Automorphisms commuting with a conditional expectation onto a subfactor with finite index

Yasuyuki Kawahigashi

Department of Mathematics, Faculty of Science
University of Tokyo, Hongo, Tokyo, 113, JAPAN

(email:d33844@tansei.cc.u-tokyo.ac.jp)
(or:d33844%tansei.cc.u-tokyo.ac.jp@cunyvm.cuny.edu)

Abstract. We show that if an automorphism of a factor fixes a subfactor with finite index globally, then it and its restriction on the subfactor are similar in terms of innerness, central triviality, approximate pointwise innerness, approximate innerness and pointwise innerness. In particular, an automorphism fixing a subfactor with finite index globally is free [resp. centrally free] if and only if its restriction on the subfactor is free [resp. centrally free]. We also show that our method is applicable for removing the irreducibility assumption $N' \cap M = \mathbf{C}$ in Loi's classification of certain free automorphisms of subfactors.

§0 Introduction

Kosaki extended the theory of Jones [J2] on indices of subfactors of factors of type II_1 to general setting for arbitrary factors in [K1] based on Connes' spatial theory [C6] and Haagerup's work on operator-valued weights [Ha2]. Since then, several results on type III index theory have been obtained in [HK1–2, Hi1–3, K2, Lo1–2, Ln]. In [Lo1] and [HK1], they showed that if a subfactor is contained in a factor of type III with finite index, then the two factors are similar in some sense. Their work is actually study of modular automorphism groups commuting with the

1980 Mathematics Subject Classification (1985 Revision). Primary 46L40; Secondary 46L55.

conditional expectations. From this viewpoint, we generalize their methods to study automorphisms and group actions commuting with the conditional expectation onto a subfactor with finite index. Connes' automorphism approach to studying factors has been so successful that we try to study subfactors via automorphisms, and this is our first step.

It has been shown that a factor and its subfactor with finite index are similar in the sense that the factor has properties like injectivity, fullness, and property T if and only if its subfactor with finite index has the same property. (See [J2], [PP1], and [PP2] respectively.) Here we show that the automorphism on the ambient factor M fixing a subfactor N with finite index globally and its restriction on N are similar in several senses like innerness, central triviality, approximate pointwise innerness, and pointwise innerness. In general, if α [resp. $\alpha|_N$] satisfies some property, then $\alpha|_N^p$ [resp. α^p] satisfies the same property for some p which is determined by the index. In particular, a single automorphism fixing a subfactor with finite index globally is free [resp. centrally free] if and only if its restriction on the subfactor is free [resp. centrally free].

After the circulation of the first version of this paper as a preprint, the author received a preprint of Loi [Lo2], in which he showed uniqueness of certain free automorphisms of an approximately finite dimensional (AFD) factor M of type II_1 fixing an subfactor N with finite index globally, assuming that N has finite depth and $N' \cap M = \mathbf{C}$ Loi studies these problems for applying them to classification of subfactors of type III_λ AFD factors, $0 < \lambda < 1$. Here we have added §3 to show that our method based on conditional expectations removes the irreducibility assumption $N' \cap M = \mathbf{C}$ in Loi's results without appealing to Ocneanu's theorem, whose proof has not been published.

Section 1 is devoted to general preliminaries on automorphisms commuting with a conditional expectation. In §2, we apply the results in §1 to study several types of automorphisms such as inner ones, centrally trivial ones, and so on. We apply our method to Loi's theory in §3.

The author is thankful to Prof. F. Hiai and Prof. E. Størmer for sending their preprints [Hi2–3] and [HS1] respectively, to Prof. H. Kosaki for sending his preprint [K2] and unpublished work, and to Prof. P. H. Loi for sending his preprints [Lo1–2] and helpful communications on the first version of this paper and [Lo2]. The author also thanks the referee for corrections of typographical errors and comments for improving the exposition.

§1 Preliminaries

In [HK1], Hamachi and Kosaki compared flows of weights for a factor and a subfactor with finite index by constructing the crossed product algebras by modular automorphism groups. (See [CT] for background on flows of weights.) We show that their method can be extended to more general setting where actions commute with conditional expectations.

In this section, we deal with the following assumption.

Assumption 1.1. Let $N \subset M$ be σ-finite von Neumann algebras, E a faithful normal conditional expectation from M onto N, and α_g an action of a separable locally compact abelian group G on M with the following properties.

(1) $\alpha_g \cdot E = E \cdot \alpha_g$.

(2) The map $x \in M_+ \mapsto E(x) - \lambda x$ is completely positive for some $\lambda > 0$.

Note that condition (1) implies that N is globally invariant under the action α_g. (A necessary and sufficient condition for existence of E with (1) is given in [Hv].) We denote $\alpha_g|_N$ simply by α_g if no confusion arises. We are interested in mainly factors, but we do not assume M and N are factors here because we will apply the results to ultraproduct algebras and crossed product algebras later. If M and N are factors, then the largest possible λ in (2) is equal to $(\mathrm{Index}\, E)^{-1}$, where $\mathrm{Index}\, E$ denotes Kosaki's index of a conditional expectation [K1], by Kosaki's unpublished work or [BDH]. (See the arguments preceding Definition 1.7). The commutativity of G is used only for considering dual actions. Note that if one conditional expectation from a factor M onto a subfactor N satisfies (2) for some $\lambda > 0$, then all the others also satisfy it for some another $\lambda > 0$ by [Hi1]. Before working on the assumptions, we list easy propositions on the above (1).

Proposition 1.2. *Suppose M and N are factors and α_g is an action of G on M which fixes N globally. Condition (1) of Assumption 1.1 holds if one of the following is valid.*

(a) $N' \cap M = \mathbf{C}$.

(b) *M is of type II and E is the conditional expectation with respect to the trace.*

(c) *E has the minimal index in the sense of Hiai* [**Hi1**].

Proof. Because $\alpha_g^{-1} \cdot E \cdot \alpha_g$ is also a conditional expectation from M onto N for each $g \in G$, we get the conclusion by uniqueness of the appropriate conditional expectations ([C1, Théorème 1.5.5], [Hi1, Theorem 1(1)]). (Also see the proof of [Hi2, Theorem 2.8] as to (c).) Q.E.D.

If one conditional expectation $E : M \to N$ has a finite index for factors M and N, then there exists the conditional expectation having the minimal index value by [Hi1], thus Lemma 1.2 (c) shows that Assumption 1.1 (1) is not very restrictive.

Proposition 1.3. *Let $N \subset M$ be σ-finite von Neumann algebras, and E a faithful normal conditional expectation from M onto N. Take a normal faithful state φ on N and set $\psi = \varphi \cdot E \in M_*^+$. With $G = \mathbf{R}$ and $\alpha_g = \sigma_t^\psi$, $g = t \in \mathbf{R}$, we get (1) in Assumption 1.1.*

Proof. Immediate by [C1, Lemme 1.4.3], [S, Corollary 10.5]. [T1]. Q.E.D.

Proposition 1.3 shows that our situation is a generalization of [Lo1], [HK1], [K2].

We now fix a normal faithful state φ on N and set $\psi = \varphi \cdot E \in M_*^+$.

Lemma 1.4. *Under Assumption 1.1, there exists a normal faithful conditional expectation $\hat{E}$ from $M \rtimes_\alpha G$ onto $N \rtimes_\alpha G$ with the following properties.*

(1) $\hat{E} \cdot \hat{\alpha}_p = \hat{\alpha}_p \cdot \hat{E}$, *where $\hat{\alpha}_p$ denotes the dual action for $p \in \hat{G}$.*

(2) $\hat{E}|_M = E$.

