x|^{-1} + x\delta(|x|^{-1}) - i\nu x|x|^{-1}\| \\
&\le \varepsilon'(1+4\varepsilon) + (1+\varepsilon)25\varepsilon' \le 2\varepsilon' + 28\varepsilon' = 30\varepsilon'.
\end{aligned}
$$

Q.E.D.

Lemma 11. *Let A, A^∞, δ be as in Lemma 10. Let $0 < \varepsilon < 1/12$, $\varepsilon' > 0$ and $B \subset A^\infty$. Let e, f be projections in $A^\infty \cap B'$ with $\|e - f\| \le \varepsilon$, $\|\delta(e)\| \le \varepsilon'$, and $\|\delta(f)\| \le \varepsilon'$. Then there exists a unitary u in $A^\infty \cap B'$ such that $ufu^* = e$, $\|u-1\| \le 5\varepsilon$, and $\|\delta(u)\| \le 60\varepsilon'$.*

Proof. We use the method of Propositions 4.3.2 and 4.6.5 in Blackadar [3].

Set $x = ((2e-1)(2f-1)+1)/2$. Then

$$
\|1-x\| = \|(2e-1)(e-f)\| \le \varepsilon < 1/12
$$

and

$$\|\delta(x)\| \leq \|2\delta(e)\|\|2f-1\|/2 + \|2e-1\|\|2\delta(f)\|/2 \leq 2\varepsilon'.$$

Note that $ex = ef = xf$. Because $x \in A^\infty \cap B'$, we can apply Lemma 10 with $\nu = 0$, and set $u = x|x|^{-1} \in A^\infty \cap B'$. Then $\|u-1\| \leq 5\varepsilon$ and $\|\delta(u)\| \leq 60\varepsilon'$. By $ex = xf$, we get $uf = eu$. Q.E.D.

Lemma 12. *Let A, A^∞, δ be as in Lemma 10. Let $0 < \varepsilon < 1/48$, $\varepsilon' > 0$ and $B \subset A^\infty$. Suppose $x = x^* \in pA^\infty p \cap B'$ for some projection $p \in A^\infty$, $\|\delta(x)\| \leq \varepsilon'$, and $\|x - f\| \leq \varepsilon$ for some projection $f \in A$. Then there exists a projection $e \in pA^\infty p \cap B'$ such that $\|x - e\| \leq 6\varepsilon$ and $\|\delta(e)\| \leq 8\varepsilon'$.*

Proof. First we have

$$\begin{aligned}\|x^2 - x\| &\leq \|(x-f+f)^2 - (x-f) - f\| \\ &\leq \|(x-f)f + f(x-f) + (x-f)^2 - (x-f)\| \leq \varepsilon^2 + 2\varepsilon \leq 3\varepsilon.\end{aligned}$$

This implies $\mathrm{Sp}(x) \subset [-6\varepsilon, 6\varepsilon] \cup [1-6\varepsilon, 1+6\varepsilon]$. Let C be a circle with the center 1 and the radius 1/2 on the complex plane. By holomorphic functional calculus, we can define a projection

$$e = \frac{1}{2\pi i}\int_C (z-x)^{-1}\,dz \in pA^\infty p \cap B'.$$

(Note that this is a functional calculus by a function φ with $\varphi(0) = 0$.) We know $\|e - x\| \leq 6\varepsilon$, and by a similar computation to the proof of Lemma 10, we get $\|\delta(e)\| \leq 2\pi(1/2)/(2\pi) \cdot 4^2\|\delta(x)\| \leq 8\varepsilon'$. Q.E.D.

§3 Splitting of a model action

We will show splitting of a product type action from our action α arising from A_θ. The next lemma is the key to our inductive construction. We first take a well-behaved projection and a unitary which almost commute with given almost matrix units. Then by the Lemmas in §2, we can change these so that they actually commute with given almost matrix units, while keeping estimates of a derivation.

