On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors

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Abstract. We will give a proof of Ocneanu's announced classification of subfactors of the AFD type II_1 factor with the principal graphs A_n, D_n, E_7, the Dynkin diagrams, and give a single explicit equation of $\exp \dfrac{\pi\sqrt{-1}}{24}$ and $\exp \dfrac{\pi\sqrt{-1}}{60}$ for each of E_6 and E_8 such that its validity is equivalent to existence of two (and only two) subfactors for these principal graphs. Our main tool is flatness of connections on finite graphs, which is the key notion of Ocneanu's paragroup theory. We give the difference between the diagrams D_{2n} and D_{2n+1} a meaning as a $\mathbf{Z}/2\mathbf{Z}$-obstruction for flatness arising in orbifold construction, which is an analogue of orbifold models in solvable lattice models.

§0 Introduction

Since the breakthrough of the index theory of V. F. R. Jones [10], more and more deep unexpected connections of subfactor theory to several branches of mathematics and physics have been found [11].

In the theory of operator algebras, the classification of the approximately finite dimensional (AFD) subfactors is one of the most important and challenging

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problems. Especially classification of subfactors with small Jones index, less than (or equal to) 4, has attracted much attention. The principal graph of a subfactor has been known as an interesting invariant, and it has been known that the principal graph of a subfactor with index less than 4 is one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8. (See [6, Chapter 4].) On this classification problem, A. Ocneanu has announced the following striking classification in [16] as the first major step in the classification of subfactors.

Classification announced by Ocneanu. *There is one subfactor for each Dynkin diagram A_n, one for each diagram D_{2n}, and a pair of opposite but non-conjugate subfactors for each diagram E_6 and E_8. These are all the subfactors of the AFD factor of type II_1 with index less than 4.*

For proving this statement, we need an analytic argument (Ocneanu's "spanning theorem") and an algebraic/combinatorial argument on the Dynkin diagrams. Ocneanu announced the above statement in 1987, but the details of his proof have not appeared yet. For analytic aspect of this approach, S. Popa has given a proof of the spanning theorem in [18] under a weaker assumption than Ocneanu's, and moreover he has recently announced a stronger result in [19], which is in the final form in this approach. That is, this approach is a classification of subfactors by higher relative commutants, and Popa's announcement gives necessary and sufficient conditions for higher relative commutants to give the original subfactor.

The purpose of this paper is to supply proofs of algebraic/combinatorial aspect of the above announcement for the first time in publication. The most mysterious part of Ocneanu's announcement is that D_{2n+1} are eliminated while D_{2n} are possible, unlike other A-D-E classifications in mathematics. We use an idea of orbifold models in solvable lattice model theory [5, 13] to this problem and get the first complete proof in publication and a clear meaning of this fact as a $\mathbf{Z}/2\mathbf{Z}$-obstruction for flatness arising in the orbifold procedure of making D_n from a $\mathbf{Z}/2\mathbf{Z}$-symmetry of A_{2n-3}. This is the main original feature of this paper.

In Ocneanu's paragroup theory, he has the principal graph with an additional group-like structure (a connection) via his "Galois functor". This graph is an analogue of a underlying set of a group and additional structure is an analogue of Lie group structure. He calls such an object *paragroup*, which is a certain quantization of finite groups. This is a (non-commutative) analogue of the Galois groups for subfields. Paragroup can be also regarded as a "discrete" analogue of compact manifolds and an analogue of solvable lattice models without a spectral parameter. (See [16, 17] for background and [1] for solvable lattice models.) We emphasize similarity between paragroups and solvable lattice models in this paper.

An analytic part of this approach consists of proving a system of increasing finite dimensional algebras constructed from the graph, Ocneanu's string algebra, approximates the original subfactor as much as one wants. This requires a very deep analysis. (See [18].) With this part completed by S. Popa, a classification of subfactors at least with a certain good property, called *finite depth*, is reduced to a classification of paragroups.

An algebraic/combinatorial aspect of the theory consists of determining all the possible paragroup structures of a given graph. This is an analogue of a problem of determining all the finite groups when its order is given. (The Jones index is an analogue of an order of a group.) If the index is less than 4, then the finite depth condition is automatically satisfied, thus we only need to classify paragroup structures on the Dynkin diagrams. This is equivalent to a classification of flat connections, whose definition and properties will be discussed below. Although it is a trivial exercise to determine all the finite groups with order less than 4, our problem requires much more detailed arguments.

The case index=4 is still tractable as in [9, 19], but when we allow the index to be bigger than 4, the situation becomes much worse suddenly as in [17, IV.4] and a complete classification seems hopeless.

The book [6] is a basic reference on the index theory and its relation to graphs, and [16] contains a very good exposition on background of the theory.

The contents of each section are as follows.

In §1, we review basics of Ocneanu's theory. One of the purposes of this section is fixing notations clearly.

In §2, we give an explanation of the key notion *flatness* of the entire theory. Though Ocneanu has given a definition in [16] in two equivalent forms, its details were unavailable.

Section 3 handles a connection on the Dynkin diagrams. Ocneanu's connection gives an analogue of multiplication of groups. Although this part is rather elementary, we think it is helpful to work out in its complete details.

Sections 4 and 5 are main parts in our original approach based on orbifold construction. We will prove a technical proposition in §4 for the diagrams A_n by induction. This will be used in §5.

Section 5 deals with the D_n diagrams, and it is the main body of this paper. Using a cell system of Roche [20] satisfying a certain star-triangle relation, we reduce the problem of D_n to that of A_{2n-3}, which is more tractable. This is an idea of orbifold, and a subfactor with the principal graph D_{2n} is realized as $N^\theta \subset M^\theta$, where $N \subset M$ is a subfactor with the principal graph A_{4n-3} and θ is an automorphism of M of order 2 with $\theta(N) = N$.

In the last section §6, we handle E_6, E_7, E_8. For each of E_6 and E_8, we get an explicit equation of $\exp \dfrac{\pi\sqrt{-1}}{24}$ and $\exp \dfrac{\pi\sqrt{-1}}{60}$ respectively. Their validity is equivalent to flatness of Ocneanu's connection, and numerical experiment on a computer strongly suggests that the equations are valid, but they are too complicated to be verified by hand. A good symbolic manipulation program may prove them, but so far the author has been unable to prove the validity. By these, we can certainly show that for each of E_6 and E_8, there are *either* two (and only two) subfactors *or* no subfactors. For E_6, Bion-Nadal announced a construction [2], thus the first case holds in this case.

Though the motivation of the author for this study comes from the theory of operator algebras, none of the proofs in this paper use operator algebraic results except for a single point (at the beginning of §4), so the author hopes that this paper may be of interest to non-operator algebraists. For example, relation of this topic to conformal field theory is discussed in Roche [20].

The author is indebted very much to Professor A. Ocneanu for exposition of his striking theory. The author learned the theory from his lectures at University of Warwick in the summer of 1987, ones at the University of Tokyo in the summer of 1990, and personal conversations during his stay in Japan. (The lectures at Warwick roughly correspond to [16] and the ones at Tokyo were complied to the lecture notes [17] by the author.) The author thanks Professor Ocneanu very much

for all of these. He also thanks Professors M. Choda, D. Evans, and T. Miwa for comments on the preprint of this paper about duality of graphs [3], orbifold models, and crossing symmetry, respectively.

Further applications of this orbifold methods are given in [4] and [9].

After the circulation of the preprint version of this paper, the author received a paper of M. Izumi [7] in which he also proves impossibility of D_{odd} and E_7 as a principal graph by a different method based on Longo's theory [14, 15] and fusion rules. Furthermore, after the submission of this paper, a preprint of Sunder-Vijayarajan [21] was circulated and it gave a similar proof to Izumi's based on bimodule approach independently. (Impossibility proof along this line was also claimed by A. Ocneanu without a proof in his Tokyo lectures.) But it seems that one cannot prove a realization of D_{2n} as a paragroup with fusion rules.

§1 Basics from Ocneanu's theory.

Here we review some of the basics of Ocneanu's theory for the convenience of readers and for fixing notations of connections. References for this section are [16, 17, 20], though our notations are often slightly different from those in [16].

We have four finite graphs $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$ with the following properties: (1) Each graph is bipartite, that is, the vertices are divided into even ones and odd ones. (2) Perron-Frobenius eigenvalues of the adjacency matrices coincide for $\mathcal{G}_2$ and $\mathcal{G}_4$ and for $\mathcal{G}_1$ and $\mathcal{G}_3$. (3) Even vertices of $\mathcal{G}_2$ and $\mathcal{G}_1$ coincide, odd vertices of $\mathcal{G}_2$ and $\mathcal{G}_3$ coincide, even vertices of $\mathcal{G}_3$ and $\mathcal{G}_4$ coincide, and odd vertices of $\mathcal{G}_4$ and $\mathcal{G}_1$ coincide. (In the next section and later in this paper, we work on the case where all of $\mathcal{G}_j$ are the same and one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8.)

Take a diagram

$$\begin{tikzcd} a \arrow[r, "\xi_2"] \arrow[d, "\xi_1"'] & b \arrow[d, "\xi_3"] \\ c \arrow[r, "\xi_4"'] & d \end{tikzcd},$$

where ξ_j's satisfy one of the following

$$\begin{cases} \xi_1 \in \mathcal{G}_1, \xi_2 \in \mathcal{G}_2, \xi_3 \in \mathcal{G}_3, \xi_4 \in \mathcal{G}_4, & \text{or} \\ \xi_1 \in \mathcal{G}_3, \xi_2 \in \mathcal{G}_2, \xi_3 \in \mathcal{G}_1, \xi_4 \in \mathcal{G}_4, & \text{or} \\ \xi_1 \in \mathcal{G}_1, \xi_2 \in \mathcal{G}_4, \xi_3 \in \mathcal{G}_3, \xi_4 \in \mathcal{G}_2, & \text{or} \\ \xi_1 \in \mathcal{G}_3, \xi_2 \in \mathcal{G}_4, \xi_3 \in \mathcal{G}_1, \xi_4 \in \mathcal{G}_2, & \end{cases}$$

and $a = s(\xi_2) = s(\xi_1)$, $b = r(\xi_2) = s(\xi_3)$, $c = r(\xi_1) = s(\xi_4)$, $d = r(\xi_3) = r(\xi_4)$. (Here $s(\xi_j)$ and $r(\xi_j)$ mean the source and range of an edge ξ_j, that is, the starting point and the ending point.) We call such a diagram *cell*. A connection W is an assignment of a complex number to each cell, and we write

$$W \left(\begin{tikzcd} a \arrow[r, "\xi_2"] \arrow[d, "\xi_1"'] & b \arrow[d, "\xi_3"] \\ c \arrow[r, "\xi_4"'] & d \end{tikzcd} \right) \in \mathbf{C}.$$

This is an analogue of a Boltzmann weight in solvable lattice model theory. If no confusion arises, we just write a cell without mentioning W to denote the value of the connection. We also make the following conventions.

$$\begin{tikzcd} a \arrow[r, "\xi_2"] \arrow[d, "\xi_1"'] & b \arrow[d, "\xi_3"] \\ c \arrow[r, "\xi_4"'] & d \end{tikzcd} = \overline{\begin{tikzcd} b \arrow[d, "\xi_3"'] & a \arrow[l, "\xi_2"'] \arrow[d, "\xi_1"] \\ d & c \arrow[l, "\xi_4"] \end{tikzcd}} = \overline{\begin{tikzcd} c \arrow[r, "\xi_4"] & d \\ a \arrow[u, "\xi_1"] \arrow[r, "\xi_2"'] & b \arrow[u, "\xi_3"'] \end{tikzcd}} = \begin{tikzcd} d & c \arrow[l, "\xi_4"'] \\ b \arrow[u, "\xi_3"] & a \arrow[l, "\xi_2"] \arrow[u, "\xi_1"'] \end{tikzcd}$$

We also require the following renormalization rule.

$$
\begin{array}{ccc} a & \xrightarrow{\xi_2} & b \\ {\scriptstyle \xi_1}\big\downarrow & & \big\downarrow{\scriptstyle \xi_3} \\ c & \xrightarrow[\xi_4]{} & d \end{array}
= \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}}\;
\overline{\begin{array}{ccc} b & \xrightarrow{\tilde{\xi}_2} & a \\ {\scriptstyle \xi_3}\big\downarrow & & \big\downarrow{\scriptstyle \xi_1} \\ d & \xrightarrow[\tilde{\xi}_4]{} & c \end{array}}
= \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}}\;
\overline{\begin{array}{ccc} c & \xrightarrow{\xi_4} & d \\ {\scriptstyle \tilde{\xi}_1}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\xi}_3} \\ a & \xrightarrow[\xi_2]{} & b \end{array}}
$$

where the notation $\tilde{\xi}_j$ means the edge with its orientation reversed and $\mu(\cdot)$ denotes an entry of the Perron-Frobenius eigenvector of the adjacency matrix of each graph. Though the Perron-Frobenius eigenvector is determined only up to a positive scalar, it does not matter because $\mu(\cdot)$ always appears on denominators and numerators at the same time. Note that this is an analogue of crossing symmetry in solvable lattice model theory [1]. (That is, commuting squares arising from the higher relative commutants correspond to crossing symmetry and more general commuting squares correspond to the second inversion relations in solvable lattice model theory. See [1] and [4].)