(3) $\hat{E} - \lambda Id$ *is completely positive on $M \rtimes_\alpha G$.*

Proof. Regarding $M \rtimes_\alpha G$ as a subalgebra of $M \bar{\otimes} L(L^2(G))$ as usual, we set $\hat{E} = (E \otimes id|_{L(L^2(G))})|_{M \rtimes_\alpha G}$ as in [Hi3, §5]. It is clear that this $\hat{E}$ satisfies (1) and (2), and Assumption 1.1 (2) implies (3) because $\hat{E}$ is a restriction of $E \otimes id|_{L(L^2(G))}$. (In [Hi3], this $\hat{E}$ is called the canonical extension of E.) Q.E.D.

Assume Assumption 1.1 (2). Represent M in a standard form $(M, H \equiv H_{\varphi \cdot E}, J, P)$ (see [Ha1], [A]), where φ is a fixed normal faithful state on N. We may assume $\varphi \cdot E = \omega_{\xi_0}$, a vector state, with a cyclic and separating vector $\xi_0 \in H$. We define

e_N and construct $M_1 = \langle M, e_N \rangle$ as in [K1, §3]. (Here M and N do not have to be factors.) Then it is shown that $E^{-1}(1)$ is a bounded element in $\mathcal{Z}(M)$ as in [BDH]. Indeed, by $uE^{-1}(1)u^* = E^{-1}(u1u^*) = E^{-1}(1)$ for $u \in \mathcal{U}(M')$, we first get $E^{-1}(1) \in \widehat{\mathcal{Z}(M)}_+$. The proof of [K1, Lemma 3.1] works even when M and N are not factors, thus we get $E^{-1}(1) \geq E^{-1}(e_N) = 1$. Then applying $E^{-1}(J \cdot J)$ to the equality in [BDH, Remarques 3.4 (iii)] with [BDH, Corollaire 2.14] and using $E^{-1}(e_N) = 1$ and [BDH, Théorème 3.5 b)], we thus get $1 \leq E^{-1}(1) \leq \lambda^{-1}$ in $\mathcal{Z}(M)_+$.

Definition 1.5. Suppose Assumption 1.1 (2) holds.

(1) Define an operator valued weight $E_M : M_1 \to M$ by

$$E_M(x) = (E^{-1}(1))^{-1} J E^{-1}(JxJ) J, \qquad x \in M_1.$$

(2) Choose the implementing unitary u_g on H for α_g as in [Ha1, Theorem 3.2] and define an action $\tilde{\alpha}$ of G by $\tilde{\alpha}_g = \mathrm{Ad}(u_g)$ for $g \in G$.

Extension of an automorphism α as above is independently studied in [Lo2]. For type II_1 case, this was also mentioned in [W, page 227].

The method of the following proof is essentially same as that of Lemma 1.3 in [K1].

Lemma 1.6. *Suppose E is a faithful normal conditional expectation from a von Neumann algebra M onto a subalgebra N and α is an automorphism of M fixing*

N globally. Then $(\alpha^{-1} \cdot E \cdot \alpha)^{-1} = \tilde{\alpha}^{-1} \cdot E^{-1} \cdot \tilde{\alpha}$*, where* $\tilde{\alpha}$ *is defined as in Definition 1.5 (2).*

Proof. Choose an implementing unitary u on H as in Definition 1.5 (2). For weights ψ on M and χ on M', set $\tilde{\psi}(x) = \psi(uxu^*)$, $x \in M_+$, $\tilde{\chi}(x) = \chi(uxu^*)$, $x \in M'_+$, and $\tilde{E}(x) = u^* E(uxu^*)u$, $x \in M$. Then by the same kind of argument as the proof of [K1, Lemma 1.3], we get $u\dfrac{d\tilde{\psi}}{d\tilde{\chi}}u^* = \dfrac{d\psi}{d\chi}$. Choosing a faithful state φ on N and applying the same kind of argument as the proof of [K1, Lemma 1.3] again, we get $(E^{-1})^{\sim} = (\tilde{E})^{-1}$ as desired. Q.E.D.

The following lemma shows that the new quadruple $(M, M_1, \hat{E}, \tilde{\alpha})$ satisfies Assumption 1.1 again.

Lemma 1.7. *Assume Assumption 1.1. For* $\tilde{\alpha}$ *in Definition 1.5, we have the following.*

(1) E_M *is a conditional expectation from* M_1 *onto* M*.*

(2) *The map* $x \in M_1 \mapsto E_M(x) - \lambda x$ *is completely positive.*

(3) $\tilde{\alpha}_g|_M = \alpha_g$.

(4) $\tilde{\alpha}_g(M_1) = M_1$.

(5) $\tilde{\alpha}_g \cdot E_M = E_M \cdot \tilde{\alpha}_g$.

(6) $\tilde{\alpha}_g(e_N) = e_N$.

(7) $\tilde{\alpha}_g(JxJ) = J\tilde{\alpha}_g(x)J$.

Proof. (1) If $x \in M$, we get

$$
\begin{aligned}
E_M(x^*x) &= (E^{-1}(1))^{-1} J E^{-1}(Jx^*xJ)J \\
&= (E^{-1}(1))^{-1} J (Jx^*J) E^{-1}(1)(JxJ)J \\
&= (E^{-1}(1))^{-1} x^* J E^{-1}(1) J x \\
&= x^*x.
\end{aligned}
$$

(2) It is enough to show $E_M(x) \geq \lambda x$ for $x \in M_1$. (Use $E \otimes id_n$ instead of E for general cases.) We may assume that x is of the form y^*y, $y = a_1 e_N b_1 + \cdots + a_k e_N b_k$, $a_1, b_1, \ldots, a_k, b_k \in M$. Then $\lambda^{-1} \geq E^{-1}(1) \in \mathcal{Z}(M)_+$ implies $E_M(y^*y) \geq \lambda y^*y$.

(3) Trivial.

(4) We get the conclusion by $M_1 = JN'J$, $u_g J = J u_g$, and $\alpha_g(N) = N$ ([K1, Lemma 3.2], [Ha1, Theorem 3.2], and Assumption 1.1).

(5) By (3), we have the desired equality on M. Thus it is enough to show $\tilde{\alpha}_g \cdot E_M(e_N) = E_M \cdot \tilde{\alpha}_g(e_N)$.

By Lemma 1.6, we get $E^{-1} = (\alpha_g^{-1} \cdot E \cdot \alpha_g)^{-1} = \tilde{\alpha}_g^{-1} \cdot E^{-1} \cdot \tilde{\alpha}_g$. Thus

$$
\begin{aligned}
E_M(\tilde{\alpha}_g(e_N)) &= (E^{-1}(1))^{-1} J E^{-1}(J u_g (e_N) u_g^* J)J \\
&= (E^{-1}(1))^{-1} J E^{-1}(\tilde{\alpha}_g(e_N))J \\
&= (E^{-1}(1))^{-1} J \tilde{\alpha}_g(E^{-1}(e_N))J \\
&= (E^{-1}(1))^{-1},
\end{aligned}
$$

by [K1, Lemma 3.1]. On the other hand, we get $\tilde{\alpha}_g(E^{-1})(1) = E^{-1}(\tilde{\alpha}_g(1)) = E^{-1}(1)$, hence

$$\tilde{\alpha}_g(E_M(e_N)) = \tilde{\alpha}_g((E^{-1}(1))^{-1}) = (E^{-1}(1))^{-1}.$$

These imply the desired equality.

