Non-commutative geometry of solenoidal manifolds

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A *solenoidal manifold* is a compact connected space, defined as the inverse limit of a tower of proper covering maps,

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \{\, q_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq 1\} \ ,$$

where each M_ℓ is a compact connected manifold, and q_ℓ is a covering map of finite degree greater than one.

Theorem: $\mathcal{S}_{\mathcal{P}}$ is a foliated space with foliation $\mathcal{F}_{\mathcal{P}}$ whose leaves have dimension $n = \dim(M_0)$

$\star$ C. McCord, *Inverse limit sequences with covering maps*, **Trans. Amer. Math. Soc.**, 114:197–209, 1965

$\Longrightarrow$ Leaves of $\mathcal{F}_{\mathcal{P}}$ have smooth structure, partitions of unity, Riemannian metric, and all associated objects, such as leafwise elliptic operators and Hermitian vector bundles.

Theme: Study $\mathcal{S}_{\mathcal{P}}$ as a generalized manifold.

⋆ D. Sullivan, *Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers*, in **Topological methods in modern mathematics (Stony Brook, NY, 1991)**, Publish or Perish, Houston, TX, 1993, pages 543–564.

⋆ D. Sullivan, *Solenoidal manifolds*, **J. Singul.**, 9:203–205, 2014.

⋆ A. Verjovsky, *Low-dimensional solenoidal manifolds*, **arXiv:2203.10032**

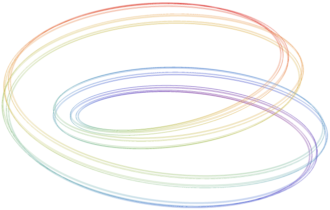
Problem: Investigate the (non-commutative) geometric properties of solenoidal manifolds - their invariants and symmetries.

Example: The Vietoris - van Dantzig Solenoid

$$\mathcal{S}(\vec{m}) = \varprojlim \{ q_\ell : \mathbb{S}^1 \to \mathbb{S}^1 \mid \ell \geq 1\}$$

where q_ℓ is a covering map of the circle $\mathbb{S}^1$ of degree $m_\ell > 1$, and $\vec{m} = (m_1, m_2, \ldots)$ denotes the collection of covering degrees.

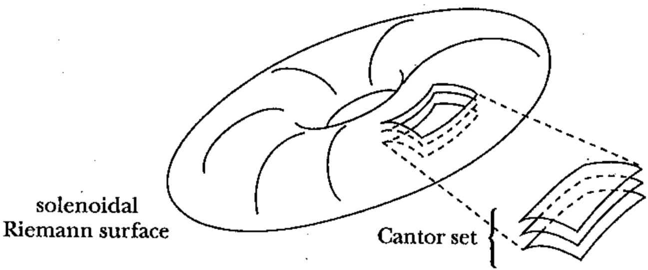
When $m_i = 2$ for all $i \geq 1$ we get the *Smale attractor*, which is a minimal set for a C^∞-foliation:



Example: Solenoidal Riemann Surface (after Sullivan)

$$\widehat{\Sigma_g} = \varprojlim \, \{ q \colon \Sigma_{g'} \to \Sigma_g \}$$

where $M_0 = \Sigma_g$ is a surface with genus $g \geq 1$, and the inverse limit is over all finite coverings of Σ_g.



Homeomorphism $\varphi\colon \mathcal{S}_{\mathcal{P}} \to \mathcal{S}'_{\mathcal{P}'}$ maps leaves of $\mathcal{F}_{\mathcal{P}}$ to leaves of $\mathcal{F}_{\mathcal{P}'}$

In both examples above, the space $\mathcal{S}_{\mathcal{P}}$ is homogeneous - the group of homeomorphisms act transitively.

Problem: Find invariants of solenoidal manifolds that do not change by a homeomorphism - a solenoidal manifold does not have a preferred presentation as a tower of coverings. Also, most solenoidal manifolds are not homogeneous.

