Title: **Classification of subfactors with the principal graph** $D_n^{(1)}$

Authors:
Masaki Izumi

Research Institute for Mathematical Sciences
Kyoto University, Kyoto, 606, JAPAN

Yasuyuki Kawahigashi

Department of Mathematics, University of California
Berkeley, CA 94720, U.S.A.

(FAX: (+1)-510-642-8204)

Proposed running head: Classification of subfactors with the principal graph $D_n^{(1)}$

Authors:
Masaki Izumi

Research Institute for Mathematical Sciences
Kyoto University, Kyoto, 606, JAPAN

Yasuyuki Kawahigashi

Department of Mathematics, University of California
Berkeley, CA 94720, U.S.A.

(FAX: (+1)-510-642-8204)

Abstract. We show that the number of the conjugacy classes of the AFD type II_1 subfactors with the principal graph $D_n^{(1)}$ is $n-2$. This gives the last missing number in the complete classification list of subfactors with index 4 by S. Popa. This also disproves an announcement of A. Ocneanu that such a subfactor is unique for each n. We give two different proofs. One is by an application of an idea of an orbifold model in solvable lattice model theory to Ocneanu's paragroup theory and the other is by reduction to classification of dihedral group actions. The latter also shows that the AFD type III_1 subfactors with the principal graph $D_n^{(1)}$ split as type II_1 subfactors tensored with the common AFD type III_1 factor. We also discuss a relation between these proofs and a construction of subfactors using Cuntz algebra endomorphisms.

Classification of subfactors with the principal graph $D_n^{(1)}$

Masaki Izumi
Research Institute for Mathematical Sciences
Kyoto University, Kyoto, 606, JAPAN

Yasuyuki Kawahigashi
Department of Mathematics, University of California
Berkeley, CA 94720, U.S.A.

Abstract. We show that the number of the conjugacy classes of the AFD type II_1 subfactors with the principal graph $D_n^{(1)}$ is $n-2$. This gives the last missing number in the complete classification list of subfactors with index 4 by S. Popa. This also disproves an announcement of A. Ocneanu that such a subfactor is unique for each n. We give two different proofs. One is by an application of an idea of an orbifold model in solvable lattice model theory to Ocneanu's paragroup theory and the other is by reduction to classification of dihedral group actions. The latter also shows that the AFD type III_1 subfactors with the principal graph $D_n^{(1)}$ split as type II_1 subfactors tensored with the common AFD type III_1 factor. We also discuss a relation between these proofs and a construction of subfactors using Cuntz algebra endomorphisms.

§0 Introduction

S. Popa announced a complete classification of subfactors of the AFD type II_1 factor with the Jones index 4, but for the case of the principal graph $D_n^{(1)}$, the extended Coxeter graph, his classification was given in terms of certain third cohomology group elements, and the number of the conjugacy classes of subfactors

1980 Mathematics Subject Classification (1985 Revision). Primary 46L55.

for each graph $D_n^{(1)}$ was unknown. We show that the number of these conjugacy classes is $n-2$ using Ocneanu's flat connection, a key notion in his combinatorial approach, and an analogue of an orbifold model in theory of solvable lattice models. Our result shows invalidity of an announcement by A. Ocneanu in the ICM-90 that there is a unique subfactor for each $D_n^{(1)}$. We also give another proof of this classification based on classification of actions of the dihedral groups which also works for type III subfactors.

The index theory of V. F. R. Jones for subfactors [J2] opened an entirely new and exciting era for the theory of operator algebras. We have been witnessing more and more surprising connections of subfactor theory to many fields in mathematics and physics.

From the operator algebraic viewpoint, one of the most important problems in the subfactor theory is classification of the approximately finite dimensional (AFD) subfactors. Classification by higher relative commutants has been very successful on this problem. In this approach, we have a so-called principal graph introduced as an invariant for a subfactor by V. Jones. Ocneanu's paragroup theory claims that for subfactors with "finite depth" and finite index this graph and the "dual" graph together with certain algebraic structure ("paragroup" which is given by a "flat connection" on the graphs) give a combinatorial characterization of the commuting squares of the tower of the higher relative commutants. (See [GHJ] for instance for the definition and significance of the finite depth condition.)

To get a complete classification in this approach, one has to prove that the tower of the relative commutants for a tunnel generated by iterated downward basic constructions generate the original factor for an appropriate choice of tunnel.

A. Ocneanu announced that this generating property is true for AFD type II_1 subfactors with finite depth, finite index, and a trivial relative commutant in [O2], but his proof has been unavailable. But S. Popa has given a proof for a stronger form of this analytic statement without assuming a trivial relative commutant property in [P1] and announced an ultimate result, necessary and sufficient conditions for this generating property, in this direction in [P2]. Thus the rest of the problem (at least for the finite depth case) in Ocneanu's paragroup approach is a classification of paragroups, which is of algebraic and combinatorial nature. (Popa's canonical commuting square in [P1] is equivalent to Ocneanu's paragroup.)

In Popa's approach, he announced a classification of AFD type II_1 subfactors with index 4 (including infinite depth subfactors) in [P2] using his striking characterization of strongly amenable subfactors. Our subfactors with the principal graph $D_n^{(1)}$ have index 4 and these correspond to Corollaire 1 (v) in [P2]. Popa's classification says these subfactors are in one-to-one correspondence to elements in $H^3(G_n, \mathbf{T})$ which vanish on the two generators of the both $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$, where G_n is a certain group quotient of $\mathbf{Z}_2 * \mathbf{Z}_2$, but it was not clear at all how many conjugacy classes of subfactors we have for each $D_n^{(1)}$. Because Popa gave the numbers of the conjugacy classes of subfactors with index 4 for the other possible principal graphs, our results here gives the last number left open in his list.

Ocneanu has made several announcements for classification of paragroups with small index in [O2, O3, O4]. In particular, he announced in his talk at ICM-90 [O4] that there is a unique AFD type II_1 subfactor for each extended Coxeter graph $D_n^{(1)}$, $n \geq 5$. Our results in this paper shows that this statement is invalid and the true number of the conjugacy classes of the AFD type II_1 subfactors with the principal graph $D_n^{(1)}$ is $n-2$.

Ocneanu notices analogy between paragroup structure and solvable lattice model theory of Andrews, Baxter, and Forrester [ABF]. (For solvable lattice model theory, see [B], [DJMO] for instance. A spectral parameter in solvable lattice model theory disappears in our paragroup context.) In this paper, we work on paragroup classification problem for the graphs $D_n^{(1)}$ with use of an idea of orbifold models in solvable lattice model theory [Kt, FG]. Usefulness of "orbifolds" was noticed in [Ka] at first and also used in [EK].

The second author worked in the case for index less than 4 in [Ka], and the first author worked for the same case in [I1, I2] using a different method based on Longo's theory [L1, L2]. This paper is a natural continuation of these papers.

The contents of each section are as follows. In §1, we classify connections of the graph $D_n^{(1)}$ up to perturbation and get a one-parameter family of connections. This means that we have a one-parameter family of commuting squares for $D_n^{(1)}$. In the next section, we work on cell systems between $D_n^{(1)}$ and $A_{2n-5}^{(1)}$. We can regard $D_n^{(1)}$ is an orbifold of $A_{2n-5}^{(1)}$. This shows we can reduce our flatness problem to a problem on the graphs $A_{2n-5}^{(1)}$. Then we classify flat connections on $D_n^{(1)}$ in §3 by working on the graphs $A_{2n-5}^{(1)}$. This means that only certain finite numbers of commuting squares in the one-parameter family give canonical commuting squares in the sense of Popa. In §4, we explain the relation between our computations here and subfactor construction by the first author based on Cuntz algebra endomorphisms. We encountered a counter example to Ocneanu's announcement in this way at first. In §5, we show that a subfactor of an *arbitrary* factor with the principal graph $D_n^{(1)}$

can be realized as the simultaneous fixed point algebras of an action of a dihedral group and subfactor classification is reduced to classification of these actions. This is a generalization of a construction by Goodman-de la Harpe-Jones [GHJ, 4.7]. This method gives another proof of our main result and a classification result for AFD type III_1 subfactors.

The book [GHJ] is a basic reference on the index theory and its relation to Coxeter graphs, and our basic references for Ocneanu's paragroup theory are [O2, O3, Ka]. In particular, [O2] contains a very good exposition on background of his method and [Ka, §1] contains a quick review on it.

We are thankful to Professor D. Bisch for helpful communications, to Professor D. Evans for explaining us the notion of orbifold models, to Professor A. Ocneanu for helpful comments on our work, to Dr. M. Okado for helpful conversations on solvable lattice models, to Professor M. Takesaki for helpful conversations on §5, and especially to Professor S. Popa for a long comment on the first draft of this paper.

§1 One-parameter family of connections on $D_n^{(1)}$

First we classify biunitary connections on the extended Coxeter graphs $D_n^{(1)}$, $n \geq 5$. We label vertices of the graph $D_n^{(1)}$ as follows.

$$D_n^{(1)}: \qquad \begin{array}{ccccccccc} b_0 & & & & & & & & b_{n-2} \\ & \diagdown & & & & & & \diagup & \\ & & b_1 - b_2 & \cdots & b_{n-4} - b_{n-3} & & & & \\ & \diagup & & & & & & \diagdown & \\ b'_0 & & & & & & & & b'_{n-2} \end{array}$$

Note that $D_n^{(1)}$ has $n+1$ vertices. First we classify a biunitary connection on $D_n^{(1)}$ by starting from the vertex b_0 as in [O3, IV.2]. (See [O3, I.3] or [O2, page 151] for the definition of biunitarity. Note that our notations here are slightly different from those in [O2] and the same as those in [O3], [Ka].) In general situations, we have two graphs $\mathcal{G}$ and $\mathcal{H}$ as in [O2, Appendix A] or [O3, Chapter I]. But in our case, the both graphs are $D_n^{(1)}$. (If the principal graph of a subfactor is $D^{(1)}$, then the "dual" graph must be $D_n^{(1)}$, too, and we can identify these two. See [O2, page 139] or [I1, Lemma 3.1] for this type of argument.) Thus we identify the two graphs by fixing an isomorphism between the two. By this remark, it will be enough to check flatness for a single graph $D_n^{(1)}$.