Lemma 13. *Let δ be the derivation of A_θ as above. Suppose we have a unital finite dimensional $*$-subalgebra F in A_θ^∞ and a self-adjoint element $h \in A_\theta^\infty$ such that $(\delta + ad(ih))(y) = 0$ for every $y \in F$. Suppose $\varepsilon > 0$, $\nu \in \mathbf{R}$ and a finite subset $B \subset A_\theta^\infty$ are given. Set $\delta' = \delta + ad(ih)$. Then there exists an ε-pair (e, w) in A_θ^∞ such that*

$$[e, y], [w, y] = 0, \qquad \textit{for every } y \in F,$$

$$\|[e, x]\|, \|[w, x]\| \le \varepsilon, \qquad \textit{for every } x \in B,$$

$$\|\delta'(e)\|, \|\delta'(w) - i\nu w\| \le \varepsilon.$$

Proof. Let G be a finite group of unitaries which generates the finite dimensional algebra F. Let ε_1 be a small enough positive number whose value will be specified later. Choose an ε-pair (e_0, w_0) in A_θ^∞ by Lemma 8 so that

$$\|\delta(e_0)\|, \|\delta(w_0) - i\nu w_0\| \le \varepsilon_1,$$

$$\|[e_0, ih]\|, \|[w_0, ih]\| \le \varepsilon_1,$$

$$\|[e_0, U]\|, \|[w_0, U]\| \le \varepsilon_1, \qquad \text{for every } U \in G,$$

$$\|[e_0, x]\|, \|[w_0, x]\| \le \varepsilon_1, \qquad \text{for every } x \in B.$$

Note that we get $\|\delta'(e_0)\|, \|\delta'(w_0) - i\nu w_0\| \leq 2\varepsilon_1$. First we set

$$\tilde{e}_1 = \frac{1}{|G|} \sum_{U \in G} U e_0 U^* \in A_\theta^\infty \cap F',$$
$$\tilde{w}_1 = \frac{1}{|G|} \sum_{U \in G} U w_0 U^* \in A_\theta^\infty \cap F'.$$

Then, we have $\|e_0 - \tilde{e}_1\| \leq \varepsilon_1$ and $\|w_0 - \tilde{w}_1\| \leq \varepsilon_1$. We also have estimates $\|\delta'(\tilde{w}_1) - i\nu\tilde{w}_1\| \leq \varepsilon_1$ and $\|\delta'(\tilde{e}_1)\| \leq \varepsilon_1$. Then by Lemma 12 and Lemma 10, we get a projection $e_1 \in A_\theta^\infty \cap F'$ and a unitary $w_1 \in A_\theta^\infty \cap F'$ with

$$\|e_1 - \tilde{e}_1\| \leq 6\varepsilon_1,$$
$$\|\delta'(e_1)\| \leq 16\varepsilon_1,$$
$$\|w_1 - \tilde{w}_1\| \leq 5\varepsilon_1,$$
$$\|\delta'(w_1) - i\nu w_1\| \leq 60\varepsilon_1.$$

Now by $e_0 w_0 e_0 w_0^* = 0$, we get

$$\begin{aligned}
&\|e_1 w_1 e_1 w_1^*\| \\
<&\|(e_1 - e_0) w_1 e_1 w_1^*\| + \|e_0 (w_1 - w_0) e_1 w_1^*\| \\
&+ \|e_0 w_0 (e_1 - e_0) w_1^*\| + \|e_0 w_0 e_0 (w_1^* - w_0^*)\| \\
\leq&(7 + 6 + 7 + 6)\varepsilon_1 = 26\varepsilon_1.
\end{aligned}$$

Setting

$$\tilde{f}_1 = (1 - e_1) w_1 e_1 w_1^* (1 - e_1) \in (1 - e_1) A_\theta^\infty (1 - e_1) \cap F',$$

we have

$$\|w_1e_1w_1^* - \tilde{f}_1\| \le \| - e_1(w_1e_1w_1^*)(1-e_1)\| + \| - (w_1e_1w_1^*)e_1\| \le 52\varepsilon_1,$$

and

$$\|\delta'(\tilde{f}_1)\| \le (16+60+16+60+16)\varepsilon_1 = 168\varepsilon_1.$$

