Centrally trivial automorphisms and an analogue of Connes' $\chi(M)$ for subfactors

Yasuyuki Kawahigashi*
Department of Mathematics
University of California, Berkeley, CA 94720
(e-mail:kawahiga@math.berkeley.edu)

April, 1992

Abstract. We study a class of centrally trivial automorphisms for subfactors, and get an upper bound for the order of the group they make (modulo normalizers) in terms of the "dual" principal graph for AFD type II_1 subfactors with trivial relative commutant, finite index and finite depth. We prove that this upper bound is attained for many known subfators. We also introduce $\chi(M,N)$ for subfactors $N \subset M$ as the relative version of Connes' invariant $\chi(M)$, and compute this group for many AFD type II_1 subfactors with finite index and finite depth including all the cases with index less than 4 and many Hecke algebra subfactors of Wenzl. In these finite depth cases, the group $\chi(M,N)$ is always finite and abelian, and we realize all the finite abelian groups as $\chi(M,N)$. Analogy between this topic and modular structure of type III factors is also discussed. As an application, we give some classification results for $\mathrm{Aut}(M,N)$. For example, for the subfactors of type A_{2n+1}, there are two and only two outer actions of $\mathbf{Z}_2$. One is of the "standard" form and the other is given by the "orbifold" action arising from the paragroup symmetry. As preliminaries, we also prove several statements on central sequence subfactors announced by A. Ocneanu.

---

*Miller Research Fellow
Permanent address: Department of Mathematics, Faculty of Science
University of Tokyo, Hongo, Tokyo, 113, JAPAN (e-mail: yasuyuki@tansei.cc.u-tokyo.ac.jp)

§1 Introduction

The aim of this paper is to exploit the notion of centrally trivial automorphisms for subfactors. We give an effective upper bound for the size of these automorphisms and their complete chracterization with an additional assumption for finite depth subfactors, introduce and compute $\chi(M,N)$, the relative version of Connes' invariant $\chi(M)$, discuss their analogy to modular automorphism groups of (injective) type III factors, and give some applications on $\mathrm{Aut}(M,N)$.

Since the breakthrough of V. Jones on index for subfactors [J3], importance of study of subfactors in the both operator algebra theory itself and other fields in mathematics has become clearer and clearer. Our aim here is extending the Connes type automorphism approach [C1, C2, C3, C4, C5] to subfactor setting, and this is a natural continuation of our orbifold construction in [EK, IK, Ka2, Ka3].

A. Ocneanu [O2] introduced a notion of *paragroup* as a combinatorial characterization of higher relative commutants of approximately finite dimensional (AFD) subfactors of type II_1 with trivial relative commutant, finite index, and finite depth. (See also [Ka2, Ka3, O3, O4].) Because S. Popa [P2, P3, P4] has proved that the higher relative commutants generate the original subfactor in a very general condition called strong amenability, combinatorial approach of paragroup gives a satisfactory classification in many cases. In particular, Ocneanu's announcement of classification of subfactors with index less than 4 without a full proof has been verified by [EG, I1, I3, Ka2, SV], and a classification for the case of index equal 4 was also obtained by [IK, P3]. (In the case of E_6, we use an earlier construction of Bion-Nadal [BN].)

A paragroup has a certain algebraic structure on two graphs. Conceptually, it can be regarded as a quantization of a Galois group, and technically as a discrete analogue of a compact manifold. In particular, an analogue of a flat connection plays a key role. Furthermore, Ocneanu [O5] recently announces that certain complex number valued topological invariants of 3-dimensional manifolds are in bijective correspondence to paragroups. Thus it is has a very deep and rich mathematical structure, but for our aim here, the most important aspect of paragroup theory is its relation to statistical mechanics. As mentioned in [O2] and explicitly clarified in [EK, Ka3], a paragroup is quite similar to an exactly solvable lattice models (IRF models) without a spectral parameter. (See [ABF, Ba, DJMO, Ji] for IRF models.) Commuting square condition in operator algebra theory corresponds to the crossing symmetry (or the second inversion relations more generally) in IRF model theory, and flatness in paragroup theory is closely related to the Yang-Baxter equation in IRF model theory.

Our idea of orbifold construction in [EK, IK, Ka2, Ka3] was that if we have a paragroup symmetry, we can make a quotient paragroup by the symmetry, but flatness axiom may not be kept in this procedure in general. It is an analogue of orbifold models in [DZ, F, FG, Kt, R] in IRF model theory, but flatness requirement makes the problem more subtle than the case of the Yang-Baxter equation.

In the construction [Ka2], we constructed subfactors with principal graph D_{2n} as simultaneous fixed point algebras by $\mathbf{Z}_2$ actions on subfactors with the principal graph A_{4n-3}. (This gave the first complete proof of realization of D_{2n} in literature, which was announced by Ocneanu. The author later learned Ocneanu's original method, too. See [Ka3, Appendix].) But on subfactors with the principal graph

A_{4n-3}, there is another (rather trivial) outer action of $\mathbf{Z}_2$. That is, this subfactor $N \subset M$ is isomorphic to $N \otimes R \subset M \otimes R$, where R is the AFD type II_1 factor. (This is so-called relative McDuff splitting as in [Bi, P1].) Then we can take an action $id \otimes \sigma$ on this splitting, where σ is a unique outer action of $\mathbf{Z}_2$ on R [C5]. Because this action gives a subfactor with the principal graph unchanged as a simultaneous fixed point algebra, the above "orbifold" action is different from this "standard" action. P. Loi [Li1, §5] introduced an invariant for group actions fixing a subfactor globally, but this invariant is always trivial for subfactors with principal graph A_n, so this invariant cannot detect the above difference. Furthermore, with a little more work, we can prove that our orbifold $\mathbf{Z}_N$ actions in [EK] on Hecke algebra subfactors of Wenzl [W] with N prime are different from actions of the "standard" form in the above sense and Loi's invariant cannot detect this difference. Thus we are naturally led to the problem why this kind of phenomena happen in subfactor setting while outer actions of $\mathbf{Z}_n$ on the AFD type II_1 factor are unique up to conjugacy by [C5].

As proved by Loi [Li1, Theorem 5.4], triviality of his invariant in the case of finite depth AFD type II_1 subfactors implies that the automorphism is approximately inner in the sense that it is of the form $\lim_n \mathrm{Ad}(u_n)$, where u_n's are unitaries in the subfactor. In the single factor case, we have another important class of automorphisms in addition to that of approximately inner automorphisms. They are centrally trivial automorphisms, which played an important role in group action theory [C1, C2, C4, C5, J1, KST, KT, O1, ST]. If we apply Connes' machinery of automorphism classification [C2, C5] to subfactor setting as pointed out in [Li1, §4], we can prove that the above "orbifold" actions give centrally trivial automorphisms in subfactor setting. That is, they act trivially on central sequences of the ambient factor in the subfactor. Indeed, if their asymptotic periods in the subfator sense are not 1, the Connes type non-commutative Rohlin machinery with approximate innerness would produce their conjugacy to actions of the standard form.

Thus we know that orbifold actions naturally give centrally trivial automorphisms in some situations, and we are led to the problem of determining the class of centrally trivial automorphisms. (This strategy parallels that of Connes for classification of automorphisms in a single factor case.) We give an effective upper bound of the size of the class of centrally trivial automorphisms in this paper for AFD type II_1 subfactors with finite depth and prove that this upper bound is attained for many known subfactors. Furthermore, with an additional assumption related to a recent work of Choda-Kosaki [CK, K], we can give a complete characterization of centrally trivial automorphisms for AFD type II_1 subfactors with finite depth. (For this purpose, we will need several statements on central sequences in subfactors announced by Ocneanu without a full proof. We will prove all the necessary statements in §2.) This situation is quite similar to that of modular automorphism groups of type III factors. We will discuss more on this similarity in §3.

Furthermore, because we have the both notions of approximately inner automorphisms and centrally trivial automorphisms, we can consider a relative version of Connes' invariant $\chi(M)$ in [C1]. We introduce basic definitions here, and give concrete results in §4. These results have some immediate corollaries on classification of $\mathrm{Aut}(M, N)$. They will be dealt with in §5.

S. Popa has been working on classification of discrete amenable group actions on strongly amenable subfactors. After completion of this work, the author learned that his theorem in [P4] states that "properly outer" actions of discrete amenable groups on strongly amenable subfactors are classified by Loi's invariant. Because it turns out that his "proper outerness" coincides with central freeness in our subfactor sense, this paper deals with the class of automorphisms Popa's current classification theorem does not cover.

Now we list basic definitions and notations. Take a II_1 subfactor $N \subset M$. We fix some basic notations as in [Li1]. We set

$$\mathrm{Aut}(M,N) = \{\alpha \in \mathrm{Aut}(M); \alpha(N) = N\},$$

$$\mathrm{Int}(M,N) = \{\mathrm{Ad}(u) \in \mathrm{Aut}(M,N); u \in \mathcal{U}(N)\}.$$

We denote the closure of $\mathrm{Int}(M,N)$ in $\mathrm{Aut}(M,N)$ by $\overline{\mathrm{Int}}(M,N)$. We say that $\alpha \in \mathrm{Aut}(M,N)$ is centrally trivial if α acts trivially on central sequences in N with respect to M. We denote by $\mathrm{Ct}(M,N)$ the subgroup of centrally trivial automorphisms. For free ultrafilter ω over $\mathbf{N}$, we have

$$\mathrm{Ct}(M,N) = \{\alpha \in \mathrm{Aut}(M,N); \alpha = \mathrm{id} \text{ on } \ N^\omega \cap M'\}$$

as in single factor cases. As a relative version of Connes' $\chi(M)$ in [C1], we set

$$\chi(M,N) = \frac{\mathrm{Ct}(M,N) \cap \overline{\mathrm{Int}}(M,N)}{\mathrm{Int}(M,N)}.$$

It is easy to see that as in the single factor case [C2], automorphisms in $\overline{\mathrm{Int}}(M,N)$ and in $\mathrm{Ct}(M,N)$ commute module $\mathrm{Int}(M,N)$, so the group $\chi(M,N)$ is always abelian.