We start from the vertex b_0. By unitarity for a 1×1-matrix, we get

$$\left|\begin{array}{ccc} b_0 & \longrightarrow & b_1 \\ \Big\downarrow & & \Big\downarrow \\ b_1 & \longrightarrow & b_2 \end{array}\right| = 1.$$

(Unitarity here corresponds to the first inversion relation in solvable lattice model theory.) By a gauge choice freedom, we may set

$$\begin{array}{ccc} b_0 & \longrightarrow & b_1 \\ \Big\downarrow & & \Big\downarrow \\ b_1 & \longrightarrow & b_2 \end{array} = 1.$$

(See [O2, page 154] or [O3, I.2, IV.2] for gauge choice. In [O2], change of gauges is called a perturbation of a connection. Because all the edges in our case are single edges, this gauge choice means an assignment of a complex number with modules 1 to each edge. Note that the two graphs $\mathcal{G}$ and $\mathcal{H}$ are the same $D_n^{(1)}$ now, but for this assignment, we regard the two graphs are different.) Next we apply the following renormalization rule.

$$
\begin{array}{ccc} a & \longrightarrow & b \\ \Big\downarrow & & \Big\downarrow \\ c & \longrightarrow & d \end{array}
= \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}}\;
\overline{\begin{array}{ccc} b & \longrightarrow & a \\ \Big\downarrow & & \Big\downarrow \\ d & \longrightarrow & c \end{array}}
= \sqrt{\frac{\mu(b)\mu(c)}{\mu(a)\mu(d)}}\;
\overline{\begin{array}{ccc} c & \longrightarrow & d \\ \Big\downarrow & & \Big\downarrow \\ a & \longrightarrow & b \end{array}},
$$

where $\mu(\cdot)$ denotes each entry of the Perron-Frobenius eigenvector of the incidence matrix of the graph. (See [O2, page 151] or [O3, I.3] for this renormalization rule. This condition together with unitarity is equivalent to the well-known commuting square condition for the tower of relative commutants. This condition corresponds to the crossing symmetry in solvable lattice model theory.) In our case, the vector μ is given by the following picture.

$$
D_n^{(1)}: \qquad
\begin{array}{l} 1 \diagdown \\ \qquad\; 2 — 2 \cdots 2 — 2 \\ 1 \diagup \end{array}
\begin{array}{l} \diagup 1 \\ \\ \diagdown 1 \end{array}
$$

By this renormalization rule, we get

$$
\begin{array}{ccc} b_1 & \longrightarrow & b_0 \\ \Big\downarrow & & \Big\downarrow \\ b_2 & \longrightarrow & b_1 \end{array} = \frac{1}{\sqrt{2}}.
$$

Repeating this procedure, we get five entries of the following 3×3-matrix.

$$
(u_{ij}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & * & * \\ \frac{1}{\sqrt{2}} & * & * \end{pmatrix},
$$

where

$$u_{ij} = \begin{array}{ccc} b_1 & \longrightarrow & c_j \\ \Big\downarrow & & \Big\downarrow \\ c_i & \longrightarrow & b_1 \end{array}, \qquad c_1 = b_0, \quad c_2 = b_0', \quad c_3 = b_2.$$

By unitarity of a 1×1-matrix, we can show $|u_{22}| = \dfrac{1}{2}$. Then by unitarity of the above 3×3-matrix, we can determine the four entries denoted by $*$, and we get the following matrix.

$$(u_{ij}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

This is a special case of Ocneanu's determinating process of a connection at a triple point. The uniqueness of the above four entries follows from the fact that Perron-Frobenius eigenvalue is 2. (See [O3, IV.2] or [Ka, Remark 3.3].) For the segment from b_1 to b_{n-3}, we get the following connection values by certain gauge choices.

$$\begin{array}{ccc} b_j & \longrightarrow & b_{j+1} \\ \Big\downarrow & & \Big\downarrow \\ b_{j+1} & \longrightarrow & b_{j+2} \end{array} = \begin{array}{ccc} b_{j+1} & \longrightarrow & b_j \\ \Big\downarrow & & \Big\downarrow \\ b_{j+2} & \longrightarrow & b_{j+1} \end{array} = \begin{array}{ccc} b_{j+1} & \longrightarrow & b_{j+2} \\ \Big\downarrow & & \Big\downarrow \\ b_j & \longrightarrow & b_{j+1} \end{array} = \begin{array}{ccc} b_{j+2} & \longrightarrow & b_{j+1} \\ \Big\downarrow & & \Big\downarrow \\ b_{j+1} & \longrightarrow & b_j \end{array} = 1,$$

$$\begin{array}{ccc} b_k & \longrightarrow & b_{k+1} \\ \Big\downarrow & & \Big\downarrow \\ b_{k+1} & \longrightarrow & b_k \end{array} = \begin{array}{ccc} b_{k+1} & \longrightarrow & b_k \\ \Big\downarrow & & \Big\downarrow \\ b_k & \longrightarrow & b_{k+1} \end{array} = 0, \qquad 1 \le j \le n-5, \quad 1 \le k \le n-4.$$

Now we work at the second triple point b_{n-3}. Setting

$$v_{ij} = \begin{array}{ccc} b_{n-3} & \longrightarrow & c_j \\ \downarrow & & \downarrow \\ c_i & \longrightarrow & b_{n-3} \end{array}, \qquad c_1 = b_{n-4}, \quad c_2 = b_{n-2}, \quad c_3 = b'_{n-2},$$

we get the following 3×3 matrix.

$$(v_{ij}) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & * & * \\ \frac{1}{\sqrt{2}} & * & * \end{pmatrix}.$$

Here the important point is that the (1,1)-entry of this matrix is 0. By this reason, the unitarity of this matrix does not determine the four entries denoted by $*$ uniquely, and the uniqueness argument breaks down. Instead, we have a parameter c for the above matrix as follows.

$$(v_{ij}) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{c}{2} & -\frac{c}{2} \\ \frac{1}{\sqrt{2}} & -\frac{c}{2} & \frac{c}{2} \end{pmatrix},$$

where c is a complex parameter with $|c| = 1$. There is no more freedom of gauge choices to make c real. The above matrix also gives the following formula by the renormalization rule.

$$\begin{array}{ccc} b_{n-2} & \longrightarrow & b_{n-3} \\ \downarrow & & \downarrow \\ b_{n-3} & \longrightarrow & b_{n-2} \end{array} = \begin{array}{ccc} b'_{n-2} & \longrightarrow & b_{n-3} \\ \downarrow & & \downarrow \\ b_{n-3} & \longrightarrow & b'_{n-2} \end{array} = \bar{c},$$

$$
\begin{array}{ccccccc}
b_{n-2} & \longrightarrow & b_{n-3} & & b'_{n-2} & \longrightarrow & b_{n-3} \\
\Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow \\
b_{n-3} & \longrightarrow & b'_{n-2} & & b_{n-3} & \longrightarrow & b_{n-2}
\end{array} = -\bar{c}.
$$

Thus we have a one-parameter family of biunitary connections on $D_n^{(1)}$, and these are the all biunitary connections up to gauge choice.

§2 One-parameter family of connections on $A_{2n-5}^{(1)}$ and cell systems between $D_n^{(1)}$ and $A_{2n-5}^{(1)}$

We have to determine for what values of c we have flatness. Our strategy for flatness is the same as in [Ka]. That is, we embed the string algebra double complex of $D_n^{(1)}$ into that of $A_{2n-5}^{(1)}$ and check commutativity of vertical and horizontal strings starting from $*$ in this larger double complex. (See §3 for details about flatness.) Thus another extended Coxeter graph we need is $A_{2n-5}^{(1)}$, which is illustrated by the following picture.

$$
A_{2n-5}^{(1)}: \qquad a_0 \begin{array}{l} \diagup a_1 - a_2 \cdots a_{n-4} - a_{n-3} \diagdown \\ \diagdown a'_1 - a'_2 \cdots a'_{n-4} - a'_{n-3} \diagup \end{array} a_{n-2}
$$

Note that $A_{2n-5}^{(1)}$ has $2n-4$ vertices. The Perron-Frobenius eigenvector for this graph is given by the following picture.

$$
A_{2n-5}^{(1)}: \qquad 2 \begin{array}{l} \diagup 2 - 2 \cdots 2 - 2 \diagdown \\ \diagdown 2 - 2 \cdots 2 - 2 \diagup \end{array} 2
$$

As in Roche [R, page 407] (see also [Ka, §5]), we construct a cell system between $D_n^{(1)}$ and $A_{2n-5}^{(1)}$. This is used to embed the string algebra of $D_n^{(1)}$ into that of $A_{2n-5}^{(1)}$.

(See [O2, pages 128–], [O3, Chapter II], or [R, pages 398–] for string algebras for graphs.) Note that the graph $D_n^{(1)}$ is an "orbifold" of $A_{2n-5}^{(1)}$ by a $\mathbf{Z}_2$-symmetry as D_n is an "orbifold" of A_{2n-3} by a $\mathbf{Z}_2$-symmetry. This is related to orbifold models in solvable lattice model theory. (See [Kt, FG] for instance.) Operator algebraic interpretation of this kind of duality was given by M. Choda [Ch].

Our cell system is given by the following. In the following squares, the left vertical edges are from the graph $D_n^{(1)}$, the right vertical edges from $A_{2n-5}^{(1)}$ and the horizontal edges connect these two graphs.

$$\begin{array}{ccc} b_1 & \longrightarrow & a_1 \\ \big\downarrow & & \big\downarrow \\ b_0 & \longrightarrow & a_0 \end{array} = \begin{array}{ccc} b_1 & \longrightarrow & a'_1 \\ \big\downarrow & & \big\downarrow \\ b_0 & \longrightarrow & a_0 \end{array} = \begin{array}{ccc} b_1 & \longrightarrow & a'_1 \\ \big\downarrow & & \big\downarrow \\ b'_0 & \longrightarrow & a_0 \end{array} = \frac{1}{\sqrt{2}},$$

$$\begin{array}{ccc} b_{n-3} & \longrightarrow & a_{n-3} \\ \big\downarrow & & \big\downarrow \\ b_{n-2} & \longrightarrow & a_{n-2} \end{array} = \begin{array}{ccc} b_{n-3} & \longrightarrow & a'_{n-3} \\ \big\downarrow & & \big\downarrow \\ b_{n-2} & \longrightarrow & a_{n-2} \end{array} = \begin{array}{ccc} b_{n-3} & \longrightarrow & a'_{n-3} \\ \big\downarrow & & \big\downarrow \\ b'_{n-2} & \longrightarrow & a_{n-2} \end{array} = \frac{1}{\sqrt{2}},$$

$$\begin{array}{ccc} b_1 & \longrightarrow & a_1 \\ \big\downarrow & & \big\downarrow \\ b'_0 & \longrightarrow & a_0 \end{array} = \begin{array}{ccc} b_{n-3} & \longrightarrow & a_{n-3} \\ \big\downarrow & & \big\downarrow \\ b'_{n-2} & \longrightarrow & a_{n-2} \end{array} = -\frac{1}{\sqrt{2}},$$

$$\begin{array}{ccc} b_0 & \longrightarrow & a_0 \\ \big\downarrow & & \big\downarrow \\ b_1 & \longrightarrow & a_1 \end{array} = \begin{array}{ccc} b_0 & \longrightarrow & a_0 \\ \big\downarrow & & \big\downarrow \\ b_1 & \longrightarrow & a'_1 \end{array} = \begin{array}{ccc} b'_0 & \longrightarrow & a_0 \\ \big\downarrow & & \big\downarrow \\ b_1 & \longrightarrow & a'_1 \end{array} = 1,$$

$$\begin{array}{ccc} b_{n-2} & \longrightarrow & a_{n-2} \\ \big\downarrow & & \big\downarrow \\ b_{n-3} & \longrightarrow & a_{n-3} \end{array} = \begin{array}{ccc} b_{n-2} & \longrightarrow & a_{n-2} \\ \big\downarrow & & \big\downarrow \\ b_{n-3} & \longrightarrow & a'_{n-3} \end{array} = \begin{array}{ccc} b'_{n-2} & \longrightarrow & a_{n-2} \\ \big\downarrow & & \big\downarrow \\ b_{n-3} & \longrightarrow & a'_{n-3} \end{array} = 1,$$

$$\begin{array}{cccccccl} b'_0 & \longrightarrow & a_0 & & b'_{n-2} & \longrightarrow & a_{n-2} & \\ \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = -1, \\ b_1 & \longrightarrow & a_1 & & b_{n-3} & \longrightarrow & a_{n-3} & \end{array}$$

$$\begin{array}{cccccccl} b_j & \longrightarrow & a_j & & b_j & \longrightarrow & a'_j & \\ \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = 1, \qquad 1 \le j \le n-4, \\ b_{j+1} & \longrightarrow & a_{j+1} & & b_{j+1} & \longrightarrow & a'_{j+1} & \end{array}$$

$$\begin{array}{cccccccl} b_{j+1} & \longrightarrow & a_{j+1} & & b_{j+1} & \longrightarrow & a'_{j+1} & \\ \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = 1, \qquad 1 \le j \le n-4. \\ b_j & \longrightarrow & a_j & & b_j & \longrightarrow & a'_j & \end{array}$$

