The E_7 commuting squares produce D_{10} as principal graph

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Abstract. We prove that the (two) connections, or commuting squares, on the Coxeter-Dynkin diagram E_7 produce a subfactor with principal graph D_{10}. This was conjectured by J.-B. Zuber in connection with modular invariants in conformal field theory, and solve the last case of computing the flat parts of the connections on Coxeter-Dynkin diagrams with index less than 4.

§0 Introduction

Since V. F. R. Jones initiated a systematic study of subfactors in [Jo], more and more connections of the subfactor theory with topology and quantum field theory have been pointed out. Our aim in this paper is to provide an evidence of a deeper relation between a notion of flatness in subfactor theory and modular invariants in conformal field theory through a computation of flat parts of a commuting square arising from the Coxeter-Dynkin diagram E_7, which was the last case left among the *A-D-E* diagrams.

In theory of operator algebras, a classification problem of subfactors (of the approximately finite dimensional factor of type II_1) have attracted much attention, and approach using higher relative commutants has been quite successful [O1, P2,

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P3]. Ocneanu's paragroup machinery [O1, O3] appeared as a combinatorial characterization of higher relative commutants and it has a strong resemblance to some combinatorial aspects of statistical mechanics and rational conformal field theory such as [ABF, Ba, DHVW, DJMO, DZ1, DZ2, DV, F, FG, Ko, So, Z, W], as noted and exploited in [BG, EK1, EK2, IK, Ka1, Ka2, Pa, R]. For example, the commuting square condition in subfactor theory [P1, GHJ, Sc] is related to the crossing symmetry or second inversion relation in solvable lattice model theory. (See [ABF, B, DJMO, Ji] for more on solvable lattice models.)

The subfactors of the approximately finite dimensional (AFD) factor of type II_1 with index less than 4 have attracted much attention. In these cases, the so-called principal graph of the inclusion must be one of the Coxeter-Dynkin diagrams of type A, D, E, as observed in [GHJ, §4.1]. (Each graph has a distinguished vertex denoted by $*$, determined as the vertex with the least entry of the Perron-Frobenius eigenvector. See [P2, Theorem 3.8], [I1].) Then for each graph, we have an additional structure — a "connection" giving a commuting square. The commuting squares can be written down explicitly as in [GHJ, §4.2] for the A-D-E graphs, and all the connections can be classified as follows [O1, O3, Ka1]: Each of the diagrams of type A or D has a unique connection, and each of the diagrams of type E has two connections. The remaining problem is to decide whether they really appear as higher relative commutants of subfactors. Alternatively, in Popa's terminology [P2], we have to determine whether or not these commuting squares are canonical commuting squares. Ocneanu claimed that this question can in general be solved by checking finitely many explicit equations, and when these are satisfied the connection is said to be *flat* [O1, O3, Ka1, Ka2]. This flatness is a key axiom in

Ocneanu's combinatorial approach, and it also plays a key role in construction of topological invariants of 3-manifolds of type Turaev-Viro [TV, DJN] from subfactors [O4, EK2]. In the case of A-D-E diagrams, Ocneanu announced the following claim [O1]: The connections on A_n, D_{2n}, E_6, E_8 are flat, but those on D_{2n+1}, E_7 are not flat. He has not shown full details of his method, but these claims have been proven by several people [I1, I2, Ka1, SV]. (The claim for A_n is easy to verify, and E_6 has been realized by [BN]. Impossibility of E_7 and D_5 is implicit in [P2, Theorem 3.8].) The classification is thus complete for index less than four, but it is natural to ask what happens in the case of non-flat connection. We have commuting squares even for non-flat connections and hence subfactors. So we have a problem of determining the principal graph. In Ocneanu's terminology [O3], this is the problem of determining a flat part for a non-flat connection.