Now suppose that $N^\omega \cap M'$ is a factor and let G be a finite group of $\mathrm{Aut}(M,N)$ with $G \cap \overline{\mathrm{Int}}(M,N) = \{1\}$. We set $K = G \cap \mathrm{Ct}(M,N)$,

$$K^\perp = \{\gamma : G \to \mathbf{T}; \gamma \text{ a character vanishing on } \ K\},$$

and L be the image of

$$G \cdot \mathrm{Ct}(M,N) \cap \overline{\{\mathrm{Ad}(u); u \in \mathcal{U}(N^G)\}}$$

in $\mathrm{Aut}(M,N)/\mathrm{Int}(M,N)$. Then we have the following theorem as in [C1, C4, 3.10].

Theorem 1.1. *In the above context, there are maps $\delta : K^\perp \to \chi(M \times G, N \times G)$ and $\pi : \chi(M \times G, N \times G) \to L$ such that the following sequence is exact.*

$$\{1\} \to K^\perp \to \chi(M \times G, N \times G) \to L \to \{1\}.$$

Definitions of δ, π and a proof of this work in the exactly same way as in [J2].

The author thanks Professor M. Choda for useful communications, Professor U. Haagerup for explaining his recent work [H] before writing, Professor M. Izumi for pointing out our mistake in Theorem 3.3 in the first version of this paper, Professor V. F. R. Jones for showing his unpublished notes [J2] and suggesting the approach of the relative version of $\chi(M)$, Professor H. Kosaki for helpful communications

including showing his work [CK, Ko] at an early stage, Professor A. Ocneanu for permitting us to include his proof of Lemma 2.5 here, and Professor S. Popa for inspiring discussions on [EK, Ka2, P4].

§2 Preliminaries on central sequences

A. Ocneanu made many striking announcements on central sequences in subfactors in [O2, O4], and we will need several of them for our arguments. But unfortunately, details of his proof have not been written, so we include proofs for his announcements in this section. All the statements in this section were more or less claimed in his Tokyo lectures in 1990, but all the proofs here except for those of Lemmas 2.5 and 2.13 are by us. His original intention of studying central sequence subfactors was for proving the generating property for AFD type II_1 subfactors with trivial relative commutant, finite index and finite depth, but Popa's proof for fully general case has appeared in [P2, P3, P4], while Ocneanu's announced result has no proof in literatures yet. So we use the generating property for finite depth subfactors, which make some arguments simpler, but many still have to be proved.

In this section, $N \subset M$ is a subfactor of an approximately finite dimensional (AFD) factor M of type II_1 with finite index, finite depth, and a trivial relative commutant. We denote $[M : N]$, the principal graph, and the "dual" principal graph by $\beta^2, \mathcal{G}, \mathcal{H}$ respectively. Set $\tilde{\tau} = \sum_{x \in \mathcal{H}^{(0)}_{\text{even}}} \mu(x)^2$, where μ denotes the Perron-Frobenius eigenvector entries normalized with $\mu(*) = 1$. We fix a free ultrafilter ω over $\mathbf{N}$. Our aim is to obtain an upper bound for higher relative commutants of the subfactor $N^\omega \cap M' \subset M_\omega$ as claimed by Ocneanu [O4, III.2] without a full proof.

Let

$$\cdots \subset M_{-2} \subset M_{-1} = N \subset M_0 = M \subset M_1 \subset M_2 \subset \cdots$$

be the tower and a choice of a generating tunnel of $N \subset M$ by [O2, P2]. We put $M_\infty = \bigvee_k M_k$, which is a II_1 factor, and denote the trace on this by τ. We set $A_{k,l} = M_k' \cap M_l$. The double sequence $\{A_{k,l}\}_{k,l}$ is given by a flat connection on the pair of $\mathcal{G}, \mathcal{H}$ as in [O2, O4, Ka2, Ka3]. We freely use string algebra expression of [E1, E2, O2, O3, Su]. We also set $A_{k,\infty} = \bigvee_l A_{k,l} = M_k' \cap M_\infty$, $A_{-\infty,l} = \bigvee_k A_{k,l} = M_l$, and $A_{-\infty,\infty} = \bigvee_{k,l} A_{k,l} = M_\infty$. Note that this labeling of $A_{k,l}$ is different from that in [O4, Ka2, Ka3].

The following corresponds to a part of [O2, page 137, Theorem b] and was used in [Li1].

Lemma 2.1. *An inclusion $N^\omega \cap M' \subset M_\omega$ gives a subfactor of type II_1 with finite index.*

Proof. Because the Pimsner-Popa estimate for this pair was given in [Ka1, Lemma 3.5] based on [P2], it is enough to show that $N^\omega \cap M'$ is a factor. First recall that $N^\omega \cap M' = \bigcap_k M_{-k}^\omega$ by [Ka1, Lemma 3.3] as claimed in [O2].

First we claim that $x = (x_n)$ is in $\bigcap_k M_{-k}^\omega$ if and only if x is represented with (x_n) such that $F_k = \{n; x_n \in M_{-k}\} \in \omega$. It is trivial that such an x is in $\bigcap_k M_{-k}^\omega$. Suppose that $x^0 = (x_n^0) \in M_{-1}^\omega \cap M'$. For all k, there is a sequence $x^k = (x_n^k) \in M_{-k}^\omega$ with $x^k = x^0$ in M_{-1}^ω. Set $F_0 = \mathbf{N}$ and

$$F_k = \{n; \|x_n^{k-1} - x_n^k\|_2 < 1/2^k\} \cap F_{k-1} \cap [k, \infty).$$

Then each F_k is in ω. Put $x_n = x_n^k$ for $n \in F_k \setminus F_{k+1}$ and set $x = (x_n) \in N^\omega$. Then on F_k, we get $\|x_n^k - x_n\|_2 \leq 1/2^k$. Thus $\|x^0 - x\| \leq 1/2^k$ for all k, which implies $x^0 = x$. Thus we have the claim.

Suppose we have $x = (x_n) \in \mathcal{Z}(N^\omega \cap M')$ with $\tau(x) = 0$. We have to show $x = 0$. We may assume that $\tau(x_n) = 0$ and $\|x_n\|_2 = 1$ for all n and will drive a contradiction. We choose (x_n) as in the above claim. We may assume that $\bigcap_k F_k = \emptyset$. Then for each $n \in F_k \setminus F_{k+1}$, we choose $y_n \in M_{-k}$ so that $\|[x_n, y_n]\|_2 \geq 1/2$ with $\|y_n\| = 1$. This is possible because each M_{-k} is a factor. Then the sequence $y = (y_n)$ is in $N^\omega \cap M'$ and $\|[x, y]\|_2 \neq 0$, which is a contradiction. Q.E.D.

The following lemma appeared in [O2, page 136] and [O4, page 42] in slightly different forms without a proof. We need the following form here.

Lemma 2.2 (Central Freedom Lemma). *Let $L \subset P \subset Q$ be finite von Neumann algebras and L be an AFD factor. The we get*

$$(L' \cap P^\omega)' \cap Q^\omega = L \vee (P' \cap Q)^\omega.$$

Proof. Note that it is trivial that the right hand side is contained in the left hand side.

First we prove the lemma for the case $L = \mathbf{C}$. Take $x = (x_n) \in (P^\omega)' \cap Q^\omega$. We prove that $x = (E_{P' \cap Q}(x_n))$. Suppose not. There exists a positive ε and a set $F \in \omega$ such that $\|x_n - E_{P' \cap Q}(x_n)\|_2 \geq \varepsilon$ for all $n \in F$. Then for all $n \in F$, there exists $y_n \in P$ with $\|y_n\| = 1$ and $\|[x_n, y_n]\|_2 \geq \varepsilon/2$. Setting $y = (y_n) \in P^\omega$, we get $\|[x, y]\|_2 \geq \varepsilon/2$, which is a contradiction.

Next assume that L is an AFD II$_1$ factor. Represent $L = \bigotimes_n M_2(\mathbf{C})$ and set $L_m = \bigotimes_{n=1}^m M_2(\mathbf{C})$. We claim that

$$(\bigcap_m (L_m' \cap P)^\omega)' \cap Q^\omega = \bigvee_m (((L_m' \cap P)^\omega)' \cap Q^\omega).$$

It is clear that the right hand side is contained in the left hand side. To prove the converse inclusion, suppose that $x = (x_n) \in Q^\omega$ satisfies $x \notin \bigvee_m (((L_m' \cap P)^\omega)' \cap Q^\omega)$. Then there exists a positive ε such that $\|x - E_{((L_m' \cap P)^\omega)' \cap Q^\omega}(x)\|_2 \geq \varepsilon$ for all m. Then we have $y^m \in (L_m' \cap P)^\omega$ with $\|y^m\| = 1$ and $2\|x\|\|y^m\|_2 \geq \|[x, y^m]\|_2 \geq \varepsilon/2$. Put $F_0 = \mathbf{N}$ and

$$F_m = \{n; \|[x_n, y_n^m]\|_2 \geq \varepsilon/2\} \cap F_{m-1} \cap [m, \infty).$$

Each F_m is in ω, and definie $y = (y_n)$ with $y_n = y_n^m$ for $n \in F_m \setminus F_{m-1}$. Then $y \in \bigcap_m (L_m' \cap P)^\omega$ and $\|[x, y]\|_2 \geq \varepsilon/2$, which complete the proof of the claim.

Then we have

$$\begin{aligned}
(L' \cap P^\omega)' \cap Q^\omega &= (\bigcap_m (L_m' \cap P^\omega))' \cap Q^\omega \\
&= (\bigcap_m (L_m' \cap P)^\omega)' \cap Q^\omega \\
&= \bigvee_m (((L_m' \cap P)^\omega)' \cap Q^\omega) \\
&= \bigvee_m (\mathbf{C} \otimes (L_m' \cap P)^\omega)' \cap (L_m \otimes (L_m' \cap Q)^\omega) \\
&= \bigvee_m (L_m \vee (P' \cap Q)^\omega) \\
&= L \vee (P' \cap Q)^\omega.
\end{aligned}$$

Q.E.D.

