Generalized Longo-Rehren subfactors and α-induction

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Abstract

We study the recent construction of subfactors by Rehren which generalizes the Longo-Rehren subfactors. We prove that if we apply this construction to a non-degenerately braided subfactor $N \subset M$ and $\alpha^{\pm}$-induction, then the resulting subfactor is dual to the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R$ arising from the entire system of irreducible endomorphisms of M resulting from $\alpha^{\pm}$-induction. As a corollary, we solve a problem on existence of braiding raised by Rehren negatively. Furthermore, we generalize our previous study with Longo and Müger on multi-interval subfactors arising from a completely rational conformal net of factors on S^1 to a net of subfactors and show that the (generalized) Longo-Rehren subfactors and α-induction naturally appear in this context.

1 Introduction

In subfactor theory initiated by V. F. R. Jones [11], Ocneanu's construction of asymptotic inclusions [22] have been studied by several people as a subfactor analogue of the quantum double construction. (See [5, Chapter 12] on general theory of asymptotic inclusions.) Popa's construction of symmetric enveloping inclusions [23] gives its generalizations and is important in the analytic aspects of subfactor theory. Longo and Rehren gave another construction of subfactors in [19] in the setting of sector theory [15, 16] and Masuda [21] has proved that the asymptotic inclusion and the Longo-Rehren subfactor are essentially the same constructions. Izumi [8, 9] gave very detailed and interesting studies of the Longo-Rehren subfactors. Recently, Rehren [25] gave a construction generalizing the Longo-Rehren subfactor and we call the resulting subfactor a *generalized Longo-Rehren subfactor*. This construction uses certain extensions of systems of endomorphisms from subfactors (of type III) to larger factors. We will analyze this construction in detail in this paper. (This construction will be explained in more detail in Section 2 below.)

Longo and Rehren also defined such an extension of endomorphisms for nets of subfactors in the same paper [19, Proposition 3.9], based on an old suggestion of

Roberts [26]. The essentially same construction of new endomorphisms was also given in Xu [27, page 372] and several very interesting properties and examples were found by him in [27, 28]. We call this extension of endomorphisms *α-induction*. In this paper, we study the generalized Longo-Rehren subfactors arising from α-induction based on the above works, Böckenhauer-Evans [1] and our previous work [2, 3, 4]. In the papers of Longo, Rehren, and Xu, they study nets of subfactors and have a certain condition arising from locality of the larger net, now called *chiral locality* as in [2, Section 3.3], but we do not assume this condition in this paper. We assume only a non-degenerate braiding in the sense of [24]. (See [2, Section 3.3] for more on this matter. We only need a braiding in order to define α-induction, but we also assume non-degeneracy in this paper. If we start with a completely rational net on the circle in the sense of [13], non-degeneracy of the braiding holds automatically by [13].) Izumi's work [8, 9] on a half-braiding is closely related to theory of α-induction and a theory of induction for bimodules generalizing these works has been recently given by Kawamuro [14].

Results in [4] suggest that if we apply the construction of the generalized Longo-Rehren subfactor to $\alpha^\pm$-induction for $N \subset M$, then the resulting subfactor $N \otimes N^{\mathrm{opp}} \subset P$ would be dual to the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R$ applied to the system of endomorphisms of M arising from $\alpha^\pm$-induction. In this paper we will prove that this is indeed the case. The proof involves several calculations of certain intertwiners related to a half-braiding in the sense of Izumi [8] arising from a relative braiding in Böckenhauer-Evans [1]. As an application, we solve a problem on existence of braiding raised by Rehren [25] negatively.

Furthermore, we generalize our previous study with Longo and Müger [13] on multi-interval subfactors arising from a completely rational conformal net of factors on S^1 to a net of subfactors. That is, we have studied "multi-interval subfactors" arising from such a net on S^1, whose definitions will be explained below, and proved that the resulting subfactor is isomorphic to the Longo-Rehren subfactor arising from all superselection sectors of the net in [13]. We apply the construction of multi-interval subfactors to conformal nets of *subfactors* with finite index and prove that the resulting subfactor is isomorphic to the Longo-Rehren subfactor arising from the system of α-induced endomorphisms. We then also explain a relation of this result to the generalized Longo-Rehren subfactors.

The results in Section 2 were announced in [12].

2 Generalized Longo-Rehren subfactors

Let $N \subset M$ be a type III subfactor with finite index and finite depth. Let ${}_N\mathcal{X}_N, {}_N\mathcal{X}_M, {}_M\mathcal{X}_N, {}_M\mathcal{X}_M$ be finite systems of irreducible morphisms of type N-N, N-M, M-N, M-M, respectively and suppose that the four systems together make a closed system under conjugations, compositions and irreducible decompositions, and the inclusion map from N into M decomposes into irreducible N-M morphisms within ${}_N\mathcal{X}_M$, as in [2, Assumption 4.1]. We assume that the system ${}_N\mathcal{X}_N$ is non-degenerately braided as in [24], [2, Definition 2.3]. Then we have positive and negative α-inductions, corresponding to positive and negative braidings, and the system ${}_M\mathcal{X}_M$ is generated by the both α-inductions because of the non-degeneracy as in [2, Theorem 5.10]. We do *not* assume the chiral locality condition, which arises from locality of the larger net of factors, in this paper. (See [3, Section 5] for more on the role of chiral locality.)

Now recall a new construction of subfactors due to Rehren [25] arising from two systems of endomorphisms and two extensions to the same factor as follows.

Let Δ be a system of endomorphisms of a type III factor N and consider a subfactor $N \subset M$ with finite index. An *extension* of Δ is a pair (ι, α) where ι is the embedding map of N into M and α is a map $\Delta \to \mathrm{End}(M)$, $\lambda \mapsto \alpha_\lambda$ satisfying the following properties.

1. Each α_λ has a finite dimension.
2. We have $\iota\lambda = \alpha_\lambda \iota$ for $\lambda \in \Delta$.
3. We have $\iota(\mathrm{Hom}(\lambda\mu, \nu)) \subset \mathrm{Hom}(\alpha_\lambda\alpha_\mu, \alpha_\nu)$ for $\lambda, \mu, \nu \in \Delta$.

Next let N_1, N_2 be two subfactors of a type III factor M, (ι_1, α^1) and (ι_2, α^2) be two extensions of finite systems Δ_1, Δ_2 of endomorphisms of N_1, N_2 to M, respectively. For $\lambda \in \Delta_1$ and $\mu \in \Delta_2$, we set $Z_{\lambda,\mu} = \dim \mathrm{Hom}(\alpha^1_\lambda, \alpha^2_\mu)$. Then Rehren proved in [25] that we have a subfactor $N_1 \otimes N_2^{\mathrm{opp}} \subset R$ such that the canonical endomorphism restricted on $N_1 \otimes N_2^{\mathrm{opp}}$ has a decomposition $\bigoplus_{\lambda\in\Delta_1,\mu\in\Delta_2} Z_{\lambda,\mu}\lambda \otimes \mu^{\mathrm{opp}}$ by constructing the corresponding Q-system explicitly. This is a generalization of the Longo-Rehren construction [19, Proposition 4.10] in the sense that if $N_1 = N_2 = M$, Rehren's Q-system coincides with the one given in [19]. We call it a generalized Longo-Rehren subfactor. The most natural example of such extensions seems to be the α-induction, and then we can take $\Delta = {}_N\mathcal{X}_N$, $\alpha^1 = \alpha^+$, $\alpha^2 = \alpha^-$ for α-induction from N to M based on a braiding $\varepsilon^\pm$ on the system ${}_N\mathcal{X}_N$ and then $Z_{\lambda,\mu}$ is the "modular invariant" matrix as in [2, Definition 5.5, Theorem 5.7].

Our aim is to study the generalized Longo-Rehren subfactor arising from ${}_N\mathcal{X}_N$ and $\alpha^\pm$-induction in this way. The result in [4, Corollary 3.11] suggests that this subfactor is dual to the Longo-Rehren subfactor arising from ${}_M\mathcal{X}_M$, and we prove this is indeed the case. For this purpose, we study the Longo-Rehren subfactor arising from ${}_M\mathcal{X}_M$ first as follows.