The biunitarity axiom states the following identities.

$$
\sum_{b,\xi_2,\xi_3}
\begin{array}{ccc} a & \xrightarrow{\xi_2} & b \\ {\scriptstyle \xi_1}\big\downarrow & & \big\downarrow{\scriptstyle \xi_3} \\ c & \xrightarrow[\xi_4]{} & d \end{array}
\cdot
\begin{array}{ccc} b & \xleftarrow{\xi_2} & a \\ {\scriptstyle \xi_3}\big\downarrow & & \big\downarrow{\scriptstyle \eta_1} \\ d & \xleftarrow[\eta_4]{} & c' \end{array}
= \delta_{\xi_1,\eta_1}\delta_{\xi_4,\eta_4}\delta_{c,c'},
$$

for each fixed $a, c, d, c', \xi_1, \xi_4, \eta_1, \eta_4$, and

$$
\sum_{c,\xi_1,\xi_4}
\begin{array}{ccc} a & \xrightarrow{\xi_2} & b \\ {\scriptstyle \xi_1}\big\downarrow & & \big\downarrow{\scriptstyle \xi_3} \\ c & \xrightarrow[\xi_4]{} & d \end{array}
\cdot
\begin{array}{ccc} b' & \xleftarrow{\eta_2} & a \\ {\scriptstyle \eta_3}\big\downarrow & & \big\downarrow{\scriptstyle \xi_1} \\ d & \xleftarrow[\xi_4]{} & c \end{array}
= \delta_{\xi_2,\eta_2}\delta_{\xi_3,\eta_3}\delta_{b,b'},
$$

for each fixed $a, b, d, b', \xi_2, \xi_3, \eta_2, \eta_3$. The prefix "bi-" means that there are two kinds of unitary matrix, that is, $\xi_1 \in \mathcal{G}_1, \xi_2 \in \mathcal{G}_2, \xi_3 \in \mathcal{G}_3, \xi_4 \in \mathcal{G}_4$ and $\xi_1 \in \mathcal{G}_3, \xi_2 \in \mathcal{G}_2, \xi_3 \in \mathcal{G}_1, \xi_4 \in \mathcal{G}_4$. (The other two cases follow from these two.) This corresponds to unitarity, or the first inversion relations, in solvable lattice model theory [1]. In the rest of this paper, we mean by "connection" a biunitary connection, that is, a connection satisfying this axiom.

Remark. The renormalization convention here is slightly different from that in [16] and the same as in [17]. In [20], there is no renormalization rule, and instead, Roche has Condition T in [20, page 404]. His Condition T corresponds to the second inversion relations and it follows from our biunitarity and the renormalization rule.

We choose the distinguished point $*$ among the even vertices of $\mathcal{G}_1$. For the above system, we can construct a double sequence of string algebras starting from $*$:

$$\begin{array}{ccccccc}
A_{0,0} & \subset & A_{0,1} & \subset & \cdots & \to & A_{0,\infty} \\
\cap & & \cap & & & & \cap \\
A_{1,0} & \subset & A_{1,1} & \subset & \cdots & \to & A_{1,\infty} \\
\cap & & \cap & & & & \cap \\
\vdots & & \vdots & & & & \vdots \\
\downarrow & & \downarrow & & & & \\
A_{\infty,0} & \subset & A_{\infty,1} & \subset & \cdots & &
\end{array}$$

(See definitions in [16, page 128] or [17, II.1–2] for more details.) Here several kinds of strings are identified by a connection W. For example, there are two kinds of expressions for strings in $A_{1,1}$. The connection W induces the identification using the unitary matrix in the biunitarity axiom. (See [16, page 130], [17, II.2], or [20, page 403] for details.) A trace compatible with the above embeddings can be defined, and $A_{0,\infty}$, etc., are the GNS-completion with respect to this trace. (See [16, page 129], [17, II.1], or [20, page 404] for this trace.) This inclusion $A_{0,\infty} \subset A_{1,\infty}$ is the string model subfactor of Ocneanu.

We also write a large diagram as follows.

$$
\begin{array}{ccccccccc}
\cdot & \xrightarrow{\xi_+} & \cdot & \longrightarrow & \cdots & \cdot & \longrightarrow & \cdot \\
{\scriptstyle \eta_+}\big\downarrow & & & & & & & \big\downarrow{\scriptstyle \eta_-} \\
\cdot & & & & & & & \cdot \\
\big\downarrow & & & & & & & \big\downarrow \\
\vdots & & & & & & & \vdots \\
\cdot & & & & & & & \cdot \\
\big\downarrow & & & & & & & \big\downarrow \\
\cdot & \xrightarrow[\xi_-]{} & \cdot & \longrightarrow & \cdots & \cdot & \longrightarrow & \cdot
\end{array}
$$

This means the following. We make all the possible fillings of cells for this diagram as follows.

$$
\begin{array}{ccccccccc}
\cdot & \xrightarrow{\xi_+} & \cdot & \longrightarrow & \cdots & \cdot & \longrightarrow & \cdot \\
{\scriptstyle \eta_+}\big\downarrow & & \big\downarrow & & & \big\downarrow & & \big\downarrow{\scriptstyle \eta_-} \\
\cdot & \longrightarrow & \cdot & \longrightarrow & \cdots & \cdot & \longrightarrow & \cdot \\
\big\downarrow & & \big\downarrow & & & \big\downarrow & & \big\downarrow \\
\vdots & & \vdots & & & \vdots & & \vdots \\
\cdot & \longrightarrow & \cdot & \longrightarrow & \cdots & \cdot & \longrightarrow & \cdot \\
\big\downarrow & & \big\downarrow & & & \big\downarrow & & \big\downarrow \\
\cdot & \xrightarrow[\xi_-]{} & \cdot & \longrightarrow & \cdots & \cdot & \longrightarrow & \cdot
\end{array}
$$

Such a choice is called a configuration. We multiply the connection values of all the cells in a configuration and sum them over all the configurations. This is the value assigned to the above large diagram, and we mean this value by the diagram.

This is an analogue of a partition function in solvable lattice model theory. If we need to specify a connection W explicitly, we write $W(\cdot)$ outside of the diagram.

Note that by our convention we get

$$\begin{tikzcd} a \arrow[r] & b \\ c \arrow[u] \arrow[r] & d \arrow[u] \end{tikzcd} = \begin{tikzcd} a \arrow[d] & b \arrow[l] \arrow[d] \\ c & d \arrow[l] \end{tikzcd} = \sqrt{\frac{\mu(a)\mu(d)}{\mu(b)\mu(c)}} \begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd}.$$

(We drop labels for edges if no confusion arises.) If we apply this rule to a large diagram, the coefficients $\sqrt{\mu(\cdot)}$ cancel out except for the terms for four corners. That is, if we reverse the orientations of all the horizontal [resp. vertical] edges, the connection value for the diagram changes by a positive scalar depending on the four corners. This fact will be used in the rest of this paper very frequently.

Ocneanu has defined the Galois functor in [16, 17] which assigns a biunitary connection on a graph to each subfactor with finite index. In this construction, the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are the same and also $\mathcal{G}_3$ and $\mathcal{G}_4$ are the same. In the case where the index is less than 4, we moreover have all the four graphs are the same, and it is one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8. (It is the principal graph of the subfactor.)

On the vertical string algebras of the graph $\mathcal{G}_1$, we define the n-th Jones projection by the following formula as in [17, II.3].

$$e_n = \sum_{\substack{|\alpha|=n-1\\|v|=|w|=1}} \frac{\mu(r(v))^{1/2}\mu(r(w))^{1/2}}{\beta\mu(r(\alpha))}(\alpha\cdot v\cdot\tilde{v}, \alpha\cdot w\cdot\tilde{w}),$$

where β is the Perron-Frobenius eigenvalue of the graph, α is any path from $*$, and v, w are chosen so that the compositions are possible, and $|\cdot|$ denote the length of a path. This satisfies the ordinary properties of the Jones projection in [10].

This e_n is in $A_{n+1,0}$. If we embed this into $A_{n+1,k}$ and move it by identification using W, we get the following form.

$$\sum_x \sum_{\substack{s(\xi)=*,r(\xi)=x\\|\xi|=k}} (\xi,\xi)\cdot \sum_{\substack{|\alpha|=n-1,s(\alpha)=x\\|v|=|w|=1}} \frac{\mu(r(v))^{1/2}\mu(r(w))^{1/2}}{\beta\mu(r(\alpha))}(\alpha\cdot v\cdot\tilde{v},\alpha\cdot w\cdot\tilde{w}),$$

where x is any even [resp. odd] vertex of $\mathcal{G}_2$ when k is even [resp. odd], v, w are chosen so that the compositions are possible, ξ is any horizontal path, and α, v, w are vertical paths. This statement can be proved by biunitarity, the renormalization rule, and graphical method, because the coefficients of the definition of the Jones projection are exactly ones in the renormalization rule. (See [17, II.5] for a proof. Also see [20, Appendix] for the graphical method.) The above form means that each e_n commutes with all the horizontal strings. This fact is quite important and will be used in this paper repeatedly. Similar statement holds for the horizontal Jones projections.

§2 Flatness of biunitary connections.

A biunitary connection arising from a subfactor via Galois functor satisfies another important condition called flatness. We explain this key notion "flatness" of Ocneanu's entire theory in this section. In this section and the next, there are some overlaps with writing of the author in [17], because the author thinks that it will be helpful for the reader. Consider a connection and choose the two distinguished points $*$ among the even vertices of $\mathcal{G}_1$ and $\mathcal{G}_3$ respectively. The connection (with the choice of $*$) is said to be *flat* if it satisfies one of the following equivalent conditions for the the string algebra double sequence and that with $\mathcal{G}_1$ and $\mathcal{G}_3$ interchanged. (In this paper, we work mainly on the Dynkin diagrams, and then $\mathcal{G}_1 = \mathcal{G}_3$.)

Theorem 2.1. *The following conditions are equivalent.*

(1) *In the string algebra double sequence, any two elements $x \in A_{\infty,0}$, the vertical string algebra, and $y \in A_{0,\infty}$, the horizontal string algebra, commute.*

(2) *For each vertical string $\rho = (\rho_+, \rho_-) \in A_{k,0}$, we get*

$$
\begin{array}{ccccc}
* & \xrightarrow{\xi} & \cdots & \longrightarrow & \cdot \\
{\scriptstyle \rho_+}\Big\downarrow & & & & \Big\downarrow{\scriptstyle \sigma_+} \\
\vdots & & & & \vdots \\
\Big\downarrow & & & & \Big\downarrow \\
\cdot & & & & \cdot \\
\Big\uparrow & & & & \Big\uparrow \\
\vdots & & & & \vdots \\
{\scriptstyle \rho_-}\Big\uparrow & & & & \Big\uparrow{\scriptstyle \sigma_-} \\
* & \xrightarrow[\eta]{} & \cdots & \longrightarrow & \cdot
\end{array}
= \delta_{\xi,\eta} C_{\rho,\sigma},
$$

where $C_{\rho,\sigma} \in \mathbf{C}$ depends only on $\rho, \sigma = (\sigma_+, \sigma_-)$.