By Lemma 12, there exists a projection $f_1 \in (1-e_1)A_\theta^\infty(1-e_1) \cap F'$ such that

$$\|f_1 - \tilde{f}_1\| \le 6 \cdot 52\varepsilon_1 \le 400\varepsilon_1,$$

$$\|\delta'(f_1)\| \le 8 \cdot 168\varepsilon_1 \le 1400\varepsilon_1.$$

Because

$$\|w_1e_1w_1^* - f_1\| \le 500\varepsilon_1,$$

$$\|\delta'(f_1)\| \le 1400\varepsilon_1,$$

$$\|\delta'(w_1e_1w_1^*)\| \le (60+16+60)\varepsilon_1 \le 200\varepsilon_1,$$

by Lemma 11, there exists a unitary $w_2 \in A_\theta^\infty \cap F'$ such that

$$\|\delta'(w_2)\| \le 90000\varepsilon_1,$$

$$\|w_2 - 1\| \le 2500\varepsilon_1,$$

$$w_2w_1e_1w_1^*w_2^* = f_1.$$

Now finally set $e = e_1$ and $w = w_2 w_1$. These are in $A_\theta^\infty \cap F'$, and we have now

$$
\begin{aligned}
ewew^* &= 0 \\
\|e - e_0\| &\leq 7\varepsilon_1, \\
\|w - w_0\| &\leq \|w_2(w_1 - w_0)\| + \|w_2 - 1\|\|w_0\| \\
&\leq 6\varepsilon_1 + 2500\varepsilon_1 \leq 2600\varepsilon_1.
\end{aligned}
$$

Setting $C = \max_{x \in B}(\|x\|, 1)$, we get

$$
\|[e, x]\| \leq \varepsilon_1 + 2C \cdot 7\varepsilon_1 \leq 15C\varepsilon_1,
$$

and $\|[w, x]\| \leq 5201C\varepsilon_1$, for every $x \in B$. We may assume $\varepsilon < 1/48$, and set $\varepsilon_1 = \varepsilon/(100000C)$. Then we get

$$
\begin{aligned}
\|[e, x]\| &\leq \varepsilon, \qquad \text{for every } x \in B, \\
\|[w, x]\| &\leq \varepsilon, \qquad \text{for every } x \in B, \\
\|\delta'(e)\| &\leq 16\varepsilon_1 \leq \varepsilon, \\
\|\delta'(w) - i\nu w\| &\leq 60\varepsilon_1 + 90000\varepsilon_1 \leq \varepsilon.
\end{aligned}
$$

Because now $e, w \in A_\theta^\infty \cap F'$, we are done. Q.E.D.

Now we can prove the splitting in two steps.

Theorem 14. *Let $\mathcal{R}$ be the weak closure of A_θ with respect to the trace τ, which is the AFD II_1 factor. Let α be the one-parameter automorphism group of $\mathcal{R}$ defined by $\alpha_t(u) = e^{i\lambda t}u$ and $\alpha_t(v) = e^{i\mu t}v$. We assume $\lambda/\mu \notin \mathbf{Q}$, and λ/μ is not in the $GL(2, \mathbf{Q})$ orbit of θ. Let β_t be an infinite tensor product type action of $\mathbf{R}$ on $\mathcal{R}$ with $\Gamma(\beta) = \mathbf{R}$. Then α_t is cocycle conjugate to $\alpha_t \otimes \beta_t$.*

Proof. Let

$$\{x_p \mid p \geq 1\} = \{\sum c_{n,m}u^n v^m \,|\, \text{finite sum}, c_{n,m} \in \mathbf{Q} + \mathbf{Q}i\}.$$

We set $C_n = \max_{p \leq n}(\|x_p\|, 3)$ and $\varepsilon_n = \min(1/(2^n \cdot 100), 1/(9C_n^2 \cdot 4^n))$. Choose a non-zero real number ν. By a repeated use of Lemma 13 for ε_n and Lemma 9, we get an ε_n-pair (e_n, w_n) and $h_n \in A_\theta^\infty$ with the following properties: Setting $f_{11}(n) = e_n$, $f_{12}(n) = e_n w_n^*$, $f_{21}(n) = w_n e_n$, and $f_{22}(n) = w_n e_n w_n^*$,