The following is a special case of Ocneanu's several algebraic lemmas such as excision lemma. Recall that our double sequence looks like the following.

$$
\begin{array}{ccccccccc}
 & & & & A_{-n,-n} & \subset \cdots \rightarrow & A_{-\infty,-n} = M_{-n} \\
 & & & & \cap & & \cap \\
 & & & & \vdots & & \vdots \\
 & & & & \cap & & \cap \\
 & & A_{0,0} & \subset \cdots \subset & A_{-n,0} & \subset \cdots \rightarrow & A_{-\infty,0} = M_0 \\
 & & \cap & & \cap & & \cap \\
 & & \vdots & & \vdots & & \vdots \\
 & & \cap & & \cap & & \cap \\
A_{n,n} & \subset \cdots \subset & A_{0,n} & \subset \cdots \subset & A_{-n,n} & \subset \cdots \rightarrow & A_{-\infty,n} = M_n \\
\cap & & \cap & & \cap & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A_{n,\infty} & \subset \cdots \subset & A_{0,\infty} & \subset \cdots \subset & A_{-n,\infty} & \subset \cdots &
\end{array}
$$

Lemma 2.3. *Let $e \in A_{-n,\infty}$ be the Jones projection for $A_{n,\infty} \subset A_{0,\infty}$. Let f be the central support of e in $A'_{n,\infty} \cap A_{-n,\infty} = A_{-n,n}$ and $p \in A_{0,\infty} \vee A_{-n,0}$ be the Jones projection for $A'_{0,n} \cap A_{0,\infty} \subset A_{0,\infty}$. Then we get the following.*

(1) $p \in A_{0,n} \vee A_{-n,0}$.

(2) $ep = p$.

(3) $E_{A_{-n,0} \vee A_{0,\infty}}(e) = E_{A_{-n,0}}(f)p$.

Proof. For each $x \in \mathcal{H}^{(0)}$ and an integer n, define a map φ_x from a horizontal string algebra starting from $*$ to x with length n to itself by the following.

$$\varphi(\xi,\eta) = \sum_{\xi',\eta'} \begin{array}{ccc} * & \xrightarrow{\alpha} & \cdots x \\ \xi\big\downarrow & & \big\downarrow\tilde{\xi}' \\ \vdots & & \vdots \\ x & & * \\ \vdots & & \vdots \\ \eta\big\uparrow & & \big\uparrow\tilde{\eta}' \\ * & \xrightarrow[\alpha]{} & \cdots x \end{array} (\xi',\eta').$$

By flatness, this is a unital homomorphism from a full matrix algebra to itself [Ka2, Theorem 2.1]. Thus this is an automorphism. (This is related to the mirroring mentioned in [O2, page 132.])

Set

$$p = \sum_{\xi,\eta} \frac{1}{|\mathrm{Path}_{*,x}(n)|}(\xi,\eta) \otimes \varphi^{-1}_{r(\xi)}(\xi,\eta) \in A_{0,n} \vee A_{-n,0},$$

where the both $A_{0,n}$ and $A_{-n,0}$ are expressed as the horizontal string algebras from $*$ with length n, and $|\mathrm{Path}_{*,x}(n)|$ denotes the number of paths with length n from $*$ to x. Then a direct computation shows that this p is the right Jones projection for $A'_{0,n} \cap A_{0,\infty} \subset A_{0,\infty}$. It is also easy to see that $p \in A_{0,n} \vee A_{-n,0}$ and $ep = p$ by direct computations.

We also have

$$E_{A_{-n,0}}(f) = \frac{1}{\beta^n} \sum_{x\in\mathcal{H}^{(0)}} \frac{\mu(*)}{\mu(x)} |\mathrm{Path}_{*,x}(n)| \sum_{\xi, s(\xi)=*, r(\xi)=x} (\xi,\xi),$$

where this is expressed as a horizontal string. By the commuting square condition, in order to prove $E_{A_{-n,0}\vee A_{0,\infty}}(e) = E_{A_{-n,0}}(f)p$, it is enough to see $\tau(e\rho) = \tau(E_{A_{-n,0}}(f)p\rho)$ for $\rho \in A_{0,n}\vee A_{-n,0}$ of the form $\rho = (\xi,\eta)\otimes\varphi_r^{-1}(\xi')(\xi',\eta')$, where these are expressed as horizontal strings. For this ρ, a direct computation shows that the both of $\tau(e\rho)$ and $\tau(E_{A_{-n,0}}(f)p\rho)$ are equal to $\delta_{\xi,\xi'}\delta_{\eta,\eta'}\mu(r(\xi))\beta^{-3n}$.

Q.E.D.

Lemma 2.4. *Let $e\in A_{-n,\infty}$ be the Jones projection for $A_{n,\infty}\subset A_{0,\infty}$. Then this e is also a Jones projection in $A_{-\infty,n}$ for $A_{-\infty,-n}\subset A_{-\infty,0}$.*

Proof. If $n=1$, this directly follows from the identification of strings explained in [Ka3, §2]. For general n, we use the formula of e in [PP2] expressed with e_j's and reduce the problem to the case $n=1$. Q.E.D.

The proof of the following lemma is due to Ocneanu. The author learned it from him in July, 1990.

Lemma 2.5. *Let n,e,f be as in Lemma 2.3. Let $0<\varepsilon<1/3$ and $\bar e\in A_{-n,\infty}$ be a projection. Suppose we have*

$$\|E_{A_{-n,0}\vee A_{0,\infty}}(\bar e)-\tilde\tau\|_1<\varepsilon\tilde\tau,$$

$$\|E_{A_{-n,0}}(f)-\tilde\tau\|<\varepsilon\tilde\tau.$$

Then for $x\in A_{-\infty,0}\cap A'_{-n,0}$ with $\|x\|\le 1$, we get

$$\|\bar e x\bar e-\bar e E_{A_{-\infty,-n}}(x)\|_2<2\varepsilon^{1/4}.$$

Proof. First choose v_i's in $A_{0,\infty}$ so that $\bar{e} = \sum_i v_i^* e v_i$, (a finite sum). By Lemmas 2.3 and 2.4 we have

$$
\begin{aligned}
\|\bar{e}x\bar{e} - \bar{e}E_{A_{-\infty,-n}}(x)\|_2^4 &\leq 2\|\bar{e}x\bar{e} - \bar{e}E_{A_{-\infty,-n}}(x)\|_1^2 \\
&= 2\|\sum v_i^* e v_i x \bar{e} - \sum v_i^* e v_i E_{A_{-\infty,-n}}(x)\|_1^2 \\
&= 2\|\sum v_i^* e x p v_i \bar{e} - \sum v_i^* e x e v_i\|_1^2 \\
&= 2\|\sum v_i^* e x (p v_i \bar{e} - e v_i)\|_1^2 \\
&\leq 2\tau(\sum v_i^* e x x^* e v_i)\tau(\sum(\bar{e} v_i^* p - v_i^* e)(p v_i \bar{e} - e v_i)) \\
&= 2\tau(\bar{e}E_{A_{-\infty,-n}}(xx^*))\tau(\sum \bar{e} v_i^* p v_i \bar{e} - \bar{e} - \bar{e} + \bar{e}) \\
&\leq 2\|\bar{e}\|_1 \tau(\bar{e}(\sum v_i^* p v_i - 1)\bar{e})).
\end{aligned}
$$

Next by Lemma 2.3.(3) we have

$$
\begin{aligned}
\tilde{\tau}\|\sum v_i^* p v_i - 1\|_1 &\leq \|\tilde{\tau}\sum v_i^* p v_i - E_{A_{-n,0}}(f)\sum v_i^* p v_i\|_1 + \|E_{A_{-n,0}}(f)\sum v_i^* p v_i - \tilde{\tau}\|_1 \\
&= \|\tilde{\tau}\sum v_i^* p v_i - E_{A_{-n,0}}(f)\sum v_i^* p v_i\|_1 + \|E_{A_{-n,0}\vee A_{0,\infty}}(\bar{e}) - \tilde{\tau}\|_1 \\
&\leq \varepsilon\tilde{\tau}\|\sum v_i^* p v_i\|_1 + \varepsilon\tilde{\tau}.
\end{aligned}
$$

∎

Setting $C = \|\sum v_i^* p v_i\|_1$, we get

$$C \leq \|\sum v_i^* p v_i - 1\|_1 + 1 \leq \varepsilon C + \varepsilon + 1,$$

which implies $C \leq (1+\varepsilon)/(1-\varepsilon) \leq 1 + 3\varepsilon$. Thus we get

$$\|\sum v_i^* p v_i - 1\|_1 \leq \varepsilon(1+3\varepsilon) + \varepsilon \leq 5\varepsilon.$$

With this, we get

$$\|\bar{e}x\bar{e} - \bar{e}E_{A_{-\infty,-n}}(x)\|_2^4 \leq 10\varepsilon,$$

which produces the desired estimate. Q.E.D.

The following corresponds to [O2, page 137, Theorem d].

Lemma 2.6. *Let $\tilde{M} \subset A_{-\infty,0} \vee A_{0,\infty} \subset M_\infty$ be a downward basic construction with the Jones projection $\tilde{e} \in M_\infty$. Then*

$$\begin{array}{ccc} \tilde{M}^\omega & \subset & (A_{-\infty,0} \vee A_{0,\infty})^\omega \\ \cup & & \cup \\ N^\omega \cap M' & \subset & M_\omega \end{array}$$

is a commuting square.

Proof. First note that $N^\omega \cap M' = M^\omega \cap M'_\infty$. Because $\tilde{e} \in M_\infty$, we get $N^\omega \cap M' \subset \tilde{M}^\omega$.

Choose a positive $\varepsilon < 1/3$. Choose n_0 so large that if $n > n_0$, we get $\|E_{A_{-n,0} \vee A_{0,\infty}}(\tilde{e}) - \tilde{\tau}\|_2 < \varepsilon\tilde{\tau}/2$ and $\|E_{A_{-n,0}}(f) - \tilde{\tau}\| < \varepsilon\tilde{\tau}$, and we can find a projection $\bar{e} \in A_{-n,\infty}$ with $\|\bar{e} - \tilde{e}\|_2 < \varepsilon\tilde{\tau}/2$. (See [C2].) Here f is the central support of the Jones projection for $A_{n,\infty} \subset A_{0,\infty}$ in $A'_{n,\infty} \cap A_{-n,\infty}$ as in Lemma 2.3. The estimate for f is possible by looking at the explicit form

$$E_{A_{-n,0}}(f) = \frac{1}{\beta^n} \sum_{x \in \mathcal{H}^{(0)}} \frac{\mu(*)}{\mu(x)} |\mathrm{Path}_{*,x}(n)| \sum_{\xi, s(\xi)=*, r(\xi)=x} (\xi, \xi)$$

with the Perron-Frobenius theory. Choose $x = (x_m) \in M_\omega$. Fix $n > n_0$ and we may assume that $x_m \in M \cap A'_{-n,0}$. Then

$$\begin{aligned}
&\|\tilde{e}E_{\tilde{M}^\omega}(x) - \tilde{e}E_{M^\omega_{-n}}(x)\|_2 \\
=&\|\tilde{e}x\tilde{e} - \bar{e}x\tilde{e} + \bar{e}x\tilde{e} - \bar{e}x\bar{e} + \bar{e}x\bar{e} - \bar{e}E_{M^\omega_{-n}}(x) + \bar{e}E_{M^\omega_{-n}}(x) - \tilde{e}E_{M^\omega_{-n}}(x)\|_2 \\
\leq&2\varepsilon^{1/4} + 3\varepsilon \leq 5\varepsilon^{1/4},
\end{aligned}$$

by Lemma 2.5. Because $E_{M^\omega_{-n}}(x)$ converges to $E_{N^\omega \cap M'}(x)$ in L^2-norm as in [Ka1, Lemma 3.3], we get

$$\|\tilde{e}E_{\tilde{M}^\omega}(x) - \tilde{e}E_{N^\omega \cap M'}(x)\|_2 \leq 5\varepsilon^{1/4}$$

for all $\varepsilon > 0$, whicn implies $E_{\tilde{M}^\omega}(x) = E_{N^\omega \cap M'}(x)$. Q.E.D.