Let $M \otimes M^{\mathrm{opp}} \subset R$ be the Longo-Rehren subfactor [19, Proposition 4.10] arising from the system ${}_M\mathcal{X}_M$ on M and (Γ, V, W) be the corresponding Q-system [17]. (Actually, the subfactor we deal with here is the dual to the original one constructed in [19, Proposition 4.10]. This dual version is called the Longo-Rehren subfactor in [4], [13].) That is, we have that $\Gamma \in \mathrm{End}(R)$ is the canonical endomorphism of the subfactor, $V \in \mathrm{Hom}(\mathrm{id}, \Gamma) \subset R$, and $W \in \mathrm{Hom}(\Gamma, \Gamma^2)$. We also have

$$\begin{aligned} W &\in M \otimes M^{\mathrm{opp}}, \\ R &= (M \otimes M^{\mathrm{opp}})V, \\ W^*V &= \Gamma(V^*)W = w^{-1/2}, \\ W^*\Gamma(W) &= WW^*, \\ \Gamma(W)W &= W^2, \end{aligned}$$

where $w = \sum_{\beta \in {}_M\mathcal{X}_M} d_\beta^2$ is the global index of the system ${}_M\mathcal{X}_M$ and equal to the index $[R : M \otimes M^{\mathrm{opp}}]$. By the definition of the original Longo-Rehren subfactor in [19], the Q-system $(\Theta, W, \Gamma(V))$ is given as follows. We have

$$\Theta(x) = \sum_{\beta \in {}_M\mathcal{X}_M} W_\beta(\beta \otimes \beta^{\mathrm{opp}}(x))W_\beta^*, \quad \text{for } x \in M \otimes M^{\mathrm{opp}},$$

where Θ is the dual canonical endomorphisms, the restriction of Γ to $M \otimes M$, the family $\{W_\beta\}$ is that of isometries with mutually orthogonal ranges satisfying $\sum_{\beta \in {}_M\mathcal{X}_M} W_\beta W_\beta^* = 1$, and also have

$$\Gamma(V) = \sum_{\beta_1, \beta_2, \beta_3 \in {}_M\mathcal{X}_M} \sqrt{\frac{d_1 d_2}{w d_3}} \Gamma(W_{\beta_2}) W_{\beta_1} T_{\beta_1\beta_2}^{\beta_3} W_{\beta_3}^*, \tag{1}$$

$$T_{\beta_1\beta_2}^{\beta_3} = \sum_{l=1}^{N_{12}^3} T_{\beta_1\beta_2,l}^{\beta_3} \otimes j(T_{\beta_1\beta_2,k}^{\beta_3}) \in M \otimes M^{\mathrm{opp}}, \tag{2}$$

by definition of the Longo-Rehren subfactor [19], where $\{T_{\beta_1\beta_2,l}^{\beta_3}\}_l$ is an orthogonal basis in $\mathrm{Hom}(\beta_3, \beta_1\beta_2) \subset M$, N_{ij}^k is the structure constant $\dim \mathrm{Hom}(\beta_k, \beta_i\beta_j)$, $d_j = d_{\beta_j}$ is the statistical dimension of β_j, and j is the anti-isomorphism $x \in M \mapsto x^* \in M^{\mathrm{opp}}$. Starting from this explicit expression of the Q-system $(\Theta, W, \Gamma(V))$, we would like to write down the Q-system (Γ, V, W) explicitly and identify it with the Q-system given by the construction of Rehren [25].

First, by [4, Theorem 3.9], we know that

$$[\Gamma] = \bigoplus_{\lambda_1, \lambda_2 \in {}_N\mathcal{X}_N} Z_{\lambda_1,\lambda_2} [\eta(\alpha_{\lambda_1}^+, +)\eta^{\mathrm{opp}}(\alpha_{\lambda_2}^-, -)],$$

where $[\quad]$ represents the sector class of an endomorphism, $\eta(\quad,\quad)$ is the extension of an endomorphism of $M \otimes M^{\mathrm{opp}}$ to R with a half-braiding by Izumi [8], $\alpha^\pm$ is the α-induction, the notations here follow those of [4], and $Z_{\lambda_1\lambda_2} = \dim \mathrm{Hom}(\alpha_{\lambda_1}^+, \alpha_{\lambda_2}^-)$ is the "modular invariant" as in [2, Definition 5.5]. (Recall that we now assume non-degeneracy of the braiding on ${}_N\mathcal{X}_N$.) Furthermore, by [4, Corollary 3.10], we have equivalence of two C^*-tensor categories of $\{\eta(\alpha_\lambda^+, +)\eta^{\mathrm{opp}}(\alpha_\mu^-, -)\}$ on R and $\{\lambda \otimes \mu^{\mathrm{opp}}\}$ on $N \otimes N^{\mathrm{opp}}$, thus the canonical endomorphisms of the two Q-systems are naturally identified. So we will next compute V, W explicitly and identify them with the intertwiners in Rehren's Q-system. (Note that it does not matter that two von Neumann algebras R and $N \otimes N^{\mathrm{opp}}$ are different, since only the equivalence class of C^*-tensor categories matters in the construction of the (generalized) Longo-Rehren subfactors.)

We next closely follow Izumi's arguments in [8, Section 7]. First we have the following lemma.

Lemma 2.1. *For an operator $X \in M \otimes M^{\mathrm{opp}}$, $XV \in R$ is in*

$$\mathrm{Hom}(\eta(\alpha^+_{\lambda_1}, +)\eta^{\mathrm{opp}}(\alpha^-_{\lambda_2}, -), \Gamma)$$

if and only if we have the following two conditions.

1. *$X \in \mathrm{Hom}(\Theta(\alpha^+_{\lambda_1} \otimes \alpha^{-,\mathrm{opp}}_{\lambda_2}), \Theta)$.*
2. *$X\Gamma(U^*)\Gamma(V) = \Gamma(V)X$, where*

$$U = \sum_{\beta \in {}_M\mathcal{X}_M} W_\beta(\mathcal{E}^+_{\lambda_1}(\beta) \otimes j(\mathcal{E}^-_{\lambda_2}(\beta)))(\alpha^+_{\lambda_1} \otimes \alpha^{-,\mathrm{opp}}_{\lambda_2})(W^*_\beta),$$

and $\mathcal{E}^\pm$ is the half-braiding defined in [4, Section 3].

Proof. By a standard argument similar to the one in the proof of [8, Proposition 7.3], we easily get the conclusion. Q.E.D.

Next, we rewrite the second condition in the above Lemma as follows. Using the definition of $\Gamma(V)$ as in (1), we have

$$\begin{aligned}
&\sum_{\beta_1,\beta_2,\beta_3 \in {}_M\mathcal{X}_M} X\Gamma(U^*)\sqrt{\frac{d_1 d_2}{w d_3}}\Gamma(W_{\beta_2})W_{\beta_1}T^{\beta_3}_{\beta_1\beta_2}W^*_{\beta_3} \\
= &\sum_{\beta_4,\beta_5,\beta_6 \in {}_M\mathcal{X}_M} \sqrt{\frac{d_4 d_5}{w d_6}}\Gamma(W_{\beta_5})W_{\beta_4}T^{\beta_6}_{\beta_4\beta_5}W^*_{\beta_6}X,
\end{aligned}$$

which is equivalent to the following equations for all $\beta_3, \beta_4, \beta_5 \in {}_M\mathcal{X}_M$.

$$\begin{aligned}
&\sum_{\beta_1,\beta_2 \in {}_M\mathcal{X}_M} W^*_{\beta_4}\Theta(W_{\beta_5})X\Theta(U^*)\sqrt{\frac{d_1 d_2}{d_3}}\Theta(W_{\beta_2})W_{\beta_1}T^{\beta_3}_{\beta_1\beta_2} \\
= &\sum_{\beta_6 \in {}_M\mathcal{X}_M} \sqrt{\frac{d_4 d_5}{d_6}}T^{\beta_6}_{\beta_4\beta_5}W^*_{\beta_6}XW_{\beta_3}.
\end{aligned}$$

Assuming the first condition in Lemma 2.1, we compute the left hand side of this equation as follows.