In the case of D_{2n+1}, we can conclude except for the D_7 case that the principal graph is A_{4n-1} by considering only the value of the Jones index. The index value $4\cos^2\dfrac{\pi}{12}$ for D_7 is realized for A_{11} and E_6, but by the compactness argument of Ocneanu [O3, II.6], we know that the higher relative commutants of this subfactor are contained in the string algebra of D_7, which eliminates the possibility of E_6. The case of E_7, however, is more subtle. By looking at the index value, we know that the principal graph is either D_{10} or A_{17}, but it seems that there is no easy way to determine which is the right one. Our object in this paper is to prove that D_{10} is the correct invariant. As far as we know, this is the first example of a direct computation of a flat part of the non-flat connection. Our method comes from Izumi's paper [I2] on the flatness of the E_8 connections.

Cappelli, Itzykson and Zuber [CIZ] and A. Kato [Kt] encountered an A-D-E classification in conformal field theory, and Zuber noticed a strong similarity to the subfactor classification mentioned above. Based on this observation, Zuber conjectured that the E_7 connections produce the D_{10} principal graph. Our result here proves this conjecture, and provides the last missing piece in similarity between the A-D-E classifications in subfactor theory and solvable lattice model theory.

After this work was completed, we learned that U. Haagerup independently and recently obtained the same result with the essentially same method.

This work was completed while the first author visited MSRI, Berkeley in the spring of 1992 . We wish to thank MSRI and Irving Kaplansky for the financial support and their hospitality, D. Bisch for informing us of Haagerup's result, U. Haagerup for explaining his method, which turned out to be the same as ours, J.-B. Zuber for making the conjecture during a visit to Swansea in February 1992, and the London Mathematical Society for financial support for making that visit possible. The second author was supported by a Miller Research Fellowship at University of California, Berkeley.

§1 Embedding of path algebras of A_{17} to that of E_7

We will freely use notations and results in string algebra approach as in [E1, E2, Ka1, Ka2, O1, O2, O3, Su]. We first look at the subalgebra generated by the Jones projections in the string algebra of E_7 because the Jones projections are always flat. Then we get a commuting square embedding of two string algebras as in [GHJ, §4.5], and the graphs for embeddings are obtained as in [EG], [Ok] or [DZ1, page 622]. We label the vertices of the Coxeter-Dynkin diagram E_7 as follows.

$$E_7: \qquad \overset{\overset{\textstyle 6}{\textstyle |}}{0\!-\!1\!-\!2\!-\!3}\!-\!4\!-\!5$$

Because we already know that the connections on E_7 are not flat, $N' \cap M_k$, $k \leq 6$, are generated by the Jones projections. So we look at the level $k = 7$. If we look at the minimal projection orthogonal to the Jones projections in $N' \cap M_7$ and check its string algebra expression in the E_7 string algebra, we know that it is a sum of three minimal projections and that they are in summands corresponding to the vertices $0, 2, 6$ respectively. (We use the preceding embedding graphs for this.) Because this projection is flat, we cut the string algebras making M_7 by this projection and get the following commuting square series, and proceed as in [I2].

$$(1.1) \qquad \begin{array}{ccccccc} B_{00} & \subset & B_{01} & \subset & B_{02} & \subset \cdots \\ \cap & & \cap & & \cap & \\ B_{10} & \subset & B_{11} & \subset & B_{12} & \subset \cdots \end{array}$$

where $B_{00} \subset B_{01} \subset \cdots$ is the series of string algebras of E_7 starting from the vertex 0 and $B_{10} \subset B_{11} \subset \cdots$ is the series of string algebras if E_7 starting from three vertices $0, 2, 6$. Next we compute the two vertical embedding graphs of (1.1). This is done just by counting the dimensions in the two Bratteli diagrams, and we get the following two graphs.

$$\boxed{\text{Figure 1.1}}$$

where we used notations $\overline{0}, \overline{1}, \ldots, \overline{6}$ for denoting the vertices of E_7 for the string

algebras $B_{10} \subset B_{11} \subset \cdots$ as in [I2].