(2)′ *For each horizontal string* $\rho = (\rho_+, \rho_-) \in A_{0,k}$*, we get*

$$\begin{array}{ccccccccc}
* & \xrightarrow{\rho_+} & \cdots & \longrightarrow & \cdot & \longleftarrow & \cdots & \xleftarrow{\rho_-} & * \\
{\scriptstyle\xi}\big\downarrow & & & & & & & & \big\downarrow{\scriptstyle\eta} \\
\vdots & & & & & & & & \vdots \\
\big\downarrow & & & & & & & & \big\downarrow \\
\cdot & \xrightarrow[\sigma_+]{} & \cdots & \longrightarrow & \cdot & \longleftarrow & \cdots & \xleftarrow[\sigma_-]{} & \cdot
\end{array} = \delta_{\xi,\eta} C_{\rho,\sigma},$$

where $C_{\rho,\sigma} \in \mathbf{C}$ *depends only on* $\rho, \sigma = (\sigma_+, \sigma_-)$*.*

(3) *For any horizontal paths* ξ_+, ξ_- *and vertical paths* η_+, η_- *with all the sources and ranges equal to* $*$*, we get*

$$\begin{array}{ccccc}
* & \xrightarrow{\xi_+} & \cdots & \longrightarrow & * \\
{\scriptstyle\eta_+}\big\downarrow & & & & \big\downarrow{\scriptstyle\eta_-} \\
\vdots & & & & \vdots \\
\big\downarrow & & & & \big\downarrow \\
* & \xrightarrow[\xi_-]{} & \cdots & \longrightarrow & *
\end{array} = \delta_{\xi_+,\xi_-} \delta_{\eta_+,\eta_-}.$$

(3)′ *For any horizontal path* ξ *and vertical path* η *with* $s(\xi) = r(\xi) = s(\eta) = r(\eta) = *$*, we get*

$$\begin{array}{ccccc}
* & \xrightarrow{\xi} & \cdots & \longrightarrow & * \\
{\scriptstyle\eta}\big\downarrow & & & & \big\downarrow{\scriptstyle\eta} \\
\vdots & & & & \vdots \\
\big\downarrow & & & & \big\downarrow \\
* & \xrightarrow[\xi]{} & \cdots & \longrightarrow & *
\end{array} = 1.$$

Proof. (1) $\Leftrightarrow$ (2) : Let $m = |\xi| = |\eta|$ in (2). The condition (2) exactly means $\rho \in A_{k,m} \cap A'_{0,m}$ via identification using the connection. Because k, m are arbitrary, we get equivalence of (1) and (2).

(1) $\Leftrightarrow$ (2)$'$: Same as above.

(2), (2)$'$ $\Rightarrow$ (3) : Suppose $\eta_+ \neq \eta_-$. Because $s(\xi_+) = r(\xi_+) = s(\xi_-) = r(\xi_-) = *$, we can write $\xi_+ = \rho_+ \cdot \tilde{\rho}_-$ and $\xi_- = \sigma_+ \cdot \tilde{\sigma}_-$ with $s(\rho_+) = s(\rho_-) = s(\sigma_+) = s(\sigma_-) = *$. Then by (2)$'$, we get

$$
\begin{tikzcd}
* \arrow[r, "\rho_+"] \arrow[d, "\eta_+"'] & \cdots \arrow[r] & \cdot & \cdots \arrow[l] & * \arrow[l, "\rho_-"'] \arrow[d, "\eta_-"] \\
\vdots \arrow[d] & & & & \vdots \arrow[d] \\
\cdot \arrow[r, "\sigma_+"'] & \cdots \arrow[r] & \cdot & \cdots \arrow[l] & \cdot \arrow[l, "\sigma_-"]
\end{tikzcd}
= 0.
$$

This implies the left hand side of the identity in (3) is 0. Similarly, if $\xi_+ \neq \xi_-$, then the formula is 0.

Suppose $\xi_+ = \xi_-$ and $\eta_+ = \eta_-$. Write $\xi = \rho_+ \cdot \tilde{\rho}_-$ with $s(\rho_+) = s(\rho_-) = *$ as above. Set

$$
C = \begin{tikzcd}
* \arrow[r, "\rho_+"] \arrow[d, "\eta_+"'] & \cdots \arrow[r] & \cdot & \cdots \arrow[l] & * \arrow[l, "\rho_-"'] \arrow[d, "\eta_-"] \\
\vdots \arrow[d] & & & & \vdots \arrow[d] \\
\cdot \arrow[r, "\rho_+"'] & \cdots \arrow[r] & \cdot & \cdots \arrow[l] & \cdot \arrow[l, "\rho_-"]
\end{tikzcd}.
$$

By (2)$'$, this C does not depend on $\eta_+ (= \eta_-)$. Let $p = \sum_{\zeta_1, \zeta_2} (\zeta_1, \zeta_1) \cdot (\zeta_2, \zeta_2)$, where ζ_1, ζ_2 are any vertical path and horizontal path with $s(\zeta_1) = r(\zeta_1) = s(\zeta_2) =$

$*$, $r(\zeta_2) = r(\rho_+)$, $|\zeta_1| = |\eta_+|$, $|\zeta_2| = |\rho_+|$. This p is a projection commuting with ρ and we know that $p\rho = C\sum_\zeta (\zeta, \zeta) \cdot \rho$, where ζ is any vertical path with $s(\zeta) = r(\zeta) = *$, $|\zeta| = |\eta_+|$. Because ρ is a partial isometry, we get $|C| = 1, 0$. Because C does not depend on the choice of η_+, η_-, we set $\eta_+ = \eta_- = \sigma \cdot \tilde{\sigma}$, where σ is any path. Then we get $C \geq 0$, hence $C = 1, 0$. If $C = 0$, we get

$$
\begin{array}{ccccccccc}
* & \xrightarrow{\rho_+} & \cdots & \longrightarrow & \cdot & \longleftarrow & \cdots & \xleftarrow{\rho_-} & * \\
\sigma\Big\downarrow & & & & & & & & \Big\downarrow\sigma \\
\vdots & & & & & & & & \vdots, \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
\cdot & \longrightarrow & \cdots & \longrightarrow & \cdot & \longleftarrow & \cdots & \longleftarrow & \cdot
\end{array}
$$

for all σ with $|\sigma| = |\eta_+|/2$ and $s(\sigma) = *$. This means that a non-zero element ρ is identified with 0 via connection, which is a contradiction. Thus we get $C = 1$, which immediately implies the desired conclusion.

$(3) \Rightarrow (2)$: Suppose $\xi \neq \eta$ in the formula in (2). Then applying condition (3) to the diagram

$$
\begin{array}{ccccccccc}
* & \xrightarrow{\xi} & \cdots & \longrightarrow & \cdot & \longleftarrow & \cdots & \xleftarrow{\xi} & * \\
\rho_+\Big\downarrow & & & & & & & & \Big\downarrow\rho_+ \\
\vdots & & & & & & & & \vdots \\
\cdot & & & & & & & & \cdot, \\
\vdots & & & & & & & & \vdots \\
\tilde{\rho}_-\Big\downarrow & & & & & & & & \Big\downarrow\tilde{\rho}_- \\
* & \xrightarrow[\eta]{} & \cdots & \longrightarrow & \cdot & \longleftarrow & \cdots & \xleftarrow[\eta]{} & *
\end{array}
$$

we get the left hand side of the identity in (2) is 0 for all σ_+, σ_-. Now we fix ξ, η with $s(\xi) = s(\eta) = *$, $r(\xi) = r(\eta)$. Then it is enough to show that

$$\sum_{\sigma_+,\sigma_-}\left|\begin{array}{ccccc} * & \xrightarrow{\xi} & \cdots & \longrightarrow & \cdot \\ \rho_+\big\downarrow & & & & \big\downarrow\sigma_+ \\ \vdots & & & & \vdots \\ \big\downarrow & & & & \big\downarrow \\ \cdot & & & & \cdot \\ \big\downarrow & & & & \big\downarrow \\ \vdots & & & & \vdots \\ \tilde{\rho}_-\big\downarrow & & & & \big\downarrow\tilde{\sigma}_- \\ * & \xrightarrow[\xi]{} & \cdots & \longrightarrow & \cdot \end{array} - \begin{array}{ccccc} * & \xrightarrow{\eta} & \cdots & \longrightarrow & \cdot \\ \rho_+\big\downarrow & & & & \big\downarrow\sigma_+ \\ \vdots & & & & \vdots \\ \big\downarrow & & & & \big\downarrow \\ \cdot & & & & \cdot \\ \big\downarrow & & & & \big\downarrow \\ \vdots & & & & \vdots \\ \tilde{\rho}_-\big\downarrow & & & & \big\downarrow\tilde{\sigma}_- \\ * & \xrightarrow[\eta]{} & \cdots & \longrightarrow & \cdot \end{array}\right|^2 = 0.$$

Expanding the left hand side, we get $1 + 1 - 1 - 1 = 0$. Indeed, for example, the first 1 is obtained as

$$\begin{array}{ccccccccc} * & \xrightarrow{\xi} & \cdots & \longrightarrow & \cdot & \longleftarrow & \cdots & \xleftarrow{\xi} & * \\ \rho_+\big\downarrow & & & & & & & & \big\downarrow\rho_+ \\ \vdots & & & & & & & & \vdots \\ \cdot & & & & & & & & \cdot \\ \vdots & & & & & & & & \vdots \\ \tilde{\rho}_-\big\downarrow & & & & & & & & \big\downarrow\tilde{\rho}_- \\ * & \xrightarrow[\xi]{} & \cdots & \longrightarrow & \cdot & \longleftarrow & \cdots & \xleftarrow[\xi]{} & * \end{array}$$

$(3) \Rightarrow (3)'$: Trivial.

$(3)' \Rightarrow (3)$: Fix η_+, ξ_-. By unitarity, the sum of the squares of the absolute values of the left hand side of the identity in (3) for all ξ_+, η_- is 1. By $(3)'$, all the terms except for $\xi_+ = \xi_-$ and $\eta_- = \eta_+$ are zero. Q.E.D.

Remark 2.2. Condition (3) was used as the definition of flatness by Ocneanu in [16, page 153] under the name "parallel transport axiom". Condition (3) can be written in the following form:

$$
\begin{array}{ccccccccc}
* & \xrightarrow{\rho_+} & \cdots & \longrightarrow & \cdot & \longleftarrow & \cdots & \xleftarrow{\rho_-} & * \\
{\scriptstyle \xi}\big\downarrow & & & & & & & & \big\downarrow{\scriptstyle \eta} \\
\vdots & & & & & & & & \vdots \\
\big\downarrow & & & & & & & & \big\downarrow \\
\cdot & \xrightarrow[\sigma_+]{} & \cdots & \longrightarrow & \cdot & \longleftarrow & \cdots & \xleftarrow[\sigma_-]{} & \cdot
\end{array}
= \delta_{\xi,\eta}\delta_{\rho,\sigma},
$$

where $\rho = (\rho_+, \rho_-)$ and $\sigma = (\sigma_+, \sigma_-)$. This means the string ρ does not change its form in the transport. (See [17, II.5] for definitions of transport.) This is the reason Ocneanu calls this condition flatness in analogy to the flatness in differential geometry. Condition (2) was mentioned in [16, page 128] and equivalence between (2) and (3) was mentioned in [16, page 154] without proof. Condition (1) is by the author and was inserted in [17, II.5] by the author. Note that flatness depends on the choice of $*$.

Significance of flatness in the theory of operator algebras is as follows. If we apply the Galois functor to a subfactor $N \subset M$ to get a graph and a connection and apply the string algebra construction, the higher relative commutant $N' \cap M_k$ is contained in $A_{k+1,0}$ by Ocneanu's compactness argument [17, II.6]. But we get the equality $N' \cap M_k = A_{k+1,0}$ by counting the dimensions of the both hand sides. (Note that we have an anti-isomorphism as in [16, page 135].) This is flatness. Similarly we get the other flatness for the case $\mathcal{G}, \mathcal{H}$ interchanged.

Moreover, if we have a (not necessarily flat) connection on a finite graph, we still can construct a subfactor by the string algebra construction, and then the tower of the relative commutants of this subfactor is obtained as the "flat part" of the string algebras. (See [17, II.6] for more precise statement and a proof.) In particular, if we start with a flat connection, we get back the original graph as an invariant. This fact was stated as the Range Theorem in [17].

The vertical [resp. horizontal] Jones projections commute with horizontal [resp. vertical] strings as noted in §1. Thus for the Jones projections, condition (1) is satisfied. This fact is referred to as flatness of the Jones projections. The depth of a graph with $*$ is defined to be the biggest distance from $*$ to a vertex on the graph. It is well known that if $k \geq$ depth, then $A_{k+1,0}$ is generated by $A_{k,0}$ and the k-th vertical Jones projection. (A similar statement holds for $A_{0,k}$.) Thus, for checking flatness, it is enough to check the identities in (2) for the diagrams with size less than (depth) $\times$ 2(depth), and similarly it is enough to check the identities in (3) for the diagrams with size less than 2(depth) $\times$ 2(depth). This implies that whether a given connection on a finite graph is flat or not can be determined by finite times of computations. Moreover, the Bratteli diagram for $A_{k,0} \subset A_{k+1,0}$ consists of a reflection of that for $A_{k-1,0} \subset A_{k,0}$ and a new part. Because the reflection part is spanned by $A_{k,0}$ and the Jones projection, we only need to check flatness for the new part. This is what the remark in [17, page 154] means.

