Longo-Rehren subfactors arising from α-induction

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Abstract

We study (dual) Longo-Rehren subfactors $M \otimes M^{\mathrm{opp}} \subset R$ arising from various systems of endomorphisms of M obtained from α-induction for some braided subfactor $N \subset M$. Our analysis provides useful tools to determine the systems of R-R morphisms associated with such Longo-Rehren subfactors, which constitute the "quantum double" systems in an appropriate sense. The key to our analysis is that α-induction produces half-braidings in the sense of Izumi, so that his general theory can be applied. Nevertheless, α-induced systems are in general not braided, and thus our results allow to compute the quantum doubles of (certain) systems without braiding. We illustrate our general results by several examples, including the computation of the quantum double systems for the asymptotic inclusion of the E_8 subfactor as well as its three analogues arising from conformal inclusions of $SU(3)_k$.

1 Introduction

There are various constructions analogous to the quantum double construction of Drinfel′d [8] in subfactor theory. The first of such constructions is Ocneanu's asymptotic inclusion (see e.g. [10]) which produces $M \vee (M' \cap M_\infty) \subset M_\infty$ from a given hyperfinite II_1 subfactor $N \subset M$ with finite index and finite depth. That is, if we compare the system of M-M bimodules (or N-N bimodules) arising from $N \subset M$ and that of M_∞-M_∞ bimodules arising from $M \vee (M' \cap M_\infty) \subset M_\infty$, then the latter can be regarded as a "quantum double" of the former due to its categorical structure. This viewpoint was noticed by Ocneanu in connection to topological quantum field theory of three dimensions, and the categorical meaning of the construction has been recently clarified by Müger [23]. (A general reference for the asymptotic inclusions and topological quantum field theories is [10, Chapter 12]. We actually need a connectedness assumption of a certain graph for the above interpretation of "quantum double." See [10, Theorem 12.29] for a precise statement.) Popa's notion of a symmetric enveloping algebra in [26] also gives a construction of a new subfactor from a given one, and if the initial subfactor $N \subset M$ is hyperfinite, of type II_1, of finite index, and of finite depth, then this construction gives a subfactor isomorphic to the asymptotic inclusion.

Later, Longo and Rehren introduced in [20] another construction of a subfactor from a given system Δ of endomorphisms, which is now called the Longo-Rehren subfactor. Masuda [22] has proved that the asymptotic inclusion and the Longo-Rehren subfactor are essentially the same constructions, though the constructions arise from very different viewpoints and appear rather unrelated at first sight. Izumi [14] has developed a general theory on the structure of sectors associated with Longo-Rehren subfactors. He introduced a notion of half-braiding and showed that the structure of the quantum double system $\mathcal{D}(\Delta)$ is closely related to half-braidings. Namely, any morphism in $\mathcal{D}(\Delta)$ is given by certain extensions of morphisms defined by means of a half-braiding. This extension will be called η-extension in this paper. Moreover, he presents various interesting applications in [15] with calculations involving Ocneanu's tube algebra handled in the setting of Longo-Rehren subfactors.

Longo and Rehren also introduced an extension formula for endomorphisms of a smaller net to a larger net for nets of subfactors in the same paper [20]. Xu [30, 31] obtained various interesting results by using essentially the same construction in connection to conformal inclusions. Two of us [1, 3] systematically analyzed the extension formula of Longo and Rehren for nets of subfactors. It was named α-induction in [1, 2, 3] in order to emphasize structural similarities with the Mackey machinery of induction and restriction of group representations and to distinguish it from the different sector induction, nevertheless. We have further studied α-induction in the very general setting of braided subfactors in [5, 6]. We identified it with Ocneanu's graphical construction of chiral generators and obtained several results by making use of his graphical methods of double triangle algebras. Izumi's work [14] shows that the study of Longo-Rehren subfactors using a half-braiding is somewhat similar to the study of α-induction. Moreover, α-induction produces interesting systems of endomorphisms which come with various half-braidings, as we will demonstrate in this paper. So it is quite natural to study their quantum doubles by means of associated Longo-Rehren subfactors and applying Izumi's general theory, and this is what we propose in this paper.