§2 Computations of connection matrices

Next we look at the series (1.1) and will determine the connection for this series

just by studying the graphs, or at least we compute several entries of the connection

matrices in a way similar to that in [I2] with the same notation as in [I2]. Of course,

there is a connection for the series (1.1) arising from the original connection, but

it is very hard to compute that explicitly. We will determine the information that

we need regarding the connection from the graphs using gauge choices without

considering the original connection E_7. This was the basic idea of Izumi [I2]. We

do not have to determine all the entries completely, so we use the symbol $*$ for

entries which we do not determine. Our aim in this section is to prove that after

certain gauge choices all the connection entries are given by the following list.

$$
(2.1) \qquad
\begin{array}{ccc}
0 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{1}
\end{array}
=
\begin{array}{c}
 \\
\overline{0} \\
\overline{2}
\end{array}
\begin{array}{c}
1_1 \quad 1_2 \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{array}
$$

$$
(2.2) \qquad
\begin{array}{ccc}
0 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{3}
\end{array}
=
\begin{array}{c}
 \\
\overline{2} \\
\overline{6}
\end{array}
\begin{array}{c}
1_1 \quad 1_2 \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{array}
$$

$$(2.3)\qquad
\begin{array}{ccc}
1 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{0}
\end{array}
\;=\;
\begin{array}{c}
 \\
\overline{{}_1 1} \\
 \\
\overline{{}_2 1}
\end{array}
\begin{pmatrix}
0 & 2 \\
1 & 0 \\
0 & *
\end{pmatrix}$$

$$(2.4)\qquad
\begin{array}{ccc}
1 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{2}
\end{array}
\;=\;
\begin{array}{c}
\overline{{}_1 1} \\
\overline{{}_2 1} \\
\overline{{}_1 3} \\
\overline{{}_2 3}
\end{array}
\begin{pmatrix}
0 & 2_1 & 2_2 & 2_3 \\
0 & 1 & 0 & 0 \\
* & 0 & 0 & * \\
* & 0 & 0 & * \\
0 & 0 & 1 & 0
\end{pmatrix}$$

$$(2.5)\qquad
\begin{array}{ccc}
1 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{6}
\end{array}
\;=\;
\begin{array}{c}
\overline{{}_1 3} \\
\overline{{}_2 3}
\end{array}
\begin{pmatrix}
0 & 2 \\
0 & * \\
1 & 0
\end{pmatrix}$$

$$(2.6)\qquad
\begin{array}{ccc}
1 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{4}
\end{array}
\;=\;
\begin{array}{c}
\overline{{}_1 3} \\
\overline{{}_2 3}
\end{array}
\begin{pmatrix}
2_1 & 2_2 \\
1 & 0 \\
0 & 1
\end{pmatrix}$$

(2.7)

$$
\begin{array}{ccc}
2 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{1}
\end{array}
\quad = \quad
\begin{array}{c}
\\
\overline{0} \\
{}_1\overline{2} \\
{}_2\overline{2} \\
{}_3\overline{2}
\end{array}
\begin{array}{c}
\begin{matrix} 1_1 & 1_2 & 3_1 & 3_2 \end{matrix} \\
\left(\begin{matrix}
0 & * & * & 0 \\
* & 0 & 0 & 0 \\
0 & 0 & 0 & * \\
0 & * & * & 0
\end{matrix}\right)
\end{array}
$$

(2.8)

$$
\begin{array}{ccc}
2 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{3}
\end{array}
\quad = \quad
\begin{array}{c}
\\
{}_1\overline{2} \\
{}_2\overline{2} \\
{}_3\overline{2} \\
\overline{6} \\
{}_1\overline{4} \\
{}_2\overline{4}
\end{array}
\begin{array}{c}
\begin{matrix} 1_1 & 1_2 & 3_1 & 3_2 & 3_3 & 3_4 \end{matrix} \\
\left(\begin{matrix}
0 & 0 & 0 & * & 0 & * \\
0 & * & 0 & * & 0 & * \\
* & 0 & * & 0 & * & 0 \\
* & 0 & * & 0 & 0 & 0 \\
* & 0 & * & 0 & * & 0 \\
0 & * & 0 & * & 0 & *
\end{matrix}\right)
\end{array}
$$

(2.9)

$$
\begin{array}{ccc}
2 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{5}
\end{array}
\quad = \quad
\begin{array}{c}
\\
{}_1\overline{4} \\
{}_2\overline{4}
\end{array}
\begin{array}{c}
\begin{matrix} 3_1 & 3_2 \end{matrix} \\
\left(\begin{matrix}
1 & 0 \\
0 & 1
\end{matrix}\right)
\end{array}
$$