Lemma 2.7. *We get that*

$$N^\omega \cap M' \subset M_\omega \subset \langle M_\omega, \tilde{e}\rangle$$

is a basic construction, where $\tilde{e}$ is regardes as an element of M^ω_∞, the subfactor $N^\omega \cap M'$ has a trivial relative commutant in M_ω, and the index $[M_\omega : N^\omega \cap M']$ is given by $\tilde{\tau}$.

Proof. The only part we have to prove on the basic construction is that the central support q of $\tilde{e}$ in $\langle M_\omega, \tilde{e}\rangle$ is 1, by [PP2, Proposition 1.2 2°]. By the central freedom lemma, we get $(M_\omega)' \cap M^\omega_\infty = M \vee A^\omega_{0,\infty} \supset M \vee A_{0,\infty}$. Then

$$\tilde{\tau} = E_{(M_\omega)' \cap M^\omega_\infty}(\tilde{e}) = E_{(M_\omega)' \cap M^\omega_\infty}(q\tilde{e}) = qE_{(M_\omega)' \cap M^\omega_\infty}(\tilde{e}) = \tilde{\tau}q.$$

(We learned this trick from Ocneanu.) Thus we get

$$[M_\omega : N^\omega \cap M'] = E_{M_\omega}(\tilde{e}) = E_{M_\omega}(E_{M \vee A_{0,\infty}}(\tilde{e})) = \tilde{\tau}.$$

By central freedom lemma (and its proof), we also get

$$(N^\omega \cap M')' \cap M_\omega = (\bigcap_k M^\omega_{-k})' \cap M^\omega \cap M' = \bigvee_k (M'_{-k} \cap M) \cap M' = M \cap M' = \mathbf{C}.$$

Q.E.D.

Lemma 2.8. *Suppose $P_0 \subset P_1 \subset P_2 \subset Q$ are II_1 factors with $[P_1 : P_0] < \infty$ and $P_0' \cap P_1 = \mathbf{C}$ and $P_0 \subset P_1 \subset P_2 = \langle P_1, e \rangle$ is standard with the Jones projection e in Q. We also assume that $P_1' \cap Q \subset P_0' \cap Q$ are II_1 factors with $[P_0' \cap Q : P_1' \cap Q] = [P_1 : P_0]$. Then*

$$P_2' \cap Q \subset P_1' \cap Q \subset P_0' \cap Q$$

is also standard.

Proof. Because

$$\begin{array}{ccc} P_1' \cap Q & \subset & P_0' \cap Q \\ \cup & & \cup \\ P_1' \cap P_2 & \subset & P_0' \cap P_2 \end{array}$$

is a commuting square by [GHJ], we get $E_{P_1' \cap Q}(e) = [P_0' \cap Q : P_1' \cap Q]^{-1}$. This makes

$$P_1' \cap Q \cap \{e\}' \subset P_1' \cap Q \subset P_0' \cap Q$$

standard by [PP1]. Q.E.D.

Lemma 2.9. *We have* $M_\infty^\omega \cap (N^\omega \cap M')' = \bigvee_k A^\omega_{-k,\infty}$.

Proof. By the central freedom lemma, the right hand side is equal to $\bigvee_k((M^\omega_{-k})' \cap M^\omega_\infty)$. It is trivial that this is contained in the left hand side, which is equal to $(\bigcap M^\omega_{-k})' \cap M^\omega_\infty$. The converse inclusion is proved as in the proof of the claim in the last step in the proof of the central freedom lemma. Q.E.D.

Let denote the graph defined as in [O4, page 40] by $\mathcal{K}$. This is given as follows. Recall that even vertices of $\mathcal{H}$ corresponds to M-M bimodules. Let $\mathcal{K}^{(0)}_{\text{even}} = \mathcal{H}^{(0)}_{\text{even}} \times \mathcal{H}^{(0)}_{\text{even}}$ and $\mathcal{K}^{(0)}_{\text{odd}} = \mathcal{H}^{(0)}_{\text{even}}$. For $(X_1, X_2) \in \mathcal{K}^{(0)}_{\text{even}}$ and $X \in \mathcal{K}^{(0)}_{\text{odd}}$, the number of edges from (X_1, X_2) to X is given by the multiplicity of the bimodule X in $X_1 \otimes_M X_2$. Note that if we fix X_1 and let X and X_2 vary, this gives a principal graph of the subfactor obtained by cutting the basic construction factor by a minimal projection corresponding to X_1. (See [I1, §3] for some examples.)

The next subfactor was called the asymptotic inclusion by Ocneanu [O4, III.1]. A part of it was claimed in [O2, page 137, Theorem a].

Lemma 2.10. *The inclusion*

$$A_{-\infty,0} \vee A_{0,\infty} \subset A_{-\infty,\infty}$$

gives a subfactor with index $\tilde{\tau}$. *The higher relative commutants of this subfactor are contained in the string algebra of the above graph* $\mathcal{K}$ *from* $*$.

Proof. It is easy to see that the commuting squares

$$\begin{array}{ccc} A_{-n,0} \vee A_{0,n} & \subset & A_{-(n+1),0} \vee A_{0,n+1} \\ \cap & & \cap \\ A_{-n,n} & \subset & A_{-(n+1),n+1} \end{array}$$

approximates the subfactor $A_{-\infty,0} \vee A_{0,\infty} \subset A_{-\infty,\infty}$. Thus, we have the four graphs as follows,

$$\begin{array}{ccc}
\mathcal{H}^{(0)}_{\mathrm{even}} \times \mathcal{H}^{(0)}_{\mathrm{even}} & \longrightarrow & \mathcal{H}^{(0)}_{\mathrm{odd}} \times \mathcal{H}^{(0)}_{\mathrm{odd}} \\
\kappa \Big\downarrow & & \Big\downarrow \\
\mathcal{H}^{(0)}_{\mathrm{even}} & \longrightarrow & \mathcal{G}^{(0)}_{\mathrm{even}}
\end{array}$$

and this is of the form Ocneanu's general machinery in [O4, II] is applied.

So his compactness argument in [O4, II.6] produces the desired inclusion. It is easy to see that the Perron-Frobenius eigenvector μ' is given by $\mu'((X_1, X_2)) = \mu(X_1)\mu(X_2)$ and $\mu'(X) = \sqrt{\sum_{Y \in \mathcal{H}^{(0)}_{\mathrm{even}}} \mu(Y)^2}\mu(X)$ and the Perron-Frobenius eigenvalue is given by $\tilde{\tau}^{1/2}$. Q.E.D.

Let

$$A_{-\infty,0} \vee A_{0,\infty} \subset A_{-\infty,\infty} \subset \langle A_{-\infty,\infty}, f_1 \rangle \subset \langle A_{-\infty,\infty}, f_1, f_2 \rangle \subset \langle A_{-\infty,\infty}, f_1, f_2, f_3 \rangle \subset \cdots$$

be the Jones tower and $f_1, f_2, \ldots$ be the Jones projections. Set $B^l_k = A_{-k,\infty} \vee \{f_1, \ldots, f_l\}$. Then by [GHJ, Proposition 4.2.2], the following double sequence gives commuting squares.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
A_{-k,0} \vee A_{0,\infty} & \subset & A_{-k,\infty} & \subset & B^1_k & \subset & B^2_k & \subset & \cdots \\
\cap & & \cap & & \cap & & \cap & & \\
A_{-(k+1),0} \vee A_{0,\infty} & \subset & A_{-(k+1),\infty} & \subset & B^1_{k+1} & \subset & B^2_{k+1} & \subset & \cdots \\
\cap & & \cap & & \cap & & \cap & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Furthemore, the vertcal limit of the double sequence gives the tower for $A_{-\infty,0} \vee A_{0,\infty} \subset A_{-\infty,\infty}$. Then we have the following lemma.

Lemma 2.11. *(1) We have the following identity.*

$$\bigvee_k (B_k^j)^\omega \cap (\bigvee_k (A_{-k,0} \vee A_{0,\infty})^\omega)' = \langle A_{-\infty,\infty}, f_1, \ldots, f_j \rangle \cap (A_{-\infty,0} \vee A_{0,\infty})'.$$

(2) The sequence

$$\bigvee_k (A_{-k,0} \vee A_{0,\infty})^\omega \subset \bigvee_k A_{-k,\infty}^\omega \subset \bigvee_k (B_k^1)^\omega \subset \bigvee_k (B_k^2)^\omega \subset \cdots$$

gives the Jones tower.