Discuss two applications:

- ⋆ "Gap Labeling" for leafwise elliptic operators
- ⋆ "Spectrum" of elements $\gamma \in \Gamma = \pi_1(M_0)$

Given the solenoidal manifold

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \ \{q_\ell \colon M_\ell \to M_{\ell-1}\} \ \subset \prod_{\ell \geq 0} M_\ell \ .$$

Projections $p_\ell \colon \mathcal{S}_{\mathcal{P}} \to M_\ell$ whose restriction to leaves of $\mathcal{F}_{\mathcal{P}}$ are covering maps.

Covering maps $\overline{q}_\ell = q_\ell \circ \cdots \circ q_1 \colon M_\ell \to M_0$.

Choose basepoint $x \in \mathcal{S}_{\mathcal{P}}$ and set $x_\ell = p_\ell(x) \in M_\ell$.

The fiber $\mathfrak{X} = p_0^{-1}(x_0)$ is a Cantor set.

Set $\Gamma_\ell = \mathrm{Image} \ \{(\overline{q}_\ell)_\# \colon \pi_1(M_\ell, x_\ell) \to \pi_1(M_0, x_0)\} = \Gamma$

Obtain a nested chain of subgroups of finite index,

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_\ell \supset \cdots$$

Set $X_\ell = \Gamma/\Gamma_\ell$ which is finite set with transitive Γ-action, and Γ-projections $\overline{p}_\ell \colon X_\ell \to X_{\ell-1}$. Obtain Γ-Cantor space

$$X_\infty \equiv \varprojlim \; \{\overline{p}_\ell \colon X_\ell \to X_{\ell-1}\} \; \subset \prod_{\ell \geq 0} X_\ell \; .$$

Action of Γ on X_∞ is conjugate to the monodromy along leaves of the foliation $\mathcal{F}_\mathcal{P}$ of $\mathcal{S}_\mathcal{P}$ for the fiber $\mathfrak{X} = p_0^{-1}(x_0)$.

The action $\Phi \colon \Gamma \times X_\infty \to X_\infty$ is minimal and equicontinuous.

Theorem: $\mathcal{S}_\mathcal{P}$ is homeomorphic to the suspension foliation for the action (X_∞, Γ, Φ). That is, $\mathcal{S}_\mathcal{P} \cong \widetilde{M} \times X_\infty/(\gamma \cdot \widetilde{z}) \sim (z, \Phi(\gamma)(x))$ where $\widetilde{M}$ is the universal covering space for M.

Quick review of some results from

$\star$ Hurder & Lukina, *Wild solenoids*, **Trans. Amer. Math. Soc.**, 371:4493-4533, 2019; arXiv:1702.03032.

Let $(\mathfrak{X}, \Gamma, \Phi)$ be a Cantor action. That is, for each $\gamma \in \Gamma$, $\Phi(\gamma)\colon \mathfrak{X} \to \mathfrak{X}$ is a homeomorphism of the Cantor space $\mathfrak{X}$.

We say that $(\mathfrak{X}, \Gamma, \Phi)$ is:

- a *minimal* action if $\mathcal{O}(x) = \{\gamma \cdot x \mid \gamma \in \Gamma\}$ is dense in $\mathfrak{X}$ for each $x \in \mathfrak{X}$.
- *equicontinuous* with respect to a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, if for all $\epsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in \mathfrak{X}$ and $\gamma \in \Gamma$ we have that $d_{\mathfrak{X}}(x, y) < \delta$ implies $d_{\mathfrak{X}}(\gamma \cdot x, \gamma \cdot y) < \epsilon$.

Let $\Phi(\Gamma) \subset \textbf{Homeo}(\mathfrak{X})$ denote the image subgroup.

$\mathfrak{G}(\Phi) = \overline{\Phi(\Gamma)} \subset \textbf{Homeo}(\mathfrak{X})$ in the *uniform topology of maps*.

$\mathfrak{G}(\Phi)$ is a separable profinite group.

$\widehat{\Phi} \colon \mathfrak{G}(\Phi) \times \mathfrak{X} \to \mathfrak{X}$ denotes the induced action of $\mathfrak{G}(\Phi)$ on $\mathfrak{X}$.

$(\mathfrak{X}, \Gamma, \Phi)$ minimal implies the group $\mathfrak{G}(\Phi)$ acts transitively on $\mathfrak{X}$.