To be more specific, we start with a subfactor $N \subset M$ with a finite braided system of N-N morphisms allowing us to apply α-induction. Then the two chiral α-inductions arising from the braiding produce chiral systems of M-M morphisms, and together they generate the full induced system. We define a system Δ to be (subsystems of) either the chiral or the full induced system and study their associated Longo Rehren inclusions $M \otimes M^{\mathrm{opp}} \subset R(\Delta)$. We construct half-braidings with respect to such systems Δ for certain classes of endomorphisms, and this enables us to apply Izumi's theory for analyzing the structure of the quantum double system $\mathcal{D}(\Delta)$ which can be given by η-extensions. The important point is that α-induced systems are not braided in general. (They can even be non-commutative [30, 31, 2, 3] and general criteria for non-commutativity were given in [5, 6].) Thus our analysis is aimed at going beyond the computation of quantum doubles of braided systems which has been carried out in [24, 9], and avoiding at the same time complex constructions like the tube algebra used in [15].

In fact, the rich structure of α-induced systems allows to derive fairly concrete results concerning the structure of the quantum doubles. Namely, we derive concrete formulae for the (dimensions of the) intertwiner spaces between various η-extensions. It is crucial that we allow the braiding on the N-N morphisms to be degenerate. However, the situation simplifies considerably whenever this braiding is non-degenerate. For example, in this case the quantum double of the full induced system is given as the direct product of the original N-N system with itself. As a corollary we obtain a new proof of Rehren's recent theorem on "generalized Longo-Rehren subfactors" in a typical case arising from α-induction. Similarly, the quantum double of the chiral system is given by the direct product of the original N-N system with the ambichiral system in the non-degenerate case. However, in the general, degenerate case the situation is more involved. More precisely, the subsystem of degenerate morphisms arranges the direct product of the N-N system with the ambichiral system into orbits whose elements have to be identified whereas fixed points split, so that the quantum double is now some kind of orbifold of the one we would have obtained in the non-degenerate case.

An orbifold phenomenon has been encountered earlier in computations of dual principal graphs and bimodule systems of asymptotic inclusions of $SU(n)_k$ subfactors which correspond to degenerately braided systems [24, 9]. Our results show that the same phenomenon shows up for quantum doubles of (in general not braided) systems arising from α-induction and having its origin in degeneracies of the braiding of the original N-N system. We illustrate this by computing the quantum doubles of several examples arising from conformal inclusions of $SU(2)$ and $SU(3)$. They correspond to the asymptotic inclusions of subfactors with principal graph E_6 and E_8 as well their three analogues from $SU(3)$ conformal inclusions.

This paper is organized as follows. In Section 2 we recall basic facts on α-induction, state our main assumption and review the results of [14] we use in the sequel. In Section 3 we consider the quantum doubles of full induced systems. We introduce half-braidings for the induced morphisms $\alpha_\lambda^\pm$ and obtain formulae for the intertwiner spaces of their η-extensions and finally consider the non-degenerate case. In Section 4 we propose the same analysis for the quantum doubles of chiral systems. Finally we treat examples arising from conformal inclusions in Section 5.

2 Preliminaries

2.1 Braided systems of morphisms and α-induction

Let A and B be type III von Neumann factors. A unital $*$-homomorphism $\rho : A \to B$ is called a B-A morphism. The positive number $d_\rho = [B : \rho(A)]^{1/2}$ is called the statistical dimension of ρ; here $[B : \rho(A)]$ is the Jones index [16] of the subfactor $\rho(A) \subset B$. If ρ and σ are B-A morphisms with finite statistical dimensions, then the vector space of intertwiners

$$\mathrm{Hom}(\rho, \sigma) = \{t \in B : t\rho(a) = \sigma(a)t\,,\ a \in A\}$$

is finite-dimensional, and we denote its dimension by $\langle \rho, \sigma \rangle$. An A-B morphism $\bar\rho$ is a conjugate morphism if there are isometries $r_\rho \in \mathrm{Hom}(\mathrm{id}_A, \bar\rho\rho)$ and $\bar r_\rho \in \mathrm{Hom}(\mathrm{id}_B, \rho\bar\rho)$ such that $\rho(r_\rho)^* \bar r_\rho = d_\rho^{-1} \mathbf{1}_B$ and $\bar\rho(\bar r_\rho)^* r_\rho = d_\rho^{-1} \mathbf{1}_A$. The map $\phi_\rho : B \to A$, $b \mapsto r_\rho^* \bar\rho(b) r_\rho$, is called the (unique) standard left inverse and satisfies

$$\phi_\rho(\rho(a) b \rho(a')) = a \phi_\rho(b) a'\,, \quad a, a' \in A\,, \quad b \in B\,. \tag{1}$$