(2.10)

$$
\begin{array}{ccc}
3 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{0}
\end{array}
\quad = \quad
\begin{array}{c}
\\
{}_1\overline{1} \\
{}_2\overline{1}
\end{array}
\begin{array}{c}
\begin{matrix} 2 & 6 \end{matrix} \\
\left(\begin{matrix}
1 & 0 \\
0 & 1
\end{matrix}\right)
\end{array}
$$

$$(2.11)\qquad
\begin{array}{c}
3 \longrightarrow \;\cdot\\[2pt]
\Big\downarrow \qquad\quad \Big\downarrow \\[2pt]
\cdot \longrightarrow \overline{2}
\end{array}
\;=\;
\begin{array}{c}
\begin{array}{cccccc} 2_1 & 2_2 & 2_3 & 6 & 4_1 & 4_2 \end{array}\\[2pt]
\begin{array}{c}
{}_1\overline{1}\\[6pt]
{}_2\overline{1}\\[6pt]
{}_1\overline{3}\\[6pt]
{}_2\overline{3}\\[6pt]
{}_3\overline{3}\\[6pt]
{}_4\overline{3}
\end{array}
\left(
\begin{array}{cccccc}
0 & 0 & * & * & * & 0\\[6pt]
0 & * & 0 & 0 & 0 & *\\[6pt]
0 & 0 & * & * & * & 0\\[6pt]
* & * & 0 & 0 & 0 & *\\[6pt]
0 & 0 & * & * & * & 0\\[6pt]
* & * & 0 & 0 & 0 & *
\end{array}
\right)
\end{array}$$

$$(2.12)\qquad
\begin{array}{c}
3 \longrightarrow \;\cdot\\[2pt]
\Big\downarrow \qquad\quad \Big\downarrow \\[2pt]
\cdot \longrightarrow \overline{6}
\end{array}
\;=\;
\begin{array}{c}
\begin{array}{cccc} 2 & 6 & 4_1 & 4_2 \end{array}\\[2pt]
\begin{array}{c}
{}_1\overline{3}\\[6pt]
{}_2\overline{3}\\[6pt]
{}_3\overline{3}\\[6pt]
{}_4\overline{3}
\end{array}
\left(
\begin{array}{cccc}
1 & 0 & 0 & 0\\[6pt]
0 & 1 & 0 & 0\\[6pt]
0 & 0 & 1 & 0\\[6pt]
0 & 0 & 0 & 1
\end{array}
\right)
\end{array}$$

$$(2.13)\qquad
\begin{array}{c}
3 \longrightarrow \;\cdot\\[2pt]
\Big\downarrow \qquad\quad \Big\downarrow \\[2pt]
\cdot \longrightarrow \overline{4}
\end{array}
\;=\;
\begin{array}{c}
\begin{array}{cccccc} 2_1 & 2_2 & 6_1 & 6_2 & 4_1 & 4_2 \end{array}\\[2pt]
\begin{array}{c}
{}_1\overline{3}\\[6pt]
{}_2\overline{3}\\[6pt]
{}_3\overline{3}\\[6pt]
{}_4\overline{3}\\[6pt]
{}_1\overline{5}\\[6pt]
{}_2\overline{5}
\end{array}
\left(
\begin{array}{cccccc}
* & 0 & * & 0 & * & 0\\[6pt]
0 & * & 0 & 0 & 0 & *\\[6pt]
* & 0 & * & 0 & * & 0\\[6pt]
0 & * & 0 & * & 0 & *\\[6pt]
* & 0 & * & 0 & * & 0\\[6pt]
0 & * & 0 & * & 0 & *
\end{array}
\right)
\end{array}$$

$$(2.14) \qquad \begin{array}{ccc} 4 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \overline{1} \end{array} = \begin{array}{c} \\ \overline{_12} \\ \overline{_22} \end{array} \begin{array}{c} 3_1 \quad 3_2 \\ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \end{array}$$