(6) Simply write α and u for α_g and u_g for a fixed $g \in G$. As in the proof of [K1, Lemma 3.1], set $K = \overline{N\xi_0}$ so that e_N is the orthogonal projection onto K. Then a vector $\eta_0 \in K$ in the self-dual cone for N exists so that $\varphi \cdot \alpha|_N = \omega_{\eta_0}$. By (1) of Assumption 1.1, we get $\varphi \cdot E \cdot \alpha = \omega_{\eta_0}$. By [T1, §4], the unitary involutions for M and N are the same, thus η_0 is in the self-dual cone for M. Because $e\eta_0 = \eta_0$, we get $\varphi \cdot E \cdot \alpha = \omega_{\eta_0}$, hence the implementing unitary u for α is given by $u(x\eta_0) = \alpha(x)\xi_0$, $x \in M$. (See [A, Theorem 11].) Then

$$\begin{aligned} ue_N u^*(x\xi_0) &= ue_N(\alpha^{-1}(x)\eta_0) \\ &= ue_N\alpha^{-1}(x)e_N\eta_0 \\ &= uE(\alpha^{-1}(x))\eta_0 \\ &= u\alpha^{-1}(E(x))\eta_0 \\ &= E(x)\xi_0 = e_N x\xi_0. \end{aligned}$$

(7) Trivial. Q.E.D.

If $G = \mathbf{R}$ and α_g is given by the modular automorphism group $\sigma^{\varphi \cdot E}$, then we have two ways of extending this to M_1; $\widetilde{\sigma^{\varphi \cdot E}}$ on M_1 as in Definition 1.5, and the modular automorphism group of a state $\varphi \cdot E \cdot E_M$ on M_1. Lemma 1.7 (6) and [K1, Lemma 5.1] show that these two extensions coincide.

Lemma 1.8. *Under Assumption 1.1, the crossed product algebra $M_1 \rtimes_{\tilde{\alpha}} G$ is the basic extension of $\hat{E} : M \rtimes_\alpha G \to N \rtimes_\alpha G$ and we get $\hat{\alpha}^\sim = \tilde{\alpha}^\frown$.*

Proof. We represent M in a standard form on $\mathcal{H}$, and consider everything in the Hilbert space $\mathcal{H} \otimes L^2(G) \cong L^2(G, \mathcal{H})$. Let $\tilde{J}$ be the modular conjugation for $M \rtimes_\alpha G$ and $\hat{e}$ be the projection corresponding to $\hat{E}$. Then it is enough to show $\tilde{J}(M_1 \rtimes_{\tilde{\alpha}} G)\tilde{J} = (N \rtimes_\alpha G)'$ by [K1, Lemma 3.2]. For $\xi \in L^2(G, \mathcal{H})$, $\tilde{J}$ is given by $(\tilde{J}\xi)(g) = u_g^* J\xi(g^{-1})$ by [Ha2, Lemma 2.8]. On the other hand, we know that $(N \rtimes_\alpha G)' = \langle N' \otimes \mathbf{C}, U^*(\mathbf{C} \otimes \mathcal{R}(G))U\rangle$ by [Ha2, Theorem 2.1], where $\mathcal{R}(G)$ is a von Neumann algebra generated by the right regular representation of G and the unitary U on $L^2(G, \mathcal{H})$ is given by $(U\xi)(g) = u_g\xi(g)$. For $x \in N'$ and $g \in G$, an easy computation shows

$$(\tilde{J}(x \otimes 1)\tilde{J}\xi)(g) = \tilde{\alpha}_g^{-1}(JxJ)\xi(g),$$

$$\tilde{J}U^*(1 \otimes \rho_g)U\tilde{J} = 1 \otimes \lambda_g,$$

where ρ and λ denote the right and left regular representation of G. Because $M_1 = JN'J$ by [K1, Lemma 3.2], this shows $\tilde{J}(N \rtimes_\alpha G)'\tilde{J} = M_1 \rtimes_{\tilde{\alpha}} G$.

It is easy to see $\hat{\alpha}^\sim = \tilde{\alpha}^\frown$ now. Q.E.D.

The following is a generalization of [HK1, Theorem]. With above preliminaries, the same method as in [HK] works, but we include a proof for the sake of completeness.

Theorem 1.9. *Under Assumption 1.1, let $\mathcal{Z}(M \rtimes_\alpha G) \cong L^\infty(X_M, \mu_M)$ and $\mathcal{Z}(N \rtimes_\alpha G) \cong L^\infty(X_N, \mu_N)$. Then there exists a non-singular action T of $\hat{G}$ on a measure space (X, μ) such that the both $L^\infty(X_M)$ and $L^\infty(X_N)$ are regarded as subalgebras of $L^\infty(X)$ and there exists a conditional expectation E_1 from $L^\infty(X)$ onto $L^\infty(X_M)$ [resp. E_2 from $L^\infty(X)$ onto $L^\infty(X_N)$], with Pimsner-Popa estimate with the constant λ, intertwining T_p and $\hat{\alpha}_p$, $p \in \hat{G}$.*

Proof. By Lemma 1.4 (3) we have the Pimsner-Popa estimate for $\hat{E}$. The restriction of this $\hat{E}$ on $\mathcal{Z}((M \rtimes_\alpha G) \cap (N \rtimes_\alpha G)') \cong L^\infty(X, \mu)$ gives a map from $\mathcal{Z}((M \rtimes_\alpha G) \cap (N \rtimes_\alpha G)')$ to $\mathcal{Z}(N \rtimes_\alpha G)$, which still satisfies the inequality.

Lemma 1.7 allows us to apply the same arguments for $M \subset M_1$. Then Lemma 1.8 implies that $(M \rtimes_\alpha G) \cap (N \rtimes_\alpha G)'$ is anti-isomorphic to $(M_1 \rtimes_{\tilde{\alpha}} G) \cap (M \rtimes_\alpha G)'$ and this anti-isomorphism intertwines α and $\tilde{\alpha}$, hence we get the conclusion.

Q.E.D.

§2 Innerness, central freeness, and so on

In this section, we apply the results in §1 to show that an automorphism on a factor M fixing a subfactor N with finite index globally is similar to its restriction on N in terms of innerness, central triviality, approximate pointwise innerness, and pointwise innerness.

First note that if the action α_g is centrally ergodic both on M and N, we get the following, which generalizes [HK1, Theorem].

Theorem 2.1. *Suppose the action α_g is centrally ergodic both on M and N in addition to Assumption 1.1. Set $\mathcal{Z}(M \rtimes_\alpha G) \cong L^\infty(X_M, \mu_M)$ and $\mathcal{Z}(N \rtimes_\alpha G) \cong$*

$L^\infty(X_N, \mu_N)$. Then there exists a non-singular action T of $\hat{G}$ on a measure space (X, μ) satisfying the following.

(1) *X is isomorphic to $X_M \times \{1, 2, \dots, m\}$ and $X_N \times \{1, 2, \dots, n\}$ as a measure space, where m, n are integers with $m, n \le \operatorname{Index} E$.*

(2) *The projection maps π_M and π_N from $X \cong X_M \times \{1, 2, \dots, m\} \cong X_N \times \{1, 2, \dots, n\}$ onto X_M and X_N intertwines T and the actions given by $\hat{\alpha}$ on X_M and X_N.*

Proof. Apply the proof of Theorem 1.9. The action T is given by the dual action of α on $\mathcal{Z}((M \rtimes_\alpha G) \cap (N \rtimes_\alpha G)')$. Then the disintegration for $\mathcal{Z}((M \rtimes_\alpha G) \cap (N \rtimes_\alpha G)') \supset \mathcal{Z}(N \rtimes_\alpha G)$ implies $L^\infty(X, \mu) = \int_{X_N}^{\oplus} \mathcal{A}(x)\, d\mu_N(x)$, where $\mathcal{A}(x)$ is an abelian von Neumann algebra for each $x \in X_N$. (See [T2, Theorem 8.21] for instance.) Let Y be the subset of $x \in X_N$ such that $\mathcal{A}(x)$ has a partition of unity into c nonzero mutually orthogonal projections with $c > \lambda^{-1}$. Then the Pimsner-Popa inequality implies $\mu_N(Y) = 0$. Hence each $\mathcal{A}(x)$ is atomic, and the number of atoms is less than or equal to $\operatorname{Index} E$. Lemma 1.4 (1) implies that the number of atoms is invariant under the ergodic action induced by $\hat{\alpha}$ on X_N. (Because α is centrally ergodic, the dual action is also centrally ergodic by [JT, Proposition 2.1.13].) Thus X is isomorphic to $X_N \times \{1, 2, \dots, n\}$ with $n \le \lambda^{-1}$. We apply the same arguments for $M \subset M_1$ again to get the conclusion. Q.E.D.