Proof. (1) We have

$$\begin{aligned}
\bigvee_k (B_k^j)^\omega \cap (\bigvee_k (A_{-k,0} \vee A_{0,\infty})^\omega)' &= \bigvee_k (B_k^j)^\omega \cap \bigcap_k ((A_{-k,0} \vee A_{0,\infty})^\omega)' \\
&= \bigcap_k (((A_{-k,0} \vee A_{0,\infty})^\omega)' \cap \bigvee_{l \geq k} (B_l^j)^\omega) \\
&= \bigcap_k \bigvee_l (((A_{-k,0} \vee A_{0,\infty})^\omega)' \cap (B_l^j)^\omega) \\
&= \bigcap_k \bigvee_l ((A_{-k,0} \vee A_{0,\infty})' \cap B_l^j)^\omega \\
&= \bigcap_k \bigvee_l ((A_{-k,0} \vee A_{0,\infty})' \cap B_l^j) \\
&= (\bigvee_k B_k^j) \cap (\bigvee_k (A_{-k,0} \vee A_{0,\infty}))' \\
&= \langle A_{-\infty,\infty}, f_1, \ldots, f_j \rangle \cap (A_{-\infty,0} \vee A_{0,\infty})'.
\end{aligned}$$

Here we used the central freedom lemma and the finite dimensionality of $(A_{-k,0} \vee A_{0,\infty})' \cap B_l^j$. This finite dimensionality follows from finite dimensionality of the center of $A_{-k,0} \vee A_{0,\infty}$ and the Pimsner-Popa estimate for the conditional expectation from B_l^j onto $A_{-k,0} \vee A_{0,\infty}$.

(2) By the same kind of argument as above, we can prove that each $\bigvee_k (B_k^l)^\omega$ is a factor. First we prove that

$$\bigvee_k (A_{-k,0} \vee A_{0,\infty})^\omega \subset \bigvee_k A^\omega_{-k,\infty} \subset \bigvee_k (B_k^1)^\omega$$

is standard. (The other parts of the proof are just repetition of the same arguments.) For the inclusion $\bigvee_k A^\omega_{-k,\infty} \subset \bigvee_k (B_k^1)^\omega$, we have the Pimsner-Popa estimate $E(x) \geq \tilde{\tau} x$ because of the commuting square condition. We have $f_1 \in \bigvee_k (B_k^1)^\omega$ with $E_{\bigvee_k A^\omega_{-k,\infty}}(f_1) = \tilde{\tau}$. This implies $[\bigvee_k (B_k^1)^\omega : \bigvee_k A^\omega_{-k,\infty}] = \tilde{\tau}$, hence the conclusion by [PP2, Proposition 1.2 2°]. Q.E.D.

Lemma 2.12. *Suppose $P_0 \subset P_1 \subset P_2 \subset Q$ are II_1 factors with $[P_1 : P_0] < \infty$ and $P_0' \cap P_1 = \mathbf{C}$ and $P_0 \subset P_1 \subset P_2$ is standard. We also assume that $P_2' \cap Q \subset P_1' \cap Q \subset P_0' \cap Q$ are II_1 factors with $[P_1' \cap Q : P_2' \cap Q] = [P_1 : P_0]$ and this sequence is standard. Then we can choose a tower*

$$P_0 \subset P_1 \subset P_2 \subset P_3 \subset P_4 \subset \cdots \subset Q$$

so that

$$\cdots P_4' \cap Q \subset P_3' \cap Q \subset P_2' \cap Q \subset P_1' \cap Q \subset P_0' \cap Q$$

is a tunnel.

Proof. Set $\lambda = [P_1 : P_0]$. Let $f_1 \in P_2$ be the Jones projection for $P_0 \subset P_1$. Then we claim that f_1 is the Jones projection in $P_0' \cap Q$ for the inclusion $P_2' \cap Q \subset P_1' \cap Q$.

Using [GHJ], we can prove that

$$\begin{array}{ccc} P_1' \cap Q & \subset & P_0' \cap Q \\ \cup & & \cup \\ P_1' \cap P_2 & \subset & P_0' \cap P_2 \end{array}$$

is a commuting square. So we get $E_{P_1' \cap Q}(f_1) = \lambda$. Then f_1 is the Jones projection by [PP2, Proposition 1.2 1°]. Take $f_2 \in P_1' \cap Q$ with $E_{P_2' \cap Q}(f_2) = \lambda$. Then

$$P_2' \cap Q \cap \{f_2\}' \subset P_2' \cap Q \subset P_1' \cap Q$$

is a downward basic construction by [PP1]. We next claim that $P_1 \subset P_2 \subset P_3 = \langle P_2, f_2 \rangle$ is standard. For $x, y \in P_1$, we get

$$f_2(xf_1y)f_2 = xf_2f_1f_2y = \lambda f_2xy = f_2E_{P_1}(xf_1y).$$

Thus it is enough to see that the central support of f_2 in P_3 is 1 by [PP2, Proposition 1.2 2°]. This can be proved as in the proof of Lemma 2.7 with $E_{P_2' \cap Q} = \lambda$. It is now trivial that $P_2' \cap Q \cap \{f_2\}' = P_3' \cap Q$. We repeat this procedure to get the conclusion. Q.E.D.

The strategy for proving the the following lemma is due to Ocneanu.

Lemma 2.13. *Set $P_0 = N^\omega \cap M'$ and $P_1 = M_\omega$. We have a Jones tower*

$$P_0 \subset P_1 \subset P_2 \subset P_3 \subset \cdots \subset M_\infty^\omega$$

so that

$$\cdots \subset P_3' \cap M_\infty^\omega \subset P_2' \cap M_\infty^\omega \subset P_1' \cap M_\infty^\omega \subset P_0' \cap M_\infty^\omega$$

is a tunnel.

Proof. By Lemma 2.9 and the central freedom lemma, we get that the inclusion $P_1' \cap M_\infty^\omega \subset P_0' \cap M_\infty^\omega$ is given by $\bigvee_k (A_{-k,0} \vee A_{0,\infty})^\omega \subset \bigvee_k A_{-k,\infty}^\omega$. By Lemma 2.11 (1), this is a subfactor and by commuting square condition and the Pimsner-Popa estimate, we get

$$[\bigvee_k A_{-k,\infty}^\omega : \bigvee_k (A_{-k,0} \vee A_{0,\infty})^\omega] \leq \tilde{\tau}.$$

By Lemma 2.11 (2), we get

$$[\bigvee_k A_{-k,\infty}^\omega : \bigvee_k (A_{-k,0} \vee A_{0,\infty})^\omega] = \tilde{\tau}.$$

We can set $P_2 = \langle P_1, \tilde{e} \rangle$ by Lemmas 2.7 and 2.8. Now we just apply Lemma 2.12 to get the conclusion. Q.E.D.

Lemma 2.14. *The "dual" higher relative commutants of $N^\omega \cap M' \subset M_\omega$ is contained in the higher relative commutants of $A_{-\infty,0} \vee A_{0,\infty} \subset A_{-\infty,\infty}$ with a trace preserving injective antihomomorphism.*

Proof. By Lemma 2.13, it is easy to see that the "dual" higher relative commutants of $N^\omega \cap M' \subset M_\omega$ is contained in the higher relative commutants of the tunnel for $\bigvee_k (A_{-k,0} \vee A_{0,\infty})^\omega \subset \bigvee_k A^\omega_{-k,\infty}$. But this latter subfactor have the trivial relative commutant by Lemma 2.11. So the relative commutants for the tunnel is antiisomrphic to the relative commutants for the tower with a trace preserving antiisomorphism by [P2, Proposition 3.2 2°]. We get the conclusion by Lemma 2.11 (1). Q.E.D.

Lemma 2.15. *The "dual" higher relative commutants of $N^\omega \cap M' \subset M_\omega$ is contained in the string algebra of the above graph $\mathcal{K}$ from $*$ with a trace preserving injective antihomomorphism.*

Proof. Clear by Lemmas 2.10 and 2.14. Q.E.D.

Remark 2.16. Though Ocneanu claimed that the equalities hold in the both inclusions in Lemmas 2.10 and 2.15, we have been unable to prove it. But we do not need these equalities in this paper. Furthermore, the author stated in Ocneanu's Tokyo lecture notes [O4, page 42] that the central sequence subfactors and the asymptotic inclusion have the same Galois invariants. This was a mistake caused by misunderstanding of the author, and the correct form should be that they have mutually dual invariants.

§3 Upper bound for the size of $\mathrm{Ct}(M, N)$ for AFD subfactors of type II_1 with finite depth

Next we get an upper bound of the size of centrally trivial automorphisms on some well-understood subfactors and with an extra assumption we give their complete characterization. In this section, we work on only AFD type II_1 subfactors with trivial relative commutants, finite index and finite depth. Note that in this situation, aperiodic automorphisms are automatically centrally free as noticed in [Li1, §4]. This suggests that the group $\mathrm{Ct}(M, N)$ is rather small. We will prove this is indeed the case.

First we set $G = \mathrm{Ct}(M, N)/\{\mathrm{Ad}(u); u \in \mathcal{N}(N)\}$, where $\mathcal{N}(N)$ denotes the set of normalizers of N in M. Note that $|\mathcal{N}(N)/\mathcal{U}(N)|$ is finite by [PP1]. This group G acts naturally on M_ω and this action is outer. Because the subfator $N^\omega \cap M'$ is contained in the fixed point algebra by this action, we get the order of G is bounded by the index $[M_\omega : N^\omega \cap M']$, which is finite by Lemma 2.7. Thus we get the following.

Proposition 3.1. *The group $Ct(M, N)/\{Ad(u); u \in \mathcal{N}(N)\}$ is finite.*

But it turns out that this upper bound is not sharp at all. Indeed, for the case of subfactors of type A_n, we will prove that this group is 0 or $\mathbf{Z}_2$ according to the parity of n, but this upper bound $[M_\omega : N^\omega \cap M']$ goes to infinity as $n \to \infty$ as in [O2, page 138]. So we work to get a a sharper bound as follows.

Theorem 3.2. *Let $N \subset M$ be an AFD type II_1 subfactor with trivial relative commutant, finite index and finite depth. Then the order of the group $Ct(M, N)/\{Ad(u); u \in \mathcal{N}(N)\}$ is bounded by the number of even vertices of the* *"dual" principal graph for $N \subset M$ with the normalized Perron-Frobenius eigenvector entries equal to 1.*

Proof. We make a basic construction to a subfactor $N^\omega \cap M' \subset M_\omega$ which contains an intermediate subfactor $(M_\omega)^G$. Then with the Jones projection e, we get $M_\omega \subset M_\omega \times G \subset \langle M_\omega, e\rangle$. If G is non-trivial, this means that the subfactor $M_\omega \subset \langle M_\omega, e\rangle$ has non-trivial normalizers. But such normalizers can be detected from the principal graph by [PP1]. Then Lemmas 2.10 and 2.15 produce the above bound. Indeed, in Lemma 2.10, we care only about vertices with the Perron-Frobenius eigenvector entries 1. So we only care about M-M bimodule arising from an automorphism of M. Then bimodule tensor product is just an automorphism multiplication, and we get the conclusion. Q.E.D.