$$(2.15) \qquad \begin{array}{ccc} 4 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \overline{3} \end{array} = \begin{array}{c} \\ \overline{_12} \\ \overline{_22} \\ \overline{_16} \\ \overline{_26} \\ \overline{_14} \\ \overline{_24} \end{array} \begin{array}{c} 3_1 \quad 3_2 \quad 3_3 \quad 3_4 \quad 5_1 \quad 5_2 \\ \begin{pmatrix} * & 0 & * & 0 & * & 0 \\ 0 & * & 0 & * & 0 & * \\ 0 & 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & * \\ * & 0 & * & 0 & * & 0 \\ 0 & * & 0 & * & 0 & * \end{pmatrix} \end{array}$$

$$(2.16) \qquad \begin{array}{ccc} 4 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \overline{5} \end{array} = \begin{array}{c} \\ \overline{_14} \\ \overline{_24} \end{array} \begin{array}{c} 3_1 \quad 3_2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$(2.17) \qquad \begin{array}{ccc} 5 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \overline{2} \end{array} = \begin{array}{c} \\ \overline{_13} \\ \overline{_23} \end{array} \begin{array}{c} 4_1 \quad 4_2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$(2.18) \qquad
\begin{array}{ccc}
5 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{6}
\end{array}
\quad = \quad
\begin{array}{c@{}c}
 & \begin{array}{cc} 4_1 & 4_2 \end{array} \\
\begin{array}{c} \overline{{}_1 3} \\[1.5em] \overline{{}_2 3} \end{array} &
\left(\begin{array}{cc} 1 & 0 \\[1em] 0 & 1 \end{array}\right)
\end{array}$$

$$(2.19) \qquad
\begin{array}{ccc}
5 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{4}
\end{array}
\quad = \quad
\begin{array}{c@{}c}
 & \begin{array}{cc} 4_1 & 4_2 \end{array} \\
\begin{array}{c} \overline{{}_1 3} \\[1.5em] \overline{{}_2 3} \end{array} &
\left(\begin{array}{cc} 1 & 0 \\[1em] 0 & 1 \end{array}\right)
\end{array}$$

$$(2.20) \qquad
\begin{array}{ccc}
6 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{1}
\end{array}
\quad = \quad
\begin{array}{c@{}c}
 & \begin{array}{cc} 3_1 & 3_2 \end{array} \\
\begin{array}{c} \overline{0} \\[1.5em] \overline{2} \end{array} &
\left(\begin{array}{cc} 0 & 1 \\[1em] * & 0 \end{array}\right)
\end{array}$$

$$(2.21) \qquad
\begin{array}{ccc}
6 & \longrightarrow & \cdot \\
\downarrow & & \downarrow \\
\cdot & \longrightarrow & \overline{3}
\end{array}
\quad = \quad
\begin{array}{c@{}c}
 & \begin{array}{cccc} 3_1 & 3_2 & 3_3 & 3_4 \end{array} \\
\begin{array}{c} \overline{2} \\[1em] \overline{6} \\[1em] \overline{{}_1 4} \\[1em] \overline{{}_2 4} \end{array} &
\left(\begin{array}{cccc} * & 0 & * & 0 \\[0.5em] 0 & 1 & 0 & 0 \\[0.5em] * & 0 & * & 0 \\[0.5em] 0 & 0 & 0 & * \end{array}\right)
\end{array}$$

$$(2.22) \qquad \begin{array}{ccc} 6 & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \overline{5} \end{array} = \begin{array}{c} \overline{14} \\ \\ \overline{24} \end{array} \begin{array}{cc} 3_1 & 3_2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

We now prove the above claim. In the first place, we make gauge choices for the following pairs of vertices in this order:

$$\begin{array}{ccccccccccc} 3 & 1 & 3 & 1 & 2 & 3 & 4 & 4 & 4 & 5 & 6 \\ | & | & | & | & | & | & | & | & | & | & | \\ 3, & 3, & 1, & 1, & 4, & 5, & 4, & 2, & 6, & 3, & 4. \end{array}$$

In this way, we fix the matrices (2.12), (2.2), (2,10), (2.1), (2.6), (2.9), (2.16), (2.17), (2.18), (2.19), and (2.22) as desired.