The next shows the relations of Connes spectra of α and $\alpha|_N$.

Corollary 2.2. *Assume α is centrally ergodic both on M and N in Assumption 1.1. Then there exists an integer k such that $k\Gamma(\alpha) \subset \Gamma(\alpha|_N)$ and $k\Gamma(\alpha|_N) \subset \Gamma(\alpha)$.*

Proof. Set $k = [\lambda^{-1}]!$. If $p \in \Gamma(\alpha|_N) \subset \hat{G}$, then disintegration in Theorem 2.1 shows that $T_{kp} = id$ on X, hence $kp \in \Gamma(\alpha)$. We get the converse inclusion, too.

Q.E.D.

Because $\hat{\alpha}$ on $\mathcal{Z}(M \rtimes_\alpha G)$ is given by the characteristic invariants (see [J1], [JT, Proposition 2.1.13]), this gives a restriction on a relation of characteristic invariants of α and $\alpha|_N$. In the simplest case, we get the following, which generalizes [Lo1, Corollary 2.5.9], a result on T-sets.

Theorem 2.3. *Suppose N is a subfactor with finite index of a factor M and α is an automorphism of M with $\alpha(N) = N$. Assume α [resp. $\alpha|_N$] is inner. Set $p = p_o(\alpha|_N)$ [resp. $p = p_o(\alpha)$], the outer period, and $\gamma = \gamma(\alpha|_N)$ [resp. $\gamma = \gamma(\alpha)$], the obstruction. (See* [**C3**] *for definitions.) Define q to be the least positive integer such that $\gamma^q = 1$, $q \le p$. Then $p > 0$ and $pq \le \operatorname{Index} E$.*

Proof. Choose a conditional expectation $E : M \to N$ with the minimal index and apply Theorem 2.1 with $\lambda = (\operatorname{Index} E)^{-1}$. The center of $N \rtimes_\alpha \mathbf{Z}$ [resp. $M \rtimes_\alpha \mathbf{Z}$] is generated by $(uU^{-p})^q$, where $\operatorname{Ad}(u) = \alpha^p|_N$ [resp. $\operatorname{Ad}(u) = \alpha^p$], and U is the implementing unitary in the crossed product algebra. Because of compactness of $\mathbf{T} = \hat{\mathbf{Z}}$, the action T of $\mathbf{T}$ in Theorem 2.1 is (translation) $\times\, id$ on $\mathbf{T} \times \{1, \dots, m\}$. By the n-to-1 projection π_N, we get a translation of $\mathbf{T}$ with speed pq on $\mathbf{T}$, thus we get $mpq = n$ [resp. $npq = m$]. Then we get the conclusion. Q.E.D.

Corollary 2.4. *Suppose N is a subfactor with finite index of a factor M and α is an automorphism of M with $\alpha(N) = N$. Then α is free if and only if $\alpha|_N$ is so.*

Proof. Immediate by Theorem 2.3. Q.E.D.

Loi independently obtained this corollary in [Lo2, Corollary 5.2] by a different method.

Next we work on centrally trivial automorphisms. Several properties of centrally trivial automorphisms were studied in [C3, KST, O, ST], and it has been known that this class of automorphisms is important for classification of group actions. (For the ultraproduct algebras, see [C2] or [O, Chapter 5].)

Definition 2.5. Let M and N be σ-finite von Neumann algebras, E a faithful normal conditional expectation from M onto N, and ω a free ultrafilter on $\mathbf{N}$. Define $M_{\omega,N}$ to be the quotient of all the bounded sequences (x_n) in M such that $\|[x_n, \varphi \cdot E]\| \to 0$, $n \to \omega$, for all $\varphi \in N_*$, by the two sided ideal of sequences converging $*$-strongly to 0 when $n \to \omega$.

Lemma 2.6. *The C^*-algebra $M_{\omega,N}$ in Definition 2.5 is a finite von Neumann algebra.*

Proof. The same proof as [C2, Theorem 2.9] works. Q.E.D.

Theorem 2.7. *Suppose N is a subfactor with finite index of a factor M and α is an automorphism of M with $\alpha(N) = N$. If α [resp. $\alpha|_N$] is centrally trivial, then there exists a positive integer $p \le \operatorname{Index} E$ with $\alpha^p|_N \in Cnt(N)$ [resp. $\alpha^p \in Cnt(M)$].*

Proof. Choose a conditional expectation $E : M \to N$ with the minimal index again. First assume $\alpha|_N \in \mathrm{Cnt}(N)$.

Note that α acts on $M_{\omega,N}$. We denote α^ω for this action. Then M_ω and N_ω are subalgebras of $M_{\omega,N}$ and $\alpha^\omega(M_\omega) = M_\omega$ and $\alpha^\omega(N_\omega) = N_\omega$. Applying E term by term, we get a normal faithful conditional expectation E^ω from $M_{\omega,N}$ onto N_ω

with the Pimsner-Popa estimate with constant $\lambda = (\operatorname{Index} E)^{-1}$. Thus Assumption 1.1 is satisfied for this E^ω. (cf. Proposition 1.11 of [PP1].)

We claim next that there exist a positive integer $p \le \operatorname{Index} E$ and a non-zero $a \in M_{\omega,N}$ such that $(\alpha^\omega)^p(x)a = ax$ for all $x \in M_{\omega,N}$. Suppose not. Then the center of the crossed product algebra $M_{\omega,N} \rtimes_{\alpha^\omega} \mathbf{Z}$ is contained in $\{\sum_{k=0,|k|>\operatorname{Index} E} a_k U^k \mid a_k \in M_{\omega,N}\}$, where U is the implementing unitary of the crossed product $M_{\omega,N} \rtimes_{\alpha^\omega} \mathbf{Z}$. Thus there exists no non-zero a in this center such that $\widehat{\alpha^\omega}_p(a) = p^k a$, $p \in \mathbf{T} \subset \mathbf{C}$, for any $0 < k \le \operatorname{Index} E$. On the other hand, the dual action of $\mathbf{T}$ on $\mathcal{Z}(N_\omega \rtimes_{\alpha_\omega} \mathbf{Z}) \cong \mathcal{Z}(N_\omega) \otimes L^\infty(\mathbf{T})$ is given by $id \times$ (translation). By compactness of $\mathbf{T}$, we know that the space X and the action T in Theorem 1.9 are of the form $X \cong Y \times \mathbf{T}$, $T = id \times$ (translation). Decomposing X_M into speed l components for each $l \in \mathbf{Z}$, we get a contradiction to the property of π_M.

The same proof as in [C3, Proposition 2.1.2] shows that $(\alpha^\omega)^p$ is trivial on M_ω now. (It does not matter that a is not in M_ω here.)

On the other hand, assume α is centrally trivial on M. Then by Lemma 1.7 and the above proof, there exists a positive integer $p \le \operatorname{Index} E$ such that $\tilde{\alpha}^p$ is centrally trivial on M_1.