We show an example of a flat connection corresponding to finite groups. (See [17, I.3] and [16, page 142].)

Example 2.3. Let G be a finite group and n its order. Then we get a flat connection from a subfactor $\mathcal{R} \subset \mathcal{R} \rtimes G$, where $\mathcal{R}$ is the AFD factor of type II_1 and we make a crossed product from an outer action of G. This connection is described as follows.

The graph $\mathcal{G}_1 = \mathcal{G}_2$ has a single odd vertex x and its has n even vertices, which are labeled by $g \in G$. The vertex labeled by 1, the multiplication unit, is the vertex $*$. Each even vertex is connected to x by a single edge. The graph $\mathcal{G}_3 = \mathcal{G}_4$ has a single odd vertex x and its even vertices are labeled by (equivalence classes of) irreducible representations of G. Each even vertex σ is connected to x by $|\sigma|$ edges, where $|\sigma|$ denotes the dimension of the irreducible representation σ. The Perron-Frobenius eigenvector μ is given by $\mu(x) = \sqrt{n}, \mu(g) = 1, \mu(\sigma) = |\sigma|$, and the connection is given by

$$\begin{tikzcd} g \arrow[r] \arrow[d] & x \arrow[d, "j"] \\ x \arrow[r, "i"'] & \sigma \end{tikzcd} = \sigma_{ij}(g).$$

It is easy to see this satisfies biunitarity axiom. Take the following 2×2-cell.

$$\begin{tikzcd} g_1 \arrow[r] \arrow[d] & x \arrow[r] & g_2 \arrow[d] \\ x \arrow[d] & & x \arrow[d] \\ g_4 \arrow[r] & x \arrow[r] & g_3 \end{tikzcd}$$

If we fix a center vertex σ for this diagram, the sum of the products of connection values for all the configurations is given by

$$\begin{aligned}
&\sum_{i,j,k,l} \sigma_{ij}(g_1)\overline{\sigma_{kj}(g_2)}\sigma_{kl}(g_3)\overline{\sigma_{il}(g_4)} \cdot \frac{|\sigma|}{n} \\
=&\sum_{i,j,k,l} \sigma_{ij}(g_1)\sigma_{jk}(g_2^{-1})\sigma_{kl}(g_3)\sigma_{li}(g_4^{-1}) \cdot \frac{|\sigma|}{n} \\
=&\frac{|\sigma|}{n}\mathrm{Tr}(\sigma(g_1 g_2^{-1} g_3 g_4^{-1})).
\end{aligned}$$

Now we let σ vary, then we get the value

$$\sum_{\sigma} \frac{|\sigma|}{n}\mathrm{Tr}(\sigma(g_1 g_2^{-1} g_3 g_4^{-1})) = \delta_{g_1 g_2^{-1} g_3 g_4^{-1},1}$$

for the above diagram by orthogonal relations in representation theory of finite groups. Because our graph has depth 2, we only need to check the following identities for all $g, h, k, l \in G$ with $h \neq k$ in order to verify flatness.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & x & \longrightarrow & g & \longleftarrow & x & \longleftarrow & 1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
x & & & & & & & & x \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
h & \longrightarrow & x & \longrightarrow & l & \longleftarrow & x & \longleftarrow & k
\end{array} = 0.$$

This easily follows from the above computation for 2×2-cells. If we interchange the two graphs, a similar computation works. Thus this connection is flat.

§3 Biunitary connections on the Dynkin diagrams.

The contents of this section are essentially included in [16, 17]. We just work out details for the Dynkin diagram cases. With Ocneanu's general method in [17], the materials here are rather easy.

In this section and later, all the four graphs in the string algebra construction are the same and one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8.

Ocneanu has claimed the following theorem in [16]. (Definition of equivalence of connections is given in [16, 17].)

Theorem 3.1. *On each A_n, there is only one connection up to equivalence. On each D_n, there are two connections up to equivalence, and these two are equivalent up to a graph isomorphism, the flip of the fork. On each of E_6, E_7, E_8, there are two connections up to equivalence.*

Here we give details of its proof. As pointed out by Ocneanu, the point is that the biunitarity axiom is strong enough to determine the connection if size of the unitary matrix is less than or equal to 3, that is, the graph has at most triple points. By this general remark, we can conclude that the number of equivalence classes of biunitary connections on each of the Dynkin diagrams is at most two. On the other hand, Ocneanu has shown two connections explicitly on each of the Dynkin diagrams. The only problem we have to work is that these two are equivalent or not.

Proof of Theorem 3.1. First we show existence of a connection. As in [16, page 159] and [17, IV.2], we set

$$\begin{array}{ccc} i & \longrightarrow & l \\ \Big\downarrow & & \Big\downarrow \\ k & \longrightarrow & j \end{array} = \delta_{kl}\varepsilon + \sqrt{\frac{\mu(k)\mu(l)}{\mu(i)\mu(j)}}\delta_{ij}\bar{\varepsilon},$$

where $\varepsilon = \sqrt{-1}\exp\dfrac{\pi\sqrt{-1}}{2N}$ and N is the Coxeter number. (Note that we fix an identification of the four graphs, though it is not canonical.) It is easy to see that this is indeed a connection. (For unitarity, we use $\varepsilon^2 + \bar\varepsilon^2 + \beta = 0$, where β is the Perron-Frobenius eigenvalue. See [17, IV.2].) Changing ε to $\bar\varepsilon$, we get another solution. (As noted in [17], this solution works even when $\beta = 2$ if we set $N = \infty, \varepsilon = \sqrt{-1}$.)

For A_n, we can determine the connection up to gauge choice, which is the equivalence relation, one by one from the endpoint as in [17, IV.2]. This procedure works until one meets a triple point. (It is an advantage of Ocneanu's approach over the commuting square picture that one-by-one construction like this is possible.)

Let

$$\begin{array}{ccc} b_1 — & a & —b_3 \cdots \\ & | & \\ & b_2 & \\ & \vdots & \end{array}$$

be one of the graphs D_n, E_6, E_7, E_8, that is the vertex a is the triple point. By an appropriate choice of gauges, we have a 3×3-unitary matrix

$$\begin{pmatrix} \dfrac{\mu(b_1)}{\mu(a)} & \dfrac{\sqrt{\mu(b_1)\mu(b_2)}}{\mu(a)} & \dfrac{\sqrt{\mu(b_1)\mu(b_3)}}{\mu(a)} \\ \dfrac{\sqrt{\mu(b_1)\mu(b_2)}}{\mu(a)} & x & y \\ \dfrac{\sqrt{\mu(b_1)\mu(b_3)}}{\mu(a)} & y & z \end{pmatrix},$$

where $|y| = \dfrac{\sqrt{\mu(b_2)\mu(b_3)}}{\mu(a)}$, $x, y, z \in \mathbf{C}$. (See [17, IV.2, IV.4].) The problem is whether a choice of x, y, z is possible or not. Setting $x_i = \dfrac{\mu(b_i)}{\mu(a)}$, $i = 1, 2, 3$, we get

$|x| = \sqrt{1 - x_2(x_1 + x_3)}, |y| = \sqrt{x_2 x_3}$ by unitarity. Because the other condition for unitarity is

$$x_1\sqrt{x_2} + \sqrt{x_2}x + \sqrt{x_3}y = 0,$$

the number of equivalence classes of possible unitary matrices is 2 if and only if we have

$$\sqrt{1 - x_2(x_1 + x_3)} + x_3 > x_1 > |x_3 - \sqrt{1 - x_2(x_1 + x_3)}|.$$

For E_6, E_7, E_8, we can check this inequality by direct computation. For D_n, we can set $\mu(b_1) = \mu(b_2) = 1/2$, $\mu(a) = \beta/2$, $\mu(b_3) = \beta^2/2 - 1$, where β is the Perron-Frobenius eigenvalue of the D_n, that is, $\beta = 2\cos\dfrac{\pi}{2n-2}$. The above inequality in this case is

$$\sqrt{1 - \frac{1}{\beta}\left(\frac{1}{\beta} + \frac{\beta^2 - 1}{\beta}\right)} + \frac{\beta^2 - 2}{\beta} > \frac{1}{\beta} > \frac{\sqrt{\beta^2 - 3}}{\beta},$$

and this is valid because $\beta^2 > 2$. Thus all of D_n, E_6, E_7, E_8 have two solutions at the triple point. Because we already know existence of a connection, we conclude that each of the above indeed has two mutually conjugate connections. We have exhibited two mutually conjugate solutions above, so these are the only connections up to the equivalence relation.

Now for proving the assertion on the flip of D_n. We choose the other endpoint than the two endpoints on the fork as the starting point and apply the method of determining a connection in the A_n diagrams until one meets the triple point. We can choose a real connection there. Then at the triple point, we have a 3×3-unitary matrix again as above, though (1,1)-entry is different from the above now. Then we get $x = \bar{y} = z$ in the above 3×3-matrix. Thus two connections are the same up to the flip. Q.E.D.

We will use the notation W for the above connection on the Dynkin diagrams.

Remark 3.2. If all the four graphs in the string algebra construction are the same, the contragredient map is trivial, the graph has a triple point, the graph has no cycles, and the Perron-Frobenius eigenvalue is bigger than 2, there are no connections on it. This is due to Ocneanu, and his proof is given in [17, IV.4].

Remark 3.3. If the graph is one of the extended Dynkin diagrams $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$, the above method still applies. In these cases, we get an equality instead of the strict inequality in the above proof. Thus we have only one biunitary connection on each graph, hence the number of subfactors with principal graphs $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ is at most one. This is an unpublished result of Ocneanu, which is used in [19, Corollaire 1 (iii)].

In the rest of this paper, we will work on flatness of the connections on the Dynkin diagrams. The results suggest that flatness is a very strong condition while biunitarity is not so strong if graphs are appropriately chosen. In a subfactor context, it means that constructions of irreducible subfactors with finite index are much easier than those of subfactors with finite (or amenable) depth.

§4 Properties of connections on the Dynkin diagrams A_n

The string algebras for A_n with one of the endpoints to be $*$ are generated by the Jones projections as noted by the original work of Jones [10, §5.2], thus flatness is trivial, and corresponding subfactor classification is already noted by Popa [18, Corollary 6.7].

If we choose a point which is not an endpoint as $*$, the connection is not flat. Indeed, suppose it is flat. Then the string algebra double sequence gives a subfactor $N \subset M$ with Jones index $[M:N] = 4\cos^2 \dfrac{\pi}{N} < 4$, where N is the Coxeter number. (See [17, II.2].) Let k be the number of vertices connected to $*$. Now we have $k > 1$ because $*$ is not an endpoint. By flatness of the connection, the relative commutant is given by $N' \cap M = \mathbf{C}^k$. (This follows from Ocneanu's computation of the tower of relative commutants based on compactness argument. See [17, II.6] for the proof.) But it was proved by Jones [10, Corollary 2.2.4], as well-known, that if $[M:N] < 4$, then $N' \cap M = \mathbf{C}$, which is a contradiction. This argument also works for D_n, E_6, E_7, E_8. We get the following theorem.

Theorem 4.1. *Each A_n has one flat connection. Though there are two choice of $*$, the two are isomorphic up to a graph isomorphism. Thus there is only one subfactor having A_n as its principal graph for each n.*

This statement for subfactors is already obtained in [18, Corollary 6.7]. Thus it is over about A_n. But in the next section, we will reduce the problem of flatness of the diagrams D_n to the A_n diagrams by orbifold method, and need a technical equality for connections on the A_n diagrams. We state the equality and prove it here. This is the most technical part of this paper as well as the next section and will show why the difference between D_{even} and D_{odd} arises.

We use the following numbering of the vertices of the diagram A_n and let W be the connection on it as defined in §3.

$$A_n : \qquad 0 — 1 \cdots n-2 — n-1$$

We will need the following proposition in the next section for D_n.

Proposition 4.2. *For each connection W on A_{2m+1}, we have the following equality.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 1 & \longrightarrow \cdots \longrightarrow & 2m-1 & \longrightarrow & 2m & & \\
\downarrow & & & & & & \downarrow & & \\
1 & & & & & & 2m-1 & & \\
\downarrow & & & & & & \downarrow & & \\
\vdots & & & & & & \vdots & & = (-1)^m. \\
\downarrow & & & & & & \downarrow & & \\
2m-1 & & & & & & 1 & & \\
\downarrow & & & & & & \downarrow & & \\
2m & \longrightarrow & 2m-1 & \longrightarrow \cdots \longrightarrow & 1 & \longrightarrow & 0 & &
\end{array}$$

The left hand side here is given by a complicated formula involving many complex numbers. To determine its value, we first show that the value has modulus 1, then we show that the value is real, and finally we determine the sign. We need a lemma at first.