By the crossing symmetry

$$\begin{array}{ccc} a & \xrightarrow{\xi_2} & b \\ \xi_1 \downarrow & & \downarrow \xi_3 \\ c & \xrightarrow[\xi_4]{} & d \end{array} = \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}} \begin{array}{ccc} b & \xrightarrow{\tilde{\xi}_2} & a \\ \xi_3 \downarrow & & \downarrow \xi_1 \\ d & \xrightarrow[\tilde{\xi}_4]{} & c \end{array}$$

and (2.1), (2.2), we can partially fill entries of (2.4). Then by a gauge choice for $\begin{array}{c} 2 \\ | \\ 2 \end{array}$, we get (2.4) as desired.

Using the crossing symmetry, (2.10), and $\dfrac{\mu(6)\mu(1)}{\mu(3)\mu(0)} = 1$, we get (2.20). Similarly, using the crossing symmetry, (2.2), and $\dfrac{\mu(6)\mu(1)}{\mu(3)\mu(0)} = 1$, we get (2.5). Similarly, using the crossing symmetry and (2.1), we get (2.3).

Next we fill several entries from the determined entries and the crossing symmetry. Using unitarity, we get (2.7) as above. After filling further entries with the crossing symmetry, we get the following expression for (2.15).

$$
\begin{array}{c}
\quad\;\; 3_1 \qquad\quad 3_2 \qquad\quad 3_3 \qquad\qquad 3_4 \qquad\qquad 5_1 \qquad\quad 5_2 \\
\begin{array}{c}
\overline{12} \\[4pt] \overline{22} \\[4pt] \overline{16} \\[4pt] \overline{26} \\[4pt] \overline{14} \\[4pt] \overline{24}
\end{array}
\left(
\begin{array}{cccccc}
-\sqrt{\frac{\mu(2)}{\mu(4)}}\,a & -\sqrt{\frac{\mu(2)}{\mu(4)}}\,\bar{b}\omega & -\sqrt{\frac{\mu(5)^2\mu(2)}{\mu(3)^2\mu(4)}} & 0 & \sqrt{\frac{\mu(5)\mu(2)}{\mu(3)\mu(4)}} & 0 \\[10pt]
-\sqrt{\frac{\mu(2)}{\mu(4)}}\,b & \sqrt{\frac{\mu(2)}{\mu(4)}}\,\bar{a}\omega & 0 & -\sqrt{\frac{\mu(5)^2\mu(2)}{\mu(3)^2\mu(4)}} & 0 & \sqrt{\frac{\mu(5)\mu(2)}{\mu(3)\mu(4)}} \\[10pt]
0 & 0 & \sqrt{\frac{\mu(6)}{\mu(4)}} & 0 & \sqrt{\frac{\mu(5)\mu(6)}{\mu(3)\mu(4)}} & 0 \\[10pt]
0 & 0 & 0 & \sqrt{\frac{\mu(6)}{\mu(4)}} & 0 & \sqrt{\frac{\mu(5)\mu(6)}{\mu(3)\mu(4)}} \\[10pt]
a & \bar{b}\omega & -\frac{\mu(5)}{\mu(3)} & 0 & \sqrt{\frac{\mu(5)}{\mu(3)}} & 0 \\[10pt]
b & -\bar{a}\omega & 0 & -\frac{\mu(5)}{\mu(3)} & 0 & \sqrt{\frac{\mu(5)}{\mu(3)}}
\end{array}
\right)
\end{array}
$$

where $|\omega| = 1$, and $a, b, \omega \in \mathbf{C}$.