We work with the setting of [5], i.e. we are working with a type III subfactor and finite system ${}_N\mathcal{X}_N \subset \mathrm{End}(N)$ of braided morphisms which is compatible with the inclusion $N \subset M$. Then the inclusion is in particular forced to have finite Jones index and also finite depth (see e.g. [10]). More precisely, we make the following

Assumption 2.1 Let $N \subset M$ be a type III subfactor together with a finite system of endomorphisms ${}_N\mathcal{X}_N \subset \mathrm{End}(N)$ in the sense of [5, Def. 2.1] which is braided in the sense of [5, Def. 2.2]. For a given subsystem ${}_N\mathcal{Y}_N \subset {}_N\mathcal{X}_N$ we assume that $\theta = \bar\iota\iota \in \Sigma({}_N\mathcal{Y}_N)$ for the injection M-N morphism $\iota : N \hookrightarrow M$ and a conjugate N-M morphism $\bar\iota$.

Here $\Sigma({}_N\mathcal{Y}_N)$ denotes the set of finite sums of morphisms in ${}_N\mathcal{Y}_N$, and we will use a similar notation for other systems.

With the braiding ε on ${}_N\mathcal{X}_N$ and its extension to $\Sigma({}_N\mathcal{X}_N)$ as in [5], one can define the α-induced morphisms $\alpha_\lambda^\pm \in \mathrm{End}(M)$ for $\lambda \in \Sigma({}_N\mathcal{X}_N)$ by the Longo-Rehren formula [20], namely by putting

$$\alpha_\lambda^\pm = \bar\iota^{-1} \circ \mathrm{Ad}(\varepsilon^\pm(\lambda, \theta)) \circ \lambda \circ \bar\iota\,,$$

where $\bar{\iota}$ denotes a conjugate morphism of the injection map $\iota : N \hookrightarrow M$. Then α_λ^+ and α_λ^- extend λ, i.e. $\alpha_\lambda^\pm \circ \iota = \iota \circ \lambda$, which in turn implies $d_{\alpha_\lambda^\pm} = d_\lambda$ by the multiplicativity of the minimal index [19]. Let $\gamma = \iota\bar{\iota}$ denote Longo's canonical endomorphism from M into N. Then there is an isometry $v \in \mathrm{Hom}(\mathrm{id}, \gamma)$ such that any $m \in M$ is uniquely decomposed as $m = nv$ with $n \in N$. Thus the action of the extensions $\alpha_\lambda^\pm$ are uniquely characterized by the relation $\alpha_\lambda^\pm(v) = \varepsilon^\pm(\lambda, \theta)^* v$ which can be derived from the braiding fusion equations (BFE's, see e.g. [5, Eq. (5)]). Moreover, we have $\alpha_{\lambda\mu}^\pm = \alpha_\lambda^\pm \alpha_\mu^\pm$ if also $\mu \in \Sigma({}_N\mathcal{X}_N)$, and clearly $\alpha_{\mathrm{id}_N}^\pm = \mathrm{id}_M$. In general one has

$$\mathrm{Hom}(\lambda, \mu) \subset \mathrm{Hom}(\alpha_\lambda^\pm, \alpha_\mu^\pm) \subset \mathrm{Hom}(\iota\lambda, \iota\mu)\,, \qquad \lambda, \mu \in \Sigma({}_N\mathcal{X}_N)\,. \tag{2}$$

The first inclusion is a consequence of the BFE's. Namely, $t \in \mathrm{Hom}(\lambda, \mu)$ obeys $t\varepsilon^\pm(\theta, \lambda) = \varepsilon^\pm(\theta, \mu)\theta(t)$, and thus

$$t\alpha_\lambda^\pm(v) = t\varepsilon^\pm(\lambda, \theta)^* v = \varepsilon^\pm(\mu, \theta)^* \theta(t) v = \varepsilon^\pm(\mu, \theta)^* v t = \alpha_\mu^\pm(v) t\,.$$

The second follows from the extension property of α-induction. Hence $\alpha_{\bar{\lambda}}^\pm$ is a conjugate for $\alpha_\lambda^\pm$ as there are $r_\lambda \in \mathrm{Hom}(\mathrm{id}, \bar{\lambda}\lambda) \subset \mathrm{Hom}(\mathrm{id}, \alpha_{\bar{\lambda}}^\pm \alpha_\lambda^\pm)$ and $\bar{r}_\lambda \in \mathrm{Hom}(\mathrm{id}, \lambda\bar{\lambda}) \subset \mathrm{Hom}(\mathrm{id}, \alpha_\lambda^\pm \alpha_{\bar{\lambda}}^\pm)$ such that $\lambda(r_\lambda)^* \bar{r}_\lambda = \bar{\lambda}(\bar{r}_\lambda)^* r_\lambda = d_\lambda^{-1}\mathbf{1}$. We also have some kind of naturality equations for α-induced morphisms,

$$x\varepsilon^\pm(\rho, \lambda) = \varepsilon^\pm(\rho, \mu)\alpha_\rho^\pm(x) \tag{3}$$

whenever $x \in \mathrm{Hom}(\iota\lambda, \iota\mu)$, $\rho \in \Sigma({}_N\mathcal{X}_N)$.