$$[f_{jk}(n), f_{j'k'}(n')] = 0, \qquad \text{for } j, k, j', k' = 1, 2,\ n \neq n',$$

$$\|[f_{jk}(n), x_p]\| \leq 3\varepsilon_n, \qquad \text{for } p = 1, \ldots, n,$$

$$\|h_n\| \leq \frac{1}{2^n},$$

$$(\delta + \mathrm{ad}(ih_1 + \cdots + ih_n))(f_{jk}(l)) = (j-k)i\nu f_{jk}(l), \qquad \text{for } l = 1, \ldots, n,$$

$$\tau(f_{11}(n) + f_{22}(n)) \geq 1 - \frac{1}{4^n}.$$

(When we have $f_{jk}(1), \ldots, f_{jk}(n)$, let F be the finite dimensional subalgebra generated by 1 and $f_{jk}(1), \ldots, f_{jk}(n)$, $j, k = 1, 2$ and set

$$h = h_1 + \cdots + h_n + (\frac{\nu}{2}f_{11}(1) - \frac{\nu}{2}f_{22}(1)) + \cdots + (\frac{\nu}{2}f_{11}(n) - \frac{\nu}{2}f_{22}(n)).$$

Then we get an ε_{n+1}-pair (e_{n+1}, w_{n+1}) by Lemma 13.) Set $h = \sum_{n=1}^{\infty} h_n \in A_\theta$, and let

$$u_t = \mathrm{Exp}_r\left(\int_0^t ; \alpha_s(ih)\, ds\right) = \sum_{n=0}^{\infty} \int_0^t \cdots \int_0^{t_{n-1}} \alpha_{t_n}(ih) \cdots \alpha_{t_1}(ih)\, dt_n \ldots dt_1.$$

(This is an expansional. See §2 of Araki [2].) We also set $\bar{\alpha}_t = \mathrm{Ad}u_t \cdot \alpha_t$. Now our almost 2×2 matrix units behave well with respect to the generator of $\bar{\alpha}$ without error terms.

So far, we have made a central sequence of mutually commuting almost 2×2 matrix units. We will make a central sequence of mutually commuting true 2×2 matrix units from them. Because our matrix units have to behave well with respect to $\bar{\alpha}_t$, we need a careful choice. We make a matrix unit by composing small pieces from countably many almost matrix units. For this purpose, we will make a double sequence as follows. Choose a bijection φ from $\mathbf{N}^2$ to $\mathbf{N}$ such that $\varphi(n,m) < \varphi(n,m+1)$ and $n \leq \varphi(n,1)$. We set

$$e_{jk}(n,1) = f_{jk}(\varphi(n,1)),$$

$$e_{jk}(n,m+1) = (1 - e_{11}(n,1) - e_{22}(n,1) - \cdots - e_{11}(n,m) - e_{22}(n,m)) f_{jk}(\varphi(n,m+1))$$

$$e_{jk}(n) = \sum_{m=1}^{\infty} e_{jk}(n,m).$$

Note that the right hand side formula of the definition of $e_{jk}(n)$ does not converge in operator norm, but does converge in L^2 norm. Thus our matrix units are not any more in the C^*-algebra A_θ, but in the von Neumann algebra $\mathcal{R}$. (Because the range of the trace of projections in A_θ does not contain any rational number, we cannot make a matrix unit in A_θ.) If $l \le n$, we have

$$\begin{aligned}\|[x_l, e_{jk}(n)]\|_2 &\le \|[x_l, e_{jk}(n,1)]\|_2 + \|[x_l, \sum_{m=2}^{\infty} e_{jk}(n,m)]\|_2 \\ &\le 3\varepsilon_{\varphi(n,1)} + 2C_n \varepsilon_{\varphi(n,1)}^{1/2} \le 3C_n\varepsilon_n^{1/2} \le \frac{1}{2^n}.\end{aligned}$$