$$\begin{aligned}
&\sum_{\beta_1,\beta_2\in {}_M\mathcal{X}_M} \sqrt{\frac{d_1d_2}{d_3}} W^*_{\beta_4} X\Theta((\alpha^+_{\lambda_1}\otimes\alpha^{-,\mathrm{opp}}_{\lambda_2})(W^*_{\beta_5})U^*W_{\beta_2})W_{\beta_1}T^{\beta_3}_{\beta_1\beta_2} \\
=&\sum_{\beta_1,\beta_2\in {}_M\mathcal{X}_M} \sqrt{\frac{d_1d_2}{d_3}} W^*_{\beta_4} XW_{\beta_1}(\beta_1\otimes\beta_1^{\mathrm{opp}})((\alpha^+_{\lambda_1}\otimes\alpha^{-,\mathrm{opp}}_{\lambda_2})(W^*_{\beta_5})U^*W_{\beta_2})T^{\beta_3}_{\beta_1\beta_2} \\
=&\sum_{\beta_1,\beta_2\in {}_M\mathcal{X}_M} \sqrt{\frac{d_1d_2}{d_3}} W^*_{\beta_4} XW_{\beta_1}(\beta_1\otimes\beta_1^{\mathrm{opp}})(\sum_{\beta\in {}_M\mathcal{X}_M}(\alpha^+_{\lambda_1}\otimes\alpha^{-,\mathrm{opp}}_{\lambda_2})(W^*_{\beta_5})(\alpha^+_{\lambda_1}\otimes\alpha^{-,\mathrm{opp}}_{\lambda_2}) \\
&\qquad\times(W_\beta)(\mathcal{E}^+_{\lambda_1}(\beta)^*\otimes j(\mathcal{E}^-_{\lambda_2}(\beta))^*)W^*_\beta W_{\beta_2})T^{\beta_3}_{\beta_1\beta_2} \\
=&\sum_{\beta_1\in {}_M\mathcal{X}_M} \sqrt{\frac{d_1d_5}{d_3}} W^*_{\beta_4} XW_{\beta_1}(\beta_1\otimes\beta_1^{\mathrm{opp}})(\mathcal{E}^+_{\lambda_1}(\beta_5)^*\otimes j(\mathcal{E}^-_{\lambda_2}(\beta_5))^*)T^{\beta_3}_{\beta_1\beta_5}.
\end{aligned}$$

That is, our equation is now

$$\begin{aligned}
&\sum_{\beta_1\in {}_M\mathcal{X}_M} \sqrt{\frac{d_1}{d_3}} W^*_{\beta_4} XW_{\beta_1}(\beta_1\otimes\beta_1^{\mathrm{opp}})(\mathcal{E}^+_{\lambda_1}(\beta_5)^*\otimes j(\mathcal{E}^-_{\lambda_2}(\beta_5))^*)T^{\beta_3}_{\beta_1\beta_5} \\
=&\sum_{\beta_6\in {}_M\mathcal{X}_M} \sqrt{\frac{d_4}{d_6}} T^{\beta_6}_{\beta_4\beta_5} W^*_{\beta_6} XW_{\beta_3} \qquad (3)
\end{aligned}$$

for all $\beta_3,\beta_4,\beta_5\in {}_M\mathcal{X}_M$. Now set $\beta_4=\mathrm{id}$ in this equation. Then on the left hand side, we have a term $W^*_0XW_{\beta_1}$, which is in $\mathrm{Hom}((\beta_1\otimes\beta_1^{\mathrm{opp}})(\alpha^+_{\lambda_1}\otimes\alpha^{-,\mathrm{opp}}_{\lambda_2}),\mathrm{id}_{M\otimes M^{\mathrm{opp}}})$.

Now setting $X_{\beta_1}=W^*_0XW_{\beta_1}$, we get

$$\sum_{\beta_1\in {}_M\mathcal{X}_M} \sqrt{\frac{d_1}{d_3}} X_{\beta_1}(\beta_1\otimes\beta_1^{\mathrm{opp}})(\mathcal{E}^+_{\lambda_1}(\beta_5)^*\otimes j(\mathcal{E}^-_{\lambda_2}(\beta_5))^*)T^{\beta_3}_{\beta_1\beta_5} = \sqrt{\frac{1}{d_5}} W^*_{\beta_5} XW_{\beta_3}$$

for any $\beta_3,\beta_5\in {}_M\mathcal{X}_M$ from the equation (3), and this implies

$$X=\sum_{\beta_1,\beta_3,\beta_5\in {}_M\mathcal{X}_M} \sqrt{\frac{d_1d_5}{d_3}} W_{\beta_5} X_{\beta_1}(\beta_1\otimes\beta_1^{\mathrm{opp}})(\mathcal{E}^+_{\lambda_1}(\beta_5)^*\otimes j(\mathcal{E}^-_{\lambda_2}(\beta_5))^*)T^{\beta_3}_{\beta_1\beta_5}W^*_{\beta_3}. \qquad (4)$$

Consider the linear map sending $X\in M\otimes M^{\mathrm{opp}}$ with

$$XV\in\mathrm{Hom}(\eta(\alpha^+_{\lambda_1},+)\eta^{\mathrm{opp}}(\alpha^-_{\lambda_2},-),\Gamma)$$

to

$$(W^*_0XW_\beta)_\beta\in\bigoplus_{\beta\in {}_M\mathcal{X}_M}\mathrm{Hom}(\beta\alpha^+_{\lambda_1},\mathrm{id})\otimes\mathrm{Hom}(\beta^{\mathrm{opp}}\alpha^{-,\mathrm{opp}}_{\lambda_2},\mathrm{id}).$$

The dimensions of the space of such X and the space

$$\bigoplus_{\beta\in {}_M\mathcal{X}_M}\mathrm{Hom}(\beta\alpha^+_{\lambda_1},\mathrm{id})\otimes\mathrm{Hom}(\beta\mathrm{opp}\alpha^{-,\mathrm{opp}}_{\lambda_2},\mathrm{id})$$

are both equal to $Z_{\lambda_1\lambda_2}$, and this map is injective by the equation (4), so this map is also surjective. That is, a general form of such an X is determined now by the equation (4), where X_β's are now arbitrary intertwiners in $\mathrm{Hom}(\beta\alpha^+_{\lambda_1}, \mathrm{id}) \otimes \mathrm{Hom}(\beta^{\mathrm{opp}}\alpha^{-,\mathrm{opp}}_{\lambda_2}, \mathrm{id})$. Fix $\lambda_1, \lambda_2 \in {}_N\mathcal{X}_N, \beta \in {}_M\mathcal{X}_M$ and l_1 and l_2 be indices in the set $\{1, 2, \ldots, \dim \mathrm{Hom}(\beta\alpha^+_{\lambda_1}, \mathrm{id})\}$, $\{1, 2, \ldots, \dim \mathrm{Hom}(\beta\alpha^-_{\lambda_2}, \mathrm{id})\}$ respectively. Following [25], we use the letter l for the multi-index $(\lambda_1, \lambda_2, \beta, l_1, l_2)$. Note that in order for us to get a non-trivial index, that is, $l_1 > 0, l_2 > 0$, the endomorphism β must be *ambichiral* in the sense that it appears in irreducible decompositions of both α^+-induction and α^--induction as in [2]. Let $\{T^+_{l_1}\}_{l_1}$ and $\{T^-_{l_2}\}_{l_2}$ be orthonormal bases of $\mathrm{Hom}(\alpha^+_{\lambda_1}, \bar\beta)$ and $\mathrm{Hom}(\alpha^-_{\lambda_2}, \bar\beta_2)$, respectively.

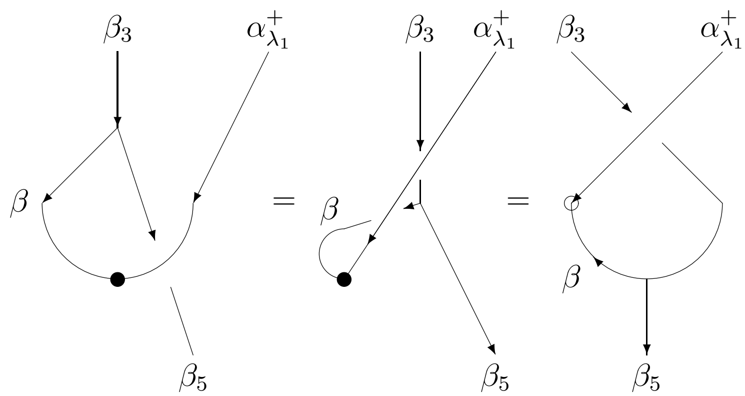

Figure 1: An application of the braiding-fusion equation

We now study some intertwiners using a graphical calculus in [2, Section 3]. First note that we have identities as in Fig. 1 by the braiding-fusion equation [8, Definition 4.2], [4, Definition 2.2 2] for a half-braiding, where crossings in the picture represent the half-braidings and the black and white small circles represent intertwiners in $\mathrm{Hom}(\beta\alpha^+_{\lambda_1}, \mathrm{id})$ and $\mathrm{Hom}(\alpha^+_{\lambda_1}, \bar\beta)$ respectively. (See [2, Section 3] for interpretations of the graphical calculus. Here and below, a triple point, a black or while small circle always represents an isometry or a co-isometry. One has to be careful that we have a normalizing constant involving the fourth roots of statistical dimensions as in [2, Figures 7,9]. From now on, we drop orientations of wires, which should causes no confusions.) We also have the following lemma to relate these two intertwiners.