Recall that the statistics phase of ω_λ for $\lambda \in {}_N\mathcal{X}_N$ is given as

$$d_\lambda \phi_\lambda(\varepsilon^+(\lambda, \lambda)) = \omega_\lambda \mathbf{1}\,.$$

The monodromy matrix Y is defined by

$$Y_{\lambda,\mu} = \sum_{\rho \in {}_N\mathcal{X}_N} \frac{\omega_\lambda \omega_\mu}{\omega_\rho} N_{\lambda,\mu}^\rho d_\rho\,, \qquad \lambda, \mu \in {}_N\mathcal{X}_N\,, \tag{4}$$

with $N_{\lambda,\mu}^\rho = \langle \rho, \lambda\mu \rangle$ denoting the fusion coefficients. Then one checks that Y is symmetric, that $Y_{\bar{\lambda},\mu} = Y_{\lambda,\mu}^*$ as well as $Y_{\lambda,0} = d_\lambda$ [27, 12, 11]. (As usual, the label "0" refers to the identity morphism $\mathrm{id} \in {}_N\mathcal{X}_N$.) Now let Ω be the diagonal matrix with entries $\Omega_{\lambda,\mu} = \omega_\lambda \delta_{\lambda,\mu}$. Putting

$$Z_{\lambda,\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle\,, \qquad \lambda, \mu \in {}_N\mathcal{X}_N\,, \tag{5}$$

defines a matrix subject to the constraints

$$Z_{\lambda,\mu} = 0, 1, 2, \ldots\,, \qquad \text{and} \qquad Z_{0,0} = 1\,,$$

and commuting with Y and Ω [5]. The Y- and Ω-matrices obey $\Omega Y \Omega Y \Omega = zY$ where $z = \sum_\lambda d_\lambda^2 \omega_\lambda$ [27, 12, 11], and this actually holds even if the braiding is degenerate (see [5, Sect. 2]). If $z \neq 0$ we put $c = 4\arg(z)/\pi$, which is defined modulo 8, and call it the "central charge". Moreover, S- and T-matrices are then defined by

$$S = |z|^{-1} Y \,, \qquad T = \mathrm{e}^{-\mathrm{i}\pi c/12} \Omega$$

and hence fulfill $TSTST = S$. One has $|z|^2 = [[{}_N\mathcal{X}_N]]$ with the global index $[[{}_N\mathcal{X}_N]] = \sum_\lambda d_\lambda^2$ and S is unitary, so that S and T are indeed the standard generators in a unitary representation of the modular group $SL(2;\mathbb{Z})$, if and only if the braiding is non-degenerate [27]. Consequently, Z gives a modular invariant in this case.

Let ${}_M\mathcal{X}_M \subset \mathrm{End}(M)$ denote a system of endomorphisms consisting of a choice of representative endomorphisms of each irreducible subsector of sectors of the form $[\iota\lambda\bar{\iota}]$, $\lambda \in {}_N\mathcal{X}_N$. We choose $\mathrm{id} \in \mathrm{End}(M)$ representing the trivial sector in ${}_M\mathcal{X}_M$. Then we define similarly the chiral systems ${}_M\mathcal{X}_M^\pm$ and the α-system ${}_M\mathcal{X}_M^\alpha$ to be the subsystems of endomorphisms $\beta \in {}_M\mathcal{X}_M$ such that $[\beta]$ is a subsector of $[\alpha_\lambda^\pm]$ and of $[\alpha_\lambda^+\alpha_\mu^-]$, respectively, for some $\lambda,\mu \in {}_N\mathcal{X}_N$. (Note that any subsector of $[\alpha_\lambda^+\alpha_\mu^-]$ is automatically a subsector of $[\iota\nu\bar{\iota}]$ for some $\nu \in {}_N\mathcal{X}_N$.) The ambichiral system is defined as the intersection ${}_M\mathcal{X}_M^0 = {}_M\mathcal{X}_M^+ \cap {}_M\mathcal{X}_M^-$, so that ${}_M\mathcal{X}_M^0 \subset {}_M\mathcal{X}_M^\pm \subset {}_M\mathcal{X}_M^\alpha \subset {}_M\mathcal{X}_M$. Thus their "global indices", i.e. the sums over the squares of the statistical dimensions of their morphisms, fulfill $1 \le [[{}_M\mathcal{X}_M^0]] \le [[{}_M\mathcal{X}_M^\pm]] \le [[{}_M\mathcal{X}_M^\alpha]] \le [[{}_M\mathcal{X}_M]] = [[{}_N\mathcal{X}_N]]$. (Throughout this paper we denote the global index of a system by use of double rectangular brackets.)