Thus $e_{jk}(n)$'s form a central sequence of mutually commuting 2×2 matrix units in $\mathcal{R}$. By Lemma 2.3.6 in Connes [7], we get a factorization $\bar{\alpha}_t = \alpha'_t \otimes \beta_t^{(\nu)}$, where

$$\beta_t^{(\nu)} = \bigotimes_{n=1}^{\infty} \operatorname{Ad}\exp it \begin{pmatrix} -\nu/2 & 0 \\ 0 & \nu/2 \end{pmatrix},$$

and α'_t is some action of $\mathbf{R}$. Because $\beta^{(\nu)} \otimes \beta^{(\nu)} \cong \beta^{(\nu)}$, we know α_t is cocycle conjugate to $\alpha_t \otimes \beta_t^{(\nu)}$. By repeating this procedure for another ν' with $\nu'/\nu \notin \mathbf{Q}$, we know that α_t is cocycle conjugate to $\alpha_t \otimes \beta_t^{(\nu)} \otimes \beta_t^{(\nu')}$. We know that $\beta_t^{(\nu)} \otimes \beta_t^{(\nu')}$ is cocycle conjugate to β_t by Corollary 1.9 in Kawahigashi [12], thus we are done.

Q.E.D.

§4 Uniqueness up to cocycle conjugacy

In this section, we prove the main result. The next theorem says that a model action can absorb a general almost periodic action. (See §7 of Olesen-Pedersen-Takesaki [14] for related definitions.)

Theorem 15. *Let $\mathcal{R}$ be the AFD II_1 factor. Let α be an almost periodic one-parameter automorphism group of $\mathcal{R}$. Let β_t be an infinite tensor product type* *action of* $\mathbf{R}$ *on* $\mathcal{R}$ *with* $\Gamma(\beta) = \mathbf{R}$. *Then* $\alpha_t \otimes \beta_t$ *is cocycle conjugate to the* β_t *on* $\mathcal{R}$.

Proof. Denote by $\mathcal{R}_0$ the AFD II_1 factor on which α acts. Because $\mathrm{Sp}_d(\alpha)$ is countable, we set $\{\lambda_k \mid k \geq 1\} = \mathrm{Sp}_d(\alpha)$. Consider the AFD II_1 factor $\mathcal{R}_1$ and a free action on it of a countable direct sum of copies of $\mathbf{Z}$. Let $\mathcal{R}_2 = \mathcal{R}_1 \rtimes \oplus_{k=1}^\infty \mathbf{Z} \cong \mathcal{R}$ be the crossed product by this action, and let u_k's, $k \geq 1$, be the implementing unitaries of this crossed product algebra. By Theorem 2.1 in Kawahigashi [12], we may assume β on $\mathcal{R}_2$ is of the following form:

$$\begin{aligned}\beta_t(x) &= x, \qquad \text{for } x \in \mathcal{R}_1,\\ \beta_t(u_k) &= e^{-i\lambda_k t} u_k.\end{aligned}$$

Because $(\mathcal{R}_2^\beta)' \cap \mathcal{R}_2 = \mathbf{C}$, we get

$$((\mathcal{R}_0 \bar{\otimes} \mathcal{R}_2)^{\alpha\otimes\beta})' \cap \mathcal{R}_0 \bar{\otimes} \mathcal{R}_2 \subset \mathcal{R}_0 \bar{\otimes} (\mathcal{R}_2 \cap (\mathcal{R}_2^\beta)') = \mathcal{R}_0 \bar{\otimes} \mathbf{C}.$$

But we know that $x \otimes u_k \in (\mathcal{R}_0 \bar{\otimes} \mathcal{R}_2)^{\alpha\otimes\beta}$ if $x \in \mathcal{R}_0^\alpha(\lambda_k)$. Thus by the almost periodicity of α, we get

$$((\mathcal{R}_0 \bar{\otimes} \mathcal{R}_2)^{\alpha\otimes\beta})' \cap \mathcal{R}_0 \bar{\otimes} \mathcal{R}_2 \subset (\mathcal{R}_0 \bar{\otimes} \mathbf{C}) \cap (\bigcup_{k\geq 1} \mathcal{R}_0^\alpha(\lambda_k) \otimes u_k)' = \mathbf{C}.$$

Because $\alpha \otimes \beta$ and β are both almost periodic actions, we can apply Theorem 2.1 in Kawahigashi [12] to conclude $\alpha \otimes \beta$ is cocycle conjugate to β. Q.E.D.