It is easy to check unitarity of this system. We also need a biunitary connection for $A^{(1)}_{2n-5}$. For a complex parameter c with $|c| = 1$ which is the same parameter as in §1, we define the following connection on $A^{(1)}_{2n-5}$.

$$\begin{array}{cccccccccccccccl} a_1 & \longrightarrow & a_0 & & a'_1 & \longrightarrow & a_0 & & a_0 & \longrightarrow & a_1 & & a_0 & \longrightarrow & a'_1 & \\ \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = 1, \\ a_0 & \longrightarrow & a'_1 & & a_0 & \longrightarrow & a_1 & & a'_1 & \longrightarrow & a_0 & & a_1 & \longrightarrow & a_0 & \end{array}$$

$$\begin{array}{cccccccccccccccl} a_j & \longrightarrow & a_{j+1} & & a_{j+2} & \longrightarrow & a_{j+1} & & a_{j+1} & \longrightarrow & a_j & & a_{j+1} & \longrightarrow & a_{j+2} & \\ \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = 1, \\ a_{j+1} & \longrightarrow & a_{j+2} & & a_{j+1} & \longrightarrow & a_j & & a_{j+2} & \longrightarrow & a_{j+1} & & a_j & \longrightarrow & a_{j+1} & \end{array}$$

$$\begin{array}{cccccccccccccccl} a'_j & \longrightarrow & a'_{j+1} & & a'_{j+2} & \longrightarrow & a'_{j+1} & & a'_{j+1} & \longrightarrow & a'_j & & a'_{j+1} & \longrightarrow & a'_{j+2} & \\ \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = & \Big\downarrow & & \Big\downarrow & = 1, \\ a'_{j+1} & \longrightarrow & a'_{j+2} & & a'_{j+1} & \longrightarrow & a'_j & & a'_{j+2} & \longrightarrow & a'_{j+1} & & a'_j & \longrightarrow & a'_{j+1} & \end{array}$$

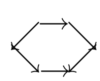
$$\begin{tikzcd} a_{n-3} \arrow[r] \arrow[d] & a_{n-2} \arrow[d] \\ a_{n-2} \arrow[r] & a'_{n-3} \end{tikzcd} = \begin{tikzcd} a'_{n-3} \arrow[r] \arrow[d] & a_{n-2} \arrow[d] \\ a_{n-2} \arrow[r] & a_{n-3} \end{tikzcd} = c,$$

$$\begin{tikzcd} a_{n-2} \arrow[r] \arrow[d] & a'_{n-3} \arrow[d] \\ a_{n-3} \arrow[r] & a_{n-2} \end{tikzcd} = \begin{tikzcd} a_{n-2} \arrow[r] \arrow[d] & a_{n-3} \arrow[d] \\ a'_{n-3} \arrow[r] & a_{n-2} \end{tikzcd} = \bar{c},$$

All the other admissible squares like $\begin{tikzcd} a_0 \arrow[r] \arrow[d] & a_1 \arrow[d] \\ a_1 \arrow[r] & a_0 \end{tikzcd}$ have connection value 0.

Here we used convention $a'_0 = a_0$, $a'_{n-2} = a_{n-2}$ and the integer j is in the interval $[0, n-4]$. By the word "admissible" we mean that all the four edges of the square come from the graph $A^{(1)}_{2n-5}$.

By the above cell system, we can embed the string algebra of $D^{(1)}_n$ into that of $A^{(1)}_{2n-5}$. But we have identification of strings by connections in both double complexes, so we have to show that the identifications are compatible with this embedding. In order to show this, it is enough to check a kind of the star triangle relation as in [Ka, Lemma 5.1]. That is, we have hexagons



where the left two downward edges are from the graph $D^{(1)}_n$, the right two downward edges from $A^{(1)}_{2n-5}$, and the two horizontal edges connect the two graphs. For each fixed hexagon, we first consider configurations $\genfrac{}{}{0pt}{}{\searrow}{\swarrow}\!\!\rightarrow$ for inside of the hexagon. For each configuration, we have a $D_n^{(1)}$ connection value for the left parallelogram and two cell system values for the right parallelograms. We multiply these three numbers and make a sum over all the configurations. Similarly we make another sum over configurations $\rightarrow\!\!\genfrac{}{}{0pt}{}{\swarrow}{\searrow}$. It is easy to see that the compatibility we have to check is equivalent to equality for these two sums for all the hexagons. If the left two downward edges of the hexagon is in between the vertices b_1 and b_{n-3}, then it is trivial that these two sums are equal. If the vertex b_{n-3} is involved in the hexagon, we need a little computations, but we can check all the cases by direct computations. Indeed, there are 34 cases, and a typical example is as follows.

$$\begin{tikzcd}[row sep=small, column sep=small]
 & b_{n-3} \arrow[r] & a_{n-3} \arrow[dr] & \\
b'_{n-2} \arrow[ur] \arrow[dr] & & & a_{n-2} \\
 & b_{n-3} \arrow[r] & a'_{n-3} \arrow[ur] &
\end{tikzcd}$$

For the configuration $\genfrac{}{}{0pt}{}{\searrow}{\swarrow}\!\!\rightarrow$, the central vertex in the hexagon is either b_{n-2} or b'_{n-2}. Thus our sum is

$$-\frac{c}{2}\cdot\frac{1}{\sqrt{2}}\cdot 1+\frac{c}{2}\cdot\left(-\frac{1}{\sqrt{2}}\right)\cdot 1=-\frac{c}{\sqrt{2}}.$$

For the other configuration $\rightarrow\!\!\genfrac{}{}{0pt}{}{\swarrow}{\searrow}$, we have only one choice, that is, the central vertex is a_{n-2}. In this case our sum is $-\dfrac{1}{\sqrt{2}}\cdot 1\cdot c=-\dfrac{c}{\sqrt{2}}$, which is equal to the above sum.

For the hexagons involving b_1, the same computation setting $c=1$ works and we get the desired conclusion.

§3 Flatness of the connections on $D_n^{(1)}$

By the result of §2, we can reduce our flatness problem for $D_n^{(1)}$ to a problem for $A_{2n-5}^{(1)}$. Here we recall some basics for flatness. (Flatness of connections was introduced in [O2, page 153] in the name of the parallel transport axiom. Details of flatness are found in [O3, II.5, II.6] and [Ka, §2].) We choose a distinguished vertex $*$ among vertices of the graph and construct a string algebra double complex as in [O3, II.2]. Then flatness means that the vertical strings from $*$ and the horizontal strings from $*$ commute. Because each horizontal line of the string algebra double complex gives a basic extension, this flatness means that the vertical string algebra from $*$ gives the tower of the higher relative commutants. In particular, we get the original graph $\mathcal{G}$ as the principal graph of the string algebra subfactor. In the string algebra, we can define the Jones projections as in [O3, II.3] and they are always flat [O3, II.5]. This means that the vertical [resp. horizontal] Jones projections commute with all the horizontal [resp. vertical] strings.

We take $b_0 = *$ for $D_n^{(1)}$. Note that this is essentially the only choice because if we take one of $b_1, \cdots b_{n-3}$ to be $*$, our connection is not flat by [Ka, a remark preceding Theorem 4.1] or [I1, 6.1 (i)].

Then note that the string algebra from $*$ is generated by the following: the Jones projections, $(b_0 — b_1 — b_0', b_0 — b_1 — b_0')$, $(b_0 — b_1 \cdots b_{n-3} — b_{n-2}, b_0 — b_1 \cdots b_{n-3} — b_{n-2})$. Because there is no problem for the Jones projections as explained above, we only have to work on the other two strings. We embed these two strings into the string algebras of $A_{2n-5}^{(1)}$.

First we show that the string $(b_0 — b_1 — b_0', b_0 — b_1 — b_0')$ does not cause any problem for commutativity for any value of the parameter c. If we embed this string into the string algebra of $A_{2n-5}^{(1)}$ using our cell system, we get the following string.

$$\frac{1}{2}((\xi_1, \xi_1) + (\xi_2, \xi_2) - (\xi_1, \xi_2) - (\xi_2, \xi_1)),$$

$$\xi_1 = (a_0 — a_1 — a_0), \quad \xi_2 = (a_0 — a_1' — a_0).$$

Note that the element

$$\frac{1}{2}((\xi_1, \xi_1) + (\xi_2, \xi_2) + (\xi_1, \xi_2) + (\xi_2, \xi_1))$$

is the first Jones projection for $A_{2n-5}^{(1)}$, thus this causes no trouble for commutativity. Thus it is enough to prove that the vertical [resp. horizontal] string (ξ_1, ξ_2) commutes with every horizontal [resp. vertical] string. For this purpose, we need parallel transports of the vertical (ξ_1, ξ_2), which is given by the following lemma.

Lemma 3.1. *We define the rotation ρ of vertices of $A_{2n-5}^{(1)}$ by $\rho(a_0) = a_1, \rho(a_1) = a_2, \dots, \rho(a_{n-3}) = a_{n-2}, \rho(a_{n-2}) = a_{n-3}', \dots, \rho(a_2') = a_1', \rho(a_1') = a_0$. Then we* *have the following equality for the connection on $A^{(1)}_{2n-5}$.*

$$\begin{tikzcd}
a_0 \arrow[r, "\eta_1"] \arrow[d] & \cdots \arrow[r] & x_0 \arrow[d] \\
a_1 \arrow[d] & & x_1 \arrow[d] \\
a_0 & & x_2 \\
a'_1 \arrow[u] & & x_3 \arrow[u] \\
a_0 \arrow[r, "\eta_2"] \arrow[u] & \cdots \arrow[r] & x_4 \arrow[u]
\end{tikzcd} = \quad \delta_{\eta_1,\eta_2}\delta_{\rho^m(a_1),x_1}\delta_{\rho^m(a_0),x_2}\delta_{\rho^m(a'_1),x_3}\delta_{\rho^m(a_0),x_4},$$

where m is given by $\rho^m(a_0) = x_0$.

Proof. If we fix three vertices of a 1×1-cell, there is only one choice of the other vertex to get a non-zero connection value for $A^{(1)}_{2n-5}$, and this non-zero value is one of $1, c, \bar{c}$. Using this fact and induction, we can prove the desired formula easily.

Q.E.D.

The above formula implies the desired conclusion for (ξ_1, ξ_2). (See [Ka, Theorem 2.1 (2)].)

By the above arguments, our flatness is now equivalent to commutativity of the vertical (ξ, ξ) and the horizontal (ξ, ξ) for the double complex of $D^{(1)}_n$, where the path ξ is given by b_0—b_1—$\cdots$—b_{n-2}. Define two paths η, ζ for $A^{(1)}_{2n-5}$ by a_0—a_1—$\cdots a_{n-3}$—a_{n-2} and a_0—a'_1—$\cdots a'_{n-3}$—a_{n-2} respectively. Then it is easy to see the image of (ξ, ξ) by our embedding using the cell system is

$$\frac{1}{2}((\eta, \eta) + (\eta, \zeta) + (\zeta, \eta) + (\zeta, \zeta)).$$

We now need the following lemma.