Lemma 2.2. *Let $T_j \in \mathrm{Hom}(\beta, \alpha^+_\lambda)$ and define $\hat T_j \in \mathrm{Hom}(\beta, \alpha^+_\lambda)$ by the graphical expression in Fig. 2. Then we have $T^*_k T_j = \hat T^*_k \hat T_j$.*

Proof. We compute as in Fig. 3. Q.E.D.

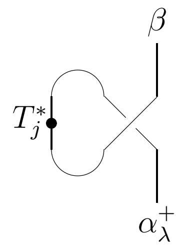

Figure 2: The intertwiner $\hat{T}_j$

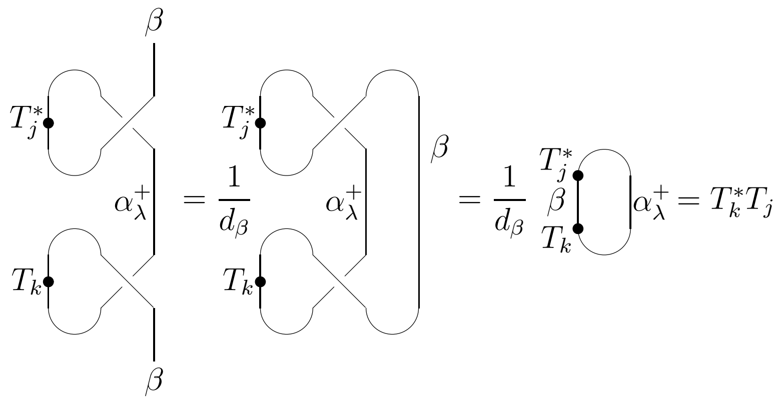

Figure 3: The inner product $\hat{T}_k^* \hat{T}_j$

Based on this, we set

$$S_{\beta_1}^{\beta_2\beta_3} = \sum_{k=1}^{N_{23}^1} (T_{\beta_2\beta_3,k}^{\beta_1})^* \otimes j(T_{\beta_2\beta_3,k}^{\beta_1})^* \in M \otimes M^{\mathrm{opp}}$$

and we now define $X_l \in M \otimes M^{\mathrm{opp}}$ as follows.

$$X_l = \sqrt{d_{\lambda_1} d_{\lambda_2}} \sum_{\beta_3,\beta_5 \in {}_M\mathcal{X}_M} \sqrt{\frac{d_3}{d_1 d_5}} W_{\beta_5} S_{\beta_5}^{\bar{\beta}_1\beta_3} (T_{l_1}^+ \otimes j(T_{l_2}^-))(\mathcal{E}_{\lambda_1}^+(\beta_3)^* \otimes j(\mathcal{E}_{\lambda_2}^-(\beta_3))^*) W_{\beta_3}^*. \tag{5}$$

Then by the equation (4), the operator $U_l \in R$ defined by $U_l = X_l V$ is in

$$\mathrm{Hom}(\eta(\alpha_{\lambda_1}^+, +)\eta^{\mathrm{opp}}(\alpha_{\lambda_2}^-, -), \Gamma)$$

and $\{U_l\}_{\beta,l_1,l_2}$ is a linear basis of this intertwiner space. We next prove that $\{U_l\}_{\beta,l_1,l_2}$ is actually an orthonormal basis with respect to the usual inner product. Recall that for

$$s, t \in \mathrm{Hom}(\eta(\alpha_{\lambda_1}^+, +)\eta^{\mathrm{opp}}(\alpha_{\lambda_2}^-, -), \Gamma),$$

we have

$$E_{M\otimes M^{\mathrm{opp}}}(st^*)=\frac{d_{\lambda_1}d_{\lambda_2}}{w}t^*s\in \mathbf{C}$$

because $d_{\eta(\alpha^+_{\lambda_1},+)\eta^{\mathrm{opp}}(\alpha^-_{\lambda_2},-)}=d_{\lambda_1}d_{\lambda_2}$. (See [8, Lemma 3.1 (i)].) We then have

$$\begin{aligned}
E_{M\otimes M^{\mathrm{opp}}}(U_lU^*_{l'}) &= \frac{1}{w}X_lX^*_{l'}\\
&= \delta_{ll'}\frac{d_{\lambda_1}d_{\lambda_2}}{w}\sum_{\beta_3,\beta_5\in {}_M\mathcal{X}_M}\frac{d_3}{d_1d_5}N^5_{13}W_{\beta_5}W^*_{\beta_5}\\
&= \delta_{ll'}\frac{d_{\lambda_1}d_{\lambda_2}}{w},
\end{aligned}$$

and this proves that $\{U_l\}_{\beta,l_1,l_2}$ is indeed an orthonormal basis. This also shows that we have

$$\phi_\Theta(X^*_mX_l)=W^*E_{\Gamma(R)}(X^*_mX_l)W=W^*\Gamma(U^*_mU_l)W=\delta_{lm},$$

where ϕ_Θ is the standard left inverse of Θ. (See [20] for a general theory of left inverses.)

Let $l=(\lambda_1,\lambda_2,\beta'_1,m_1,m_2)$, $m=(\mu_1,\mu_2,\beta''_1,m_1,m_2)$, $n=(\nu_1,\nu_2,\beta_1,n_1,n_2)$ be multi-indices as above. We compute $E_{\Gamma(R)}(X^*_mX^*_lX_n)$ as follows.

$$\begin{aligned}
E_{\Gamma(R)}(X^*_mX^*_lX_n) &= \Gamma(V^*X^*_mX^*_lX_nV)\\
&= \Gamma(w^{1/2}V^*X^*_m\Gamma(V^*)WX^*_lX_nV)\\
&= \Gamma(w^{1/2}V^*X^*_m\Gamma(V^*X^*_l)WX_nV)\\
&= \Gamma(w^{1/2}U^*_m\Gamma(U^*_l)WU_n).
\end{aligned}$$

Based on this, we set

$$Y^n_{lm}=w^{-1/2}V^*X^*_mX^*_lX_nV=U^*_m\Gamma(U^*_l)WU_n\in R$$

and then this is an element in

$$\mathrm{Hom}(\eta(\alpha^+_{\nu_1},+)\eta^{\mathrm{opp}}(\alpha^-_{\nu_2},-),\eta(\alpha^+_{\mu_1},+)\eta^{\mathrm{opp}}(\alpha^-_{\mu_2},-)\eta(\alpha^+_{\lambda_1},+)\eta^{\mathrm{opp}}(\alpha^-_{\lambda_2},-)),$$

which is then contained in

$$\mathrm{Hom}(\nu_1,\mu_1\lambda_1)\otimes\mathrm{Hom}(\nu_2,\mu_2\lambda_2)^{\mathrm{opp}}\subset N\otimes N^{\mathrm{opp}}\subset M\otimes M^{\mathrm{opp}}$$

by [4, Theorem 3.9]. That is, we now have

$$E_{\Gamma(R)}(X^*_mX^*_lX_n)=w^{1/2}\Theta(Y^n_{lm})\in\Theta(M\otimes M^{\mathrm{opp}})$$

and

$$\phi_\Theta(X^*_mX^*_lX_n)=V^*X^*_mX^*_lX_nV. \tag{6}$$

Proposition 2.3. *In the above setting, the Q-system* (Γ, V, W) *is given as follows.*

$$\begin{aligned} \Gamma(x) &= \sum_l U_l(\eta(\alpha^+_{\lambda_1},+)\eta^{\mathrm{opp}}(\alpha^-_{\lambda_2},-))(x)U_l^*, \quad \text{for } x \in R, && (7)\\ V &= U_{(0,0,0,1,1)}, && (8)\\ W &= \sum_{l,m,n} \Gamma(U_l)U_m Y^n_{lm} U_n^*. && (9)\end{aligned}$$

Proof. Since $\{U_l\}_{\beta_1,l_1,l_2}$ is an orthonormal basis of

$$\mathrm{Hom}(\eta(\alpha^+_{\lambda_1},+)\eta^{\mathrm{opp}}(\alpha^-_{\lambda_2},-),\Gamma),$$

we get the fist identity (7). By the definition (5) of X_l, we have $X_{(0,0,0,1,1)} = 1$, hence $U_{(0,0,0,1,1)} = V$, which is (8). Since $Y^n_{lm} = U^*_m\Gamma(U^*_l)WU_n$, we get (9). Q.E.D.