Note that we can apply this theorem to our action α arising from A_θ. We do not need the assumption that λ/μ is not in the $GL(2, \mathbf{Q})$ orbit of θ here. Thus this theorem is valid even if α_t is inner for some $t \neq 0$. The next is the main theorem.

Theorem 16. *Let $\mathcal{R}$ be the weak closure of A_θ with respect to the trace τ, which is the AFD II_1 factor. Let α be the one-parameter automorphism of $\mathcal{R}$ defined by $\alpha_t(u) = e^{i\lambda t}u$ and $\alpha_t(v) = e^{i\mu t}v$. We assume $\lambda/\mu \notin \mathbf{Q}$, and λ/μ is not in the $GL(2,\mathbf{Q})$ orbit of θ. Then α_t is cocycle conjugate to an infinite tensor product type action β of $\mathbf{R}$ on $\mathcal{R}$ with $\Gamma(\beta) = \mathbf{R}$, which is unique up to cocycle conjugacy.*

Proof. Immediate by Theorems 14 and 15. Q.E.D.

For the relative commutant, we get the following. (See Proposition 3.2 in Kawahigashi [12].)

Corollary 17. *For the action α in Theorem 16, we have the trivial relative commutant property: $\mathcal{R}' \cap (\mathcal{R} \rtimes_\alpha \mathbf{R}) = \mathbf{C}I$.*

Proof. This is immediate by Theorem 16 and Proposition 3.2 in Kawahigashi [12].

It is also possible to give a direct proof of this statement by a similar computation to the proof of Lemma 1. Q.E.D.

§5 A remark on C^∞ projections

Set

$$T = \{\tau(p) \,|\, p \text{ is a } C^\infty \text{ projection with respect to } \delta\}.$$

We know that $(\mathbf{Z}+\theta\mathbf{Z}) \cap [0,1] \subseteq T \subseteq [0,1]$. And the fact that T is dense in $[0,1]$ was the key in §3. In this sense, it is a desirable property that T is a large set in $[0,1]$. It is a problem how large this set T is for a general one-parameter automorphism group. Here we will show $T = [0,1]$ for our one-parameter automorphism group. It is enough to show $h = \sum_{n=1}^\infty h_n$ in the proof of Theorem 14 is C^∞ with respect to our δ. Because each h_n is in A_θ^∞, the only problem is the validity of termwise differentiation. We claim h_n in the proof of Theorem 14 can be chosen so that

$$(*) \qquad \|h_n\|, \|\delta(h_n)\|, \ldots, \|\delta^n(h_n)\| \le \frac{1}{2^n}.$$

It is easy to see this implies we can apply δ to h term by term for arbitrarily many times. We provide the outline of the proof of this claim in four steps.

STEP 1. Suppose $h_1, \ldots, h_{n-1}$ are given, and we set

$$\begin{aligned} h' =& h_1 + \cdots + h_{n-1} + \\ & (\frac{\nu}{2} f_{11}(1) - \frac{\nu}{2} f_{22}(1)) + \cdots + (\frac{\nu}{2} f_{11}(n-1) - \frac{\nu}{2} f_{22}(n-1)) \end{aligned}$$

as in the proof of Theorem 14. We choose e, w with the additional conditions:

$$(**) \qquad \begin{aligned} \|\delta(e)\|, \ldots, \|\delta^{n+1}(e)\| \le \varepsilon, \\ \|\delta(w) - i\nu w\|, \ldots, \|\delta^n(\delta(w) - i\nu w)\| \le \varepsilon, \\ \|[h', e]\|, \|[\delta(h'), e]\|, \ldots, \|[\delta^n(h'), e]\| \le \varepsilon, \\ \|[h', w]\|, \|[\delta(h'), w]\|, \ldots, \|[\delta^n(h'), w]\| \le \varepsilon, \end{aligned}$$

where ε is some small positive number whose value will be specified later. This is possible by the essentially same proof as that of Lemma 8.