Lemma 3.2. *We have the following identities for the connection in $A^{(1)}_{2n-5}$ containing a complex parameter c.*

$$\begin{tikzcd}
a_0 \arrow[r, "\eta"] \arrow[d, "\eta"'] & \cdots \arrow[r] & a_{n-2} \arrow[d, "\sigma_2"] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
a_{n-2} \arrow[r, "\sigma_1"'] & \cdots \arrow[r] & a_0
\end{tikzcd} = \delta_{\sigma_1,\tilde{\zeta}}\delta_{\sigma_2,\tilde{\zeta}}c^{n-2},$$

$$\begin{tikzcd}
a_0 \arrow[r, "\zeta"] \arrow[d, "\zeta"'] & \cdots \arrow[r] & a_{n-2} \arrow[d, "\sigma_2"] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
a_{n-2} \arrow[r, "\sigma_1"'] & \cdots \arrow[r] & a_0
\end{tikzcd} = \delta_{\sigma_1,\tilde{\eta}}\delta_{\sigma_2,\tilde{\eta}}c^{n-2},$$

$$\begin{tikzcd}
a_0 \arrow[r, "\zeta"] \arrow[d, "\eta"'] & \cdots \arrow[r] & a_{n-2} \arrow[d, "\sigma_2"] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
a_{n-2} \arrow[r, "\sigma_1"'] & \cdots \arrow[r] & a_0
\end{tikzcd} = \delta_{\sigma_1,\tilde{\eta}}\delta_{\sigma_2,\tilde{\zeta}},$$

$$\begin{tikzcd}
a_0 \arrow[r, "\eta"] \arrow[d, "\zeta"'] & \cdots \arrow[r] & a_{n-2} \arrow[d, "\sigma_2"] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
a_{n-2} \arrow[r, "\sigma_1"'] & \cdots \arrow[r] & a_0
\end{tikzcd} = \delta_{\sigma_1,\tilde{\zeta}}\delta_{\sigma_2,\tilde{\eta}},$$

where ˜ *denotes the reversed paths.*

Proof. The same kind of computation as in the proof of Lemma 3.1 gives the conclusion. Q.E.D.

Let $\sigma = (\eta,\eta)+(\eta,\zeta)+(\zeta,\eta)+(\zeta,\zeta)$. By Lemma 3.2, we can compute the parallel transport $T_{\nu_1,\nu_2}(\sigma)$ of the vertical strings σ along horizontal paths (ν_1,ν_2). (Here we use a notation in [O2, page 127].) We may assume $|\nu_1| = |\nu_2| = n-2$. If the both of the endpoints $r(\nu_1)$ and $r(\nu_2)$ are different from a_{n-2}, such a term does not cause any trouble for commutativity. If one of $r(\nu_1)$ and $r(\nu_2)$ is a_{n-2} and the other is not, then we get $T_{\nu_1,\nu_2}(\sigma) = 0$ by Lemma 3.2. If the both of $r(\nu_1)$ and $r(\nu_2)$ are a_{n-2}, we get the following by direct computation using Lemma 3.2.

$$T_{\eta,\eta}(\sigma) = (\tilde{\zeta},\tilde{\zeta}) + c^{n-2}(\tilde{\zeta},\tilde{\eta}) + c^{-n+2}(\tilde{\eta},\tilde{\zeta}) + (\tilde{\eta},\tilde{\eta}),$$

$$T_{\zeta,\zeta}(\sigma) = (\tilde{\zeta},\tilde{\zeta}) + c^{-n+2}(\tilde{\zeta},\tilde{\eta}) + c^{n-2}(\tilde{\eta},\tilde{\zeta}) + (\tilde{\eta},\tilde{\eta}),$$

$$T_{\eta,\zeta}(\sigma) = 0,$$

$$T_{\zeta,\eta}(\sigma) = 0.$$

Then it is easy to see that commutativity of these and the horizontal strings σ is equivalent to $c^{n-2} = c^{-n+2}$, that is, $c^{2(n-2)} = 1$. Thus we have proved the following.

Proposition 3.3. *The connection on $D_n^{(1)}$ defined in §1 is flat if and only if the complex parameter c satisfies $c^{2(n-2)} = 1$.*

Thus there are $2(n-2)$ flat connections on $D_n^{(1)}$. But there is a graph automorphism of order 2 on $D_n^{(1)}$ fixing the distinguished point $*$. It is the flip exchanging b_{n-2} and b'_{n-2}. By this flip, we can identify a connection for c and that for $-c$, and this is the only graph automorphism fixing $*$. Thus our conclusion on subfactor classification is as follows.

Theorem 3.4. *The number of the conjugacy classes of the AFD type II_1 subfactors having the principal graph $D_n^{(1)}$, $n \geq 5$, is $n-2$.*

We list three remarks here on the above theorem.

Remark 3.5. In [O2, page 159] and [O3, IV.2], Ocneanu showed the following formula for connections on the Coxeter graphs A_n, D_n, E_6, E_7, E_8 and said the proof for biunitarity also works for $D_n^{(1)}$, $A_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, and $E_8^{(1)}$ by setting $N = \infty$, $\varepsilon = \sqrt{-1}$.

$$\begin{array}{ccc} i & \longrightarrow & l \\ \Big\downarrow & & \Big\downarrow \\ k & \longrightarrow & j \end{array} = \delta_{kl}\varepsilon + \sqrt{\frac{\mu(k)\mu(l)}{\mu(i)\mu(j)}}\delta_{ij}\bar{\varepsilon},$$

where $\varepsilon = \sqrt{-1}\exp\dfrac{\pi\sqrt{-1}}{2N}$ and N is the Coxeter number. For the case index< 4, all the flat connections are represented by this formula, although the above connections are not always flat. But in our cases for $D_n^{(1)}$, this formula gives only our flat connection with $c = 1$, and the other flat connections are not represented by this formula.

Remark 3.6. If $n = 4$ then the graph arises from a group of order 4. It is well known that there are two flat connections, one for $\mathbf{Z}_4$, and the other for $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Thus the conclusion in the theorem is also valid for $n = 4$.

Remark 3.7. The subfactor for the graph $A^{(1)}_{2n-5}$ is of the form

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & \alpha(x) \end{pmatrix} \,\middle|\, x \in \mathcal{R} \right\} \subset M_2(\mathbf{C}) \otimes \mathcal{R},$$

where α is an automorphism of the AFD type II_1 factor $\mathcal{R}$ with $\alpha^{n-2} \in \mathrm{Int}(\mathcal{R})$. (See [P2].) Our c^2 for $A^{(1)}_{2n-5}$ corresponds to the obstruction of this automorphism α ([O4], [C2, page 41]).

§4 Comments on subfactor construction using Cuntz algebra endomorphisms

The first author encountered a counter example to Ocneanu's announcement at first by working on subfactor construction for $D^{(1)}_5$ using an endomorphism of the Cuntz algebra [Cu] and producing 3 different subfactors in this case. The details of this work will be presented in [I2], but here we give a brief comment on the relation between the above computation in this paper and the Cuntz algebra approach.

Let O_3 be the Cuntz algebra generated by 3 isometries S_1, S_2, S_3 satisfying $S_1S_1^* + S_2S_2^* + S_3S_3^* = 1$. For $a \in \mathbf{T}$, we define endomorphism ρ_a on O_3 as follows.

$$\begin{aligned}
\rho_a(S_1) &= \frac{S_1 + S_2}{2} + \frac{S_3S_3}{\sqrt{2}}, \\
\rho_a(S_2) &= \left(\frac{S_1 + S_2}{2} - \frac{S_3S_3}{\sqrt{2}} \right) U, \\
\rho_a(S_3) &= \bar{a}\frac{S_1 - S_2}{\sqrt{2}} S_3^* + aS_3(S_1S_1^* - S_2S_2^*),
\end{aligned}$$

where $U = S_1S_1^* + S_2S_2^* - S_3S_3^*$ is a unitary. We also define a $\mathbf{Z}_2$ action α and a $\mathbf{T}$ action γ as follows.

$$\alpha(S_1) = S_2, \qquad \alpha(S_2) = S_1, \qquad \alpha(S_3) = -S_3,$$

$$\gamma_t(S_1) = e^{2it}S_1, \qquad \gamma_t(S_2) = e^{2it}S_2, \qquad \gamma_t(S_3) = e^{it}S_3,$$

Then these satisfy the following relations.

$$\gamma_t \cdot \rho = \rho \cdot \gamma_t, \qquad \alpha \cdot \rho = \rho, \qquad \rho \cdot \alpha = \mathrm{Ad}U \cdot \rho,$$

$$S_1 x = \rho_a^2(x)S_1, \quad S_2\alpha(x) = \rho_a^2(x)S_2, \quad S_3\rho_{\bar{a}^2}(x) = \rho_a^2(x)S_3, \quad S_3\rho_a(x) = \rho_{\bar{a}^2} \cdot \rho_a(x)S_3,$$

If we define a conditional expectation E from O_3 to $\rho_a(O_3)$ by $E(x) = \rho_a(S_1^*\rho_a(x)S_1)$, $x \in O_3$, then $(2S_1^*, 2S_1)$ is a quasi-basis in the sense of Watatani [Wt, §1.2] and the index of E is 4.

There is a unique KMS state for γ, and by the G.N.S. construction for this state, we obtain inclusions of type $\mathrm{III}_{1/2}$ AFD factors. We denote by $M \supset N_a$ these inclusions.

Let O_3^γ be the fixed point algebra of O_3 under γ. We can show that O_3^γ is isomorphic to the string algebra of $D_5^{(1)}$ and $M^\gamma \supset N_a^\gamma$ is a pair of II_1 factors. We can also show that $\rho_{a|O_3^\gamma}$ comes from a connection on $D_5^{(1)}$. More precisely, the complex number a^2 here corresponds to c in §1. Then the principal graph of $M^\gamma \supset N_a^\gamma$ is $D_5^{(1)}$ if and only if $a^{12} = 1$ by Proposition 3.3. This can be proved by direct computation for $\rho_{\bar{a}^2} \cdot \rho_{a|O_3^\gamma}$, too. We also have the following fact.

Fact. (1) *For $a, b \in \mathbf{T}$ satisfying $a^6 = b^6 = 1$, the inclusion $M \supset N_a$ is isomorphic to $M \supset N_b$ if and only if $a^2 = b^2$.*

(2) *The principal graph of $M \supset N_a$ is $D_5^{(1)}$ if and only if $a^6 = 1$.*

If $a^{12} = 1$ and $a^6 \neq 1$, then the principal graphs of $M \supset N_a$ and $M^\gamma \supset N_a^\gamma$ are different. In this case, the inclusion $M \supset N_a$ does not split into a type II_1 inclusion.

§5 Reduction to classification of actions of dihedral groups

Goodman, de la Harpe, and Jones constructed a subfactor with the principal graph $D_n^{(1)}$ (and $A_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $A_{\infty,\infty}$, A_∞, D_∞) in [GHJ, 4.7.d]. Their construction gives a subfactor with the principal graph $D_n^{(1)}$ as $(\mathcal{R} \otimes \mathbf{C})^{\mathbf{D}_{n-2}} \subset (\mathcal{R} \otimes M_2(\mathbf{C}))^{\mathbf{D}_{n-2}}$ for a product type action of the dihedral group $\mathbf{D}_{n-2}$ on the AFD type II_1 factor $\mathcal{R}$. Here we show that a subfactor of an *arbitrary* factor with the principal graph $D_n^{(1)}$ can be realized as $(P \otimes \mathbf{C})^{\mathbf{D}_m} \subset (P \otimes M_2(\mathbf{C}))^{\mathbf{D}_m}$ for an action of the dihedral group $\mathbf{D}_m$ for some m on some factor P and classification of these subfactors can be reduced to classification of these group actions. (For subfactor theory for arbitrary factors, see [Ks1, L1, L2].) Invoking classification of finite group actions on the AFD factors [J1], [KST], we get another proof of our main result and also prove that a subfactor of the AFD type III_1 factor with the principal graph $D_n^{(1)}$ is a tensor product of an AFD type II_1 subfactor with the principal graph $D_n^{(1)}$ and a common AFD type III_1 factor.