Note that $\alpha|_N$ is conjugate to $\tilde{\alpha}^{e_N}$ by Lemma 1.7 (6). If N is of type II_1, then M_1 is also of type II_1, and [C4, Proposition 4.2, Theorem 4.3] imply that $\alpha^p|_N$ is centrally trivial. If N is of type II_∞, then M_1 is also of type II_∞ and $\mathrm{tr}_{M_1}(e_N) = \infty$, hence e_N is equivalent to 1 in M_1. If N is of type III, then e_N is equivalent to 1 in M_1 again. In the both cases, choose a partial isometry $v \in M_1$ with $vv^* = e_N$ and $v^*v = 1$. Then $\mathrm{Ad}(v^*\tilde{\alpha}(v)) \cdot \tilde{\alpha}$ is conjugate to $\tilde{\alpha}^{e_N}$, hence to $\alpha|_N$. This implies that $\alpha^p|_N$ is centrally trivial. Q.E.D.

Corollary 2.8. *Suppose N is a subfactor with finite index of a factor M and α is an automorphism of M with $\alpha(N) = N$. Then α is centrally free if and only if $\alpha|_N$ is centrally free.*

Proof. Immediate by Theorem 2.7. Q.E.D.

Connes and Takesaki introduced a continuous homomorphism "module" from $\mathrm{Aut}(M)$ to $\mathrm{Aut}(\mathcal{F}(M))$ in [CT]. (Here M denotes a infinite separable factor and $\mathcal{F}(M)$ its flow of weights.) For AFD factors, the module is important as a tool for distinguishing approximately inner automorphisms. We show that this *mod* is compatible with the common finite extension of flows of weights of [HK1].

Theorem 2.9. *Suppose N is a subfactor with finite index of a separable factor M of type III and α is an action of the discrete abelian group G on M with $\alpha_g(N) = N$ for all g. Set $\mathcal{Z}(M \rtimes_\sigma \mathbf{R}) \cong L^\infty(X_M, \mu_M)$ and $\mathcal{Z}(N \rtimes_\sigma \mathbf{R}) \cong L^\infty(X_N, \mu_N)$, where σ denotes the modular automorphism groups $\sigma^{\psi \cdot E}$ for a weight ψ on N. Then there exists a non-singular action T of $G \times \mathbf{R}$ on a measure space (X, μ) satisfying the following.*

(1) *X is isomorphic to $X_M \times \{1, 2, \ldots, m\}$ and $X_N \times \{1, 2, \ldots, n\}$ as a measure space, where m, n are integers with $m, n \leq \mathrm{Index}\, E$.*

(2) *The projection maps π_M and π_N from $X \cong X_M \times \{1, 2, \ldots, m\} \cong X_N \times \{1, 2, \ldots, n\}$ onto X_M and X_N intertwines T and the actions given by the product of mod(α) and the flow of weights on X_M and X_N.*

First, we show a lemma.

Lemma 2.10. *Assume the assumption of Theorem 2.9. Without loss of generality, we may assume that there is a dominant weight ψ on N with the following properties:*

(1) *The weight ψ is invariant under $\alpha|_N$.*

(2) *The weight $\psi \cdot E$ is dominant on M and invariant under α.*

(3) *The action α extends to a pair $\tilde{M} \equiv M \rtimes_{\sigma^{\psi \cdot E}} \mathbf{R} \supset \tilde{N} \equiv N \rtimes_{\sigma^{\psi}} \mathbf{R}$ so that the extended action satisfies Assumption 1.1 with $\hat{E}$ of Lemma 1.4.*

Proof. Choose a conditional expectation $E : M \to N$ with the minimal index again. By Lemma 1.4, we may replace α, E, M, and N by $\hat{\hat{\alpha}}$, $\hat{\hat{E}}$, $M \bar{\otimes} L(L^2(G))$, and $N \bar{\otimes} L(L^2(G))$, respectively. Then apply [ST, Lemma 5.10] to get a dominant weight ψ on N which is invariant under the (new perturbed) action $\alpha|_N$. (The cocycle to perturb α is chosen within N, hence the perturbed action still satisfies Assumption 1.1 (1).) Then $\psi \sim \lambda\psi$ for all $\lambda > 0$ implies $\psi \cdot E \sim \lambda\psi \cdot E$ on M, hence $\psi \cdot E$ is also dominant. (See [CT, Theorem II.1.1, Definition II.1.2].) The action α now extends to the crossed product by $\mathbf{R}$ as in [HS1, Lemma 13.2]. It is easy to see that this satisfies (3). Q.E.D.

Proof of Theorem 2.9. We may assume (1), (2), and (3) of Lemma 2.10. The action α extends to M_1 as in Lemma 1.7 and then it extends to the crossed product $\tilde{M}_1$ by the modular automorphism group as in [HS1, Lemma 13.2]. On the other hand, α extends as in (3) of Lemma 2.10 and it extends to $\tilde{M}_1$ (see Lemma 1.8) as in Lemma 1.7. It is easy to see that these two extensions coincide. Then we can apply the argument of [HK1] (or Theorem 2.1 here) to get the conclusion. Q.E.D.

Haagerup and Størmer showed that $\mathrm{mod}(\alpha)$ is trivial if and only if α is approximately pointwise inner in [HS1, Corollary 13.5]. (See [HS1, Definition 12.3] for the definition.) Thus we get the following corollary.

Corollary 2.11. *Suppose N is a subfactor with finite index of a separable factor M of type III and α is an action of the discrete abelian group G on M with $\alpha_g(N) = N$ for all g. Define*

$$P(\alpha) = \{g \in G \mid \alpha_g \text{ is approximately pointwise inner.}\}.$$

Then there exists an integer k such that $kP(\alpha) \subset P(\alpha|_N)$ and $kP(\alpha|_N) \subset P(\alpha)$.

Proof. Apply the same argument as the proof of Corollary 2.2. Q.E.D.

For AFD factors, the module is trivial if and only if the automorphism is approximately inner as announced in [C5, section 3.8]. (See [KST, Theorem 1 (i)] for the proof.) Thus we get the following corollary immediately.

Corollary 2.12. *Suppose N is an AFD subfactor with finite index of a separable factor M of type III and α is an action of the discrete abelian group G on M with $\alpha_g(N) = N$ for all g. Define $A(\alpha) = \{g \in G \mid \alpha_g \text{ is approximately inner.}\}$. Then there exists an integer k such that $kA(\alpha) \subset A(\alpha|_N)$ and $kA(\alpha|_N) \subset A(\alpha)$.*

Haagerup and Stømer also introduced the notion of pointwise inner automorphisms in [HS1, Definition 12.3] and studied them in [HS2]. In particular, they showed that an automorphism α of a separable factor M of type III_λ, $0 \le \lambda < 1$, is pointwise inner if and only if its canonical extension $\tilde{\alpha}$ to $M \rtimes_\sigma \mathbf{R}$ is inner ([HS2, Theorem 5.2]) and that an automorphism α of a separable factor of type III is an extended modular automorphism up to inner perturbation if and only if $\tilde{\alpha}$ is inner ([HS2, Proposition 5.4]). On the other hand, the flow of weights of the crossed product of a factor M of type III by an action α of a discrete group G is determined by the flow of weights of M,

$$H = \{g \in M \mid \alpha_g \text{ is an extended modular automorphism up to inner perturbation.}\},$$

and the restriction of α on H. (See [Se, Theorem] and [KT, Theorem 3.3]. In [KT], injectivity of the factor is assumed, but it is unnecessary.) This shows importance of this type of automorphisms. Note that for an automorphism of AFD factors, it is an extended modular automorphism up to inner perturbation if and only if it is centrally trivial as announced in [C5, section 3.8] (see [KST, Theorem 1 (ii)] for the proof). Now we get the following theorem for this type of automorphisms.