STEP 2. Set $\delta' = \delta + \mathrm{ad}(ih')$. Then by Step 1, we get

$$(***) \qquad \begin{aligned} \|\delta'(e)\|, \ldots, \|{\delta'}^{n+1}(e)\| \le C\varepsilon, \\ \|\delta'(w) - i\nu w\|, \ldots, \|{\delta'}^n(\delta'(w) - i\nu w)\| \le C\varepsilon, \end{aligned}$$

for some positive number C. We change e, w as in the proof of Lemma 13 so that these commute with the given $n-1$ almost matrix units. Because holomorphic functional calculus can be carried out with estimates of higher derivatives, we get the same type of estimate as $(***)$ for a different positive number from C, which we denote by the same symbol C. (The proof is essentially same as that of Lemma 13.)

STEP 3. For a new pair (e, w), we still have the same type of estimates as $(**)$, if we replace ε on the right hand sides by $C\varepsilon$ for another different positive number C.

STEP 4. Our h_n is defined by the formula in the proof of Lemma 9 for δ'. Then each of $h_n, \ldots, \delta^n(h_n)$ can be expressed by a finite sum, each term of which contains one of the following:

$$\begin{aligned}
&\delta(e), \ldots, \delta^{n+1}(e), \\
&\delta(w) - i\nu w, \ldots, \delta^n(\delta(w) - i\nu w), \\
&[h', e], [\delta(h'), e], \ldots, [\delta^n(h'), e], \\
&[h', w], [\delta(h'), w], \ldots, [\delta^n(h'), w].
\end{aligned}$$

Thus we have estimates

$$\|h_n\|, \|\delta(h_n)\|, \ldots, \|\delta^n(h_n)\| \leq C\varepsilon,$$

for another positive number C. Thus if we choose ε small enough at the beginning, the estimates $(*)$ can be achieved.

References

[1] H. Araki, Asymptotic ratio set and property L'_λ, *Publ. RIMS Kyoto Univ.* **6** (1970/71), 443-460.

[2] H. Araki, Expansional in Banach algebras, *Ann. Sci. École Norm. Sup.* **6** (1973), 67–84.

[3] B. Blackadar, K-theory for Operator Algebras, Springer 1986.

[4] O. Bratteli, Derivations, Dissipations, and Group Actions on C^*-algebras, Lecture Notes in Math. No. 1229, Springer 1986.

[5] B. Brenken, Representations and automorphisms of the irrational rotation algebra, *Pacific J. Math.* **111** (1984), 257–282.

[6] A. Connes, Une classification des facteurs de type III, *Ann. Sci. École Norm. Sup.* **6** (1973), 133–252.

[7] A. Connes, Outer conjugacy classes of automorphisms of factors, *Ann. Sci. École Norm. Sup.* **8** (1975), 383–419.

[8] A. Connes, Factors of type III_1, property L'_λ and closure of inner automorphisms, *J. Operator Theory* **14** (1985), 189–211.

[9] U. Haagerup, Connes' bicentralizer problem and uniqueness of the injective factor of type III_1, *Acta Math.* **158** (1987), 95–147.

[10] V. F. R. Jones & M. Takesaki, Actions of compact abelian groups on semifinite injective factors, *Acta Math.* **153** (1984), 213–258.

[11] Y. Kawahigashi, Centrally ergodic one-parameter automorphism groups on semifinite injective von Neumann algebras, (To appear in *Math. Scand.*)

[12] Y. Kawahigashi, One-parameter automorphism groups of the hyperfinite type II_1 factor, (preprint, 1988).

[13] A. Ocneanu, Actions of discrete amenable groups on factors, Lecture Notes in Math. No. 1138, Springer 1985.

[14] D. Olesen, G. K. Pedersen, & M. Takesaki, Ergodic actions of compact abelian groups, *J. Operator Theory* **3** (1980), 237–269.

[15] M. Rieffel, C^*-algebras associated with irrational rotations, *Pacific J. Math.* **93** (1981), 415–429.

[16] M. Takesaki, Cocycle conjugacy of group actions on factors, (To appear in Proc. Sympos. Pure Math., Amer. Math. Soc.)

[17] Y. Watatani, Toral Automorphisms on irrational rotation algebras, *Math. Japonica* **26** (1981), 479–484.