Let a, b be the generators of the dihedral group $\mathbf{D}_m$ with the relations $a^m = b^2 = (ab)^2 = 1$. We define the following equivalence relation for actions of $\mathbf{D}_m$.

Definition 5.1. Let θ, θ' be actions of the dihedral group $\mathbf{D}_m$ on a factor. These are said to be equivalent if and only if there exists an integer k such that θ and $\theta' \cdot \pi^k$ are conjugate, where π is an automorphism $\mathbf{D}_m$ defined by $\pi(a) = a, \pi(b) = ab$.

Let M be a factor and α, β automorphisms of M with $\alpha^m = \beta^2 = (\alpha\beta)^2 = id$, $m \geq 3$. This defines an action θ of the dihedral group $\mathbf{D}_m$ on M by $\theta_a = \alpha, \theta_b = \beta$. We also denote this action by $\langle \alpha, \beta \rangle$. We call this action *minimal periodic* if α has a minimal period m in the sense of [Co2, page 46] and β is outer. Note that if an action θ of $\mathbf{D}_m$ is minimal periodic and π is an automorphism of $\mathbf{D}_m$, then $\theta \cdot \pi$ is also minimal periodic.

Lemma 5.2. *(1) Two cocycle conjugate minimal periodic automorphisms are conjugate.*

(2) The fixed point algebra of a minimal periodic action of $\mathbf{D}_m$ *is a factor.*

(3) Two cocycle conjugate minimal periodic actions of $\mathbf{D}_m$ *are conjugate.*

Proof. (1) This is [Co2, Corollary 2.6 (b)].

(2) Let $\theta = \langle \alpha, \beta \rangle$ be a minimal periodic action of $\mathbf{D}_m$ on a factor M. Then a direct computation shows that $M \rtimes_\theta \mathbf{D}_m$ is a factor because $\alpha^k \cdot \beta$ is outer for each k and $M \rtimes_{\theta'} \mathbf{Z}_m$ is a factor, where θ' is a restriction of θ on $\mathbf{Z}_m = \mathbf{Z}/m\mathbf{Z}$ generated by a. We get the conclusion by [Pa, Corollary 3.2].

(3) Thanks to (2), the usual one-cohomology vanishing trick of [Co2, Corollary 2.6] works. Q.E.D.

Lemma 5.3. *Let M be a factor and α be a minimal periodic automorphism of M with a period m. Then there exists a minimal periodic automorphism $\alpha_0 \in Aut(M^\alpha)$ with a period m such that (M, α) is conjugate to $(M^\alpha \rtimes_{\alpha_0} \mathbf{Z}_m, \hat{\alpha}_0)$, where $\hat{\alpha}_0$ is the dual action of α_0.*

Proof. Taking a system of matrix units $\{e_{ij}\}_{i,j=1}^m \subset M^\alpha$, we may identify (M, α) with $(eMe \otimes M_m(\mathbf{C}), \alpha^e \otimes id)$, $e = e_{11}, \alpha^e = \alpha|_{eMe}$. By Takesaki duality [T, Theorem 4.5] and Lemma 5.2 (1), we obtain the result. Q.E.D.

The following is a generalization of [GHJ, Lemma 4.7.1]. (Also see Invariance Principle of A. J. Wassermann [Ws, page 227].)

Lemma 5.4. *Let $M \supset N$ be a pair of factors and $E \in E(M, N)$ have a finite index, where $E(M, N)$ denotes the set of faithful normal conditional expectations from M onto N. In case that M is of type II_1, we assume that E comes from the unique trace. Let M_1 be the basic extension of M by N, e the Jones projection for E, and $E_1 \in E(M_1, M)$ the expectation constructed by E^{-1} as in* [Ks1]. *We assume that there exists an action θ of a group G on M_1 satisfying the following conditions.*

(1) $E_1 \cdot \theta_g = \theta_g \cdot E_1, E \cdot \theta_g|_M = \theta_g \cdot E,$ *for* $g \in G$.

(2) $\theta_g(e) = e,$ *for* $g \in G$.

(3) *The fixed point algebras M_1^θ, M^θ, and N^θ are factors and $Ind(F) = Ind(F_1) = Ind(E)$, where $F = E|_{M^\theta} \in E(M^\theta, N^\theta)$, $F_1 = E_1|_{M_1^\theta} \in E(M_1^\theta, M^\theta)$.*

Then $M_1^\theta = \langle M^\theta, e\rangle$ *and* M_1^θ *is the basic extension of* M^θ *by* N^θ.

Proof. By the downward basic construction, we have a projection $f \in M_1^\theta$ such that $M_1^\theta = \langle M^\theta, f\rangle$ and $F_1(f) = \mathrm{Ind}(F_1)$. By $F_1(e) = E_1(e) = \mathrm{Ind}(E) = \mathrm{Ind}(F_1)$, we have a unitary $u \in M_1^\theta$ satisfying $ufu^* = e$ as in the proof of [Ks2, Theorem 2]. Because $exe = E(x)e = F(x)e$ for $x \in M^\theta$, we get the results. Q.E.D.

We generalize a construction of an AFD type II_1 subfactor with the principal graph $D_n^{(1)}$ in [GHJ, 4.7] as follows.

Theorem 5.5. *Let* M *be a factor and* $\alpha, \beta \in Aut(M)$ *satisfy* $\alpha^m = \beta^2 = (\alpha\beta)^2 = id$, $m \geq 3$. *We assume that* $\langle \alpha, \beta\rangle$ *is a minimal periodic action of* $\mathbf{D}_m$ *and the outer invariants of* α *are* (p_o, γ). *If we define an action of* $\mathbf{D}_m$ *on* $M \otimes M_2(\mathbf{C})$ *by*

$$\left\langle \alpha \otimes Ad\begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2}\end{pmatrix}, \beta \otimes Ad\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right\rangle, \quad \omega = e^{2\pi\sqrt{-1}/m}, \omega^{1/2} = e^{\pi\sqrt{-1}/m},$$

then we get the following.

(1) *The principal graph of a subfactor* $(M \otimes \mathbf{C})^{\mathbf{D}_m} \subset (M \otimes M_2(\mathbf{C}))^{\mathbf{D}_m}$ *is* $D_{p_o+2}^{(1)}$.

(2) *The parameter* c *of the connection determined by the tower of higher relative commutants as in §1 satisfies the relation that* c^2 *is given by the obstruction of* $\hat{\alpha}$.

Proof. By Lemma 5.3, there exists a minimal periodic automorphism $\alpha_0 \in \mathrm{Aut}(M^\alpha)$ such that

$$M = M^\alpha \rtimes_{\alpha_0} \mathbf{Z}_m = \langle M^\alpha, \lambda\rangle,$$

$$\alpha(\lambda) = \omega\lambda,$$

where λ is the implementing unitary for α_0. Note that α_0 and α have the same outer periods. Let u be a unitary in M^α satisfying $\alpha_0^{p_o} = \mathrm{Ad}(u)$. Then we have

$$M \cap (M^\alpha)' = \langle u\lambda^{-p_o} \rangle. \tag{5.5.1}$$

Because β is outer, both $M \rtimes \mathbf{D}_m$ and $M \rtimes_\alpha \mathbf{Z}_m$ are factors, hence we obtain the following. (See [A, II.3].)

$$\begin{aligned} M \cap (M^{\mathbf{D}_m})' &= J(J(M \cap (M^{\mathbf{D}_m})')J)J = J(M' \cap (M \rtimes \mathbf{D}_m))J \\ &= J(M' \cap (M \rtimes_\alpha \mathbf{Z}_m))J = M \cap (M^{\mathbf{Z}_m})', \end{aligned} \tag{5.5.2}$$

where J is the modular conjugation. By direct computations, we get the following.

(5.5.3)
$$(M \otimes M_2(\mathbf{C}))^{\mathbf{Z}_m} = \left\{ \begin{pmatrix} a & b\lambda^{-1} \\ \lambda c & d \end{pmatrix} \; ; \; a, b, c, d \in M^\alpha \right\},$$

(5.5.4)
$$(M \otimes M_2(\mathbf{C}))^{\mathbf{Z}_m} \cap ((M \otimes \mathbf{C})^{\mathbf{Z}_m})' = \left\{ \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \; ; \; c_1, c_2 \in \mathbf{C} \right\},$$

(5.5.5)
$$(M \otimes M_2(\mathbf{C}))^{\mathbf{D}_m} = \left\{ \begin{pmatrix} a & b\lambda^{-1} \\ -\beta(b\lambda^{-1}) & \beta(a) \end{pmatrix} \; ; \; a, b \in M^\alpha \right\},$$

(5.5.6)
$$(M \otimes M_2(\mathbf{C}))^{\mathbf{D}_m} \cap ((M \otimes \mathbf{C})^{\mathbf{D}_m})' = \mathbf{C}.$$

For (5.5.4) and (5.5.6), we used (5.5.1) and (5.5.2).

We investigate the structure of the inclusion $(M \otimes \mathbf{C})^{\mathbf{Z}_m} \subset (M \otimes M_2(\mathbf{C}))^{\mathbf{Z}_m}$.

Let

$$\rho = \mathrm{Ad} \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{Aut}(M \otimes M_2(\mathbf{C})).$$

Then we have the following.

(5.5.7)
$$\rho((M\otimes M_2(\mathbf{C}))^{\mathbf{Z}_m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; ; \; a,b,c,d \in M^\alpha \right\} = M^\alpha \otimes M_2(\mathbf{C}),$$

(5.5.8)
$$\rho((M\otimes \mathbf{C})^{\mathbf{Z}_m}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \alpha_0^{-1}(a) \end{pmatrix} \; ; \; a \in M^\alpha \right\},$$

Because the outer period of α^0 is p_o, this means that the principal graph of $(M\otimes \mathbf{C})^{\mathbf{Z}_m} \subset (M\otimes M_2(\mathbf{C}))^{\mathbf{Z}_m}$ is $A^{(1)}_{2p_o-1}$.