In Ocneanu's bimodule approach ([O2, O4, Y]), these vertices of the "dual" principal graph corresponds to automorphisms of M as in the above proof. Then Choda-Kosaki [CK, Ko] showed that if these automorphisms can be chosen in $\mathrm{Aut}(M, N)$, they are characterized by the following algebraic property. An automorphism $\alpha \in \mathrm{Aut}(M, N)$ is of the above type if and only if there exists a $k > 0$ and non-zero $a \in M_k$ with $\alpha(x)a = ax$ for all $x \in N$.

Because $N^\omega \cap M' = N^\omega \cap M'_k$ for all k, it is easy to see all of these automorphisms are centrally trivial. (The same proof as in [C2] works.) Thus the above upper bound is attained if the automorphisms appearing at the even vertices of the "dual" principal graph with the normalized Perron-Frobenius eigenvector entries equal to 1 are chosen in $\mathrm{Aut}(M, N)$.

Theorem 3.3. *Suppose that automorphisms σ_x of M appearing at the even vertices x of the "dual" principal graph with the normalized Perron-Frobenius entries equal to 1 can be chosen in $Aut(M,N)$ in the above context. Then, an automorphism $\alpha \in Aut(M,N)$ acts trivially on central sequences of M in N if and only if it is of the form $Ad(u)\cdot\sigma_x$, where u is a normalizer of N in M, and σ_x is as above.*

In the first version of this paper, the author missed the assumption of the above theorem in misunderstanding the results of Choda-Kosaki. We thank M. Izumi for pointing out this mistake to us.

Here we list an example of the graph $\mathcal{K}$ for the subfactor of type A_5. The circled vertex gives the nontrivial normalizer of $N^\omega \cap M' \subset M_\omega$ and we can prove that it gives the non-trivial element of $\mathrm{Ct}(M,N)/\{\mathrm{Ad}(u); u\in\mathcal{N}(N)\} = \mathbf{Z}_2$.

$\boxed{\text{Figure 3.1}}$

More examples of this group $\mathrm{Ct}(M,N)$ will be given in the next section.

We now discuss analogy between $\mathrm{Ct}(M,N)$ and modular automorphism groups of type III (injective) factors.

As pointed out in [Ka1] and noticed by several people independently, paragroups are similar to flows of weights of typer III factors in [CT]. If we have an automorphism of a type III factor, it naturally induces an action on the flow of weights and it is called Connes-Takesaki module [CT]. This is similar to Loi's invariant in [Li1, §5]. Loi proved that for irreducible AFD II_1 factors with finite depth, the kernel of his invariant is characterized by approximate innerness in the sense that an automorphism is of the form $\lim_n \mathrm{Ad}(u_n)$, where u_n's are unitaries of the subfactor.

This is an analogue of the fact that the kernel of the Connes-Takesaki module is characterized by approximate innerness for injective type III factors, which was announced by Connes [C4, section 3.8] without a proof and proved by Sutherland, Takesaki, and the author in [KST, Theorem 1]. Also in the course of Loi's proof, we see similarity between the crossed product by modular automorphism group and the Jones tower, and between the centralizer and the tunnel construction.

Next recall that centrally trivial automorphisms of injective type III factors are characterized as (extended) modular automorphisms up to inner perturbation. This was also announced by [C4, section 3.8] without a proof and proved by [KST, Theorem 1]. If we compare this to Theorem 3.3, we are led to an idea that this subgroup of $\mathrm{Aut}(M,N)/\mathrm{Int}(M,N)$ appearing at the even vertices of the "dual" principal graph with the normalized Perron-Frobenius eigenvector entries equal to 1 is a discrete analogue of modular automorphism groups of type III factors. From this viewpoint, we can get a natural interpretation of Choda-Kosaki's result as follows.

Choda-Kosaki's result means that an automorphism of a subfactor comes from this discrete analogue of the modular automorphism group if and only if it becomes "almost" inner when it is extended to the Jones tower. In the case of separable type III factors, Haagerup-Størmer [HS, Proposition 5.4] proved in connection to their pointwise inner automorphisms that an automorphism comes from the (extended) modular automorphism group if and only if it becomes inner when it is extended to the crossed products by the modular automorphism group for the dominant weight. So analogy holds again.

Furthermore, in type III subfactor theory, these even vertices correspond to automorphisms appearing in decomposition of the powers of Longo's canonical endomorphism [Ln2, Ln3]. Longo already noticed in [Ln1] that his canonical endomorphisms are similar to modular automorphism groups. This also supports the above similarity. We believe that more and more analogy will hold.

Popa's recent classification in [P4] of group actions on subfactors gives the following classification. "Properly outer" discrete amenable group actions on strongly amenable subfactors are classified by Loi's invariant. Because his notion "proper outerness" turns out to coincide with central freeness in our subfactor sense, this is again an analogue of a classification result in injective type III factors. That is, centrally free actions of discrete amenable groups on injective type III factors are classified by modules [C4, KST, O1, ST]. Our result here deals with the opposite class of automorphisms.

From this analogy viewpoint, the construction of D_{2n} subfactors form A_{4n-3} subfactors in [Ka1] and the orbifold constructions in [EK] are regarded as analogues of the change of the flows of weights in crossed products by the modular automorphism groups [KT, Se].

The orbifold $\mathbf{Z}_2$ actions on A_{4n-3} and the $\mathbf{Z}_2$ actions flipping the two tails of D_{2n} are in duality. This is regarded as an analogue of the duality between the Connes-Takesaki module [CT] and the Sutherland-Takesaki modular invariant [ST].

Note that importance of these automorphisms appearing at the even vertices of the "dual" principal graph was noticed in different contexts such as Izumi's work [I1, I2] on subfactors with the principal graph A_5 and Haagerup's construction of finite depth subfactors with small index [H].

§4 Computation of $\chi(M,N)$ for AFD subfactors of type II_1 with finite depth

In this section, we work on only AFD type II_1 subfactors with trivial relative commutants, finite index and finite depth again. In the single factor case, it is rather difficult to construct a II_1 factor with non-trivial $\chi(M)$ as in [C1, C4]. But we will prove that in subfactor setting, all the finite abelian groups naturally as $\chi(M,N)$ even in the above fairly simple settings.

First note that

$$|\chi(M,N)| \le |\mathrm{Ct}(M,N)/\{\mathrm{Ad}(u); u \in \mathcal{N}(N)\}| \cdot |\mathcal{N}(N)/\mathcal{U}(N)|,$$

and $|\mathcal{N}(N)/\mathcal{U}(N)|$ is finite by [PP1]. So we have a upper bound for the order of $\chi(M,N)$ as follows.

Theorem 4.1. *Let $N \subset M$ be as above. Then we have an upper bound for the order of $\chi(M,N)$ as follows.*

$$|\chi(M,N)| \le |\mathcal{N}(N)/\mathcal{U}(N)| \cdot K_{M,N},$$

where $K_{M,N}$ is the number of even vertices of the "dual" principal graph for $N \subset M$ with the normalized Perron-Frobenius eigenvector entries equal to 1.

Note that the right hand side of the above inequality is 1 in many cases, and it forces $\chi(M,N)$ to be 0.

Remark 4.2. If we drop the finite depth assumption in the above, we do not get finiteness of $\chi(M,N)$ in general. Take the following subfactor $N \subset M$ with the principal graph $A_{\infty,\infty}$.

$$N = \begin{pmatrix} x & 0 \\ 0 & \alpha(x) \end{pmatrix} \subset M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c, d, x are in R, the AFD II_1 factor and α is a free action of $\mathbf{Z}$ on R. The automorphism α itself can be regarded as an element of $\mathrm{Aut}(M,N)$. By stability of α on R [C2], we can take u_n in R so that $\alpha = \lim_n \mathrm{Ad}(u_n)$ on R and $\lim_n \|u_n - \alpha(u_n)\|_2 = 0$. This implies that the above $\alpha \in \mathrm{Aut}(M,N)$ is approximated by adjoints of $\begin{pmatrix} u_n & 0 \\ 0 & \alpha(u_n) \end{pmatrix} \in N$, and hence $\alpha \in \overline{\mathrm{Int}}(M,N)$. Next look at $N^\omega \cap M'$. It is easy to see that this algebra is given by the elements of the form

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \qquad x \in (R_\omega)^\alpha.$$

Then all α^n give elements in $\mathrm{Ct}(M,N) \cap \overline{\mathrm{Int}}(M,N)$ and all these are different modulo $\mathrm{Int}(M,N)$. Thus $\chi(M,N)$ contains a copy of $\mathbf{Z}$.

Next we show that the bound in Theorem 4.1 is attained in the case with index less than 4. A classification of AFD type II_1 subfactors with index less than 4 was announced by Ocneanu without a proof in [O2]. Now the generating property was proved by Popa [P2], and algebraic and combinatorial parts of the paragroup theory have been all verified by [EG, I1, I3, Ka2, SV]. Only non-trivial cases for $\chi(M,N)$ in Theorem 4.1 are A_{2n+1} and E_6. In these cases, we have $\mathrm{Aut}(M,N) = \overline{\mathrm{Int}}(M,N)$ by [Li1, §5], but we also give a direct proof of approximate innerness here so that the same method works in the Hecke algebra subfactor case. For the case of A_{2n+1}, let α be an element in $\mathrm{Aut}(M, N)$ with order 2 arising from the $\mathbf{Z}_2$ paragroup symmetry considered in [Ka2, §5].

Lemma 4.3. *This α gives an element of order 2 in $\chi(M, N)$.*

Proof. As in [Ka2, §5], take a tower of double sequence string algebras with the double starting points. As in a similar way to that in [Ka3, §2], we also construct a natural tunnel

$$\cdots \supset N_3 \supset N_2 \supset N_1 \supset N \supset M.$$

with double starting points. (Note that this tunnel does not have a generating property.) First α acts trivially on the higher relative commutants, so $\alpha \in \overline{\mathrm{Int}}(M, N)$. Note that $N^\omega \cap M' \subset N_n^\omega$. The algebra N_n has a two minimal projections $p, 1-p$ corresponding to the two starting ponts at the first stage. Let $x \in N^\omega \cap M'$. Then we may assume that $x \in (pN_n p)^\omega + ((1-p)N_n(1-p))^\omega$ and x is decomposed as $x_1 + x_2$. Now x must commute with N hence with vertical strings with length n, so looking at the vertical parallel transports from the two endpoints to the middle points of A_{2n+1}, we can conclude that $\alpha(x_1) = x_2$. This means that α acts trivially on $N^\omega \cap M'$. Because $\alpha \notin \mathrm{Int}(M, N)$, we get the conclusion. Q.E.D.