Next we further compute Y^n_{lm}. We first have

$$\begin{aligned} Y^n_{lm} &= W^*\Theta(Y^n_{lm})W\\ &= w^{-1/2}W^*E_{\Gamma(R)}(X^*_m X^*_l X_n)W\\ &= w^{-1/2}\phi_\Theta(X^*_m X^*_l X_n)\\ &= \sum_{\beta\in {}_M\mathcal{X}_M} \frac{d_\beta^2}{w^{3/2}}(\phi_\beta\otimes\phi_\beta^{\mathrm{opp}})(W^*_\beta X^*_m X^*_l X_n W_\beta),\end{aligned}$$

where ϕ_β is the standard left inverse of β. In this expression, we compute the term $W^*_\beta X^*_m X^*_l X_n W_\beta$ as follows.

$$\begin{aligned} &W^*_\beta X^*_m X^*_l X_n W_\beta\\ =\ &\sqrt{d_{\lambda_1}d_{\lambda_2}d_{\mu_1}d_{\mu_2}d_{\nu_1}d_{\nu_2}} \sum_{\beta'_3,\beta_5\in {}_M\mathcal{X}_M} \frac{d_\beta}{d_{\beta_5}\sqrt{d_{\beta''_1}d_{\beta'_1}d_{\beta_1}}}\\ &\qquad \times(\mathcal{E}^+_{\mu_1}(\beta)\otimes j(\mathcal{E}^-_{\mu_2}(\beta)))((T^+_{m_1})^*\otimes j(T^-_{m_2})^*)(S^{\bar\beta''_1\beta}_{\beta'_3})^*\\ &\qquad \times(\mathcal{E}^+_{\lambda_1}(\beta'_3)\otimes j(\mathcal{E}^-_{\lambda_2}(\beta'_3)))((T^+_{l_1})^*\otimes j(T^-_{l_2})^*)(S^{\bar\beta'_1\beta'_3}_{\beta_5})^*\\ &\qquad \times S^{\bar\beta_1\beta}_{\beta_5}(T^+_{n_1}\otimes j(T^-_{n_2}))(\mathcal{E}^+_{\nu_1}(\beta)^*\otimes j(\mathcal{E}^-_{\nu_2}(\beta))^*\end{aligned}$$

Our aim is to show that our Y^n_{lm} coincides with Rehren's $\mathcal{T}^n_{lm}$ in [25, page 400]. Our Y^n_{lm} is already in $\mathrm{Hom}(\nu_1,\mu_1\lambda_1)\otimes\mathrm{Hom}(\nu_2,\mu_2\lambda_2)^{\mathrm{opp}}$ as in Rehren's $\mathcal{T}^n_{lm}$. So we expand our Y^n_{lm} with respect to the basis $\{\tilde T_e = T^1_{e_1}\otimes j(T^2_{e_2})\}_{e=(e_1,e_2)}$, where $\{T^1_{e_1}\}_{e_1}$, $\{T^2_{e_2}\}_{e_2}$ are bases for $\mathrm{Hom}(\nu_1,\mu_1\lambda_1)$, $\mathrm{Hom}(\nu_2,\mu_2\lambda_2)$, respectively. We will prove that the coefficients of Y^n_{lm} for such an expansion coincide with Rehren's coefficients ζ^n_{lm,e_1,e_2} in [25, page 400].

Let $S^+_l = S^+_{\beta_1,\lambda_1,l_1} \in \mathrm{Hom}(\beta_1,\alpha^+_{\lambda_1})$ be isometries so that $\{S^+_{\beta_1,\lambda_1,l_1}\}_{l_1}$ gives an orthonormal basis in $\mathrm{Hom}(\beta_1,\alpha^+_{\lambda_1})$. Similarly we choose $S^-_l = S^-_{\beta_1,\lambda_2,l_2}\in\mathrm{Hom}(\beta_1,\alpha^-_{\lambda_2})$.

Rehren puts an inner product in $\mathrm{Hom}(\alpha^+_{\lambda_1}, \alpha^-_{\lambda_2})$ in [25, page 400]. When we decompose this space as $\bigoplus_{\beta\in {}_M\mathcal{X}^0_M} \mathrm{Hom}(\alpha^+_{\lambda_1}, \beta) \otimes \mathrm{Hom}(\beta, \alpha^-_{\lambda_2})$, Rehren's normalization implies that his orthonormal basis consists of intertwiners of the form $\sqrt{d_{\lambda_1}/d_\beta} S^-_l S^{+*}_l$, where $S^\pm_l$ are isometries as above. This implies that Rehren's ζ^n_{lm,e_1,e_2} is given as follows.

$$\sqrt{\frac{d_{\lambda_1} d_{\lambda_2} d_{\mu_1} d_{\mu_2} d_{\beta''_1}}{w d_{\nu_1} d_{\nu_2} d_{\beta_1} d_{\beta'_1}}} S^{+*}_n (T^2_{e_1})^* ((S^+_l S^{-*}_l) \times (S^+_m S^{-*}_m)) T^1_{e_2} S^-_n. \tag{10}$$

Note that we have

$$E_{M\otimes M^{\mathrm{opp}}}(X_n V \tilde{T}_e V^* X^*_m X^*_l) = \frac{d_{\nu_1} d_{\nu_2}}{w} \tilde{T}_e V^* X^*_m X^*_l X_n V, \tag{11}$$

where we have $\tilde{T}_e = T^1_{e_1} \otimes j(T^2_{e_2})$ as above. (See [8, Lemma 3.1].)

We expand our Y^n_{lm} with respect to the basis $\{\tilde{T}_e\}_e$. Then the coefficient is given as follows using the relations (6), (11).

$$\begin{aligned}
(Y^n_{lm})^* \tilde{T}_e &= w^{-1/2} \phi_\Theta (X^*_n X_l X_m) \tilde{T}_e \\
&= w^{-1/2} V^* X^*_n X_l X_m V \tilde{T}_e \\
&= \frac{w^{1/2}}{d_{\nu_1} d_{\nu_2}} E_{M\otimes M^{\mathrm{opp}}}(X_l X_m V \tilde{T}_e V^* X^*_n) \\
&= \frac{1}{w^{1/2} d_{\nu_1} d_{\nu_2}} X_l X_m \Theta(\tilde{T}_e) X^*_n
\end{aligned} \tag{12}$$

We represent X_l graphically as in Fig. 4, where we follow the graphical convention of [2, Section 3], and $\{T_i\}_i$ is an orthonormal basis of $\mathrm{Hom}(\beta_5, \beta_1\beta_3)$. After this figure, we drop the symbols T_i, $S^{\pm *}_l$, and the summation $\sum_{T_i}$ for simplicity.

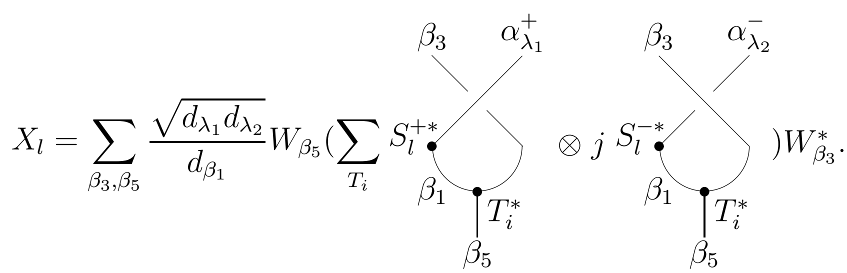

Figure 4: A graphical expression for X_l

We next have a graphical expression for $X_l X_m$ as in Fig. 5, where we have used a braiding-fusion equation for the half-braiding.

Here we prepare two lemmas.

Lemma 2.4. *For an intertwiner in* $\mathrm{Hom}(\beta_1\beta_2, \beta_3) \otimes \mathrm{Hom}(\beta_3, \beta_1\beta_2)$*, the application of the left inverse* ϕ_{β_1} *is given as in Fig. 6.*

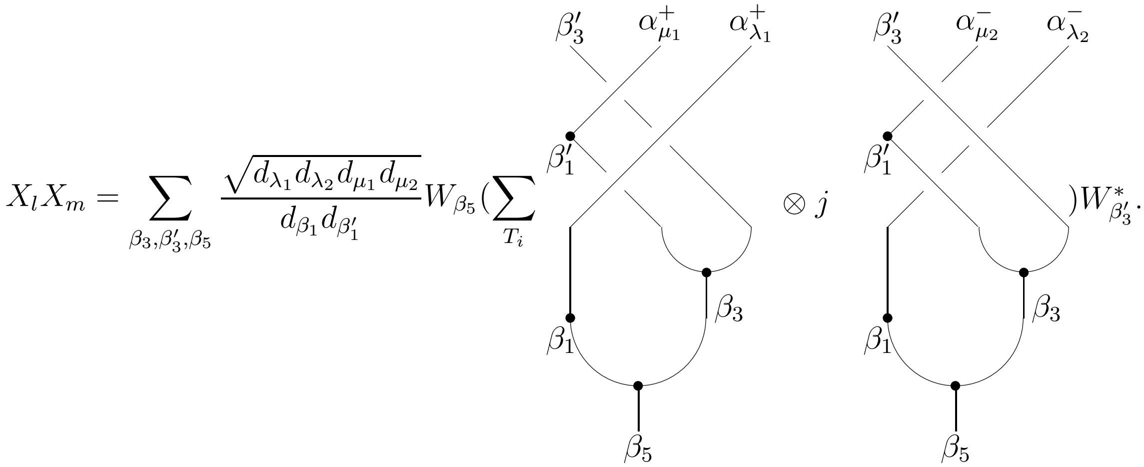

Figure 5: A graphical expression for $X_l X_m$

Proof. Immediate by [8, Lemma 3.1. (i)] and our graphical normalization convention. Q.E.D.