Thanks to Lemma 5.4 and the argument in [GHJ, 4.7.d], the towers associated with $(M\otimes \mathbf{C})^{\mathbf{Z}_m} \subset (M\otimes M_2(\mathbf{C}))^{\mathbf{Z}_m}$ and $(M\otimes \mathbf{C})^{\mathbf{D}_m} \subset (M\otimes M_2(\mathbf{C}))^{\mathbf{D}_m}$ are as follows.

$$(M\otimes \mathbf{C})^{\mathbf{Z}_m} \subset (M\otimes M_2(\mathbf{C}))^{\mathbf{Z}_m} \subset (M\otimes M_2(\mathbf{C})\otimes M_2(\mathbf{C}))^{\mathbf{Z}_m} \subset \cdots$$
$$(M\otimes \mathbf{C})^{\mathbf{D}_m} \subset (M\otimes M_2(\mathbf{C}))^{\mathbf{D}_m} \subset (M\otimes M_2(\mathbf{C})\otimes M_2(\mathbf{C}))^{\mathbf{D}_m} \subset \cdots$$

where the group actions are defined by

$$\tilde{\alpha} = \alpha \otimes \mathrm{Ad}\left(\begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} \otimes \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix} \otimes \cdots \right)$$
$$\tilde{\beta} = \beta \otimes \mathrm{Ad}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \cdots \right)$$

Let

$$P_k = (M \otimes \overbrace{M_2(\mathbf{C}) \otimes \cdots \otimes M_2(\mathbf{C})}^{k \text{ times}})^{\mathbf{Z}_m} \cap ((M \otimes \mathbf{C})^{\mathbf{Z}_m})'$$
$$Q_k = (M \otimes \overbrace{M_2(\mathbf{C}) \otimes \cdots \otimes M_2(\mathbf{C})}^{k \text{ times}})^{\mathbf{Z}_m} \cap ((M \otimes M_2(\mathbf{C}))^{\mathbf{Z}_m})'$$
$$R_k = (M \otimes \overbrace{M_2(\mathbf{C}) \otimes \cdots \otimes M_2(\mathbf{C})}^{k \text{ times}})^{\mathbf{D}_m} \cap ((M \otimes \mathbf{C})^{\mathbf{D}_m})'$$
$$S_k = (M \otimes \overbrace{M_2(\mathbf{C}) \otimes \cdots \otimes M_2(\mathbf{C})}^{k \text{ times}})^{\mathbf{D}_m} \cap ((M \otimes M_2(\mathbf{C}))^{\mathbf{D}_m})'$$
$$A = M \cap (M^{\mathbf{Z}_m})' = M \cap (M^{\mathbf{D}_m})'.$$

Then we have the following.

$$\begin{aligned} P_k &= (A \otimes \overbrace{M_2(\mathbf{C}) \otimes \cdots \otimes M_2(\mathbf{C})}^{k \text{ times}})^{\mathbf{Z}_m} \\ R_k &= (A \otimes \overbrace{M_2(\mathbf{C}) \otimes \cdots \otimes M_2(\mathbf{C})}^{k \text{ times}})^{\mathbf{D}_m} \\ &= P_k^{\tilde{\beta}} \end{aligned}$$

In the same way, we also have $S_k = Q_k^{\tilde{\beta}}$. Since $\tilde{\beta}$ preserves P_k and the Jones projections, and exchanges two minimal projections in P_1, the automorphism $\tilde{\beta}|_{\bigcup P_k}$ comes from the unique non-trivial graph automorphism of $A^{(1)}_{2p_o-1}$ fixing the $*$. So $\bigcup R_k$ is the string algebra of $D^{(1)}_{p_o+2}$ and the parameter c of the connection in §1 is given by the square root of the obstruction of α_0. Q.E.D.

We list three remarks on this construction.

Remark 5.6. We study the relation between $\beta_0 = \beta|_{M^\alpha}$ and α_0. By

$$\alpha \cdot \beta(\lambda) = \beta \cdot \alpha^{-1}(\lambda) = \omega^{-1}\beta(\lambda),$$

we have $\beta(\lambda) = w\lambda^{-1}$ for some unitary $w \in M^\alpha$. Then

$$\begin{aligned}\beta_0 \cdot \alpha_0(x) &= \beta(\lambda x \lambda^{-1}) = w\lambda^{-1}\beta_0(x)\lambda w^* \\ &= w\alpha_0^{-1} \cdot \beta_0(x) w^* \quad \text{for } x \in M^\alpha.\end{aligned}$$

By direct computations, we obtain the following.

$$\tag{5.6.1} \begin{aligned} &\rho \cdot \left(\beta \otimes \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \cdot \rho^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ =&\rho \cdot \left(\beta \otimes \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\begin{pmatrix} a & b\lambda^{-1} \\ \lambda c & \alpha_0(d) \end{pmatrix} \\ =&\rho\begin{pmatrix} \beta \cdot \alpha_0(d) & -\beta(\lambda c) \\ -\beta(b\lambda^{-1}) & \beta(a) \end{pmatrix} \\ =&\begin{pmatrix} \beta \cdot \alpha_0(d) & -\beta(\lambda c)\lambda \\ -\lambda^{-1}\beta(b\lambda^{-1}) & \alpha_0^{-1}\beta(a) \end{pmatrix} \\ =&\begin{pmatrix} w\alpha_0^{-1} \cdot \beta_0(d)w^* & -w\lambda^{-1}\beta_0(c)\lambda \\ -\lambda^{-1}\beta_0(b)\lambda w^* & \alpha_0^{-1}\beta_0(a) \end{pmatrix} \\ =&\begin{pmatrix} w\alpha_0^{-1} \cdot \beta_0(d)w^* & -w\alpha_0^{-1} \cdot \beta_0(c) \\ -\alpha_0^{-1} \cdot \beta_0(b)w^* & \alpha_0^{-1}\beta_0(a) \end{pmatrix}, \end{aligned}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M^\alpha \otimes M_2(\mathbf{C})$. Because $\rho \cdot \beta \cdot \rho^{-1}$ has a period 2, we have

$$(\alpha_0^{-1} \cdot \beta_0)^2 = \mathrm{Ad}(w^*), \quad \alpha_0^{-1} \cdot \beta_0(w) = w.$$

Remark 5.7. By (5.5.7), (5.5.8), and (5.6.1), we obtain the following.

(5.7.1)

$$N = \rho((M \otimes M_2(\mathbf{C}))^{\mathbf{D}_m}) = \left\{ \begin{pmatrix} a & b \\ -\alpha_0^{-1} \cdot \beta_0(b)w^* & \alpha_0^{-1} \cdot \beta_0(a) \end{pmatrix} \; ; \; a, b \in M^\alpha \right\},$$

(5.7.2)

$$L = \rho((M \otimes \mathbf{C})^{\mathbf{D}_m}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \alpha_0^{-1}(a) \end{pmatrix} \; ; \; a \in (M^\alpha)^{\beta_0} \right\}.$$

Because $[M^\alpha : (M^\alpha)^{\beta_0}]_0 = 2$, there exists a unitary $S \in M^\alpha$ with order 2 such that S normalizes $(M^\alpha)^{\beta_0}$ and

$$M^\alpha = (M^\alpha)^{\beta_0} + (M^\alpha)^{\beta_0} S,$$

$$\beta_0(S) = -S.$$

We define unitaries v, t as follows.

$$v = \begin{pmatrix} S & 0 \\ 0 & \alpha_0^{-1} \cdot \beta_0(S) \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & -\alpha_0^{-1}(S) \end{pmatrix} \in N \setminus L,$$

$$t = \begin{pmatrix} S & 0 \\ 0 & \alpha_0^{-1}(S) \end{pmatrix} \in \rho((M \otimes C)^{\mathbf{Z}_m}) \setminus L.$$

Because t normalizes N and L, we define a $\mathbf{Z}_2$-action τ on N by $\tau = \mathrm{Ad}\, t|_N$. It is easy to show $\rho((M \otimes M_2(\mathbf{C}))^{\mathbf{Z}_m}) = N + Nt$, $\rho((M \otimes \mathbf{C})^{\mathbf{Z}_m}) = L + Lt$. So we have

$$\begin{array}{ccccccc} \rho((M \otimes \mathbf{C})^{\mathbf{Z}_m}) & \subset & \rho((M \otimes M_2(\mathbf{C}))^{\mathbf{Z}_m}) & & L \rtimes_\tau \mathbf{Z}_2 & \subset & N \rtimes_\tau \mathbf{Z}_2 \\ \cup & & \cup & \cong & \cup & & \cup \\ L & \subset & N & & L & \subset & N \end{array}.$$

Because $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = vt$ also normalizes N, we define $\theta \in \mathrm{Aut}(N)$ by $\theta = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Big|_N$. Then it is easy to show that

$$\tau = \mathrm{Ad}(v) \cdot \theta,$$

$$N^\theta = L + Lv.$$

Remark 5.8. The conjugacy class of a subfactor $(M \otimes \mathbf{C})^{\mathbf{D}_m} \subset (M \otimes M_2(\mathbf{C}))^{\mathbf{D}_m}$ does not depend on the choice of the generator β. That is, two equivalent minimal periodic actions in the sense of Definition 5.1 produce conjugate subfactors. This can be proved as follows. If we take $\alpha^k \cdot \beta$ instead of β, then the action on $M \otimes M_2(\mathbf{C})$ turns into

$$\begin{aligned} &\left\langle \alpha \otimes \mathrm{Ad}\begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix}, (\alpha^k \beta) \otimes \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \\ =&\left\langle \alpha \otimes \mathrm{Ad}\begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix}, \beta \otimes \mathrm{Ad}\begin{pmatrix} 0 & \omega^{k/2} \\ -\omega^{-k/2} & 0 \end{pmatrix} \right\rangle. \end{aligned}$$

By

$$\begin{pmatrix} 1 & 0 \\ 0 & \omega^{k/2} \end{pmatrix}\begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \omega^{-k/2} \end{pmatrix} = \begin{pmatrix} \omega^{1/2} & 0 \\ 0 & \omega^{-1/2} \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 \\ 0 & \omega^{k/2} \end{pmatrix}\begin{pmatrix} 0 & \omega^{k/2} \\ -\omega^{-k/2} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \omega^{-k/2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we get the results.

Note that the conjugacy class of the subfactor $(M \otimes \mathbf{C})^{\mathbf{D}_m} \subset (M \otimes M_2(\mathbf{C}))^{\mathbf{D}_m}$ depends on the choice of α because the outer invariants depends on the choice of the generators of $\mathbf{Z}_m$.

Lemma 5.9. *Let P be a factor and $\alpha_0, \beta_0 \in Aut(P)$ satisfy*

$$\alpha_0^m = \beta_0^2 = id, \quad (m \geq 3),$$

$$\beta_0 \cdot \alpha_0 \cdot \beta_0 = Ad(w) \cdot \alpha_0^{-1}, \quad w \in P; \textit{unitary},$$

$$\alpha_0(w) = \beta_0(w).$$

If α_0 has a minimal period m and β_0 is outer, then there exists $\beta \in Aut(P \rtimes_{\alpha_0} \mathbf{Z}_m)$ such that $\langle \hat{\alpha}_0, \beta \rangle$ is a minimal periodic action of $\mathbf{D}_m$ and $\beta|_P = \beta_0$. If $\beta_1 \in Aut(P \rtimes_{\alpha_0} \mathbf{Z}_m)$ is another extension of β_0 so that $\langle \hat{\alpha}_0, \beta_1 \rangle$ is an action of $\mathbf{D}_m$, then $\beta_1 = \hat{\alpha}_0^k \cdot \beta$ for some k.