Using this, first we get the following expression for (2.13).

$$
\begin{array}{c}
\quad\;\; 2_1 \qquad\; 2_2 \qquad 6_1 \qquad 6_2 \qquad 4_1 \qquad 4_2 \\
\begin{array}{c}
\overline{13} \\[6pt] \overline{23} \\[6pt] \overline{33} \\[6pt] \overline{43} \\[6pt] \overline{15} \\[6pt] \overline{25}
\end{array}
\left(
\begin{array}{cccccc}
* & 0 & x & y & \bar{a} & \bar{b} \\[8pt]
w & z & 0 & 0 & b\bar{\omega} & -a\bar{\omega} \\[8pt]
* & * & * & * & * & 0 \\[8pt]
* & * & * & * & 0 & * \\[8pt]
\sqrt{\frac{\mu(5)\mu(2)}{\mu(3)\mu(4)}} & 0 & \sqrt{\frac{\mu(5)\mu(6)}{\mu(3)\mu(4)}} & 0 & \sqrt{\frac{\mu(5)}{\mu(3)}} & 0 \\[10pt]
0 & \sqrt{\frac{\mu(5)\mu(2)}{\mu(3)\mu(4)}} & 0 & \sqrt{\frac{\mu(5)\mu(6)}{\mu(3)\mu(4)}} & 0 & \sqrt{\frac{\mu(5)}{\mu(3)}}
\end{array}
\right)
\end{array}
$$

where $x, y, z, w \in \mathbf{C}$. Using the orthogonality of the first and second rows and the fact that the $(1,1)$-entry is not zero, which comes from (2.8), we can conclude that $w = 0$. Then by the orthogonality of the second and the fifth rows, we can conclude that $b = 0$. By the orthogonality of the first and sixth rows, we next get $y = 0$.

Then (2.15) is of the desired form. In (2.8), the entry $({}_1\overline{4}, 3_2)$ is 0, and this implies

that the entry $({}_3\overline{2}, 3_2)$ is 0 by orthogonality of the first and the fourth columns.

Then the orthogonality relations give (2.11) in the desired form. Finally we get

(2.14), (2.8), (2.13), and (2.21) in this order, and we are done.

§3 Computing the flat part and the Goodman-de la Harpe-Jones subfactor for

A_{17} and E_7

We look at the following diagram for computing the flat part.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 6 \\
\big\downarrow & & & & & & & & \big\downarrow \\
a & \longrightarrow & b & \longrightarrow & c & \longrightarrow & d & \longrightarrow & e
\end{array} .
$$

For each choice of vertex e, we determine the vertex a for which the above

diagram can have a non-zero value. This is done as in [I2] with the matrices in §2,

and we get the following.

(1) If e is $\overline{0}$, then a is $\overline{6}$.

(2) If e is $\overline{2}$, then a is $\overline{2}$.

(3) If e is $\overline{6}$, then a is $\overline{0}$ or $\overline{6}$.

(4) If e is $\overline{4}$ and the vertical edge from 6 to 4 is labeled as 1, then a is $\overline{2}$.

(5) If e is $\overline{4}$ and the vertical edge from 6 to 4 is labeled as 2, then a is $\overline{0}$ or $\overline{6}$.

This shows that the vertical string $(0 \to \overline{2}, 0 \to \overline{2})$ commutes with all the horizon-

tal strings on E_7 starting from the vertex 0, because the horizontal string algebras

are generated by the Jones projections and a single projection $(0 \to 1 \to 2 \to 3 \to$

$6, 0 \to 1 \to 2 \to 3 \to 6)$. This means that the higher relative commutants arising

from the E_7 connections are strictly larger than the string algebras of A_{17}. We have thus proved that the principal graph of the subfactors arising from the E_7 connections is D_{10} as desired. Note that there are two (mutually non-equivalent) connections on the graph E_7, but both give the same subfactor, since the subfactor with the principal graph D_{10} is unique.

In general, we have the following proposition for the flat parts.

Proposition 3.1. *Suppose we have a double complex (A_{kl}) of string algebras arising from four graphs as in [O3, Ka2]. Set the flat parts $B_k = A_{k,\infty} \cap A'_{0,\infty}$. Then the following is a series of commuting squares of period 2.*

$$
\begin{array}{ccc}
B_k & \subset & A_{k,0} \\
\cap & & \cap \\
B_{k+1} & \subset & A_{k+1,0}
\end{array}
$$

Proof. Let e_k be the k-th vertical Jones projection in $A_{k+1,0}$. Then for $x \in B_{k+1}$, we get $e_k x e_k = E_{B_k}(x)e_k = E_{A_k}(x)e_k$, hence $E_{B_k}(x) = E_{A_k}(x)$. This gives the conclusion.   Q.E.D.