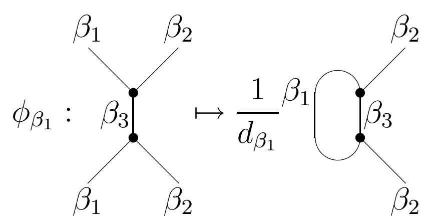

Figure 6: A graphical expression for the left inverse

Lemma 2.5. *For a change of bases, we have a graphical identity as in Fig. 7, where we have summations over orthonormal bases of (co)-isometries for small black circles.*

Proof. The change of bases produces quantum $6j$-symbols, and their unitarity gives the conclusion. Q.E.D.

Then next we compute $X_l X_m \Theta(\tilde{T}_e) X_n^*$. It is expressed as

$$\begin{aligned} & X_l X_m \Theta(\tilde{T}_e) X_n^* \\ = \; & \frac{\sqrt{d_{\lambda_1} d_{\lambda_2} d_{\mu_1} d_{\mu_2} d_{\nu_1} d_{\nu_2}}}{d_{\beta_1} d_{\beta'_1} d_{\beta''_1}} \left(\frac{d_{\nu_1} d_{\nu_2}}{d_{\lambda_1} d_{\lambda_2} d_{\mu_1} d_{\mu_2}} \right)^{1/4} \\ & \qquad \times \sum_{\beta'_3, \beta_5, \tilde{\beta}_3} W_{\beta_5} (\text{graphical expression of Fig. (8)}) W^*_{\beta_5}, \qquad (13) \end{aligned}$$

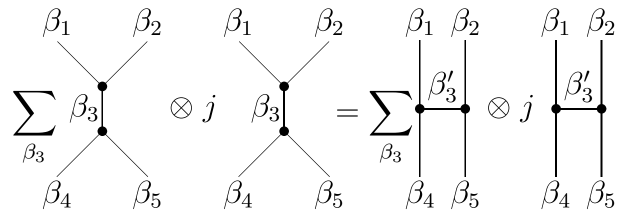

Figure 7: A change of orthonormal bases

where small white circles represent intertwiners corresponding to $T^1_{e_1}, T^2_{e^2}$ regarded as elements in M, we have applied ϕ_Θ graphically using Lemma 2.4, changed the orthonormal bases in the space $\mathrm{Hom}(\beta_1\beta_1'\beta_3', \beta_5)$ using Lemma 2.5 and thus we now have a summation over $\tilde{\beta}_3$ rather than over β_3.

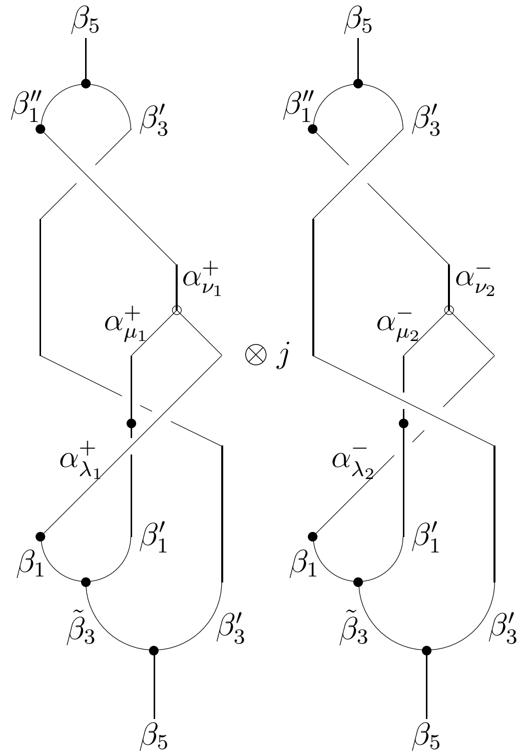

Figure 8: A graphical expression for $X_l X_m \Theta(\tilde{T}_e) X_n^*$

Then the complex number value represented by Fig. 8 can be computed as in Fig. 9, where we have used the braiding-fusion equation for a half-braiding twice.

Here we have the following lemma.

Lemma 2.6. *Let β, β' be ambichiral and choose isometries $T \in \mathrm{Hom}(\beta, \alpha^+_\lambda)$, $S \in \mathrm{Hom}(\beta', \alpha^+_\mu)$. Then we have the identity as in Fig. 10.*

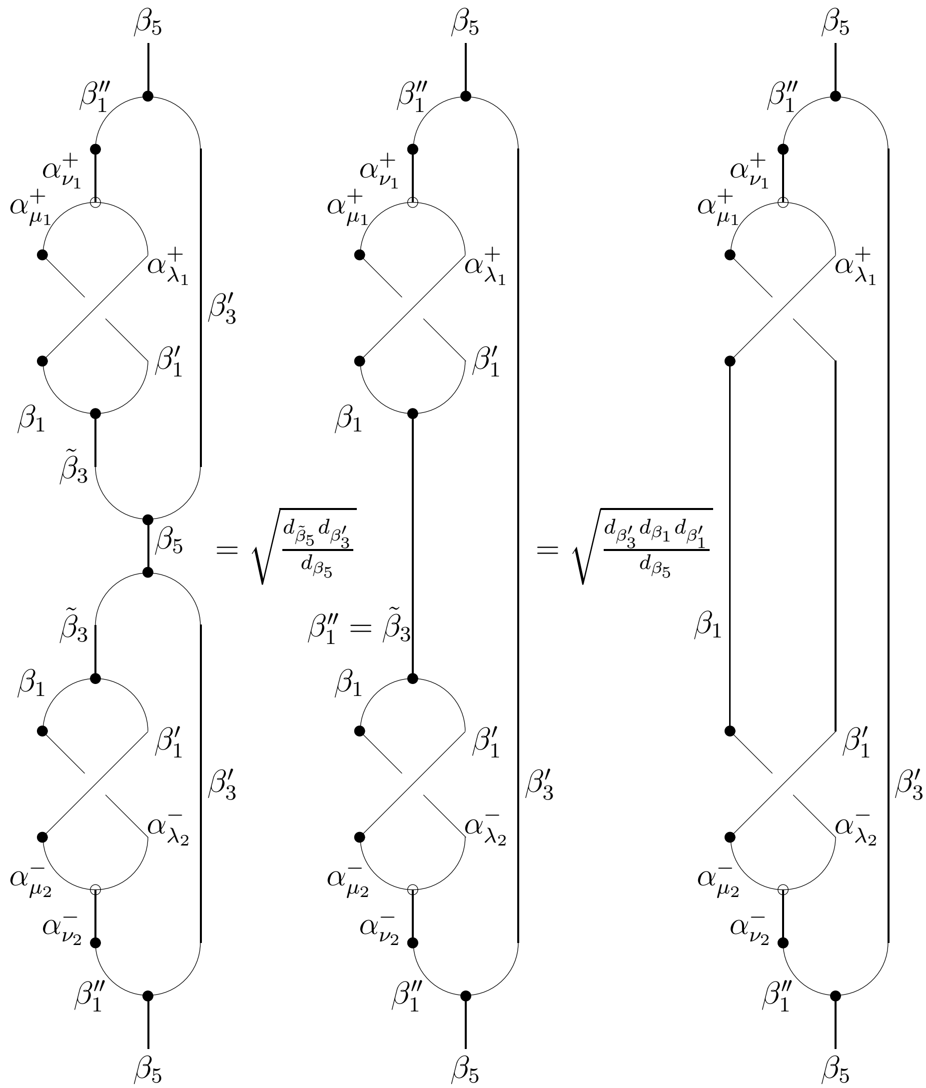

Figure 9: The value of Fig. 8

Proof. We compute the both hand sides by the definitions of the half and the relative braidings in [4, (10)] and [1, Subsection 3.3], respectively, and then we get $\beta'(T^*)S^*\varepsilon^+(\lambda,\mu)TT^*$, where we have used $\varepsilon^+(\lambda,\mu)\alpha^+_\lambda(SS^*) = SS^*\varepsilon^+(\lambda,\mu)$, which follows from the arguments and the figure in [27, page 377]. (The chiral locality is not used in the argument in [27, page 377].) Q.E.D.