Isotropy group $\mathfrak{D}(\Phi, x) = \{\widehat{\gamma} \cdot x = x \mid \widehat{\gamma} \in \mathfrak{G}(\Phi)\}$

Then $\mathfrak{X} \cong \mathfrak{G}(\Phi)/\mathfrak{D}(\Phi, x)$ as $\mathfrak{G}(\Phi)$-spaces. That is, $\mathfrak{X}$ is a homogeneous $\mathfrak{G}(\Phi)$-space.

$\mathfrak{X}$ is profinite group $\Longleftrightarrow$ isotropy $\mathfrak{D}(\Phi, x)$ is trivial.

A minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, then either:

- ⋆ it is homogeneous if $\mathfrak{D}(\Phi, x)$ is trivial, or
- ⋆ it has $\widehat{\gamma} \in \mathfrak{G}(\Phi)$ with $\widehat{\Phi}(\widehat{\gamma})(x) = x$ and $\widehat{\gamma} \neq id$

McCord answered a question of Bing, showing:

Theorem: The solenoidal manifold $\mathcal{S}_{\mathcal{P}}$ is homogeneous if and only if the action of $\mathfrak{G}(\Phi)$ on $\mathfrak{X}$ has trivial isotropy.

Problem: Study the properties of solenoidal manifolds, and their corresponding Cantor monodromy actions, when the isotropy group $\mathfrak{D}(\Phi, x)$ is non-trivial.

If $\mathfrak{D}(\Phi, x)$ is non-trivial, then have non-trivial adjoint action

$$\mathrm{Adj} \colon \mathfrak{D}(\Phi, x) \times \mathfrak{X} \to \mathfrak{X}$$

Principle: The "geometry" of the Γ-space $\mathfrak{X}$ is strongly influenced by the properties of the adjoint action of $\mathfrak{D}(\Phi, x)$.

Theme: For a solenoidal manifold $\mathcal{S}_{\mathcal{P}}$, invariants of (X_∞, Γ, Φ) determine invariants of the solenoidal manifold.

The *Steinitz number* defined by a sequence of integers $\vec{m} = (m_1, m_2, \ldots)$ with $m_\ell > 1$ is the *supernatural number*

$$\xi(\vec{m}) = \mathrm{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\} \ ,$$

where lcm denotes the least common multiple of the collection of integers. A Steinitz number ξ can be written uniquely as the formal product over the set of primes Π,

$$\xi = \prod_{p \in \Pi} \ p^{\chi_\xi(p)}$$

The *characteristic function* $\chi_\xi \colon \Pi \to \{0, 1, \ldots, \infty\}$ counts the multiplicity with which a prime p appears in the infinite product ξ.

Two Steinitz numbers ξ and ξ' are said to be *asymptotically equivalent* if there exists finite integers $m, m' \geq 1$ such that $m \cdot \xi = m' \cdot \xi'$, and we then write $\xi \overset{a}{\sim} \xi'$

Definition: The *type* associated to a Steinitz number ξ is the asymptotic equivalence class of ξ, denoted by $\tau[\xi]$.

Lemma. ξ and ξ' satisfy $\xi \overset{a}{\sim} \xi'$ if and only if their characteristic functions χ, χ' satisfy

- $\chi(p) = \chi'(p)$ for all but finitely many primes $p \in \Pi$
- $\chi(p) = \infty$ if and only iff $\chi'(p) = \infty$ for all primes $p \in \Pi$.

Steinitz order

Let $\mathfrak{G}$ be a separable profinite group, and $\mathfrak{U} \subset \mathfrak{G}$ an open subgroup. Then $\mathfrak{U}$ has finite index.

The *Steinitz order* of $\mathfrak{G}$ is the supernatural number associated to a presentation of $\mathfrak{G}$ as an inverse limit of finite groups, defined by:

$$\xi(\mathfrak{G}) = \mathrm{lcm}\{\#\ \mathfrak{G}/\mathfrak{N} \mid \mathfrak{N} \subset \mathfrak{G} \text{ open normal subgroup}\}$$

The *type* of $\mathfrak{G}$ is the asymptotic equivalence class $\tau[\mathfrak{G}]$ of $\xi(\mathfrak{G})$.