Note that the same proof works for E_6, which also has a $\mathbf{Z}_2$ symmetry of the paragroup. Theorem 4.3 gives that these orbifold $\mathbf{Z}_2$ actions coincide with the action appearing at the even vertices of A_{2n+1} and E_6 with the normalized Perron-Frobenius eigenvector entries equal to 1, because they are not strongly outer in the sense of [CK], and hence centrally trivial.

Thus we get the following.

Proposition 4.4. *In the case of AFD subfactors with index less than 4, we get the following.*

$$\chi(M,N)=\begin{cases} 0, & \textit{for } A_{2n}, D_{2n+4}, E_8, \textit{ with } n\geq 1\\ \mathbf{Z}_2, & \textit{for } A_{2n+3}, E_6, \textit{ with } n\geq 1\\ \mathbf{Z}_2\oplus\mathbf{Z}_2, & \textit{for } A_3\\ \mathbf{Z}_3\oplus\mathbf{Z}_3, & \textit{for } D_4.\end{cases}$$

Proof. Theorem 4.1 and Lemma 4.3 give the conclusion except for A_3 and D_4. These are group crossed product by $\mathbf{Z}_2$ and $\mathbf{Z}_3$ respectively. In the case of $\mathbf{Z}_2$, $N = R \subset R\times_\alpha \mathbf{Z}_2 = M$. Let u be the implementing unitary of the crossed product. Then $\mathrm{Ad}(u), \hat\alpha, \mathrm{Ad}(u)\cdot\hat\alpha$ give three non-trivial element in $\chi(M,N)$, so Theorem 4.1 gives the conclusion. The case for D_4 is similar. Q.E.D.

In the case of Wenzl's Hecke algebra subfactors with index $\dfrac{\sin^2(N\pi/k)}{\sin^2(\pi/k)}$ in [W], we considered a $\mathbf{Z}_N$ action arising from a paragroup symmetry for orbifold construction in [EK, §4] for the case $k\equiv 0 \pmod N$, based on Jimbo-Miwa-Okado solution of the Yang-Baxter equation [JMO1, JMO2]. As noted in [EK, Remark 5.9], these automorphisms are in $\overline{\mathrm{Int}}(M,N)$ if N is prime. (The same kind of partition function computation as in [EK, §5] produces this.) Then the same kind of arguments as in the proof of Lemma 4.3 shows that these are in $\mathrm{Ct}(M,N)$. With Theorem 4.1 and description of the dual principal graphs for these subfactors given in [EK, §3], we get the following.

Proposition 4.5. *For Wenzl's Hecke algebra subfactors $N \subset M$ with index $\dfrac{\sin^2(N\pi/k)}{\sin^2(\pi/k)}$ with N prime, we get the following.*

$$\chi(M,N) = \begin{cases} 0, & \text{if } k \not\equiv 0 \ (mod\, N) \\ \mathbf{Z}_N, & \text{if } k \equiv 0 \ (mod\, N). \end{cases}$$

This shows our $\mathbf{Z}_N$ actions in [EK, §4] give elements in $\chi(M,N)$ and they are the actions with special meaning for working on the simultaneous fixed point algebras. Furthermore, to define the actions, we used the original sequence of commuting squares of period N of Wenzl in [EK], and in [EK, Remark 5.11] we showed an alternate construction of the action using the period 2 commuting square sequence, which is the canonical commuting square. Because the both give elements of $\chi(M,N)$ as above, we can show the both give the same subgroup of order N in $\mathrm{Aut}(M,N)$.

Next, we will prove that all the finite abelian groups are realized as $\chi(M,N)$ for some AFD type II_1 subfactor $N \subset M$ with finite index, finite depth, and trivial relative commutants. Our strategy is that we first construct $\mathbf{Z}_N$ with Theorem 1.1 and Theorem 4.1 using the orbifold construction in [Ka3, §3] and then get an arbitrary finite abelian groups using tensor products. Connes' type arguments as in [C3] shows that tensor products of two centrally trivial automorphisms are centrally trivial. Because we have an upper bound for the order of χ and this bound is multiplicative with respect to tensor products, we get the desired realization.

First we point out the following follows from Connes' method [C3].

Proposition 4.6. *For subfactors $N_1 \subset M_1$ and $N_2 \subset M_2$, and automorphisms $\alpha_1 Aut(M_1, N_1)$ and $\alpha_2 \in Aut(M_2, N_2)$, we get $\alpha_1 \otimes \alpha_2 \in Ct(M_1 \bar{\otimes} M_2, N_1 \bar{\otimes} N_2)$ if and only if $\alpha_1 \in Ct(M_1, N_1)$ and $\alpha_2 \in Ct(M_2, N_2)$.*

Note that "only if" part is easy. Connes' machinery for proving "if" part in the case $M_1 = N_1$ and $M_2 = N_2$ in [C3] works in our case, too. This Theorem gives an inclusion $\chi(M_1, N_1) \times \chi(M_2, N_2) \subset \chi(M_1 \bar{\otimes} M_2, N_1 \bar{\otimes} N_2)$. In the following case, we get the equality of this inclusion.

Proposition 4.7. *Let $N_1 \subset M_1$ and $N_2 \subset M_2$ be AFD II_1 factors with trivial relative commutants, finite index, and finite depth. We assume that for the both subfactors, the principal graphs have no vertices with the normalized Perron-Frobenius eigenvector entry equal to 1 and distance 2 from $*$. Suppose that the upper bound for χ in Theorem 4.1 is attained for the both $\chi(M_1, N_1)$ and $\chi(M_2, N_2)$. Then we get*

$$\chi(M_1, N_1) \times \chi(M_2, N_2) = \chi(M_1 \bar{\otimes} M_2, N_1 \bar{\otimes} N_2).$$

Proof. By the condition on the principal graphs, subfactors $N_1 \subset M_1$, $N_2 \subset M_2$ and $N_1 \bar{\otimes} N_2 \subset M_1 \bar{\otimes} M_2$ have no nontrivial normalizers by [PP1]. Then the upper bound in Theorem 4.1 is multiplicative with respect to tensor products of subfactors. So the above inclusion implies the conclusion. Q.E.D.

With Proposition 4.7, it is enough for realization of all the finite abelian group as $\chi(M, N)$ to construct $\mathbf{Z}_n$ for all n for subfactors with condition as in Proposition 4.7. We will do this by the orbifold construction.

Let R be the AFD type II_1 factor. Take a subfactor $N = R \subset M = R \times \mathbf{Z}_m$, where $\mathbf{Z}_m$ acts outerly on R. For n with $(m,n) = 1$, we construct an action of $\mathbf{Z}_n$ on this subfactor as in [Ka3, §3]. Then this simultaneous crossed product $N \times \mathbf{Z}_n \subset M \times \mathbf{Z}_n$ has $\chi = \mathbf{Z}_n$ and satisfies the condition of Proposition 4.7. This is seen as follows.

The principal graph of this simultaneous crossed product is given as in [Ka3, §3]. (Also see [Ch, EK, §1] for this algorithm and [Li2, Lemma 4.2] for a related statement.) Here is an example.

$\boxed{\text{Figure 4.1}}$

Then it is easy to see that the condition on the principal graph is satisfied, and the order of χ is bounded by n by Theorem 4.1. Theorem 1.1 gives computation of χ, but the notes [J2] is unpublished, so we include a sketch of a direct computation here.

It is enough to prove that the dual action of this $\mathbf{Z}_n$ action gives an element of

$$\mathrm{Ct}(M \times \mathbf{Z}_n, N \times \mathbf{Z}_n) \cap \overline{\mathrm{Int}}(M \times \mathbf{Z}_n, N \times \mathbf{Z}_n).$$

Note that the $\mathbf{Z}_n$ action α is centrally free and not approximately inner in the subfactor sense. Then Connes' type argument [C2] produces an M-central sequence (u_n) of unitaries of N with $\alpha(u_n) \sim e^{2\pi i/n} u_n$. Then $\mathrm{Ad}(u_n^*)$ approximate the dual action on $N \times \mathbf{Z}_n \subset M \times \mathbf{Z}_n$. Furthermore, we can prove that $(N^\omega \times \mathbf{Z}_n) \cap (M \times \mathbf{Z}_n)'$ is contained in N^ω using the fact that α is not approximately inner. Thus we are done.

Thus we have proved the following.

Theorem 4.8. *All the finie abelian group are realized as $\chi(M,N)$ for an AFD type II_1 subfactor $N \subset M$ with finite index, finite depth and no non-trivial normalizers.*

§5 Applications to classification of $\mathrm{Aut}(M,N)$

As Connes' $\chi(M)$ gives information on classification of $\mathrm{Aut}(M)$, we get some classification result on $\mathrm{Aut}(M,N)$ using $\chi(M,N)$ easily. (Here we work on the situation of the relative McDuff splitting [Bi, P1] instead of the McDuff splitting [M].) We just list immediate corollaries for the cases index less than 4.

First, note that subfactors $N \subset M$ with the principal graphs A_{2n} or E_8 satisfies $\mathrm{Aut}(M,N) = \overline{\mathrm{Int}}(M,N)$ and $\mathrm{Ct}(M,N) = \mathrm{Int}(M,N)$. Furthermore, $\alpha \in \mathrm{Aut}(M,N)$ is inner on M if and only if it is inner on N. So Connes' classification of automorphisms on single factors works completely, and we get the following Corollary.

Corollary 5.1. *For an AFD type II_1 subfactor $N \subset M$ with principal graph A_{2n} or E_8, outer conjugacy classification of Aut(M,N) is given by outer period and Connes' obstruction.*

Another immediate corollary is the following.

Corollary 5.2. *On an AFD type II_1 factor $N \subset M$ with the principal graph A_{2n+1} $(n > 2)$ or E_6, there are two and only two outer $\mathbf{Z}_2$ actions. One is given by the form $\sigma \otimes id$ on splitting $R \otimes N \subset R \otimes M$, where R is the AFD type II_1 factor, and the other is given by paragroup symmetry.*

Proof. We can consider an asymptotic period as in [C2]. If it is 2, then Connes arguments of [C2] works and produces the splitting $\sigma \otimes id$ as pointed out in [Li1, §4]. If it is 1, it is centrally trivial and thus it must be the orbifold action. Q.E.D.