Let us now consider the subsystem ${}_N\mathcal{Y}_N$ appearing in Assumption 2.1. If the inclusion ${}_N\mathcal{Y}_N \subset {}_N\mathcal{X}_N$ is proper, then we may play the same game considering α-induction for exclusively $\lambda \in {}_N\mathcal{Y}_N$. This way we will obtain α-induced systems which are contained in the α-induced systems associated to ${}_N\mathcal{X}_N$, i.e. we have the following scheme of inclusions:

$$\begin{array}{ccccccc}
{}_M\mathcal{X}_M^0 & \subset & {}_M\mathcal{X}_M^\pm & \subset & {}_M\mathcal{X}_M^\alpha & \subset & {}_M\mathcal{X}_M \\
\cup & & \cup & & \cup & & \cup \\
{}_M\mathcal{Y}_M^0 & \subset & {}_M\mathcal{Y}_M^\pm & \subset & {}_M\mathcal{Y}_M^\alpha & \subset & {}_M\mathcal{Y}_M
\end{array}$$

We will use these systems for the construction of Longo-Rehren subfactors and for the analysis of sectors associated to them. We are particularly interested in examples where (at least) the braiding on the subsystem ${}_N\mathcal{Y}_N$ may be degenerate. Let ${}_N\mathcal{Y}_N^{\mathrm{deg}}$ denote the system of degenerate morphisms, i.e.

$${}_N\mathcal{Y}_N^{\mathrm{deg}} = \{\nu \in {}_N\mathcal{Y}_N \mid \varepsilon^+(\nu,\rho) = \varepsilon^-(\nu,\rho) \quad \text{for all} \quad \rho \in {}_N\mathcal{Y}_N\} \,.$$

Clearly, the braiding on ${}_N\mathcal{Y}_N$ is non-degenerate (in the sense of [27] or [5, Def. 2.3]) if and only if ${}_N\mathcal{Y}_N^{\mathrm{deg}} = \{\mathrm{id}\}$. Note that since θ decomposes by Assumption 2.1 only into morphisms of ${}_N\mathcal{Y}_N$ and since $\alpha_\lambda^\pm(v) = \varepsilon^\pm(\lambda,\theta)^* v$ for any $\lambda \in {}_N\mathcal{X}_N$ we find $\alpha_\rho^+ = \alpha_\rho^-$ whenever $\rho \in {}_N\mathcal{Y}_N^{\mathrm{deg}}$. Finally we introduce

$${}_N\mathcal{Y}_N^{\mathrm{per}} = \{\lambda \in {}_N\mathcal{X}_N \mid \varepsilon^+(\lambda,\rho) = \varepsilon^-(\lambda,\rho) \quad \text{for all} \quad \rho \in {}_N\mathcal{Y}_N\}$$

and call it the *relative permutant* of ${}_N\mathcal{Y}_N$ in ${}_N\mathcal{X}_N$. Clearly, $\alpha_\lambda^+ = \alpha_\lambda^-$ whenever $\lambda \in {}_N\mathcal{Y}_N^{\mathrm{per}}$.