$\star$ Wilson, **Profinite groups**, London Mathematical Society Monographs. New Series, Vol. 19, 1998.

Relative Steinitz order

Let $\mathfrak{D} \subset \mathfrak{G}$ be a closed subgroup, and $\mathfrak{N} \subset \mathfrak{G}$ is an open normal subgroup, then $\mathfrak{N} \cdot \mathfrak{D}$ is an open subgroup of $\mathfrak{G}$.

Definition: Let $\mathfrak{D} \subset \mathfrak{G}$ be a closed subgroup of profinite group $\mathfrak{G}$. The *Relative Steinitz order* of the pair $(\mathfrak{G}, \mathfrak{D})$ is defined by :

$$\xi(\mathfrak{G} : \mathfrak{D}) = \mathrm{lcm}\{\# \ \mathfrak{G}/(\mathfrak{N} \cdot \mathfrak{D}) \mid \mathfrak{N} \subset \mathfrak{G} \text{ open normal subgroup}\}$$

Steinitz orders satisfy the Lagrange identity, where the multiplication is taken in the sense of supernatural numbers

$$\xi(\mathfrak{G}) = \xi(\mathfrak{G} : \mathfrak{D}) \cdot \xi(\mathfrak{D}) \ .$$

$\tau[\mathfrak{G} : \mathfrak{D}]$ is the asymptotic equivalence class of $\xi(\mathfrak{G} : \mathfrak{D})$.

Typesets

Let $\mathfrak{D} \subset \mathfrak{G}$ be a closed subgroup.

For $\widehat{\gamma} \in \mathfrak{G}$, let $\mathfrak{G}_{\widehat{\gamma}} = \overline{\{\widehat{\gamma}^n \mid n \in \mathbb{Z}\}} \subset \mathfrak{G}$.

Then $\mathfrak{G}_{\widehat{\gamma}}$ is an abelian Cantor group, and define

$$\xi(\mathfrak{G} : \mathfrak{D}, \widehat{\gamma}) = \mathrm{lcm}\{\# \left(\mathfrak{G}_{\widehat{\gamma}}/\mathfrak{G}_{\widehat{\gamma}} \cap \mathfrak{D} \cdot \mathfrak{N}\right) \mid \mathfrak{N} \text{ open normal subgroup}\}$$

$\tau[\mathfrak{G} : \mathfrak{D}, \widehat{\gamma}]$ is the asymptotic equivalence class of $\xi(\mathfrak{G} : \mathfrak{D}, \widehat{\gamma})$.

Definition: The *typeset* of $(\mathfrak{G}, \mathfrak{D})$ is the collection of types

$$\tau[\mathfrak{G} : \mathfrak{D}] = \{\tau[\mathfrak{G} : \mathfrak{D}, \widehat{\gamma}] \mid \widehat{\gamma} \in \mathfrak{G}\} \ .$$

Find more discussion and results in the work

$\star$ Hurder & Lukina, *Type invariants for solenoidal manifolds*, arXiv:2305.00863

Application to solenoidal manifolds.

Given a nested chain of subgroups of finite index,

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_\ell \supset \cdots$$

Let $C_\ell \subset \Gamma_\ell$ be the normal core; that is, the largest normal subgroup in Γ_ℓ. Then define the Cantor groups

$$\widehat{\Gamma}_\infty \equiv \varprojlim \; \{\overline{p}_\ell \colon \Gamma/C_\ell \to \Gamma/C_{\ell-1}\} \; \subset \prod_{\ell \geq 0} \Gamma/C_\ell \; .$$

$$\widehat{D}_\infty \equiv \varprojlim \; \{\overline{p}_\ell \colon \Gamma_\ell/C_\ell \to \Gamma_{\ell-1}/C_{\ell-1}\} \; \subset \prod_{\ell \geq 0} \Gamma_\ell/C_\ell \; .$$

Proposition: The relative Steinitz order satisfies

$$\begin{aligned}\xi(\widehat{\Gamma}_\infty : \widehat{D}_\infty) &= \mathrm{lcm}\ \{\# X_\ell \mid \ell = 1, 2, \cdots\} \\ &= \mathrm{lcm}\ \{\deg\{\overline{p}_\ell \colon M_\ell \to M_0\} \mid \ell = 1, 2, \cdots\}\end{aligned}$$

That is, the Steinitz order of a tower of coverings is equal to a relative Steinitz order of profinite groups associated to the tower.