From this general statement and the above computation, we can conclude existence of a commuting square which embeds the string algebra of D_{10} from $*$ to the string algebra of E_7 from $*$. By direct computations, we can see that the two graphs for embedding are also D_{10} and E_7 respectively in this case.

Goodman, de la Harpe, and Jones [GHJ, §4.5] constructed subfactors from commuting squares embedding the string algebras of A_n from $*$ to the string algebras of other A-D-E graphs, and Okamoto [Ok] computed the principal graphs of these subfactors. In the case of E_7 with the choice of $*$ as the starting vertex, the above

result shows there is an intermediate subfactor for the Goodman-de la Harpe-Jones subfactor. Their numerical index value [GHJ, Proposition 4.5.2] for this subfactor is really $8\cos^2\dfrac{\pi}{18}$, and the above intermediate subfactor splits this number as $2 \times 4\cos^2\dfrac{\pi}{18}$. Existence of such an intermediate subfactor can be also seen from the principal graph of Okamoto [Ok, DZ1] and a result of Pimsner-Popa on existence of normalizers in [PP, Proposition 1.7].

§4 Comments on conformal field theory

We now discuss meaning of the above computations from the viewpoint of conformal field theory. These observations were bases of Zuber's conjecture mentioned in the introduction.

In the A-D-E classification of [CIZ, Kt] on level k $SU(2)$ WZW-models, the A_n, D_{2n}, E_6, E_8 models have block-diagonal modular invariants and the D_{2n+1} and E_7 models have off-diagonal modular invariants. Comparing this with the A-D-E classification of subfactors announced by A. Ocneanu, we easily notice a rather mysterious similarity between block diagonal modular invariants and flat connections. Furthermore, this similarity still holds between our work on orbifold subfactors from Hecke algebras [EK1] and modular invariants of the orbifold graphs $\mathcal{D}^{(n)}$ [DZ2, Tables II, III].

Dijkgraaf-Verlinde [DV] studied the relation between off-diagonal modular invariants and non-trivial automorphism of the fusion algebra of conformal field theory. They further discussed the following E_7 pairing in [DV, pages 95–96]. The partition functions labeled by D_{10} and E_7 are given as follows.

$$D_{10}: \quad Z = |\chi_1 + \chi_{17}|^2 + |\chi_3 + \chi_{15}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + 2|\chi_9|^2,$$

$$E_7: \quad Z = |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + |\chi_9|^2 + [(\chi_3 + \chi_{15})\bar{\chi}_9 + c.c.].$$

Both have the same chiral algebra containing the $SU(2)_{k=16}$ KM-algebra and the characters are given by $\chi_1 + \chi_{17}$, $\chi_3 + \chi_{15}$, $\chi_5 + \chi_{13}$, $\chi_7 + \chi_{11}$, χ_9^+, and χ_9^-. Then the corresponding operators are denoted as 1, φ_1, φ_2, φ_3, φ_4^+ and φ_4^-. In this setting, the off-diagonal pairing is given by the exchange of φ_1 and φ_4^+. They also computed the same fusion rules of these models with S-matrix and observed that the pairing exchanging φ_1 and φ_4^+ gives a $\mathbf{Z}_2$ symmetry of the fusion rules. Or, in the notation of [Z, page 315, Table I], this automorphism of order 2 exchanges the block with exponents $(3, 15)$ with the one of exponent 9. (This fusion rule is the same as the one obtained with the N-N bimodules arising from the subfactor with D_{10} principal graph. See [I1] for fusion rule computations.) With this symmetry, one can twist the characters χ with respect to the the conjugated characters $\bar{\chi}$ and get the E_7 invariant which is not block diagonal from the D_{10} invariant which is block diagonal. With this observation and similarity between block diagonal modular invariants and flat connection, Zuber conjectured that the (non-flat) connections on the E_7 graph would give principal graph D_{10}. Our computations proved this conjecture and gave a further evidence on this similarity. Our commuting square obtained at the end of §3 clarifies a relation between the two diagrams suggested by this pairing.

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