Theorem 2.13. *Suppose N is a subfactor with finite index of a separable factor M of type III and α is an action of the discrete abelian group G on M with $\alpha_g(N) = N$ for all g. Define*

$$D(\alpha) = \{g \in G \mid \alpha_g \textit{ is an extended modular automorphism up to inner perturbation.}\}.$$

Then there exists an integer k such that $kD(\alpha) \subset D(\alpha|_N)$ and $kD(\alpha|_N) \subset D(\alpha)$.

Proof. Extend α to $\tilde{M}$ and $\tilde{N}$ as in Lemma 2.10. Now we can apply Theorem 1.9 as in the proof of Theorem 2.7. By [KT, Lemma 3.2] and [HS2, Proposition 5.4], we conclude that if $\alpha_g|_N$ is an extended modular automorphism up to inner perturbation, then so is α_g^p on M for some $p \leq \operatorname{Index} E$. The conclusion now easily follows. Q.E.D.

Remark 2.14. It follows from Theorem 2.13 that for an automorphism α of a separable factor M of type III fixing a subfactor N with finite index globally, none of its nontrivial power is an extended modular automorphism up to inner perturbation if and only if we have the same property for $\alpha|_N$. An approach based on [Se] or [KT, Theorem 3.1, Theorem 4.1] can also be used for this. For example, suppose M is of type III_1 and none of non-trivial powers of α are extended modular automorphisms up to inner perturbation. Then α is free on M, hence free on N by Corollary 2.4. Now $M \rtimes_\alpha \mathbf{Z}$ and $N \rtimes_\alpha \mathbf{Z}$ are both factors and the index of the pair is finite. By [Se, Theorem], $M \rtimes_\alpha \mathbf{Z}$ is of type III_1, hence Loi's result [Lo1] (or Corollary 2.2 here) implies that $N \rtimes_\alpha \mathbf{Z}$ is also of type III_1. If $\alpha^p|_N = \mathrm{Ad}(u) \cdot \sigma_t$ for some $p > 0$, then the flow of weights of $N \rtimes_\alpha \mathbf{Z}$ is given by $L^\infty(\mathbf{T})^S$ for some rational rotation S as in the proof of [KT, Theorem 4.1], hence $N \rtimes_\alpha \mathbf{Z}$ is of type III_λ for some $\lambda \in]0, 1[$, which is a contradiction.

Remark 2.15. For AFD factors, an automorphism is centrally trivial if and only if it is an extended modular automorphism up to inner perturbation. ([C5, section 3.8], [KST, Theorem 1 (ii)].) Of course, above Theorem 2.13 is compatible with Theorem 2.7 for this case.

By [HS2, Theorem 5.2], we get the following corollary immediately.

Corollary 2.16. *Suppose N is a subfactor with finite index of a separable factor M of type III_λ, $0 \leq \lambda < 1$, and α is an action of the discrete abelian group G on M with $\alpha_g(N) = N$ for all g. Define $I(\alpha) = \{g \in G \mid \alpha_g$ is pointwise inner.$\}$. Then there exists an integer k such that $kI(\alpha) \subset I(\alpha|_N)$ and $kI(\alpha|_N) \subset I(\alpha)$.*

§3 Application to Loi's classification of automorphisms of subfactors

This section consists of remarks on Loi's paper [Lo2] to the effect that the irreducibility assumption $N' \cap M = \mathbf{C}$ in his results can be removed.

Basic observation in [Lo2] is that Connes' method in [C3] to show uniqueness of a centrally free and approximately inner automorphism of a McDuff factor works in the subfactor setting where $M \bar{\otimes} \mathcal{R} \cong M$, $\overline{\mathrm{Int}}(M)$, $\mathrm{Aut}(M)$ and $M' \cap M^\omega$ are replaced by $(N \subset M) \cong (N \bar{\otimes} \mathcal{R} \subset M \bar{\otimes} \mathcal{R})$, $\overline{\mathrm{Int}}(M, N)$, $\mathrm{Aut}(M, N)$, and $M' \cap N^\omega$ respectively. (Here $\mathrm{Aut}(M, N) = \{\alpha \in \mathrm{Aut}(M) \mid \alpha(N) = N\}$ and $\overline{\mathrm{Int}}(M, N)$ is a closure of $\{\mathrm{Ad}(u) \mid u \in \mathcal{U}(N)\}$ in $\mathrm{Aut}(M, N)$, and $\mathcal{R}$ denotes the AFD type II_1 factor.)

In [Lo2, Proposition 4.4], Loi proved the following proposition under an additional condition $N' \cap M = \mathbf{C}$ for applying Connes' non-commutative Rohlin method in [C3] to $M' \cap N^\omega$. This is used in [Lo2, Proposition 6.1] to show splitting of the AFD factor of type III_λ. (See Proposition 3.7 below.) The readers is referred to [O2] or [P] for definitions of finite depth, tunnel, and other related notions.

Proposition 3.1. *Let $N \subset M$ be AFD factors of type II_1 of finite index and finite depth, and $\alpha \in Aut(M, N)$. If α is free on M, then the restriction of α_ω onto $M' \cap N^\omega$ is also free.*

For the proof, Loi uses a theorem of Ocneanu announced in [O2, page 137, Theorem b)], whose proof has not yet been published, to the effect that $M' \cap N^\omega$ is a subfactor of M_ω with finite index. (Ocneanu recently announced more detailed study of the inclusion $M' \cap N^\omega \subset M_\omega$.) Here we give a more direct proof to the above proposition based on our results and Popa's approach [P] without assuming $N' \cap M = \mathbf{C}$.

Assume $N \subset M$ be AFD factors of type II_1 with finite index. We choose and fix a tunnel $\cdots N_2 \subset N_1 \subset N = N_0 \subset M = N_{-1}$ and Jones projections $\{e_{-j}\}_{j=0,1,2\cdots} \subset M$ such that $e_{-j} \in N_{j-1}$ and $N_j \cap \{e_{-j}\}' = N_{j+1}$. Then we get the following lemma first.

Lemma 3.2. *Fix a free ultrafilter ω on $\mathbf{N}$. Then $\cdots N_2^\omega \subset N_1^\omega \subset N^\omega \subset M^\omega$ is a tunnel for the ultraproduct II_1 factor M^ω and Jones projections $\{e_{-j}\}_{j=0,1,2\cdots} \subset M^\omega$ satisfy $N_j^\omega \cap \{e_{-j}\}' = N_{j+1}^\omega$.*

Proof. We prove this by induction on j. By [PP1, Proposition 1.10], we get

$$E_{N_{j+1}^\omega}(e_{-j-1}) = [M : N]^{-1} = [N_j^\omega : N_{j+1}^\omega]^{-1}.$$

Thus if we set $P = N_{j+1}^\omega \cap \{e_{-j-1}\}'$, then this is a II_1 subfactor of N_{j+1}^ω with the index equal to $[M : N]$ by [PP1, Corollary 1.8]. Now N_{j+2}^ω is included in P and have the same index in N_{j+1}^ω as P. This means $N_{j+2}^\omega = P = N_{j+1}^\omega \cap \{e_{-j-1}\}'$.

Q.E.D.

The following corresponds to (a part of) [O2, page 137, Theorem d)].