We give an example in the case of Vietoris solenoids.

Choose a sequence of integers $\vec{m} = (m_1, m_2, \ldots)$ with $m_\ell > 1$.

Form the chain of coverings of the circle

$$\mathbb{S}^1 \xleftarrow{m_1} \mathbb{S}^1 \xleftarrow{m_2} \mathbb{S}^1 \xleftarrow{m_3} \mathbb{S}^1 \xleftarrow{m_4} \cdots$$

$$\mathcal{S}(\vec{m}) = \varprojlim \; \{m_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1\} \subset \prod_{\ell \geq 0} \mathbb{S}^1$$

Definition: The type $\tau[\mathcal{S}(\vec{m})]$ is the asymptotic equivalence class of $\xi(\mathcal{S}(\vec{m})) = \mathrm{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\}$.

For example, the type of the Smale solenoid is $\{2^\infty\}$.

Theorem. [Bing, 1960], [McCord, 1965] 1-dimensional solenoids $\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{m}')$ are homeomorphic if and only if $\xi(\vec{m}) \overset{a}{\sim} \xi(\vec{m}')$.

$\Longrightarrow$ A Vietoris solenoid is completely determined up to homeomorphism by the type $\tau[\mathcal{S}(\vec{m})]$.

The integer Heisenberg group is the simplest non-abelian nilpotent group, represented as the upper triangular matrices in $\mathrm{GL}(3, \mathbb{Z})$.

$$\Gamma = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Theorem: Given any function $\chi \colon \Pi \to \{0, 1, \ldots, \infty\}$, there is a minimal equicontinuous action $(\mathfrak{X}, \Gamma, \Phi)$ of the Heisenberg group with Steinitz order $\xi[\mathfrak{X}, \Gamma, \Phi]$ having characteristic function χ.

The proof uses the fact that a profinite nilpotent group is isomorphic to the product of its Sylow p-subgroups, and the judicious construction of subgroup chains in Γ.

$\star$ Hurder & Lukina, *Prime spectrum and dynamics for nilpotent Cantor actions*, arXiv:2305.00896

"Gap Labeling" refers to the evaluation of index classes for elliptic operators on the tiling space for an aperiodic tiling on $\mathbb{R}^n$

Bellisard gave talks on "Gap-Labeling" in the 1980's and proposed the problem of showing that the labeling system was exactly a subgroup of $\mathbb{R}$ determined by the dynamics of the foliation.

⋆ Bellissard, *Gap labeling theorems for Schrödinger operators*, **From number theory to physics (Les Houches, 1989)**, Springer, 1992.

⋆ Benameur & Oyono-Oyono, *Gap-labelling for quasi-crystals (proving a conjecture by J. Bellissard)*, **Operator algebras and mathematical physics (Constanța, 2001)**, Theta, Bucharest, 2003.

⋆ Kaminker & Putnam, *A proof of the gap labeling conjecture*, **Michigan Math. J.**, 51:537–546, 2003.

⋆ Bellissard, Benedetti & Gambaudo, *Spaces of tilings, finite telescopic approximations and gap-labeling*, **Comm. Math. Phys.**, 261:1–41, 2006.

Again, consider a solenoidal manifold $\mathcal{S}_{\mathcal{P}}$ which determines a nested chain of subgroups of finite index,

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_\ell \supset \cdots$$

Define the finite set $X_\ell = \Gamma/\Gamma_\ell$ with projections $\overline{p}_\ell \colon X_\ell \to X_{\ell-1}$, then we have the Cantor space

$$X_\infty \equiv \varprojlim \; \{\overline{p}_\ell \colon X_\ell \to X_{\ell-1}\} \; \subset \prod_{\ell \geq 0} X_\ell \; .$$

There is a unique Γ-invariant probability measure μ on X_∞ which induces a trace $\mathrm{Tr}_\mu \colon K_0(C(X_\infty)) \to \mathbb{R}$.