Then the value $(Y^n_{lm})^*\tilde T_e$ is computed with the coefficients in the equations (12),

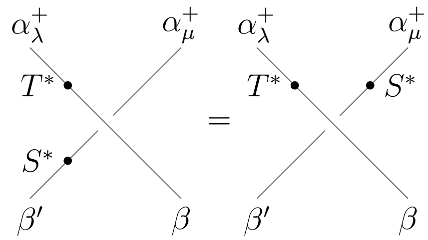

Figure 10: A naturality equation

(13), and Fig. 9. The coefficient is now

$$
\begin{aligned}
&\frac{w^{1/2}}{d_{\nu_1}d_{\nu_2}}\frac{\sqrt{d_{\lambda_1}d_{\lambda_2}d_{\mu_1}d_{\mu_2}d_{\nu_1}d_{\nu_2}}}{d_{\beta_1}d_{\beta'_1}d_{\beta''_1}}\left(\frac{d_{\nu_1}d_{\nu_2}}{d_{\lambda_1}d_{\lambda_2}d_{\mu_1}d_{\mu_2}}\right)^{1/4}\\
&\qquad\times\sqrt{\frac{d_{\beta'_3}d_{\beta_1}d_{\beta'_1}}{d_{\beta_5}}}\sqrt{\frac{d_{\beta''_1}d_{\beta_5}}{d_{\beta'_3}}}d_{\beta''_1}\\
=\;& w^{-1/2}\sqrt{\frac{d_{\lambda_1}d_{\lambda_2}d_{\mu_1}d_{\mu_2}}{d_{\nu_1}d_{\nu_2}}}\sqrt{\frac{d_{\beta''_1}}{d_{\beta_1}d_{\beta'_1}}}\left(\frac{d_{\nu_1}d_{\nu_2}}{d_{\lambda_1}d_{\lambda_2}d_{\mu_1}d_{\mu_2}}\right)^{1/4} \qquad (14)
\end{aligned}
$$

and this is multiplied with the intertwiner in Fig. 11, where the two crossings of the two wires labeled with β_1, β'_1 represent the "ambichiral braiding" studied in [1, Subsection 3.3].

Then the monodromy of β'_1 and β_1 in Fig. 11 acts on $\mathrm{Hom}(\beta'_1\beta_1, \beta''_1)$ as a scalar arising from "conformal dimensions" of $\beta_1, \beta'_1, \beta''_1$ in the ambichiral system. (See [5, Figure 8.30].) So up to this scalar, we have Fig. 12. Since the fourth root in (14) comes from our normalization for the graphical expression (see [2, Figures 7, 9]) and we can absorb the above scalar arising from the conformal dimensions by changing the bases $\{\tilde{T}_e\}_e$, our coefficient multiplied with the number represented by Fig. 11 now coincides with Rehren's coefficient computed as in (10). (Actually, λ_j and μ_j are interchanged and also α^+ and α^- are interchanged, but these are just matters of convention.)

Now with [4, Corollary 3.10], we have proved the following theorem.

Theorem 2.7. *The generalized Longo-Rehren subfactor arising from $\alpha^\pm$-induction with a non-degenerate braiding on ${}_N\mathcal{X}_N$ is isomorphic to the dual of the Longo-Rehren subfactor arising from ${}_M\mathcal{X}_M$.*

At the end of [25], Rehren asks for an Izumi type description [8] of irreducible endomorphisms of P arising from the generalized Longo-Rehren subfactor $N \otimes N^{\mathrm{opp}} \subset P$ and in particular, he asks whether a braiding exists or not on this system of endomorphisms of P. The above theorem in particular shows that the system of endomorphisms of P is isomorphic to the direct product system of ${}_M\mathcal{X}_M$ and ${}_M\mathcal{X}_M^{\mathrm{opp}}$ and thus we solve these problems and the answer to the second question is negative, since

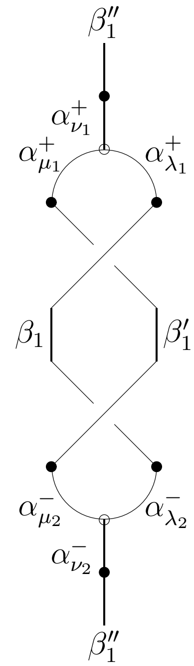

Figure 11: The remaining intertwiner

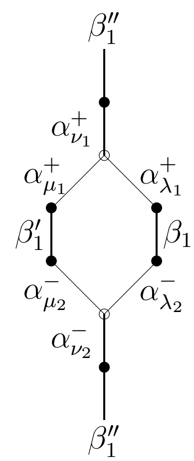

Figure 12: The new form of the remaining intertwiner

this system does not have a braiding in general and it can be even non-commutative. (Note that [2, Corollary 6.9] gives a criterion for such non-commutativity.)

Remark 2.8. If $N = M$ in the above setting, our result implies [8, Proposition 7.3], of course, but a remark on [8, page 171] gives a "twisted Longo-Rehren subfactor" rather than the usual Longo-Rehren subfactor. This is due to the monodoromy operator similar to the one in Fig. 11, but as pointed by Rehren, one can always eliminate such a twist and then the "twisted Longo-Rehren subfactor" is actually isomorphic to the Longo-Rehren subfactor. (See "Added in proof" of [8] on this point.) We also had a similar twist in our results here, originally, but we have eliminated it thanks to this remark of Rehren.

In the above setting, we can also set $N_1 = N$, $N_2 = M$, $\Delta_1 = {}_N\mathcal{X}_N$, $\Delta_2 = {}_M\mathcal{X}^0{}_M$, $\alpha^1_\lambda = \alpha^+_\lambda$, $\alpha^2_\tau = \tau$ in the construction of the generalized Longo-Rehren subfactor. Then the resulting subfactor $M \otimes N^{\mathrm{opp}} \subset R$ has a dual canonical endomorphism $\bigoplus_{\lambda\in{}_N\mathcal{X}_N,\tau\in{}_M\mathcal{X}^0{}_M} b^+_{\tau,\lambda}\lambda \otimes \tau^{\mathrm{opp}}$, where $b^+_{\tau,\lambda} = \dim \mathrm{Hom}(\alpha^+_\lambda, \tau)$ is the *chiral branching coefficient* as in [3, Subsection 3.2]. Now using the results in [4, Section 4] and arguments almost identical to the above, we can prove the following theorem.

Theorem 2.9. *The generalized Longo-Rehren subfactor $M \otimes N^{\mathrm{opp}} \subset R$ arising from α^+-induction as above with a non-degenerate braiding on ${}_N\mathcal{X}_N$ is isomorphic to the dual of the Longo-Rehren subfactor arising from ${}_M\mathcal{X}^+{}_M$.*

3 Nets of subfactors on S^1

In this section, we study multi-interval subfactors for completely rational nets of subfactors, which generalizes the study in [13].

Let $\{M(I)\}_{I\subset S^1}$ be a completely rational net of factors of S^1 in the sense of [13], where an "interval" I is a non-empty, non-dense connected open subset of S^1. (That is, we assume isotony, conformal invariance, positivity of the energy, locality, existence of the vacuum, irreducibility, the split property, strong additivity, and finiteness of the μ-index. See [6, 13] for the detailed definitions.) We also suppose to have a conformal subnet $\{N(I)\}_{I\subset S^1}$ of $\{M(I)\}_{I\subset S^1}$ with finite index as in [18]. The main result in [18] says that the subnet $\{N(I)\}_{I\subset S^1}$ is also completely rational.

Let $E = I_1 \cup I_3$ be a union of two intervals I_1, I_3 such that $\bar{I}_1 \cap \bar{I}_3 = \varnothing$. Label the interiors of the two connected components of $S \setminus E$ as I_2, I_4 so that I_1, I_2, I_3, I_4 appear on the circle in a counterclockwise order. We set $N_j = N(I_j)$, $M_j = M(I_j)$, for $j = 1, 2, 3, 4$. (This numbering should not be confused with the basic construction.) We also set $N = N_1, M = M_1$.