Lemma 3.3. *Let N be a subfactor of an AFD II_1 factor M with finite index and the generating property in the sense* [**P**] *that there exists a tunnel with $\bigvee_j (N_j' \cap M) = M$. We fix such a tunnel. Then $M' \cap N^\omega = \{e_0, e_{-1}, e_{-2}, \ldots\}' \cap N^\omega$ in M^ω.*

Proof. Because Jones projections are in M, it is trivial that we get

$$M' \cap N^\omega \subset \{e_0, e_{-1}, e_{-2}, \ldots\}' \cap N^\omega.$$

Choose an $x \in \{e_0, e_{-1}, e_{-2}, \dots\}' \cap N^\omega$, and $y \in N_j' \cap M$ for some j. Then $x \in N^\omega \cap \{e_0, \cdots, e_{-j+1}\}' = N_j^\omega$ by Lemma 3.2. This implies $xy = yx$. By $\bigvee_j (N_j' \cap M) = M$, we get $x \in M'$. Q.E.D.

We consider the conditional expectation $E_{M' \cap N^\omega}$ on M^ω with respect to the trace in the above situation and denote its restriction on M_ω just by E. For this E, Popa's method gives a Pimsner-Popa estimate as follows.

Lemma 3.4. *Let N be a subfactor of an AFD II_1 factor M with finite index and finite depth. We fix an arbitrary tunnel $\{N_k\}$. There exists a positive constant c such that for any positive element x in $N_k \vee \mathcal{Z}(N_k' \cap M)$, we get $E_{N_k}(x) \geq cx$.*

Proof. Let $\boldsymbol{n}^{(j)} = (n_k^{(j)})$ and $\boldsymbol{p}^{(j)} = (p_k^{(j)})$ be the vectors denoting the size and the trace of the minimal projection in each irreducible component of $N_j' \cap M$ respectively.

By [P, Theorem 3.8], there exists j_0 such that $p_k^{(2j+1)} = [M : N]^{j_0 - j} p_k^{(2j_0+1)}$ for $j \geq j_0$. Because $\boldsymbol{n}^{(j)} \cdot \boldsymbol{p}^{(j)} = 1$, the vector $[M : N]^{j_0 - j} \boldsymbol{n}^{(2j+1)}$ approaches to a Perron-Frobenius eigenvector of AA^t as $j \to \infty$, where A denotes the inclusion matrix as in [P, Corollary 2.3].

For $p_k^{(2j)}$, we have a similar result. Then we set $c = \inf_{j,k} n_k^{(j)} p_k^{(j)}$, which is positive. Q.E.D.

Lemma 3.5. *Let N be a subfactor of an AFD II_1 factor M with finite index and finite depth. We fix an arbitrary tunnel $\{N_k\}$. Then there is a positive constant c such that $E - c\mathrm{Id}$ is completely positive on M_ω.*

Proof. Without loss of generality, we may assume the tunnel $(N_j)_j$ has the generating property. (Such a tunnel can be chosen by [P, Theorem 4.9].)

First note that $(E \otimes id_n)(x) = \lim_{j\to\infty} E_{N_j^\omega \otimes M_n(\mathbf{C})}(x)$ for $x \in M^\omega \otimes M_n(\mathbf{C})$ by Lemmas 3.2 and 3.3. (Here all the conditional expectations are in M^ω with respect to the trace. This equality with $n = 1$ proves the formula in [O2, page 137, Theorem d)] without the assumption $N' \cap M = \mathbf{C}$.)

Fix j, n. Then there exists a positive constant c such that $E_{(N_j \vee (N_j' \cap M))^\omega \otimes M_n(\mathbf{C})}(x) \geq cx$ for all $x \in (M_\omega \otimes M_n(\mathbf{C}))_+$ by [P, Theorem 4.3], because $(N_j \otimes M_n(\mathbf{C}))_j$ is a tunnel in $M \otimes M_n(\mathbf{C})$. Because $x \in M_\omega \otimes M_n(\mathbf{C})$, we get $E_{(N_j \vee (N_j' \cap M))^\omega \otimes M_n(\mathbf{C})}(x) = E_{(N_j \vee \mathcal{Z}(N_j' \cap M))^\omega \otimes M_n(\mathbf{C})}(x)$. Then we get

$$E_{N_j^\omega \otimes M_n(\mathbf{C})}(x) = E_{N_j^\omega \otimes M_n(\mathbf{C})}(E_{(N_j \vee \mathcal{Z}(N_j' \cap M))^\omega \otimes M_n(\mathbf{C})}(x)) \geq cc'x,$$

where c' is given by Lemma 3.4 applied to the tunnel $(N_j \otimes M_n(\mathbf{C}))_j$ in $M \otimes M_n(\mathbf{C})$. This proves the conclusion. Q.E.D.

Now we can prove the Proposition.

Proof of Proposition 3.1. Because $\alpha \in \mathrm{Aut}(M, N)$, we know that α_ω acts on both M_ω and $M' \cap N^\omega$. The conditional expectation $E : M_\omega \to M' \cap N^\omega$ commutes with α_ω because the trace is α_ω-invariant, thus Assumption 1.1 is satisfied by Lemma 3.5. Suppose there exists $k > 0$ such that α_ω^k is not properly outer on $M' \cap N^\omega$. Then [Lo2, Proposition 4.2] implies that α_ω^k is trivial on $M' \cap N^\omega$, hence

$$(M' \cap N^\omega) \rtimes_{\alpha_\omega^k} \mathbf{Z} \cong (M' \cap N^\omega) \mathbin{\bar\otimes} L^\infty(\mathbf{T}).$$

On the other hand, we know that α_ω on M_ω is free by [C3], thus Theorem 1.9 implies that the dual action on $\mathcal{Z}((M' \cap N^\omega) \rtimes_{\alpha_\omega^k} \mathbf{Z})$ is trivial, which is a contradiction. (Also see the proof of Theorem 2.7.) Q.E.D.

Assume N is a subfactor of a II_1 factor M with finite index and fix a tunnel $\{N_j\}_{j\geq 0}$. To apply Connes' method in [C3], one needs a characterization of $\overline{\mathrm{Int}}(M,N)$. For this purpose, Loi defined a homomorphism Φ from $\mathrm{Aut}(M,N)$ to the group $\mathcal{G}$ of a system of automorphisms $\{\alpha_k\}_{k\geq 0}$ with

(1) each α_k is a trace preserving automorphism of $N_k' \cap M$;

(2) α_k preserves the inclusion $N_j' \cap N \subset N_j' \cap M$ for $0 \leq j \leq k$;

(3) α_k extends α_{k-1};

(4) $\alpha_k(e_{-j}) = e_{-j}$ for $0 \leq j \leq k-1$,

by fixing e_j's successively by inner perturbation in [Lo2, §5]. He showed the continuity of Φ under the assumption of $N' \cap M = \mathbf{C}$ in [Lo2, Proposition 5.3] to get an exact sequence characterizing $\overline{\mathrm{Int}}(M,N)$. (See Proposition 3.8 below.) The topology of $\mathcal{G}$ here is given by convergence for each k. We show this continuity without assuming $N' \cap M = \mathbf{C}$ as follows. Note that this Φ reminds us the module of Connes-Takesaki for automorphisms of type III factors [CT, page 554]. (See also [Lo2, Theorem 5.4] for similarity between the two.) The following proof is an analogue of the proof of Connes-Takesaki for continuity of *mod*.

Proposition 3.6. *The above $\Phi : Aut(M,N) \to \mathcal{G}$ is continuous if $[M:N] < +\infty$.*

Proof. The map $u \in \mathcal{U}(N) \to ue_0u^*$ is a continuous surjection onto

$$\{p \in \mathrm{Proj}(M) \mid E_N(p) = [M:N]^{-1}\}$$

by [PP1, Proposition 1.2] Because the both spaces are Polish, we get a Borel cross section Ψ such that $\Psi(p) \in \mathcal{U}(N)$ and $\Psi(p)e_0\Psi(p)^* = p$ by von Neumann measurable cross section Theorem, [T2, Theorem A.16]. Then α_1 is given by the restriction of $\mathrm{Ad}(\Psi(\alpha(e_0))^*)\cdot\alpha$ on the relative commutant $N_1'\cap M$. Because both $\mathrm{Aut}(M,N)$ and the group of α_1 are Polish groups, the map $\alpha\mapsto\alpha_1$ is continuous. (See [C2, Lemma 3.4] for example.) This method works for any k. Q.E.D.