For the Cantor dynamical system (X_∞, Γ, Φ) define:

$$\xi(\widehat{\Gamma}_\infty : \widehat{D}_\infty)^{-1} = \varinjlim \{\frac{1}{m}\mathbb{Q} \mid m \text{ divides } \xi(\widehat{\Gamma}_\infty : \widehat{D}_\infty)\} \subset \mathbb{Q} \ .$$

Theorem: $\mathrm{Tr}_\mu \colon K_0(C(X_\infty)) \to \xi(\widehat{\Gamma}_\infty : \widehat{D}_\infty)^{-1} \subset \mathbb{R}$.

The proof follows from the observation that an idempotent in $C(X_\infty)$ has support in a clopen subset of X_∞, and a clopen set in X_∞ is a finite union of adapted clopen sets.

The Gap Labeling Theorems are extensions of this result from $C(X_\infty)$ to the cross product algebra $C^*(\mathcal{F}_\mathcal{P}) \cong C(X_\infty) \rtimes \Gamma$.

Let $\mathcal{S}_\mathcal{P}$ be solenoidal manifold with even-dimensional leaves, with projection $p_0 \colon \mathcal{S}_\mathcal{P} \to M_0$.

Assume that M_0 is a $\mathrm{Spin}_\mathbb{C}$-manifold, with Dirac operator $\mathcal{D}$.

Then $\mathcal{D}$ lifts to a leafwise Dirac operator $\mathcal{D}_\mathcal{F}$ for $\mathcal{F}_\mathcal{P}$.

Given a K-theory class $[E] \in K^0(\mathcal{S}_\mathcal{P})$ we can form the pairing $\mathcal{D}_\mathcal{F} \otimes [E]$ then take the leafwise index class

$$\mathrm{Ind}_\mathcal{F}(\mathcal{D}_\mathcal{F} \otimes [E]) \in K_0(C^*(\mathcal{F}_\mathcal{P})) \ .$$

The invariant measure μ on X_∞ induces a holonomy invariant transverse measure $\mu_\mathcal{F}$ for $\mathcal{F}_\mathcal{P}$, which defines a trace

$$\mathrm{Tr}_\mu \colon C^*(\mathcal{F}_\mathcal{P}) \to \mathbb{R} \ .$$

Theorem: (Gap Labeling for Solenoidal Manifolds)

$$\mathrm{Tr}_\mu(\mathrm{Ind}_{\mathcal{F}}(\mathcal{D}_{\mathcal{F}} \otimes [E])) \in \xi(\widehat{\Gamma}_\infty : \widehat{D}_\infty)^{-1} \subset \mathbb{R}$$

The basic idea of the proof is to use

$$K^0(\mathcal{S}_{\mathcal{P}}) = K^0(\varprojlim \{p_\ell \colon M_\ell \to M_{\ell-1}\}) = \varinjlim K^0(M_\ell)$$

so that $[E] \in K^0(\mathcal{S}_{\mathcal{P}})$ is the lift to $\mathcal{S}_{\mathcal{P}}$ of a class $[E_\ell] \in K^0(M_\ell)$.
The index pairing $\mathrm{Ind}(\mathcal{D}_\ell \otimes [E_\ell]) \in K_0(M_\ell)$ and the result follows.

There is an analogous result for odd-dimensions:

Theorem: (Gap Labeling for Spectral Flow)

$$\mathrm{Tr}_\mu \circ \mathrm{Ind}_{\mathcal{F}} \colon K_1(\mathcal{S}_{\mathcal{P}}) \otimes K^1(\mathcal{S}_{\mathcal{P}}) \to \xi(\widehat{\Gamma}_\infty : \widehat{D}_\infty)^{-1} \ .$$

Remark: Previous work proved that the odd foliation index theorem measures the spectral flow for the leafwise Dirac operator. The above result is an analog for towers of coverings of the results for flat $\mathrm{U(n)}$-bundles in the work:

⋆ Hurder, *Eta invariants and the odd index theorem for coverings*, **Geometric and topological invariants of elliptic operators, 1988)**, Contemp. Math., Vol. 105, Amer. Math. Soc., 1990.