Lemma 4.3. *The connection W on A_n can be written as follows.*

$$\begin{tikzcd} j \arrow[r] \arrow[d] & j \pm 1 \arrow[d] \\ j \mp 1 \arrow[r] & j \end{tikzcd} = \frac{\sqrt{\mu(j+1)\mu(j-1)}}{\mu(j)}\bar{\varepsilon}, \qquad j = 1, \dots, n-2,$$

$$\begin{tikzcd} j \arrow[r] \arrow[d] & j + 1 \arrow[d] \\ j + 1 \arrow[r] & j \end{tikzcd} = (-1)^{j+1}\frac{\mu(0)}{\mu(j)}\bar{\varepsilon}^{2j+3}, \qquad j = 0, \dots, n-2,$$

$$\begin{tikzcd} j \arrow[r] \arrow[d] & j - 1 \arrow[d] \\ j - 1 \arrow[r] & j \end{tikzcd} = (-1)^{j}\frac{\mu(0)}{\mu(j)}\varepsilon^{2j+1}, \qquad j = 1, \dots, n-1,$$

$$\begin{tikzcd} j \pm 1 \arrow[r] \arrow[d] & j \arrow[d] \\ j \arrow[r] & j \mp 1 \end{tikzcd} = \varepsilon, \qquad j = 1, \dots, n-2,$$

Proof. Just by direct computation using $\mu(j) = \sin \dfrac{j+1}{N}$, where $N = n+1$, the Coxeter number. Q.E.D.

We would like to change W to be real by gauge choice. The following choice is used.

Lemma 4.4. *For each edge e on A_n, we assign a complex number $\varphi(e)$ as follows.*

$$\begin{aligned}
\varphi(j+1 \longrightarrow j) &= \delta^{-2j-3}, \qquad & j &= 0,\ldots,n-2 \\
\varphi(j \longrightarrow j-1) &= \delta^{-2j-1}, \qquad & j &= 1,\ldots,n-1 \\
\varphi(j-1 \longrightarrow j) &= \delta^{2j+1}, \qquad & j &= 1,\ldots,n-1 \\
\varphi(j \longrightarrow j+1) &= \delta^{2j+3}, \qquad & j &= 0,\ldots,n-2,
\end{aligned}$$

where $\delta = \exp(\pi\sqrt{-1}/4N)$, $N = n+1$. Then this satisfies $\varphi(\tilde{e}) = \overline{\varphi(e)}$, where $\tilde{e}$ is the edge e with the orientation reversed. And a new connection W' defined by

$$W'\begin{pmatrix} i & \longrightarrow & l \\ \big\downarrow & & \big\downarrow \\ k & \longrightarrow & j \end{pmatrix} = \varphi(l \longrightarrow i)\varphi(k \longrightarrow j) W \begin{pmatrix} i & \longrightarrow & l \\ \big\downarrow & & \big\downarrow \\ k & \longrightarrow & j \end{pmatrix}$$

is real and satisfies the following.

$$\begin{matrix} j & \longrightarrow & j\pm 1 \\ \big\downarrow & & \big\downarrow \\ j\mp 1 & \longrightarrow & j \end{matrix} = \frac{\sqrt{\mu(j+1)\mu(j-1)}}{\mu(j)}, \qquad j = 1,\ldots,n-2,$$

$$\begin{matrix} j & \longrightarrow & j+1 \\ \big\downarrow & & \big\downarrow \\ j+1 & \longrightarrow & j \end{matrix} = (-1)^{j+1}\frac{\mu(0)}{\mu(j)}, \qquad j = 0,\ldots,n-2,$$

$$\begin{matrix} j & \longrightarrow & j-1 \\ \big\downarrow & & \big\downarrow \\ j-1 & \longrightarrow & j \end{matrix} = (-1)^{j}\frac{\mu(0)}{\mu(j)}, \qquad j = 1,\ldots,n-1,$$

$$
\begin{array}{ccc}
j \pm 1 & \longrightarrow & j \\
\Big\downarrow & & \Big\downarrow \\
j & \longrightarrow & j \mp 1
\end{array}
= 1, \qquad j = 1, \dots, n-2,
$$

Proof. Direct computation using $\delta^2 = \varepsilon$ shows the desired result easily. Q.E.D.

The reason for the choice of the above W' is as follows.

Lemma 4.5. *We have the following equality.*

$$
\begin{aligned}
&W \left(
\begin{array}{ccccccccc}
0 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & 2m-1 & \longrightarrow & 2m \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
1 & & & & & & & & 2m-1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
\vdots & & & & & & & & \vdots \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
2m-1 & & & & & & & & 1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots & \longrightarrow & 1 & \longrightarrow & 0
\end{array}
\right) \\
&=(-1)^m W' \left(
\begin{array}{ccccccccc}
0 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & 2m-1 & \longrightarrow & 2m \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
1 & & & & & & & & 2m-1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
\vdots & & & & & & & & \vdots \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
2m-1 & & & & & & & & 1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots & \longrightarrow & 1 & \longrightarrow & 0
\end{array}
\right).
\end{aligned}
$$

Proof. For each configuration, values of φ inside of the big square cancel out. Thus we only need to compute the product of φ for edges on the top and the bottom horizontal edges. It is equal to

$$\begin{aligned}
&\varphi(2m \longrightarrow 2m-1)^2 \cdots \varphi(1 \longrightarrow 0)^2 \\
=&\delta^{2((-4m-1)+\cdots+(-7)+(-5)+(-3))} = \delta^{-8m(m+1)} \\
=&\exp(-8m(m+1)\pi\sqrt{-1}/8(m+1)) \\
=&(-1)^m,
\end{aligned}$$

because the Coxeter number is $2m+2$. Q.E.D.

The above lemma shows why the factor $(-1)^m$ arises in Proposition 4.2. Now all we have to prove for Proposition 4.2 is the equality

$$W'\left(\begin{array}{ccccccc}
0 & \longrightarrow & 1 & \longrightarrow \cdots \longrightarrow & 2m-1 & \longrightarrow & 2m \\
\downarrow & & & & & & \downarrow \\
1 & & & & & & 2m-1 \\
\downarrow & & & & & & \downarrow \\
\vdots & & & & & & \vdots \\
\downarrow & & & & & & \downarrow \\
2m-1 & & & & & & 1 \\
\downarrow & & & & & & \downarrow \\
2m & \longrightarrow & 2m-1 & \longrightarrow \cdots \longrightarrow & 1 & \longrightarrow & 0
\end{array}\right) = 1$$

We prove the following lemma, which is a stronger version of the proposition. (Indeed, the following lemma with $k=0$ implies the proposition because of the renormalization rule and the equality $\mu(2m) = \mu(0)$.

Lemma 4.6. *For the connection W' on A_{2m}, we have the equality*

$$W' \left(\begin{array}{ccccccccc} 2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots & \longrightarrow & k+1 & \longrightarrow & k \\ \downarrow & & & & & & & & \downarrow \\ 2m-1 & & & & & & & & k+1 \\ \downarrow & & & & & & & & \downarrow \\ \vdots & & & & & & & & \vdots \\ \downarrow & & & & & & & & \downarrow \\ k+1 & & & & & & & & 2m-1 \\ \downarrow & & & & & & & & \downarrow \\ k & \longrightarrow & k+1 & \longrightarrow & \cdots & \longrightarrow & 2m-1 & \longrightarrow & 2m \end{array} \right) = 1.$$

(The diagram is of size $(2m-k) \times (2m-k)$.)

We will prove this lemma by induction on k. For it, we need two lemmas.

Lemma 4.7. *For any connection on any graph, we get the following inequality.*

$$\left| \begin{array}{ccccccc} & & \cdot & \xrightarrow{\xi_1} & \cdots & \longrightarrow & \cdot \\ & & \xi_6 \downarrow & & & & \downarrow \xi_2 \\ \cdot & \xrightarrow{\xi_5} & \cdot & & & & \cdot \\ \xi_4 \downarrow & & & & & & \downarrow \\ \vdots & & & & & & \vdots \\ \downarrow & & & & & & \downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdots & \xrightarrow[\xi_3]{} & \cdot \end{array} \right| \leq 1.$$

Proof. The square of the above value is bounded by the following diagram.

$$
\begin{tikzcd}
 & \cdot \arrow[r, "\xi_1"] \arrow[d, "\xi_6"'] & \cdots \arrow[r] & \cdot \arrow[d, "\xi_2"] \\
\cdot \arrow[r, "\xi_5"] \arrow[d, "\xi_4"'] & \cdot & & \cdot \arrow[d] \\
\vdots \arrow[d] & & & \vdots \arrow[d] \\
\cdot & & & \cdot \\
\vdots \arrow[u] & & & \vdots \arrow[u] \\
\cdot \arrow[r, "\xi_5"'] \arrow[u, "\xi_4"] & \cdot & & \cdot \arrow[u] \\
 & \cdot \arrow[r, "\xi_1"'] \arrow[u, "\xi_6"] & \cdots \arrow[r] & \cdot \arrow[u, "\xi_2"']
\end{tikzcd}
$$

That is, this diagram is obtained by attaching the same diagram reversed vertically to the original one and remove the middle horizontal arrow ξ_3. Graphical method using biunitarity gives the desired bound. (That is, if we let ξ_4, ξ_6 vary, then the number gets bigger and this new sum is equal to 1 by biunitarity. See II.5 of [17] or [20] for graphical method.) Q.E.D.

Lemma 4.8. *For any connection on A_n, we have the following equality.*

$$
\left|\begin{array}{ccccccccc}
n-1 & \longrightarrow & n-2 & \longrightarrow & \cdots & \longrightarrow & k+1 & \longrightarrow & k \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
n-2 & & & & & & & & k+1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
\vdots & & & & & & & & \vdots \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
k+1 & & & & & & & & n-2 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
k & \longrightarrow & k+1 & \longrightarrow & \cdots & \longrightarrow & n-2 & \longrightarrow & n-1
\end{array}\right| = 1,
$$

where $k = 0, \dots, n-2$.

Proof. It is enough to show that the value of the diagram

$$
\begin{array}{ccccccccccccc}
n-1 & \longrightarrow & n-2 & \longrightarrow & \cdots & \longrightarrow & k & \longleftarrow & \cdots & \longleftarrow & n-2 & \longleftarrow & n-1 \\
\Big\downarrow & & & & & & & & & & & & \Big\downarrow \\
n-2 & & & & & & & & & & & & n-2 \\
\Big\downarrow & & & & & & & & & & & & \Big\downarrow \\
\vdots & & & & & & & & & & & & \vdots \\
\Big\downarrow & & & & & & & & & & & & \Big\downarrow \\
k & \longrightarrow & k+1 & \longrightarrow & \cdots & \longrightarrow & n-1 & \longleftarrow & \cdots & \longleftarrow & k+1 & \longleftarrow & k
\end{array}
$$

is equal to 1. Take a horizontal string

$$(n — n-1 — \cdots — k, n — n-1 — \cdots — k).$$

This is a projection corresponding to the orthogonal of the algebra

$$\langle e_1, \dots e_{n-k-3} \rangle e_{n-k-2} \langle e_1, \dots e_{n-k-3} \rangle,$$

where e_j is the j-th Jones projection on the horizontal string algebra with $* = n-1$. Thus if we embed this string to the algebra $n-1-k$ steps down and move it with a connection, we get an operator of the form

$$\sum_l \sum_{\substack{|\xi|=n-1-k \\ s(\xi)=n-1 \\ r(\xi)=l}} (\xi, \xi) \cdot p_l,$$

where each ξ is a vertical path and p_l is a projection orthogonal to

$$\langle e_1^{(l)}, \dots e_{n-k-3}^{(l)} \rangle e_{n-k-2}^{(l)} \langle e_1^{(l)}, \dots e_{n-k-3}^{(l)} \rangle,$$

where $e_j^{(l)}$ is the j-th Jones projection on the horizontal string algebra with $* = l$, by flatness of the Jones projections. Because the string

$$(k—k+1—\cdots—n-1, k—k+1—\cdots—n-1)$$

is a projection orthogonal to

$$\langle e_1^{(k)}, \dots e_{n-k-3}^{(k)} \rangle e_{n-k-2}^{(k)} \langle e_1^{(k)}, \dots e_{n-k-3}^{(k)} \rangle,$$

the above number is 1. Q.E.D.