Proof. By

$$id = (\beta_0 \cdot \alpha_0 \cdot \beta_0)^m = \mathrm{Ad}(w\alpha_0^{-1}(w)\alpha_0^{-2}(w) \cdots \alpha_0^{-(m-1)}(w)),$$

we may assume that $w\alpha_0^{-1}(w)\alpha_0^{-2}(w) \cdots \alpha_0^{-(m-1)}(w) = 1$. Let λ be the implementing unitary for α_0 in $P \rtimes_{\alpha_0} \mathbf{Z}_m$. Suppose that β is an extension of β_0 to $P \rtimes_{\alpha_0} \mathbf{Z}_m$ so that $\langle \hat{\alpha}_0, \beta \rangle$ is an action of $\mathbf{D}_m$ on $P \rtimes_{\alpha_0} \mathbf{Z}_m$. Then

$$\begin{aligned} \mathrm{Ad}(w) \cdot \alpha_0^{-1}(x) &= \beta_0 \cdot \alpha_0 \cdot \beta_0(x) \\ &= \beta(\lambda \beta_0(x) \lambda^{-1}) \\ &= \beta(\lambda) x \beta(\lambda^{-1}), \quad \text{for } x \in P. \end{aligned}$$

So $\beta(\lambda) = cw\lambda^{-1}$ for some $c \in P' \cap P \rtimes_{\alpha_0} \mathbf{Z}_m$. By $\hat{\alpha}_0 \cdot \beta(\lambda) = \beta \cdot \hat{\alpha}_0^{-1}(\lambda)$, we get $\hat{\alpha}_0(c) = c$, which implies $c \in P$. Thus c is a complex scalar. By

$$1 = \beta(\lambda^m) = \beta(\lambda)^m = c^m w\alpha_0^{-1}(w)\alpha_0^{-2}(w)\cdots\alpha_0^{-(m-1)}(w) = c^m,$$

we get $c^m = 1$. On the other hand, if $c^m = 1$, we can define $\beta \in \mathrm{Aut}(P \rtimes_{\alpha_0} \mathbf{Z}_m)$ by

$$\beta(x) = \begin{cases} \beta_0(x), & \text{for } x \in P \\ cw\lambda^{-1}, & \text{for } x = \lambda. \end{cases}$$

By $\beta_0(w) = \alpha_0(w)$, we have

$$\begin{aligned} \beta^2(\lambda) &= \beta(cw\lambda^{-1}) = c\beta_0(w)(cw\lambda^{-1})^* = \beta_0(w)\lambda w^* \\ &= \beta_0(w)\alpha_0(w^*)\lambda = \lambda. \end{aligned}$$

So β has a period 2. It is easy to see that $\langle \hat{\alpha}_0, \beta \rangle$ is an action of $\mathbf{D}_m$ on $P \rtimes_{\alpha_0} \mathbf{Z}_m$. We must show that β is outer. Suppose that $x = \sum a_k \lambda^k$, $a_k \in P$, satisfies $\beta = \mathrm{Ad}\ x$. Then for all $y \in P$, we get $a_k \alpha_0^k(y) = \beta_0(y) a_k$. Because $\alpha_0^k \cdot \beta_0$ is outer, we obtain $a_k = 0$, which is a contradiction. The second statement follows from the above arguments. Q.E.D.

The following is a generalization of a statement made in the proof of [P2, Corollaire 1 (v)].

Lemma 5.10. *Let $L \subset N$ be a pair of factors with the principal graph $D_n^{(1)}$, $n \geq 3$. Then the following hold.*

(1) *Let G be the group of automorphisms of N fixing L pointwise. Then $G \cong \mathbf{Z}_2$. We denote the generator of G by θ.*

(2) *There exists $\theta_0 \in Aut(L)$ such that $N^\theta = L \rtimes_{\theta_0} \mathbf{Z}_2$. The automorphism θ_0 is determined up to cocycle conjugacy.*

Proof. By $N' \cap L = \mathbf{C}$, there is a unique conditional expectation $E \in E(N, L)$. Let

$$L \subset N \subset N_1 \subset N_2 \cdots$$

be the tower associated with $L \subset N$ and E_i be the unique element in $E(N_i, N_{i-1})$, $(N_0 = N)$. Because the principal graph of $L \subset N$ is $D_n^{(1)}$, $L' \cap N_1 = \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C}$. Denote the set of the minimal projections in $N_1 \cap L'$ by $\{e_L, p_1, p_2\}$. Note that $E \cdot E_1 \cdots E_k$ converges to the unique trace on $\bigvee_k (N_k \cap L')$. Then the values of E_1 on $N_1 \cap L'$ are determined by the Perron-Frobenius eigenvector of $D_n^{(1)}$ and we obtain

$$\{E_1(p_1), E_1(p_2)\} = \{\frac{1}{4}, \frac{1}{2}\},$$
$$\#\{p \in N_1 \cap L'; E_1(p) = E_1(e_L) = \frac{1}{4}, p : \text{projection}\} = 2.$$

Let $\mathcal{U}(L)$ be the unitary group of L and $\mathcal{N}(L) = \{u \in \mathcal{U}(N); uLu^* = L\}$. In the same way as in [Ks2, Lemma 5], we can show that the order of the Weyl group of $L \subset N$ is 2, that is, $\#(\mathcal{N}(L)/\mathcal{U}(L)) = 2$. So there exists a unique $\theta_0 \in \mathrm{Aut}(L)$ up to cocycle conjugacy such that $L \rtimes_{\theta_0} \mathbf{Z}_2 \subset N$. Thanks to $[N : L \rtimes_{\theta_0} \mathbf{Z}_2]_0 = 2$, there exists a unique automorphism θ of N with a period 2 such that $L \rtimes_{\theta_0} \mathbf{Z}_2 = N^\theta$. By $|G| \leq [N : L]_0 = 4$, we have $|G| = 2$ or $|G| = 4$. If $|G| = 4$, then L is the fixed point algebra under the action of G, but this is impossible and we get the results. Q.E.D.

Now we reach the main result of this section.

Theorem 5.11. *Let $L \subset N$ be a subfactor with the principal graph $D_n^{(1)}$. Then there exists a factor M and $\alpha, \beta \in Aut(M)$ satisfying the conditions of Theorem 5.5 such that*

$$L \subset N \;\cong\; (M \otimes \mathbf{C})^{\mathbf{D}_m} \subset (M \otimes M_2(\mathbf{C}))^{\mathbf{D}_m},$$

where the $\mathbf{D}_m$-action is constructed as in Theorem 5.5. Moreover, the conjugacy class of a subfactor $N \subset L$ is in a bijective correspondence to a pair of an isomorphism class of M and an equivalence class of a minimal periodic action $\langle \alpha, \beta \rangle$ of $\mathbf{D}_m$ on M in the sense of Definition 5.1.

Proof. By Lemma 5.10, there exist $\theta \in \mathrm{Aut}(N)$ and $\theta_0 \in \mathrm{Aut}(L)$ with periods 2 such that $N^\theta = L \rtimes_{\theta_0} \mathbf{Z}_2$. Let $v \in L \rtimes_{\theta_0} \mathbf{Z}_2$ be the implementing unitary for θ_0. We define $\tau \in \mathrm{Aut}(N)$ by $\tau = \mathrm{Ad}(v) \cdot \theta$. Then we have

$$\tau^2 = \mathrm{Ad}(v\theta(v)) \cdot \theta^2 = \mathrm{Ad}(v^2) = id,$$

$$\tau|_L = \mathrm{Ad}(v)|_L = \theta_0.$$

We consider the following inclusions of factors.

$$\begin{array}{ccc} L \rtimes_\tau \mathbf{Z}_2 & \subset & N \rtimes_\tau \mathbf{Z}_2 \\ \cup & & \cup \\ L & \subset & N \end{array}.$$

Note that these inclusions and the dual action $\hat{\tau}$ do not depend on the choice of θ_0.

Let $t \in L \rtimes_\tau \mathbf{Z}_2$ be the implementing unitary for τ. Then $(vt)^2 = v\tau(v) = v\mathrm{Ad}(v) \cdot \theta(v) = v^2 = 1$. We write $vt = f_1 - f_2$, where f_1 and f_2 are mutually orthogonal non-zero projections. Direct computation shows

$$N \rtimes_\tau \mathbf{Z}_2 \cap (L \rtimes_\tau \mathbf{Z}_2)' = \langle vt \rangle = \mathbf{C}f_1 + \mathbf{C}f_2.$$

If N is of type II_1, then $f_1 \sim f_2$ because of $[N \rtimes_\tau \mathbf{Z}_2 : L \rtimes_\tau \mathbf{Z}_2] = 4$, and if N is of type II_∞, then $f_1 \sim f_2$ also holds because f_1 and f_2 are both infinite projections. In any case, we have $f_1 \sim f_2$. So we identify $L \rtimes_\tau \mathbf{Z}_2 \subset N \rtimes_\tau \mathbf{Z}_2$ with

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & \alpha_0^{-1}(x) \end{pmatrix} ; x \in P \right\} \subset P \otimes M_2(\mathbf{C}),$$

where $P = f_1(N \rtimes_\tau \mathbf{Z}_2)f_1$.

First, we assume that the outer period of α_0 is $p_o \neq 0$. Thanks to [Co2, Proposition 2.3], we may assume that α_0 is minimal periodic by change of the partial isometries between f_1 and f_2, if necessary. We assume this and let m be the period of α_0. Note that α_0 is uniquely determined up to cocycle conjugacy.

We investigate the relation between $\hat{\tau}$ and α_0. By $\hat{\tau}(f_1) = f_2$ and $\hat{\tau}(f_2) = f_1$, the dual action $\hat{\tau}$ is written as follows.

$$\hat{\tau}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w\sigma(d)w^* & -w\sigma(c) \\ -\sigma(b)w^* & \sigma(a) \end{pmatrix},$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P \otimes M_2(\mathbf{C}), w \in \mathcal{U}(P), \sigma \in \mathrm{Aut}(P).$$

By $\hat{\tau}^2 = id$, we have the following relations.

$$\sigma^2 = \mathrm{Ad}\, w^*, \sigma(w) = w. \tag{5.11.1}$$

Because $\hat{\tau}$ preserves $N \rtimes_\tau \mathbf{Z}_2$, we obtain $\alpha_0^{-1} \cdot \mathrm{Ad}(w) \cdot \sigma \cdot \alpha_0^{-1} = \sigma$ from

$$\hat{\tau}\begin{pmatrix} a & 0 \\ 0 & \alpha_0^{-1}(a) \end{pmatrix} = \begin{pmatrix} w\sigma \cdot \alpha_0^{-1}(a)w^* & 0 \\ 0 & \sigma(a) \end{pmatrix}.$$

So

$$(\alpha_0 \cdot \sigma)^2 = id, \tag{5.11.2}$$

$$\sigma \cdot \alpha_0^{-1}\sigma^{-1} = \mathrm{Ad}(w^*) \cdot \alpha_0, \tag{5.11.3}$$

where we use (5.11.1). We define $\beta_0 \in \mathrm{Aut}(P)$ by $\beta_0 = \alpha_0 \cdot \sigma$. Then β_0 satisfies the following.

$$\beta_0^2 = id, \tag{5.11.4}$$

$$\beta_0 \cdot \alpha_0 \cdot \beta_0 = \mathrm{Ad}(w) \cdot \alpha_0^{-1}, \tag{5.11.5}$$

$$\beta_0(w) = \alpha_0(w), \tag{5.11.6}$$

$$\hat{\tau}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w\alpha_0^{-1} \cdot \beta_0(d)w^* & -w\alpha_0^{-1} \cdot \beta_0(c) \\ -\alpha_0^{-1} \cdot \beta_0(b)w^* & \alpha_0^{-1} \cdot \beta_0(a) \end{pmatrix} \tag{5.11.7}$$

From (5.11.5), we may assume that $w\alpha_0^{-1}(w)\cdots\alpha_0^{-(m-1)}(w) = 1$ by changing the partial isometries between f_1 and f_2, if necessary. Now we get the following.

(5.11.8)

$$N \rtimes_\tau \mathbf{Z}_2 = P \otimes M_2(\mathbf{C}),$$

(5.11.9)

$$L \rtimes_\tau \mathbf{Z}_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & \alpha_0^{-1}(a) \end{pmatrix} ; a \in P \right\},$$

(5.11.10)

$$N = (P \otimes M_2(\mathbf{C}))^{\hat{\tau}} = \left\{ \begin{pmatrix} a & b \\ -\alpha_0^{-1} \cdot \beta_0(b)w^* & \alpha_0^{-1} \cdot \beta_0(a) \end{pmatrix} ; a, b \in P \right\},$$

(5.11.11)

$$L = N \cap (L \rtimes_\tau \mathbf{Z}_2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & \alpha_0^{-1}(a) \end{pmatrix} ; a \in P, \beta_0(a) = a \right\}.$$

We define M by $M = P \rtimes_{\alpha_0} \mathbf{Z}_m$. By Lemma 5.9, we have $\alpha = \hat{\alpha}_0 \in \mathrm{Aut}(M)$ and $\beta \in \mathrm{Aut}(M)$, an extension of β_0, such that $\langle \alpha, \beta \rangle$ satisfies the conditions of Theorem 5.5. By (5.7.1), (5.7.2), (5.11.10), and (5.11.11), we get the first part of the statement.