Aside: Let M be a closed Riemannian manifold, with basepoint $x_0 \in M_0$ and fundamental group $\Gamma = \pi_1(M_0, x_0)$. Each $\gamma \in \Gamma$ is represented by a closed curve $c_\gamma \colon \mathbb{S}^1 \to M_0$ with $c_\gamma(0) = x_0$.

Let $\|\gamma\|$ denote the shortest curve in the free homotopy class of c_γ.

Definition: The unmarked length spectrum of M is the collection

$$\sigma(M) = \{\|\gamma\| \mid \gamma \in \Gamma\}$$

.

The collection $\sigma(M)$ does not determine M up to isometry, even in the case where Σ_g is a Riemann surface with genus $g \geq 2$, though it does severely restrict the possibilities.

Question: Are there invariants for solenoidal manifolds analogous to the length spectra for compact manifolds?

Let $\mathcal{S}_{\mathcal{P}}$ be a solenoidal manifold with base Riemannian manifold M. We are interested in invariants of $\mathcal{S}_{\mathcal{P}}$ up to homeomorphism.

In many cases, the leaves of $\mathcal{F}_{\mathcal{P}}$ are simply connected, so it does not make sense to consider minimal length closed curves in $\mathcal{S}_{\mathcal{P}}$.

The space $\mathcal{S}_{\mathcal{P}}$ is compact, so the lift of a geodesic $\sigma\colon \mathbb{R} \to \mathcal{S}_{\mathcal{P}}$ does satisfy a recurrence property, so one can consider the length of approximately closed paths.

Since the length of a curve in $\mathcal{S}_{\mathcal{P}}$ is not a diffeomorphism invariant, we must measure the "return times" to an open set in the transversal $\mathfrak{X}$ in terms of an algebraic quantity that is invariant.

This is the idea behind the *typeset spectrum* of $\mathcal{S}_{\mathcal{P}}$.

Consider a solenoidal manifold $\mathcal{S}_{\mathcal{P}}$ which determines a nested chain of subgroups of finite index,

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_\ell \supset \cdots$$

with the associated group action $\Phi \colon \Gamma \times X_\infty \to X_\infty$.

Obtain a homomorphism $\Phi \colon \Gamma \to \Phi(\Gamma) \subset \widehat{\Gamma}_\infty \subset \mathbf{Homeo}(X_\infty)$ and the isotropy subgroup $\widehat{D}_\infty \subset \widehat{\Gamma}_\infty$.

For $\widehat{\gamma} \in \widehat{\Gamma}$, the closure of the orbit, $X_{\widehat{\gamma}} = \overline{\{\widehat{\gamma}^n \cdot x \mid n \in \mathbb{Z}\}}$, has an action of $\mathbb{Z}$, and so by the suspension construction we obtain a "solenoidal 1-cycle", $\phi_{\widehat{\gamma}} \colon \mathfrak{T}_{\widehat{\gamma}} \to \mathcal{S}_{\mathcal{P}}$.

The relative Steinitz order $\xi(\widehat{\Gamma}_\infty : \widehat{D}_\infty, \widehat{\gamma})$ is the Steinitz order of the 1-dimensional solenoid $\mathfrak{T}_{\gamma}$.

Definition: The *typeset spectrum* of $\mathcal{S}_{\mathcal{P}}$ is the collection of types

$$\tau[\mathcal{S}_{\mathcal{P}}] = \{\tau[\widehat{\Gamma}_\infty : \widehat{D}_\infty, \widehat{\gamma}] \mid \widehat{\gamma} \in \widehat{\Gamma}\} \ .$$

Question: What aspects of the non-commutative geometry of a solenoidal manifold $\mathcal{S}_{\mathcal{P}}$ are determined by its type spectrum $\tau[\mathcal{S}_{\mathcal{P}}]$?