The assumption $N'\cap M=\mathbf{C}$ is used only in [Lo2, Propositions 4.4, 5.3]. Thus we get the following generalization immediately.

By our Proposition 3.1 instead of [Lo2, Proposition 4.4], we get the following, which generalizes [Lo2, Proposition 6.1], with the same proof.

Proposition 3.7. *Let $N\subset M$ be an inclusion of AFD III_λ factors of finite index and finite depth with common discrete decompositions. If the tower of the higher relative commutants of $N\subset M$ is equal to that of the corresponding type II_1 inclusion, then there exist AFD factors $B\subset A$ of type II_1 such that $N\subset M$ is isomorphic to $B\mathbin{\bar\otimes}R_\lambda\subset A\mathbin{\bar\otimes}R_\lambda$.*

By Proposition 3.6 instead of [Lo2, Proposition 5.3], we get the following.

Proposition 3.8. *Let $N\subset M$ be an inclusion of AFD II_1 factors of finite index and finite depth. The following sequence is split exact.*

$$1\to\overline{Int}(M,N)\to Aut(M,N)\overset{\Phi}{\to}\mathcal{G}\to 1.$$

This shows that a free automorphism in $\mathrm{Ker}(\Phi)$ is unique up to outer conjugacy without assuming $N'\cap M=\mathbf{C}$.

References

[A] H. Araki, *Some properties of modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodym theorem with a chain rule*, Pacific J. **50** (1974), 309–354.

[BDH] M. Baillet, Y. Denizeau, & J.-F. Havet, *Indice d'une esperance conditionnelle*, Compos. Math. **66** (1988), 199–236.

[C1] A. Connes, *Une classification des facteurs de type III*, Ann. Sci. École Norm. Sup. **6** (1973), 133–252.

[C2] A. Connes, *Almost periodic states and factors of type* III_1, J. Funct. Anal. **16** (1974), 415–445.

[C3] A. Connes, *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. École Norm. Sup. **8** (1975), 383–419.

[C4] A. Connes, *Classification of injective factors, Cases* II_1*,* II_∞*,* III_λ*,* $\lambda \neq 1$, Ann. Math. **104** (1976), 73–115.

[C5] A. Connes, *On the classification of von Neumann algebras and their automorphisms* Symposia Math. XX (1976), 435–478.

[C6] A. Connes, *Spatial theory of von Neumann algebras*, J. Funct. Anal. **35** (1980), 153–164.

[CT] A. Connes & M. Takesaki, *The flow of weights on factors of type III*, Tohoku Math. J. **29** (1977), 473–555.

[Ha1] U. Haagerup, *The standard form of von Neumann algebras*, Math.Scand. **37** (1975), 270–283.

[Ha2] U. Haagerup, *Operator valued weights in von Neumann algebras I*, J. Funct. Anal. **32** (1979), 175–206; *II*, ibid., **33** (1979), 339–361.

[Hv] J.-F. Havet, *Espérance conditionnelles permutables à un group d'automorphismes sur une algébre de von Neumann*, C. R. Acad. Sc. Paris **282** (1976), 1095–1098.

[HS1] U. Haagerup & E. Størmer, *Equivalences of normal states on von Neumann algebras and the flow of weights*, Adv. Math. **83** (1990), 180–262.

[HS2] U. Haagerup & E. Størmer, *Pointwise inner automorphisms of von Neumann algebras* with an appendix by C. Sutherland, J. Funct. Anal. **92** (1990), 177–201.

[HK1] T. Hamachi & H. Kosaki, *Index and flow of weights of factors of type III*, Proc. Japan Academy **64A** (1988), 11–13.

[HK2] T. Hamachi & H. Kosaki, *Inclusions of type III factors constructed from ergodic flows*, Proc. Japan Academy **64A** (1988), 195–197.

[Hi1] F. Hiai, *Minimizing indices of conditional expectations onto a subfactor*, Publ. RIMS, Kyoto Univ., **24** (1988), 673–678.

[Hi2] F. Hiai, *Minimum index for subfactors and entropy*, to appear in J. Operator Theory.

[Hi3] F. Hiai, *Minimum index for subfactors and entropy, II*, J. Math. Soc. Japan **43** (1991), 347–380.

[J1] V. F. R. Jones, *Actions of finite groups on the hyperfinite type* II_1 *factor*, Mem. Amer. Math. Soc. **237** (1980).

[J2] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–15.

[JT] V. F. R. Jones & M. Takesaki, *Actions of compact abelian groups on semifinite injective factors*, Acta Math. **153** (1984), 213–258.

[KST] Y. Kawahigashi, C. E. Sutherland, & M. Takesaki, *The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions*, (to appear in Acta Math.).

[KT] Y. Kawahigashi & M. Takesaki, *Compact abelian group actions on injective factors*, (to appear in J. Funct. Anal.).

[K1] H. Kosaki, *Extension of Jones' theory on index to arbitrary factors*, J. Funct. Anal. **66** (1986), 123–140.

[K2] H. Kosaki, *Index theory for type III factors*, (to appear in Proceedings of U.S.-Japan Seminar, 1988).

[Lo1] P. H. Loi, *On the theory of index and type III factors*, Thesis, Pennsylvania State University, 1988.

[Lo2] P. H. Loi, *On automorphisms of subfactors*, preprint, 1990, UCLA.

[Ln] R. Longo, *Index of subfactors and statistics of quantum fields I*, Comm. Math. Phys. **126** (1989), 217–247.

[O1] A. Ocneanu, "Actions of discrete amenable groups on factors," Lecture Notes in Math. No. 1138, Springer, Berlin, 1985.

[O2] A. Ocneanu, *Quantized group string algebras and Galois theory for algebras*, in "Operator algebras and applications, Vol. 2 (Warwick, 1987)", London Math. Soc. Lect. Note Series Vol. 136, Cambridge University Press, 1988, pp. 119–172.

[PP1] M. Pimsner & S. Popa, *Entropy and index for subfactors*, Ann. Scient. Éc. Norm. Sup. **19** (1986), 57–106.

[PP2] M. Pimsner & S. Popa, *Sur les sous-facteurs d'indice finite d'un facteur de* II_1 *ayant la propriete T*, C. R. Acad. Sc. Paris **303** (1986), 359–361.

[P] S. Popa, *Classification of subfactors: reduction to commuting squares*, Invent. Math. **101** (1990), 19–43.

[Se] Y. Sekine, *Flows of weights of crossed products of type III factors by discrete groups*, Publ. RIMS Kyoto Univ. **26** (1990), 655–666.

[S] Ş. Strătilă, “Modular Theory in Operator Algebras,” Editura Academiei and Abacus Press, Tunbridge Wells, 1981.

[ST] C. E. Sutherland & M. Takesaki, *Actions of discrete amenable groups on injective factors of type III_λ, $\lambda \neq 1$*, Pacific J. **137** (1989), 405–444.

[T1] M. Takesaki, *Conditional expectation in von Neumann algebras*, J. Funct. Anal. **9** (1972), 306–321.

[T2] M. Takesaki, “Theory of operator algebras I,” Springer, Berlin, 1979.

[W] A. Wassermann, *Coactions and Yang-Baxter equations for ergodic actions and subfactors*, in “Operator algebras and applications, Vol. 2 (Warwick, 1987)”, London Math. Soc. Lect. Note Series Vol. 136, Cambridge University Press, 1988, pp. 203–236.