Now we can give a proof of Lemma 4.6.

Proof of Lemma 4.6. We prove the lemma by induction on k. If $k = 2m-1$, then we get

$$W'\begin{pmatrix} 2m & \longrightarrow & 2m-1 \\ \big\downarrow & & \big\downarrow \\ 2m-1 & \longrightarrow & 2m \end{pmatrix} = (-1)^{2m}\mu(0)/\mu(2m) = 1.$$

Assuming the equality for k, we prove the equality for $k-1$. In the diagram,

$$\begin{array}{ccccccc} 2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots & \longrightarrow & k-1 \\ \big\downarrow & & \big\downarrow & & & & \big\downarrow \\ 2m-1 & \longrightarrow & \cdot & \longrightarrow & \cdots & \longrightarrow & k \\ \big\downarrow & & \big\downarrow & & & & \big\downarrow \\ \vdots & & \vdots & & & & \vdots \\ \big\downarrow & & \big\downarrow & & & & \big\downarrow \\ k-1 & \longrightarrow & k & \longrightarrow & \cdots & \longrightarrow & 2m \end{array}$$

the vertex $\cdot$ is either $2m$ or $2m-2$. Thus our number is the sum of two terms, one with $\cdot = 2m$ and the other with $\cdot = 2m-2$. Let denote these by A, B respectively. If $\cdot = 2m$, then the second rows and columns of the vertices are determined uniquely, and the formulas for W' and the induction hypothesis imply that $A > 0$. By Lemma 4.4 and Lemma 4.8, we know that $A + B = \pm 1$ and $|B| \leq 1$. Thus by $A > 0$, we can conclude that $A + B = 1$ as desired. Q.E.D.

We show another lemma for the next section.

Lemma 4.9. *For each connection W on A_{2m+1}, we have the following equalities.*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 1 & \longrightarrow & \cdots\ m\ \cdots & \longrightarrow & 1 & \longrightarrow & 0 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
1 & & & & & & & & 1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
\vdots & & & & & & & & \vdots \\
m & & & & & & & & m \\
\vdots & & & & & & & & \vdots \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
2m-1 & & & & & & & & 2m-1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots\ m\ \cdots & \longrightarrow & 2m-1 & \longrightarrow & 2m
\end{array}
\quad = 1,
$$

$$
\begin{array}{ccccccccc}
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots\ m\ \cdots & \longrightarrow & 2m-1 & \longrightarrow & 2m \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
2m-1 & & & & & & & & 2m-1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
\vdots & & & & & & & & \vdots \\
m & & & & & & & & m \\
\vdots & & & & & & & & \vdots \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
1 & & & & & & & & 1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
0 & \longrightarrow & 1 & \longrightarrow & \cdots\ m\ \cdots & \longrightarrow & 1 & \longrightarrow & 0
\end{array}
\quad = 1,
$$

Proof. It is enough to prove the first equality. Because $\mu(0) = \mu(2m)$, it is enough to show

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 1 & \longrightarrow & \cdots\ m\ \cdots & \longleftarrow & 1 & \longleftarrow & 0 \\ \downarrow &&&&&&&& \downarrow \\ 1 &&&&&&&& 1 \\ \downarrow &&&&&&&& \downarrow \\ \vdots &&&&&&&& \vdots \\ m &&&&&&&& m \\ \vdots &&&&&&&& \vdots \\ \downarrow &&&&&&&& \downarrow \\ 2m-1 &&&&&&&& 2m-1 \\ \downarrow &&&&&&&& \downarrow \\ 2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots\ m\ \cdots & \longleftarrow & 2m-1 & \longleftarrow & 2m \end{array} = 1.$$

Then as in the proof of Lemma 4.8, we get the equality using the flatness of the Jones projections. Q.E.D.

§5 Flatness of connections on the Dynkin diagrams D_n

Here we come to the main part of this paper. We will show that we have two flat connections on the diagrams D_{2n} and no flat ones on D_{2n+1}.

First we show that if we choose the distinguished point $*$ to be one of the two endpoints of the forked branches, then the connections are not flat, unless the graph is D_4. Number the vertices of D_n as follows. (This numbering is different from that in the other places of this paper.)

$$D_n : \qquad \begin{array}{l} 0 \diagdown \\ \qquad 2 — 3 \cdots n-1, \\ 1 \diagup \end{array}$$

Set the vertex 0 to be the $*$. The Perron-Frobenius eigenvector is chosen so that $\mu(0) = 1$, $\mu(1) = 1$, $\mu(2) = \beta$, $\mu(3) = \beta^2 - 1, \ldots$, where β is the Perron-Frobenius eigenvalue. We show that the following diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 2 \arrow[r] & 1 & 2 \arrow[l] & 0 \arrow[l] \arrow[d] \\
2 \arrow[d] & & & & 2 \arrow[d] \\
1 \arrow[r] & 2 \arrow[r] & 0 & 2 \arrow[l] & 3 \arrow[l]
\end{tikzcd}$$

does not have value 0, which is enough for non-flatness by Theorem 2.1 $(2)'$. Note that the middle vertical arrows should be $1 \to 2 \to 0$, and we have three configurations for each

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 2 \arrow[r] & 1 \arrow[d] \\
2 \arrow[d] & & 2 \arrow[d] \\
1 \arrow[r] & 2 \arrow[r] & 0
\end{tikzcd}, \quad
\begin{tikzcd}
1 \arrow[d] & 2 \arrow[l] & 0 \arrow[l] \arrow[d] \\
2 \arrow[d] & & 2 \arrow[d] \\
0 & 2 \arrow[l] & 3 \arrow[l]
\end{tikzcd},$$

that is, the center points in each 2×2-cell can be the vertices $0, 1, 3$. A direct computation using the definition of the connection W shows that the value for the first 2×2-cell is $-1 + 2\cos\dfrac{2\pi}{2n-2}$ and the value for the second is $\sqrt{\beta^2 - 2}(-1 + \exp\dfrac{-2\pi\sqrt{-1}}{2n-2})$. The first one is equal to 0 if and only if $n = 4$ and the second one is never zero. Thus we get the desired non-flatness. The case $* = 1$ can be proved in the same way. (A recent result of M. Izumi [7, Theorem 5.1] rejects these two cases more easily.)

Next we consider the "right choice" of the $*$. Our idea is that we embed the string algebra of D_n to the string algebra of A_{2n-3} with the double starting points. (Note that these two have the same Coxeter number $2n-2$.) For this purpose, we use the "cell system" given by Roche [20, page 407]. That is, if we have a connection between two graphs, we can construct embedding of the two corresponding string algebras as in [17, II.3] and [20, page 403]. The connection is explicitly given in [20, page 407], but we list it here because our numbering system of the vertices is different from that in [20] and we reverse the order of the two graphs. We will use the following numbering of vertices of D_n and A_{2n-3}.

$$D_n: \qquad 0' - 1' \cdots n-4' - n-3' \begin{array}{l} \diagup n-2' \\ \diagdown n-1' \end{array}$$

$$A_{2n-3}: \qquad 0 - 1 \cdots n-2 - 2n-4$$

The connection is given by the following.

$$\begin{array}{ccc} j' & \longrightarrow & j \\ \big\downarrow & & \big\downarrow \\ j+1' & \longrightarrow & j+1 \end{array} = \begin{array}{ccc} j' & \longrightarrow & 2n-4-j \\ \big\downarrow & & \big\downarrow \\ j+1' & \longrightarrow & 2n-5-j \end{array} = 1, \qquad 0 \le j \le n-3,$$

$$\begin{array}{ccc} j+1' & \longrightarrow & j+1 \\ \big\downarrow & & \big\downarrow \\ j' & \longrightarrow & j \end{array} = \begin{array}{ccc} j+1' & \longrightarrow & 2n-5-j \\ \big\downarrow & & \big\downarrow \\ j' & \longrightarrow & 2n-4-j \end{array} = 1, \qquad 0 \le j \le n-3,$$

$$\begin{array}{ccc} n-3' & \longrightarrow & n-3 \\ \big\downarrow & & \big\downarrow \\ n-2' & \longrightarrow & n-2 \end{array} = \frac{1}{\sqrt{2}}, \quad \begin{array}{ccc} n-3' & \longrightarrow & n-1 \\ \big\downarrow & & \big\downarrow \\ n-2' & \longrightarrow & n-2 \end{array} = \frac{1}{\sqrt{2}},$$

$$\begin{tikzcd} n-3' \arrow[r] \arrow[d] & n-3 \arrow[d] \\ n-1' \arrow[r] & n-2 \end{tikzcd} = -\frac{1}{\sqrt{2}}, \quad \begin{tikzcd} n-3' \arrow[r] \arrow[d] & n-1 \arrow[d] \\ n-1' \arrow[r] & n-2 \end{tikzcd} = \frac{1}{\sqrt{2}},$$

$$\begin{tikzcd} n-2' \arrow[r] \arrow[d] & n-2 \arrow[d] \\ n-3' \arrow[r] & n-3 \end{tikzcd} = 1, \quad \begin{tikzcd} n-2' \arrow[r] \arrow[d] & n-2 \arrow[d] \\ n-3' \arrow[r] & n-1 \end{tikzcd} = 1,$$

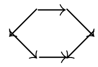
$$\begin{tikzcd} n-1' \arrow[r] \arrow[d] & n-2 \arrow[d] \\ n-3' \arrow[r] & n-3 \end{tikzcd} = -1, \quad \begin{tikzcd} n-1' \arrow[r] \arrow[d] & n-2 \arrow[d] \\ n-3' \arrow[r] & n-1 \end{tikzcd} = 1.$$

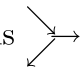
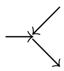
In the above diagrams, the left vertical edges are in D_n, the right vertical edges are in A_{2n-3}, and the horizontal edges connect these two graphs. Because we changed the order of the two graphs from that in [20], Roche's condition $C_{*_1,i} = 1 \Leftrightarrow i = *_2$ in the last line of [20, page 401] is not satisfied. That is, our $* = 0'$ of D_n is connected to two vertices $0, 2n-4$ of A_{2n-3}, but it does not matter.

We have string algebra double sequences for both D_n and A_{2n-3}. By general theory, it is easy to see that double sequence embeddings and the embedding given by the above are compatible if the star triangle relation, as in [20, page 404, Proposition 5] is satisfied.

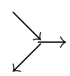
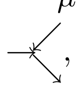
Lemma 5.1. *The connections W on D_n and A_{2n-3} given as in §3 and the connections between D_n and A_{2n-3} given as above satisfy the star-triangle relation mentioned above.*

Proof. We have to show that for each hexagon



where the left two edges are in D_n, the right two edges are in A_{2n-3} and the horizontal two edges connect these two graphs, the two ways of configurations  and  give the same value. If the numberings of the right three vertices of the hexagon are all less than $n-2$ or all bigger than $n-2$, then we get the equality trivially, because the connecting cell values are all 1 and the corresponding vertices have the same Perron-Frobenius eigenvector entries.

There are 34 hexagons involving the vertex $n-2$ on A_{2n-3}. For each case, we can check the equality directly and easily. A typical computation example is given below.

Let the left three vertices of the hexagon be $n-3', n-2', n-3'$ on D_n from the top to the bottom and the right three vertices of the hexagon are $n-1, n-2, n-1$ on A_{2n-3} from the bottom to the top. Then there are two configurations for , that is, the center point of the hexagon can be $n-2'$ or $n-1'$. The product of the three cell values for the first configuration is equal to $(\varepsilon + \frac{\mu(n-2')}{\mu(n-3')}\bar{\varepsilon}) \cdot \frac{1}{\sqrt{2}} \cdot 1$, and the value for the second is $\frac{\sqrt{\mu(n-1')\mu(n-2')}}{\mu(n-3')} \cdot \bar{\varepsilon} \cdot \frac{1}{\sqrt{2}}$. On the other hand, there is only one configuration for , that is, the center point is $n-2$. The value for this configuration is $\frac{1}{\sqrt{2}} \cdot 1 \cdot (\varepsilon + \frac{\mu(n-2)}{\mu(n-1)}\bar{\varepsilon})$, which is equal to the sum of the above two values. Q.E.D.