The solenoidal manifolds with base the n-torus $\mathbb{T}^n$ are determined up to homeomorphism by a dense subgroup of $\mathbb{Q}^n$.

⋆ Giordano, Putman & Skau, *$\mathbb{Z}^d$-odometers and cohomology*, **Groups Geom. Dyn.**, 13:909–938, 2019.

The notion of typesets was introduced by Baer for the classification of dense subgroups of $\mathbb{Q}^n$, and studied in depth in works by D. Arnold, M.C.R. Butler and L. Fuchs.

For $n > 1$ this problem is “unsolvable” in the sense of Descriptive Set Theory, but the special case of Butler subgroups are reducible to a countable equivalence using the typeset invariants above:

⋆ Thomas, *The classification problem for finite rank Butler groups*, in **Models, modules and abelian groups**, Walter de Gruyter, Berlin, 2008

Let $\mathcal{S}_{\mathcal{P}}$ be a solenoidal manifold with base manifold $M = \mathbb{T}^n$. Then for each $\gamma \in \mathbb{Z}^n \subset \widehat{\Gamma}$ we obtain a solenoidal 1-cycle $\phi_\gamma \colon \mathfrak{T}_\gamma \to \mathcal{S}_{\mathcal{P}}$ with a foliation denoted by $\mathcal{F}_\gamma$.

The leafwise Dirac operator ∂_γ on the 1-dimensional solenoid $\mathfrak{T}_\gamma$ defines an odd index class $[\partial_\gamma] \in K_1(\mathcal{F}_\gamma)$. The foliation index theorem for odd degree operators yields a pairing

$$\mathrm{Tr}_\mu \circ \mathrm{Ind}_{\mathcal{F}_\gamma} \colon K_1(\mathcal{F}_\gamma) \times K^1(\mathcal{S}_{\mathcal{P}}) \to \mathbb{R}$$

Theorem: For each $[u] \in K^1(\mathcal{S}_{\mathcal{P}})$ we have

$$\mathrm{Tr}_\mu \circ \mathrm{Ind}_{\mathcal{F}_\gamma}(\partial_\gamma \otimes [u]) \in \tau[\mathcal{S}_{\mathcal{P}}]$$

Definition: The *non-commutative type* of $\gamma \in \Gamma$ is the collection

$$\tau_{nc}[\mathcal{S}_{\mathcal{P}}, \gamma] = \tau[\{\mathrm{Ind}_{\mathcal{F}_\gamma}(\partial_\gamma \otimes [u]) \mid [u] \in K^1(\mathcal{S}_{\mathcal{P}})\}] \ .$$

The *non-commutative typeset* of $\mathcal{S}_{\mathcal{P}}$ is the collection of types:

$$\tau_{nc}[\mathcal{S}_{\mathcal{P}}] = \{\tau[\{\mathrm{Ind}_{\mathcal{F}_\gamma}(\partial_\gamma \otimes [u]) \mid [u] \in K^1(\mathcal{S}_{\mathcal{P}})\}] \mid \gamma \in \Gamma\} \ .$$

Problem: Study the properties of the non-commutative types for solenoidal manifolds.

Final Remark: For $M = \Sigma_2$ a Riemann surface of genus $g = 2$, for any $m \geq 1$, the fundamental group Γ_g contains a subgroup of finite index which is isomorphic to the free group on m-generators, $\mathbb{F}_m \cong \mathbb{Z} \star \cdots \star \mathbb{Z}$.

Given any minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ where Γ is finitely generated with m-generators, then there is a surjection $\mathbb{F}_m \to \Gamma$. We can then then use this to define a finite covering M_0 of Σ_2 over which there is a solenoidal manifold $\mathcal{S}_{\mathcal{P}}$ with base manifold M_0 whose associated monodromy action is $(\mathfrak{X}, \Gamma, \Phi)$.

Theorem: Every minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ with Γ finitely generated can be realized by a solenoidal manifold with base Σ_2.

Corollary: Every type and typeset invariants for Cantor actions can be realized by 2-dimensional solenoidal manifolds.

Thank you for your attention!