We have a finite system of mutually inequivalent irreducible DHR endomorphisms $\{\lambda\}$ for the net $\{N(I)\}$ by complete rationality. We may and do regard this as a braided system of endomorphisms of $N = N_1$. By [13, Corollary 37], this braiding is non-degenerate. We write ${}_N\mathcal{X}_N$ for this system. As in [2], we can apply $\alpha^\pm$-induction to get systems ${}_M\mathcal{X}_M, {}_M\mathcal{X}^+{}_M, {}_M\mathcal{X}^-{}_M, {}_M\mathcal{X}^0{}_M$ of irreducible endomorphisms of M. That is, they are the systems of irreducible endomorphisms of M arising from $\alpha^\pm$-induction, α^+-induction, α^--induction, and the "ambichiral" system, respectively. Since the braiding on ${}_N\mathcal{X}_N$ is non-degenerate, [2, Theorem 5.10] and [1, Proposition 5.1] imply that the ambichiral system ${}_M\mathcal{X}^0{}_M$ is given by the irreducible DHR endomorphisms of the net $\{M(I)\}$. By the inclusions ${}_M\mathcal{X}^0{}_M \subset {}_M\mathcal{X}^\pm{}_M \subset {}_M\mathcal{X}_M$ and the Galois correspondence of [8, Theorem 2.5] (or by the characterization of the

Longo-Rehren subfactor in [13, Appendix A]), we have inclusions of the corresponding Longo-Rehren subfactors $M \otimes M^{\rm opp} \subset R$, $M \otimes M^{\rm opp} \subset R^{\pm}$, $M \otimes M^{\rm opp} \subset R^0$ with $R^0 \subset R^{\pm} \subset R$. We study these Longo-Rehren subfactors in connection to the results in Section 2.

As in [13], we make identification of S^1 with $\mathbf{R} \cup \{\infty\}$, and as in [13, Proposition 36], we may and do assume that $I_1 = (-b, -a), I_3 = (a, b)$, with $0 < a < b$. Take a DHR endomorphism λ localized in I_1 for the net $\{N(I)\}$. Let $P = M(\tilde{I})$, where $\tilde{I} = (-\infty, 0)$. Let J be the modular conjugation for P with respect to the vacuum vector. We consider endomorphisms of the C^*-algebras $\overline{\bigcup_{I \subset (-\infty,\infty)} M(I)}$ and $\overline{\bigcup_{I \subset (-\infty,\infty)} N(I)}$. The canonical endomorphism γ and the dual canonical endomorphism θ are regarded as endomorphisms of these C^*-algebras. We regard α_λ^+ as an endomorphism of the former C^*-algebra as in [19], and then it is not localized in I_1 any more, but it is localized in $(-\infty, -a)$ by [19, Proposition 3.9]. We study an irreducible decomposition of α_λ^+ as an endomorphism of M_1 and choose β appearing in such an irreducible decomposition of α_λ^+ regarded as an endomorphism of M_1. That is, we choose an isometry $W \in M_1$ with $W^*W \in \alpha_\lambda^+(M)' \cap M$, $\beta(x) = W^* \alpha_\lambda^+(x) W$. Using this same formula, we can regard β as endomorphism of the C^*-algebra $\overline{\bigcup_{I \subset (-\infty,\infty)} M(I)}$. We next regard β as an endomorphism of P and let V_β be the isometry standard implementation of $\beta \in \mathrm{End}(P)$ as in [6, Appendix]. We now set $\bar{\beta} = J\beta J$. Then for any $X \in P \vee P'$, we have $\beta\bar{\beta}(X)V_\beta = V_\beta X$ as in the proof of [13, Proposition 36] since $JV_\beta J = V_\beta$. By strong additivity, we have this for all local operators X. Since $\lambda, \bar{\lambda} = J\lambda J, \beta, \bar{\beta}$ are localized in $(-\infty, a)$, (a, ∞), I_1, I_3, respectively, we know that $V_\beta \in (M_2 \vee N_4)'$. Consider the subfactor $M_1 \vee M_3 \subset (M_2 \vee N_4)'$. By Frobenius reciprocity [7], we know that the dual canonical endomorphism for the subfactor $M_1 \vee M_3 \subset (M_2 \vee N_4)'$ contains $\beta \otimes \beta^{\rm opp}$, where $M_3 = JM_1J$ is now regarded as $M_1^{\rm opp}$ and $M_1 \vee M_3$ is regarded as $M_1 \otimes M_1^{\rm opp}$, for all $\beta \in {}_M\mathcal{X}^+{}_M$. We now compute the index of the subfactor $M_1 \vee M_3 \subset (M_2 \vee N_4)'$ in two ways. On one hand, it has an intermediate subfactor $(M_2 \vee M_4)'$ and the index for $M_1 \vee M_3 \subset (M_2 \vee M_4)'$ is the global index of the ambichiral system by [13, Theorem 33]. The index of $(M_2 \vee M_4)' \subset (M_2 \vee N_4)'$ is simply that of the net $\{N(I) \subset M(I)\}$ of subfactors. We also have

$$\frac{w_+}{w_0} = \frac{w}{w_+} = \sum_{\lambda \in {}_N\mathcal{X}_N} d_\lambda Z_{\lambda 0} = d_\theta = [M(I) : N(I)],$$

where w, w_+, w_0 are the global indices of ${}_M\mathcal{X}_M, {}_M\mathcal{X}^+{}_M, {}_M\mathcal{X}^0{}_M$, respectively, by [3, Theorem 4.2, Proposition 3.1], [27, Theorem 3.3 (1)]. (Here we have used the chiral locality condition arising from the locality of the net $\{M(I)\}$. Without the chiral locality, the results in this section would not hold in general.) These imply that

$$[(M_2 \vee N_4)' : M_1 \vee M_3] = w_+. \tag{15}$$

On the other hand, the dual canonical endomorphism for the subfactor $M_1 \vee M_3 \subset (M_2 \vee N_4)'$ contains $\bigoplus_{\beta \in {}_M\mathcal{X}^+{}_M} \beta \otimes \beta^{\rm opp}$ from the above considerations since each β is irreducible as an endomorphism of M, thus the index value is at least $\sum_{\beta \in {}_M\mathcal{X}^+{}_M} d_\beta^2 =$

w_+. Together with (15), we know that the dual canonical endomorphism is indeed equal to $\bigoplus_{\beta \in {}_M\mathcal{X}^+_M} \beta \otimes \beta^{\mathrm{opp}}$.

Put $R_\beta = \sqrt{d_\beta} V_\beta \in (M_2 \vee N_4)'$. As in the proof of [13, Proposition 36], we now conclude that the subfactor $M_1 \vee M_3 \subset (M_2 \vee N_4)'$ is isomorphic to the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R^+$. Similarly, we know that the subfactor $M_1 \vee M_3 \subset (N_2 \vee M_4)'$ is isomorphic to the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R^-$. These two isomorphisms are compatible on $(M_2 \vee M_4)'$ and they give an isomorphism of $M_1 \vee M_3 \subset (M_2 \vee M_4)'$ to the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R^0$. We finally look at the inclusions

$$\begin{array}{ccccc} M \otimes M^{\mathrm{opp}} & \subset & R^0 & \subset & R^+ \\ & & \cap & & \cap \\ & & R^- & \subset & R. \end{array}$$

The right square is a commuting square by [18, Lemma 1] and thus R is generated by R^+ and R^-. (Or [2, Theorem 5.10] and [8, Proposition 2.4, Theorem 2.5] also give this generating property.) It means that the above isomorphisms give the following theorem.

Theorem 3.1. *Under the above setting, the following system of algebras arising from four intervals on the circle is isomorphic to the system of algebras (16) arising as Longo-Rehren subfactors.*

$$\begin{array}{ccccc} M_1 \vee M_3 & \subset & (M_2 \vee M_4)' & \subset & (M_2 \vee N_4)' \\ & & \cap & & \cap \\ & & (N_2 \vee M_4)' & \subset & (N_2 \vee N_4)'. \end{array}$$

Remark 3.2. Passing to the commutant, we also conclude that the subfactor $N_1 \vee N_3 \subset (M_2 \vee M_4)'$ is isomorphic to the dual of $M \otimes M^{\mathrm{opp}} \subset R$ and thus isomorphic to the generalized Longo-Rehren subfactor arising from the $\alpha^\pm$-induction studied in Section 2. In the example of the conformal inclusion $SU(2)_{10} \subset Spin(5)_1$ in [27, Section 4.1], this fact was first noticed by Rehren and it can be proved also in general directly by computing the corresponding Q-system.

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