If the minimal period of α_0 is 0, then we define M by $M = P \rtimes_{\alpha_0} \mathbf{Z}$. In the same way, we have an outer action of $\mathbf{T} \rtimes \mathbf{Z}_2$ on M and the subfactor $L \subset N$ is conjugate to $(M \otimes \mathbf{C})^{\mathbf{T} \rtimes \mathbf{Z}_2} \subset (M \otimes M_2(\mathbf{C}))^{\mathbf{T} \rtimes \mathbf{Z}_2}$. But in this case, the principal graph of $L \subset N$ is D_∞ as in [GHJ, 4.7.d], which is a contradiction. (Use Lemma 5.4 instead of [GHJ, Lemma 4.7.1].)

Let u be α_0^{-1}-cocycle. We define $\tilde{\alpha}_0$ by $\tilde{\alpha}_0 = \alpha_0 \cdot \mathrm{Ad}(u^*)$. If we take $\tilde{\alpha}_0$ instead of α_0, then σ, w, β_0 turn into $\tilde{\sigma} = \mathrm{Ad}(u) \cdot \sigma$, $\tilde{w} = w\sigma(u^*)u^*$, $\tilde{\beta}_0 = \beta_0$. Let λ be the implementing unitary for α_0. Then the implementing unitary for $\tilde{\alpha}_0$ is $\tilde{\lambda} = \lambda u^*$. If β and $\tilde{\beta} \in \mathrm{Aut}(M)$ are extensions of β_0 defined by $\beta(\lambda) = w\lambda^{-1}$ and $\tilde{\beta}(\tilde{\lambda}) = \tilde{w}\tilde{\lambda}^{-1}$, it is easy to show that β and $\tilde{\beta}$ coincide. So the above construction of M and the $\mathbf{D}_m$-action does not depend on the choice of α_0. The claim for the bijective correspondence follows from Remark 5.7 and Lemma 5.9. Q.E.D.

With this theorem, we can give a different proof of our main result, Theorem 3.4, as follows.

Another proof of Theorem 3.4. By Lemma 5.2 (3), we can use classification of finite group actions on the AFD type II_1 factor up to cocycle conjugacy, [J1]. We study characteristic invariants of [J1, 1.2]. Now G be the dihedral group $\mathbf{D}_m$ generated by a, b with $a^m = b^2 = (ab)^2 = 1$. For an action σ of $\mathbf{D}_m$ on the AFD type II_1 factor σ, denote σ_a, σ_b by α, β respectively. Because $n-2$ is now the outer period of α, it is a divisor of m. Let $m = l(n-2)$. Let γ be the obstruction of α. Then $\gamma^{n-2} = 1$ and the order of γ is l by [Co2, Proposition 2.3]. Fix γ. The inverse image N of the inner automorphism group by σ is $\{a, a^{n-2}, a^{2(n-2)}, \dots a^{(l-1)(n-2)}\} \subset G$. We study a characteristic invariant, a pair $(\lambda, \mu) \in \Lambda(G, N)$ We may assume that μ is trivial, $\mu(g) = 1$ for all $g \in N$. Note that the map $\lambda(g, \cdot)$ gives an element of $\hat{N}$ for each $g \in G$ because μ is trivial by [J1, (1.2.2)]. Set $c = \lambda(b, a^{n-2}) \in \mathbf{C}$. Then $c^l = 1$ and all the values of λ is determined by this c and $\gamma = \lambda(a, a^{n-2})$. For example, $\lambda(a^k b, a^{n-2}) = \gamma^k c$ by [J1, (1.2.3)]. For different values of c, the characteristic invariants (λ, μ) may or may not be equivalent in $\Lambda(G, N)$, but the corresponding actions are equivalent in the sense of Definition 5.1, because an automorphism of $\mathbf{D}_m$ defined by $a \mapsto a$, $b \mapsto a^k b$, changes c into $\gamma^k c$ and all the l-th roots of unity are realized in this way. (Note that γ has an order l.) Thus for fixed γ, there is a unique equivalence class of actions. Because the number of possible γ is $n-2$, the number of conjugacy classes of our subfactors is also $n-2$. Q.E.D.

For the principal graph D_∞, we can prove the following in the same way.

Theorem 5.12. *Let $L \subset N$ be a pair of factors with the principal graph D_∞. Then there exist a factor M and an outer action of $\mathbf{T} \rtimes \mathbf{Z}_2$ on M such that $L \subset N$ is conjugate to $(M \otimes \mathbf{C})^{\mathbf{T} \rtimes \mathbf{Z}_2} \subset (M \otimes M_2(\mathbf{C}))^{\mathbf{T} \rtimes \mathbf{Z}_2}$.*

For the AFD type III_1 factor, Kosaki and Longo conjectured that its subfactor splits as a tensor product of an AFD type II_1 subfactor and a common AFD type III_1 factor under a "good" condition, such as a finite depth and a trivial relative commutant. If a subfactor arises as a fixed point algebra or a crossed product algebra of an outer action of a finite group, this is true, and these are considered as rather trivial examples of the conjecture. (Finite group actions on the AFD type III_1 factor have been classified in [KST, Theorem 20] based on Jones-Ocneanu classifications [J1, O1] and Connes' announcement on centrally trivial automorphisms and approximately inner automorphisms, which was announced in [Co1, section 3.8] and proved in [KST].) The first author showed in [I1, §5] that an AFD type III_1 subfactor with the principal graph A_5 gives the first non-trivial supporting example of the above conjecture. Now we can give the second non-trivial example as follows.

Corollary 5.13. *Let $L \subset N$ be a pair of AFD type III_1 factors with the principal graph $D_n^{(1)}$. Then there exists a pair of AFD type II_1 factors $P \subset R$ such that $L \subset N$ is conjugate to $P \otimes R_\infty \subset R \otimes R_\infty$, where R_∞ is the AFD type III_1 factor.*

Proof. By [KST, Theorem 20], any finite group action σ on the AFD type III_1 factor is cocycle conjugate to $\sigma' \otimes id$, where σ' is an action on the AFD type II_1 factor and id is the trivial action on the AFD type III_1 factor. Now Lemma 5.2 (3) and Theorem 5.11 give the conclusion. Q.E.D.

[ABF] Andrews, G. E., Baxter, R. J., & Forrester, P. J.: Eight vertex SOS model and generalized Rogers-Ramanujan type identities, J. Stat. Phys. **35**, 193–266 (1984)

[A] Aubert, P.-L.: Théorie de Galois pour une W^*-algèbre, Comment. Math. Helv. **39 (51)**, 411–433 (1976)

[B] Baxter, R. J.: "Exactly solved models in statistical mechanics", Academic Press, New York, 1982.

[Ch] Choda, M.: Duality for finite bipartite graphs, to appear in Pac. J. Math.

[Co1] Connes, A.: Outer conjugacy of automorphisms of factors, Symposia Math. XX, 149–160 (1976)

[Co2] Connes, A.: Periodic automorphisms of the hyperfinite factor of type II_1, Acta Sci. Math. **39**, 39–66 (1977)

[Cu] Cuntz, J.: Simple C^*-algebras generated by isometries, Comm. Math. Phys. **57**, 173–185 (1977)

[DJMO] Date, E., Jimbo, M., Miwa, T., & Okado, M.: Solvable lattice models, in "Theta functions — Bowdoin 1987, Part 1," Proc. Sympos. Pure Math. Vol. 49, Amer. Math. Soc., Providence, R.I., pp. 295–332.

[EK] Evans, D. E., & Kawahigashi, Y.: Orbifold subfactors from Hecke algebras, preprint.

[FG] Fendley, P., & Ginsparg, P.: Non-critical orbifolds, Nucl. Phys. B **324**, 549–580 (1989)

[GHJ] Goodman, F., de la Harpe, P., & Jones, V. F. R.: "Coxeter graphs and towers of algebras", MSRI publications 14, Springer, (1989)

[I1] Izumi. M.: Application of fusion rules to classification of subfactors, Publ. RIMS Kyoto Univ. **27**, 953–994 (1991)

[I2] Izumi, M.: Subalgebras of infinite C^*-algebras with finite Watatani indices I. Cuntz algebras, preprint, (1992)

[J1] Jones, V. F. R.: Actions of finite groups on the hyperfinite type II_1 factor, Mem. Amer. Math. Soc. **237**, (1980)

[J2] Jones, V. F. R.: Index for subfactors, Invent. Math. **72**, 1–15 (1983)

[Ka] Kawahigashi, Y.: On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors, preprint, (1990)

[KST] Kawahigashi, Y., Sutherland, C. E., & Takesaki, M.: The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions, (to appear in Acta Math.).

[Ks1] Kosaki, H.: Extension of Jones' theory on index to arbitrary factors, J. Funct. Anal. **66**, 123–140 (1986)

[Ks2] Kosaki, H.: Characterization of crossed product (properly infinite case), Pac. J. Math. **137**, 159–167 (1989)

[Kt] Kostov, I.: Free field presentation of the A_n coset models on the torus, Nucl. Phys. B **300**, 559–587 (1988)

[L1] Longo, R.: Index of subfactors and statistics of quantum fields I, Comm. Math. Phys. **126**, 217–247 (1989)

[L2] Longo, R.: Index of subfactors and statistics of quantum fields II, Comm. Math. Phys. **130**, 285–309 (1990)

[O1] Ocneanu, A.: "Actions of discrete amenable groups on factors," Lecture Notes in Math. No. 1138, Springer, Berlin, (1985)

[O2] Ocneanu, A.: Quantized group string algebras and Galois theory for algebras, in "Operator algebras and applications, Vol. 2 (Warwick, 1987)," London Math. Soc. Lect. Note Series Vol. 136, Cambridge University Press, pp. 119–172 (1988)

[O3] Ocneanu, A.: "Quantum symmetry, differential geometry of finite graphs and classification of subfactors", University of Tokyo Seminary Notes 45, (Notes recorded by Y. Kawahigashi), (1991)

[O4] Ocneanu, A.: Quantum symmetry, classification of subfactors and quantum field theory, talk at ICM-90, (1990)

[Pa] Paschke, W.: Inner product modules arising from compact automorphism groups on von Neumann algebras, Trans. Amer. Math. Soc. **224**, 87–102 (1976)

[P1] Popa, S.: Classification of subfactors: reduction to commuting squares, Invent. Math. **101**, 19–43 (1990)

[P2] Popa, S.: Sur la classification des sousfacteurs d'indice fini du facteur hyperfini, C. R. Acad. Sc. Paris. **311**, 95–100 (1990)

[R] Roche, Ph.: Ocneanu cell calculus and integrable lattice models, Comm. Math. Phys. **127**, 395–424 (1990)

[T] Takesaki, M.: Duality for crossed products and the structure of von Neumann algebras of type III, Acta Math. **131**, 249–310 (1973)

[Ws] A. Wassermann, Coactions and Yang-Baxter equations for ergodic actions and subfactors, in "Operator algebras and applications, Vol. 2 (Warwick, 1987)", London Math. Soc. Lect. Note Series Vol. 136, Cambridge University Press, pp. 203–236 (1988)

[Wt] Watatani, Y.: "Index for C^*-subalgebras", Mem. Amer. Math. Soc. **83** No. 424, (1990)