Note that in the above Corollary, two actions give the different crossed product only in the case A_{4n-3} by [Ka2, §5]. (See also [Ch].) In the case of E_6, there are two subfactors ([BN, O2, Ka2]). It is easy to see that the orbifold action does not change the conjugacy class in the simultaneous $\mathbf{Z}_2$ crossed products.

Also note that if the asymptotic period is odd in the above arguments, method of [C4, page 466] works, and we get a classification up to outer conjugacy.

In the case of D_{2n} with $n > 2$, Loi's invariant may be non-trivial. But if it is trivial, then $\alpha \in \mathrm{Aut}(M, N)$ is in $\overline{\mathrm{Int}}(M, N)$, and $\chi(M, N) = 0$, so the above method for classification works.

References

[ABF] G. E. Andrews, R. J. Baxter, & P. J. Forrester, *Eight vertex SOS model and generalized Rogers-Ramanujan type identities*, J. Stat. Phys. **35** (1984), 193–266.

[Ba] R. J. Baxter, "Exactly solved models in statistical mechanics", Academic Press, New York, 1982.

[BN] J. Bion-Nadal, *An example of a subfactor of the hyperfinite II_1 factor whose principal graph invariant is the Coxeter graph E_6*, in "Current Topics in Operator Algebras", World Scientific Publishing, 1991, pp. 104–113.

[Bi] D. Bisch, *On the existence of central sequences in subfactors*, Trans. Amer. Math. Soc. **321** (1990), 117–128.

[Ch] M. Choda, *Duality for finite bipartite graphs (with applications to II_1 factors)*, Pac. J. Math. **158** (1993), 49–65.

[CK] M. Choda & H. Kosaki, *Strongly outer actions for inclusion of factors*, preprint.

[C1] A. Connes, *Sur la classification des facteurs de type II*, C. R. Acad. Sc. Paris **281** (1975), 13–15.

[C2] A. Connes, *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. École Norm. Sup. **8** (1975), 383–419.

[C3] A. Connes, *Classification of injective factors, Cases* II_1*,* II_∞*,* III_λ*,* $\lambda \neq 1$, Ann. Math. **104** (1976), 73–115.

[C4] A. Connes, *On the classification of von Neumann algebras and their automorphisms*, Symposia Math. XX (1976), 435–478.

[C5] A. Connes, *Periodic automorphisms of the hyperfinite factor of type* II_1, Acta Sci. Math. **39** (1977), 39–66.

[CT] A. Connes & M. Takesaki, *The flow of weights on factors of type III*, Tohoku Math. J. **29** (1977), 473–555.

[DJMO] E. Date, M. Jimbo, T. Miwa, & M. Okado, *Solvable lattice models*, in "Theta functions — Bowdoin 1987, Part 1," Proc. Sympos. Pure Math. Vol. 49, Amer. Math. Soc., Providence, R.I., pp. 295–332.

[DZ] P. Di Francesco & J.-B. Zuber, $SU(N)$ *lattice integrable models associated with graphs*, Nucl. Phys. **B338** (1990), 602–646.

[E1] D. E. Evans, *The* C^**-algebras of topological Markov chains*, Tokyo Metropolitan University Lecture Notes, 1984.

[E2] D. E. Evans, *Quasi-product states on* C^**-algebras*, in "Operator algebras and their connections with topology and ergodic theory", Springer Lecture Notes in Math., 1132 (1985), 129–151.

[EG] D. E. Evans & J. D. Gould, in preparation.

[EK] D. E. Evans & Y. Kawahigashi, *Orbifold subfactors from Hecke algebras*, preprint, 1992.

[F] P. Fendley, *New exactly solvable orbifold models*, J. Phys. A **22** (1989), 4633–4642.

[FG] P. Fendley & P. Ginsparg, *Non-critical orbifolds*, Nucl. Phys. B **324** (1989), 549–580.

[GHJ] F. Goodman, P. de la Harpe, & V. F. R. Jones, Coxeter graphs and towers of algebras, MSRI publications 14, Springer, 1989.

[H] U. Haagerup, in preparation.

[HS] U. Haagerup & E. Størmer, *Pointwise inner automorphisms of von Neumann algebras* with an appendix by C. Sutherland, J. Funct. Anal. **92** (1990), 177–201.

[I1] M. Izumi, *Application of fusion rules to classification of subfactors*, Publ. RIMS Kyoto Univ. **27** (1991), 953–994.

[I2] M. Izumi, *Goldman's type theorem for index 3*, Publ. RIMS Kyoto Univ. **28** (1992), 833–843.

[I3] M. Izumi, *On flatness of the Coxeter graph* E_8, to appear in Pac. J. Math.

[I4] M. Izumi, in prepration.

[IK] M. Izumi & Y. Kawahigashi, *Classification of subfactors with the principal graph* $D_n^{(1)}$, J. Funct. Anal. **112** (1993), 257–286.

[JMO1] M. Jimbo, T, Miwa, & M. Okado, *Solvable lattice models whose states are dominant integral weights of* $A_{n-1}^{(1)}$, Lett. Math. Phys. **14** (1987), 123–131.

[JMO2] M. Jimbo, T, Miwa, & M. Okado, *Solvable lattice models related to the vector representation of classical simple Lie algebras*, Comm. Math. Phys. **116** (1988), 507–525.

[J1] V. F. R. Jones, *Actions of finite groups on the hyperfinite type II_1 factor*, Mem. Amer. Math. Soc. **237** (1980).

[J2] V. F. R. Jones. *Notes on Connes' invariant $\chi(M)$*, unpublished.

[J3] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–15.

[Ka1] Y. Kawahigashi, *Automorphisms commuting with a conditional expectation onto a subfactor with finite index*, (to appear in J. Operator Theory).

[Ka2] Y. Kawahigashi, *On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors*, preprint, 1990.

[Ka3] Y. Kawahigashi, *Exactly solvable orbifold models and subfactors*, in "Functional Analysis and Related Topics", Lect. Notes in Math. (Springer Verlag) **1540** (1992), 127–147.

[KST] Y. Kawahigashi, C. E. Sutherland, & M. Takesaki, *The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions*, Acta Math. **169** (1992), 105–130.

[KT] Y. Kawahigashi & M. Takesaki, *Compact abelian group actions on injective factors*, J. Funct. Anal. **105** (1992), 112–128.

[Ko] H. Kosaki, *Automorphisms in irreducible decompositions of sectors*, preprint.

[Kt] I. Kostov, *Free field presentation of the A_n coset models on the torus*, Nucl. Phys. B **300** (1988), 559–587.

[Li1] P. H. Loi, *On automorphisms of subfactors*, preprint, 1990, UCLA.

[Li2] P. H. Loi, *On the derived tower of certain inclusions of type III_λ factors of index 4*, to appear in Pac. J. Math.

[Ln1] R. Longo, *Simple injective subfactors*, Adv. Math. **63** (1987), 152–171.

[Ln2] R. Longo, *Index of subfactors and statistics of quantum fields I*, Comm. Math. Phys. **126** (1989), 217–247.

[Ln3] R. Longo, *Index of subfactors and statistics of quantum fields II*, Comm. Math. Phys. **130** (1990), 285–309.

[M] D. McDuff, *Central sequences and the hyperfinite factor*, Proc. London Math. **21** (1970), 443–461.

[O1] A. Ocneanu, "Actions of discrete amenable groups on factors," Lecture Notes in Math. No. 1138, Springer, Berlin, 1985.

[O2] A. Ocneanu, *Quantized group string algebras and Galois theory for algebras*, in "Operator algebras and applications, Vol. 2 (Warwick, 1987)," London Math. Soc. Lect. Note Series Vol. 136, Cambridge University Press, 1988, pp. 119–172.

[O3] A. Ocneanu, *Graph geometry, quantized groups and nonamenable subfactors*, Lake Tahoe Lectures, June–July, 1989.

[O4] A. Ocneanu, "Quantum symmetry, differential geometry of finite graphs and classification of subfactors", University of Tokyo Seminary Notes 45, (Notes recorded by Y. Kawahigashi), 1991.

[O5] A. Ocneanu, *An invariant coupling between* 3*-manifolds and subfactors, with connections to topological and conformal quantum field theory*, unpublished announcement.

[PP1] M. Pimsner & S. Popa, *Entropy and index for subfactors*, Ann. Scient. Éc. Norm. Sup. **19** (1986), 57–106.

[PP2] M. Pimsner & S. Popa, *Iterating the basic constructions*, Trans. Amer. Math. Soc. **310** (1988), 127–134.

[P1] S. Popa, *Relative dimension, towers of projections and commuting squares of subfactors*, Pac. J. Math. **137** (1989), 181-207.

[P2] S. Popa, *Classification of subfactors: reduction to commuting squares*, Invent. Math. **101** (1990), 19–43.

[P3] S. Popa, *Sur la classification des sousfacteurs d'indice fini du facteur hyperfini*, C. R. Acad. Sc. Paris. **311** (1990), 95–100.

[P4] S. Popa, *Classification of actions of discrete amenable groups on amenable subfactors of of type II*, IHES preprint, 1992.

[R] Ph. Roche, *Ocneanu cell calculus and integrable lattice models*, Comm. Math. Phys. **127** (1990), 395–424.

[Se] Y. Sekine, *Flows of weights of crossed products of type III factors by discrete groups*, Publ. RIMS Kyoto Univ. **26** (1990), 655–666.

[Su] V. S. Sunder, *A model for AF-algebras and a representation of the Jones projections*, J. Operator Theory **18** (1987), 289–301.

[SV] V. S. Sunder & A. K. Vijayarajan, *On the non-occurrence of the Coxeter graphs* β_{2n+1}*,* E_7*,* D_{2n+1} *as principal graphs of an inclusion of* II_1 *factors*, (to appear in Pac. J. Math.).

[ST] C. E. Sutherland & M. Takesaki, *Actions of discrete amenable groups on injective factors of type* III_λ*,* $\lambda \neq 1$, Pacific J. Math. **137** (1989), 405–444.

[W] H. Wenzl, *Hecke algebras of type A and subfactors*, Invent. Math. **92** (1988), 345–383.

[Y] S. Yamagami, *A report on Ocneanu's lecture*, preprint, 1991.