Now we can embed the string algebra double sequence of D_n into that of A_{2n-3}, but note that the string algebras of A_{2n-3} have double starting points 0 and $2n-4$ because the vertex $* = 0'$ of D_n is connected to two points $0, 2n-4$ of A_{2n-3}. Thus a general element in our string algebra A_{2n-3} is a linear combination of elements of the form (ξ, η), where $s(\xi), s(\eta) = 0, 2n-4$, and $s(\xi)$ and $s(\eta)$ does *not* have to be equal. The operations are defined in the same way as in the ordinary string algebra. (Because $s(\xi)$ and $s(\eta)$ can be different the name "string" may be inappropriate. If one is unhappy with this, one can add an extra starting point $*$ and two paths from $*$ to 0 and $*$ to $2n-4$.) Note that this construction realizes the double sequence of D_n as the fixed point algebras of the double sequence of A_{2n-3} with double starting points by an automorphism of order 2 induced by the flip of the graph A_{2n-3}.

For checking flatness of the connection on D_n, it is enough to see whether the string $(0'—1'—\cdots n-1', 0'—1'—\cdots n-1')$ in the vertical string algebra and the same form of the string in the horizontal string algebra commute. (This is because the Jones projections take care of the other parts. See a remark preceding Example 2.3.) We embed the both into A_{2n-3} algebra and check the commutativity in this algebra. Then we get the strings $\rho = \frac{1}{2}(\xi,\xi) + \frac{1}{2}(\eta,\eta) - \frac{1}{2}(\xi,\eta) - \frac{1}{2}(\eta,\xi)$, where paths ξ, η are defined to be

$$\begin{cases} \xi = & 0—1—\cdots n-2, \\ \eta = & 2n-4—2n-3—\cdots n-2, \end{cases}$$

on the A_{2n-3} string algebra. Thus we now have to see whether the above strings ρ in the horizontal string algebra and the vertical string algebra commute. Now we prove the next lemma.

Lemma 5.2. *The strings ρ in the vertical string algebra and in the horizontal string algebras commute if and only if the real part of the value for the following diagram is 1.*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & 2n-5 & \longrightarrow & 2n-4 \\
\downarrow & & & & & & & & \downarrow \\
1 & & & & & & & & 2n-5 \\
\downarrow & & & & & & & & \downarrow \\
\vdots & & & & & & & & \vdots \\
\downarrow & & & & & & & & \downarrow \\
2n-5 & & & & & & & & 1 \\
\downarrow & & & & & & & & \downarrow \\
2n-4 & \longrightarrow & 2n-5 & \longrightarrow & \cdots & \longrightarrow & 1 & \longrightarrow & 0
\end{array}
$$

Proof. In order to simplify notations, we write m for $n-2$. Let the strings ρ in the vertical string algebra move to the horizontal one via connection W. Then by Lemma 4.9 and unitarity, we get the equalities

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 1 & \longrightarrow & \cdots\ x\ \cdots & \longleftarrow & 1 & \longleftarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
1 & & & & & & & & 1 \\
\downarrow & & & & & & & & \downarrow \\
\vdots & & & & & & & & \vdots \\
m & & & & & & & & m \\
\vdots & & & & & & & & \vdots \\
\downarrow & & & & & & & & \downarrow \\
2m-1 & & & & & & & & 2m-1 \\
\downarrow & & & & & & & & \downarrow \\
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots\ m\ \cdots & \longleftarrow & 2m-1 & \longleftarrow & 2m
\end{array}
\quad = 0,
$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 1 & \longrightarrow & \cdots\ m\ \cdots & \longleftarrow & 1 & \longleftarrow & 0 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
1 & & & & & & & & 1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
\vdots & & & & & & & & \vdots \\
m & & & & & & & & m \\
\vdots & & & & & & & & \vdots \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
2m-1 & & & & & & & & 2m-1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots\ x\ \cdots & \longleftarrow & 2m-1 & \longleftarrow & 2m
\end{array}
\quad = 0,
$$

where x is a vertex different from m. Thus we get

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \cdot & \longrightarrow & \cdots & \longrightarrow & x \\
\Big\downarrow & & & & & & \Big\downarrow \\
1 & & & & & & \cdot \\
\Big\downarrow & & & & & & \Big\downarrow \\
\vdots & & & & & & \vdots \\
m & & & & & & \cdot \\
\vdots & & & & & & \vdots \\
\Big\uparrow & & & & & & \Big\uparrow \\
2m-1 & & & & & & \cdot \\
\Big\uparrow & & & & & & \Big\uparrow \\
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots & \longrightarrow & m
\end{array}
\quad = 0,
$$

and

$$
\begin{array}{ccccccc}
0 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & m \\
\Big\downarrow & & & & & & \Big\downarrow \\
1 & & & & & & \cdot \\
\Big\downarrow & & & & & & \Big\downarrow \\
\vdots & & & & & & \vdots \\
m & & & & & & \cdot \\
\vdots & & & & & & \vdots \\
\Big\uparrow & & & & & & \Big\uparrow \\
2m-1 & & & & & & \cdot \\
\Big\uparrow & & & & & & \Big\uparrow \\
2m & \longrightarrow & \cdot & \longrightarrow & \cdots & \longrightarrow & x
\end{array}
= 0.
$$

These imply that for checking of the commutativity, we only need to see commutativity of the horizontal strings ρ and the part of the vertical ρ of the following form $M_2(\mathbf{C}) \otimes \mathrm{String}_m^{(m)}$ after identification using W. Here $M_2(\mathbf{C})$ is generated by the matrix units $(\xi,\xi), (\xi,\eta), (\eta,\xi), (\eta,\eta)$ in the horizontal string algebra and $\mathrm{String}_m^{(m)}$ is a vertical m-string algebra starting from the vertex m. Then it is easy to see that the commutativity holds if and only if we have the following equalities for every paths ξ, η with $|\xi| = |\eta| = m$, $s(\xi) = s(\eta) = m$, and $r(\xi) = r(\eta)$.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & m \\
\Big\downarrow & & & & & & \Big\downarrow \xi \\
1 & & & & & & \cdot \\
\vdots & & & & & & \vdots \\
m & & & & & & x \\
\vdots & & & & & & \vdots \\
1 & & & & & & \cdot \\
\Big\uparrow & & & & & & \Big\uparrow \eta \\
0 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & m
\end{array}
=
\begin{array}{ccccccc}
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots & \longrightarrow & m \\
\Big\downarrow & & & & & & \Big\downarrow \xi \\
2m-1 & & & & & & \cdot \\
\vdots & & & & & & \vdots \\
m & & & & & & x \\
\vdots & & & & & & \vdots \\
2m-1 & & & & & & \cdot \\
\Big\uparrow & & & & & & \Big\uparrow \eta \\
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots & \longrightarrow & m
\end{array},
$$

$$
\begin{array}{ccccccc}
0 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & m \\
\Big\downarrow & & & & & & \Big\downarrow{\scriptstyle \xi} \\
1 & & & & & & \cdot \\
\vdots & & & & & & \vdots \\
m & & & & & & x \\
\vdots & & & & & & \vdots \\
2m-1 & & & & & & \cdot \\
\Big\uparrow & & & & & & \Big\uparrow{\scriptstyle \eta} \\
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots & \longrightarrow & m
\end{array}
\;=\;
\begin{array}{ccccccc}
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots & \longrightarrow & m \\
\Big\downarrow & & & & & & \Big\downarrow{\scriptstyle \xi} \\
2m-1 & & & & & & \cdot \\
\vdots & & & & & & \vdots \\
m & & & & & & x \\
\vdots & & & & & & \vdots \\
1 & & & & & & \cdot \\
\Big\uparrow & & & & & & \Big\uparrow{\scriptstyle \eta} \\
0 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & m
\end{array},
$$

where we write x for $r(\xi) = r(\eta)$. First, we show that the first equality is always valid. To prove this, it is enough to show

$$
\sum_{\xi,\eta}
\left|
\begin{array}{ccccc}
0 & \longrightarrow & 1\cdots & \longrightarrow & m \\
\Big\downarrow & & & & \Big\downarrow{\scriptstyle \xi} \\
1 & & & & \cdot \\
\vdots & & & & \vdots \\
m & & & & x \\
\vdots & & & & \vdots \\
1 & & & & \cdot \\
\Big\downarrow & & & & \Big\downarrow{\scriptstyle \tilde{\eta}} \\
0 & \longrightarrow & 1\cdots & \longrightarrow & m
\end{array}
\;-\;
\begin{array}{ccccc}
2m & \longrightarrow & 2m-1\cdots & \longrightarrow & m \\
\Big\downarrow & & & & \Big\downarrow{\scriptstyle \xi} \\
2m-1 & & & & \cdot \\
\vdots & & & & \vdots \\
m & & & & x \\
\vdots & & & & \vdots \\
2m-1 & & & & \cdot \\
\Big\downarrow & & & & \Big\downarrow{\scriptstyle \tilde{\eta}} \\
2m & \longrightarrow & 2m-1\cdots & \longrightarrow & m
\end{array}
\right|^2 = 0.
$$

Expanding the left hand side, we get that it is equal to the following.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 1 & \longrightarrow & \cdots m \cdots & \longleftarrow & 1 & \longleftarrow & 0 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
1 & & & & & & & & 1 \\
\vdots & & & & & & & & \vdots \\
m & & & & & & & & m \\
\vdots & & & & & & & & \vdots \\
1 & & & & & & & & 1 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
0 & \longrightarrow & 1 & \longrightarrow & \cdots m \cdots & \longleftarrow & 1 & \longleftarrow & 0
\end{array}
$$

$$
+\;\begin{array}{ccccccccc}
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots m \cdots & \longleftarrow & 2m-1 & \longleftarrow & 2m \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
\begin{matrix} 2m-1 \\ \vdots \\ m \\ \vdots \\ 2m-1 \end{matrix} & & & & & & & & \begin{matrix} 2m-1 \\ \vdots \\ m \\ \vdots \\ 2m-1 \end{matrix} \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots m \cdots & \longleftarrow & 2m-1 & \longleftarrow & 2m
\end{array}
$$

$$
-\;\begin{array}{ccccccccc}
0 & \longrightarrow & 1 & \longrightarrow & \cdots m \cdots & \longleftarrow & 2m-1 & \longleftarrow & 2m \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
\begin{matrix} 1 \\ \vdots \\ m \\ \vdots \\ 1 \end{matrix} & & & & & & & & \begin{matrix} 2m-1 \\ \vdots \\ m \\ \vdots \\ 2m-1 \end{matrix} \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
0 & \longrightarrow & 1 & \longrightarrow & \cdots m \cdots & \longleftarrow & 2m-1 & \longleftarrow & 2m
\end{array}
$$

$$
-\;\begin{array}{ccccccccc}
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots m \cdots & \longleftarrow & 1 & \longleftarrow & 0 \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
\begin{matrix} 2m-1 \\ \vdots \\ m \\ \vdots \\ 2m-1 \end{matrix} & & & & & & & & \begin{matrix} 1 \\ \vdots \\ m \\ \vdots \\ 1 \end{matrix} \\
\Big\downarrow & & & & & & & & \Big\downarrow \\
2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots m \cdots & \longleftarrow & 1 & \longleftarrow & 0
\end{array}.
$$

Flatness of A_{2m+1} with $* = 0, 2m$ and the same argument as in the proof of Lemma 4.9 give that the above is equal to $1 + 1 - 1 - 1 = 0$. The similar argument to the above shows that the second equality is equivalent to the following.

$$2-2\,\mathrm{Re}\ \begin{array}{ccccccccc} 0 & \longrightarrow & 1 & \longrightarrow & \cdots m\cdots & \longrightarrow & 2m-1 & \longrightarrow & 2m \\ \downarrow & & & & & & & & \downarrow \\ 1 & & & & & & & & 2m-1 \\ \vdots & & & & & & & & \vdots \\ m & & & & & & & & m \\ \vdots & & & & & & & & \vdots \\ 2m-1 & & & & & & & & 1 \\ \downarrow & & & & & & & & \downarrow \\ 2m & \longrightarrow & 2m-1 & \longrightarrow & \cdots m\cdots & \longrightarrow & 1 & \longrightarrow & 0 \end{array} = 0.$$

(We can reverse the orientation of the horizontal arrows in the right half by the renormalization rule.) This is the desired formula. Q.E.D.

Now we can prove the following theorem.

Theorem 5.3. *None of D_{2n+1} have flat connections. Each of D_{2n} has two flat connections, and these are equivalent up to a graph isomorphism fixing $*$. Thus there are no subfactors having D_{2n+1} as principal graphs, and there is only one subfactor having D_{2n} as its principal graph for each n.*

Proof. Now this is immediate by Theorem 3.1, Proposition 4.2, Lemma 5.2.

Q.E.D.

Remark 5.4. On a proof of flatness of the connections on D_{2n}, Ocneanu mentioned an outline as follows. (See [17, IV.3].) First one shows that

$$\mathrm{Re}\ \begin{array}{ccccccccc} 0 & \longrightarrow & \cdots & \longrightarrow & n-2 & \longleftarrow & \cdots & \longleftarrow & 0 \\ \downarrow & & & & & & & & \downarrow \\ \vdots & & & & & & & & \vdots \\ \downarrow & & & & & & & & \downarrow \\ n-2 & \longrightarrow & \cdots & \longrightarrow & \cdot & \longleftarrow & \cdots & \longleftarrow & n-1 \end{array} = 0$$

for all the D_n's using the graph symmetry flip. Secondly, one shows that one can choose gauges for D_{2n} so that all the 2×2-cells have real values. Flatness for D_{2n} follows from these two claims. But the author has been unable to give a complete proof along this line. (The second claim was also mentioned in [16, page 160].) So the author has given the above proof using orbifold method, which has an advantage of handling all the D_n's equally and constructing an interesing automorphism fixing a subfactor globally. (See the following remark.) (After the sumission of this paper, A. Ocneanu showed to the author the full proof along this suggested line in October, 1991. We will present his original proof in the appendix of [12].)

Remark 5.5. Ocneanu's compactness argument [17, II.6] shows that the above double starting points construction gives a subfactor with the principal graph A_{2n-3}. Thus if n is even, we know that a subfactor with the principal graph D_n is realized as $N^\theta \subset M^\theta$, where $N \subset M$ is a subfactor with the principal graph A_{2n-3} and θ is an automorphism of M of order 2 and with $\theta(N) = N$. M. Choda asked the author whether this is valid after seeing our construction. We thank her for this question.

§6 Flatness of connections on the Dynkin diagrams E_6, E_7, E_8

We finally work on the diagrams E_6, E_7, E_8. Because there are only three diagrams, everything is essentially a matter of finite times of computation. We will show some numerical computation on a computer for non-flatness and give an explicit equation of algebraic integers for each E_6 and E_8 such that validity of the equation is equivalent to flatness. Unfortunately, the equations are so complicated that the author has been unable to verify them.

At first, we fix the following numbering of the vertices of the diagrams E_6, E_7, E_8.

$$E_6: \qquad \begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & & | & & & & \\ 0 & — & 1 & — & 2 & — & 4 & — & 5 \end{array}$$

$$E_7: \qquad \begin{array}{ccccccccccc} & & & & & & 4 & & & & \\ & & & & & & | & & & & \\ 0 & — & 1 & — & 2 & — & 3 & — & 5 & — & 6 \end{array}$$

$$E_8: \qquad \begin{array}{ccccccccccccc} & & & & & & & & 5 & & & & \\ & & & & & & & & | & & & & \\ 0 & — & 1 & — & 2 & — & 3 & — & 4 & — & 6 & — & 7 \end{array}$$

For non-flatness, we show the following computations. (See Theorem 2.1 $(2)'$.)

$$E_6, * = 3: \qquad \begin{tikzcd} 3 \arrow[r] \arrow[d] & 2 \arrow[r] & 1 & 2 \arrow[l] & 3 \arrow[l] \arrow[d] \\ 2 \arrow[d] & & & & 2 \arrow[d] \\ 1 \arrow[r] & 2 \arrow[r] & 3 & 2 \arrow[l] & 4 \arrow[l] \end{tikzcd} \simeq 0.500000i,$$

$E_7, * = 0:$

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 & 3 \arrow[l] & 2 \arrow[l] & 1 \arrow[l] & 0 \arrow[l] \arrow[d] \\
1 \arrow[d] & & & & & & & & 1 \arrow[d] \\
2 \arrow[d] & & & & & & & & 2 \arrow[d] \\
3 \arrow[d] & & & & & & & & 3 \arrow[d] \\
4 \arrow[r] & 3 \arrow[r] & 2 \arrow[r] & 1 \arrow[r] & 0 & 1 \arrow[l] & 2 \arrow[l] & 3 \arrow[l] & 5 \arrow[l]
\end{tikzcd}
\simeq -0.141915 - 0.169128i,
$$

$$
E_7, * = 6: \quad
\begin{tikzcd}
6 \arrow[r] \arrow[d] & 5 \arrow[r] & 3 \arrow[r] & 4 & 3 \arrow[l] & 5 \arrow[l] & 6 \arrow[l] \arrow[d] \\
2 \arrow[d] & & & & & & 5 \arrow[d] \\
3 \arrow[d] & & & & & & 3 \arrow[d] \\
4 \arrow[r] & 3 \arrow[r] & 5 \arrow[r] & 6 & 5 \arrow[l] & 3 \arrow[l] & 2 \arrow[l]
\end{tikzcd}
\simeq -0.051362 - 0.423344i,
$$

$$
E_7, * = 4: \quad
\begin{tikzcd}
4 \arrow[r] \arrow[d] & 3 \arrow[r] & 2 & 3 \arrow[l] & 4 \arrow[l] \arrow[d] \\
3 \arrow[d] & & & & 3 \arrow[d] \\
2 \arrow[r] & 3 \arrow[r] & 4 & 3 \arrow[l] & 5 \arrow[l]
\end{tikzcd}
\simeq 0.016011 + 0.432120i,
$$

$$E_8, * = 7: \quad \begin{tikzcd} 7 \arrow[r] \arrow[d] & 6 \arrow[r] & 4 \arrow[r] & 5 & 4 \arrow[l] & 6 \arrow[l] & 7 \arrow[l] \arrow[d] \\ 6 \arrow[d] & & & & & & 6 \arrow[d] \\ 4 \arrow[d] & & & & & & 4 \arrow[d] \\ 3 \arrow[r] & 4 \arrow[r] & 6 \arrow[r] & 7 & 6 \arrow[l] & 4 \arrow[l] & 5 \arrow[l] \end{tikzcd} \simeq -0.150142+0.462088i,$$

$$E_8, * = 5: \quad \begin{tikzcd} 5 \arrow[r] \arrow[d] & 4 \arrow[r] & 6 & 4 \arrow[l] & 5 \arrow[l] \arrow[d] \\ 4 \arrow[d] & & & & 4 \arrow[d] \\ 3 \arrow[r] & 4 \arrow[r] & 5 & 4 \arrow[l] & 6 \arrow[l] \end{tikzcd} \simeq -0.008996-0.292574i,$$

where i is the imaginary unit. These computations were done by a C program with double precision on a Sun by the author. Though the author has rounded the numbers to 6 decimals, an easy error estimate shows that all of the above diagrams indeed have non-zero values. For example, the first diagram for E_7 has the biggest estimated error. The middle vertical line should be $4 \to 3 \to 2 \to 1 \to 0$, thus we can divide the diagram into two pieces, and each has 67 configurations. For each configuration, we have to multiply 16 complex numbers, connections, which has errors less than 10^{-14}. Because each connection value is less than or equal to 1 in its absolute value, we can conclude that the final computation error is less than

10^{-10}. The choice $* = 5$ for E_6 is also non-flat by symmetry. (Recent result of M. Izumi [7, Theorem 3.7, Theorem 5.1] reject these cases more easily.)

Thus the only left cases are $* = 0$ for E_6 and $* = 0$ for E_8. These are the "right" choices of $*$ Ocneanu announced in [16]. For E_6, we only need to check the following equality for flatness. (The reason is the same as in the D_n cases. See a remark preceding Example 2.3.)

$$
(*) \qquad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 2 \arrow[r] & 1 \arrow[r] & 0 \arrow[d] \\
1 \arrow[d] & & & & & & 1 \arrow[d] \\
2 \arrow[d] & & & & & & 2 \arrow[d] \\
3 \arrow[d] & & & & & & 3 \arrow[d] \\
2 \arrow[d] & & & & & & 2 \arrow[d] \\
1 \arrow[d] & & & & & & 1 \arrow[d] \\
0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 2 \arrow[r] & 1 \arrow[r] & 0
\end{tikzcd}
= 1.
$$

For E_8, we only need to check the following equality for flatness.

$$(**)\qquad \begin{array}{ccccccccc}
0 & \longrightarrow & 1 & \longrightarrow & 2 \cdots 5 \cdots 2 & \longrightarrow & 1 & \longrightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
1 & & & & & & & & 1 \\
\downarrow & & & & & & & & \downarrow \\
2 & & & & & & & & 2 \\
\vdots & & & & & & & & \vdots \\
5 & & & & & & & & 5 \\
\vdots & & & & & & & & \vdots \\
2 & & & & & & & & 2 \\
\downarrow & & & & & & & & \downarrow \\
1 & & & & & & & & 1 \\
\downarrow & & & & & & & & \downarrow \\
0 & \longrightarrow & 1 & \longrightarrow & 2 \cdots 5 \cdots 2 & \longrightarrow & 1 & \longrightarrow & 0
\end{array} = 1.$$

These two are explicit equations of $\exp \dfrac{\pi\sqrt{-1}}{24}$ and $\exp \dfrac{\pi\sqrt{-1}}{60}$ respectively, because the connection W is explicitly given, each entry of the Perron-Frobenius eigenvector can be represented by the Perron-Frobenius eigenvalue β, and $\beta = -\varepsilon^2 - \bar{\varepsilon}^2$. But unfortunately, these equations are too complicated and the author has been unable to prove them. But numerical computation on a computer with error estimate as above has shown that the above equalities are valid up to error 10^{-5}. Our theorem is now as follows.

Theorem 6.1. *The Dynkin diagram E_7 does not have a flat connection on it. Each of E_6 and E_8 has two flat connections, assuming the above two equalities are valid. Thus there are no subfactors having D_7 as principal graphs, and there are only two* *subfactors having E_6, E_8 as their principal graph for each E_6 and E_8, assuming the validity of the above two equalities.*

We explain more about the equations for E_6 and E_8. Because there is a symmetry of four corners in $(*)$ and $(**)$, we only need to compute the following numbers. Define matrices $C(x), D(x)$ by

$$C(x)_{\xi,\eta} = \begin{array}{ccccccc} 0 & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & 3 \\ \downarrow & & & & & & \downarrow \eta \\ 1 & & & & & & \cdot \\ \downarrow & & & & & & \downarrow \\ 2 & & & & & & \cdot \\ \downarrow & & & & & & \downarrow \\ 3 & \underset{\xi}{\longrightarrow} & \cdot & \longrightarrow & \cdot & \longrightarrow & x \end{array},$$

$$D(x)_{\xi,\eta} = \begin{array}{cccccccc} 0 & \longrightarrow & 1 & \longrightarrow & 2 & \cdots & \longrightarrow & 5 \\ \downarrow & & & & & & & \downarrow \eta \\ 1 & & & & & & & \cdot \\ \downarrow & & & & & & & \downarrow \\ 2 & & & & & & & \cdot \\ \vdots & & & & & & & \vdots \\ 5 & \underset{\xi}{\longrightarrow} & \cdot & \longrightarrow & \cdot & \cdots & \longrightarrow & x \end{array},$$

where x is one of $0, 2, 5$ [resp. $0, 2, 4, 7$] and ξ, η are any paths from 3 [resp. 5] to x in the first [resp. second] case. It is easy to see that the left hand sides of $(*)$ and

(**) are computed as follows.

$$\frac{\mu(0)}{\mu(3)^2} \sum_{x=0,2,5} \mu(x)\mathrm{Tr}(C(x)C(x)^* C(x)C(x)^*),$$

$$\frac{\mu(0)}{\mu(5)^2} \sum_{x=0,2,4,7} \mu(x)\mathrm{Tr}(D(x)D(x)^* D(x)D(x)^*).$$

Numerical computation on a computer suggests the following.

(1) $C(0), C(5), D(0), D(4)$ are rank 1 partial isometries.

(2) $C(2), D(2), D(7)$ are all zero.

Because $C(0), C(5), D(0)$ are just 1×1-matrices, (1) means that these are complex numbers with modules 1. The matrix $D(4)$ is a symmetric 11×11-matrix and for computation of its 121 entries we have to compute the cell values for at most 1879 configurations. Unfortunately, the computation is still too complicated and the author has not been able to verify the above suggested claim. Note that if (1) is valid, then (2) automatically follows and everything is over because the left hand sides of $(*)$ and $(**)$ are positive numbers between 0 and 1 by unitarity.

Note that for each of E_6 and E_8, if the above identity is valid, then there are two (and only two) subfactors for the principal graph, and if it is invalid, there are no subfactors. These two are the only possibilities. In [2], Bion-Nadal constructed a subfactor with E_6 as its principal graph. This means there is a flat connection on E_6, thus the connection of her subfactor must be one of the above two, and this proves indirectly that the both connections are flat. M. Izumi also constructed a subfactor with the principal graph E_6 in [8] by a different method based on the Cuntz algebra and type III subfactors.

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