2.2 Longo-Rehren subfactors, half-braidings and η-extensions

Let M be a type III factor with a finite system $\Delta \subset \mathrm{End}(M)$ of endomorphisms. Let M^{opp} denote the opposite algebra of M and consider $M \otimes M^{\mathrm{opp}}$. By constructing a "Q-system", Longo and Rehren showed in [20, Prop. 4.10] that there is a (type III) subfactor $B \subset M \otimes M^{\mathrm{opp}}$ with canonical endomorphism $\Theta \in \mathrm{End}(M \otimes M^{\mathrm{opp}})$ decomposing as a sector as

$$[\Theta] = \bigoplus_{\beta \in \Delta} [\beta \otimes \beta^{\mathrm{opp}}] \,.$$

Here $\beta^{\mathrm{opp}} = j \circ \beta \circ j^{-1}$ where $j : M \to M^{\mathrm{opp}}$ is the anti-linear isomorphism. The subfactor $B \subset M \otimes M^{\mathrm{opp}}$ is now called the Longo-Rehren subfactor. For reasons of convenience, we consider in this paper the dual subfactor $M \otimes M^{\mathrm{opp}} \subset R$ and call it the Longo-Rehren subfactor as well. (This convention is compatible with [17].) That is, $B \subset M \otimes M^{\mathrm{opp}} \subset R$ is a Jones extension and Θ is then the dual canonical endomorphism of $M \otimes M^{\mathrm{opp}} \subset R$.

The following is a slight variation of Izumi's definition [14, Def. 4.2] of a half-braiding.

Definition 2.2 Let Φ be a system of morphisms in $\mathrm{End}(M)$ and $\Delta \subset \Phi$ a subsystem. For $\sigma \in \Sigma(\Phi)$ we call a family of unitary operators $\mathcal{E}_\sigma = \{\mathcal{E}_\sigma(\beta)\}_{\beta \in \Delta}$ a *half-braiding* of σ with respect to Δ if it satisfies the following two conditions:

1. $\mathcal{E}_\sigma(\beta) \in \mathrm{Hom}(\sigma\beta, \beta\sigma)$ for all $\beta \in \Delta$.
2. Whenever $\beta_1, \beta_2, \beta_3 \in \Delta$ then

   $$X \mathcal{E}_\sigma(\beta_3) = \beta_1(\mathcal{E}_\sigma(\beta_2)) \mathcal{E}_\sigma(\beta_1) \sigma(X)$$

   holds for every $X \in \mathrm{Hom}(\beta_3, \beta_1\beta_2)$,

Two pairs $(\sigma, \mathcal{E}_\sigma)$, $(\sigma', \mathcal{E}'_{\sigma'})$ of morphisms $\sigma, \sigma' \in \Sigma(\Phi)$ with respective half-braidings $\mathcal{E}_\sigma, \mathcal{E}'_{\sigma'}$ are said to be *equivalent* if there is unitary $u \in \mathrm{Hom}(\sigma', \sigma)$ such that

$$\mathcal{E}_\sigma(\beta) = \beta(u) \mathcal{E}'_\sigma(\beta) u^*$$

for all $\beta \in \Delta$.

Note that our definition of equivalence is slightly more general than the one in [14, Def. 4.2] because we choose the σ's from a generically larger set $\Phi \supset \Delta$. We then define an extension $\eta(\sigma, \mathcal{E}_\sigma)$ of the endomorphism $\sigma \otimes \mathrm{id}$ of $M \otimes M^{\mathrm{opp}}$ to R as in the following definition, which is just the dual version of Izumi's definition of $\widetilde{(\sigma, \mathcal{E}_\sigma)}$ in [14, Def. 4.4]. This extension is somewhat similar to α-induction. Izumi's important observation is that we need only "half" the properties of a usual braiding for this extension. We need some preparation. Let $W_\beta \in \mathrm{Hom}(\beta\otimes\beta^{\mathrm{opp}},\Theta)$, $\beta\in\Delta$, be isometries so that $W_\beta^* W_{\beta'} = \delta_{\beta,\beta'}\mathbf{1}$ and $\sum_{\beta\in\Delta} W_\beta W_\beta^* = \mathbf{1}$. (Note that for a Longo-Rehren subfactor with given Θ each W_β is unique up to a phase.) Let $\iota_{\mathrm{LR}} : M\otimes M^{\mathrm{opp}} \hookrightarrow R$ denote the inclusion homomorphism so that the dual canonical endomorphism is given by $\Theta = \bar\iota_{\mathrm{LR}}\iota_{\mathrm{LR}}$, and then $\Gamma = \iota_{\mathrm{LR}}\bar\iota_{\mathrm{LR}}$ is a canonical endomorphism. Then there is [18] an isometry $V \in \mathrm{Hom}(\mathrm{id},\Gamma)$ such that $W_{\mathrm{id}}^* V = [R : M\otimes M^{\mathrm{opp}}]^{-1/2}\mathbf{1}$, and note that $[R : M\otimes M^{\mathrm{opp}}] = \sum_{\beta\in\Delta} d_\beta^2$. Moreover, for each $X\in R$ there is a unique $a\in M\otimes M^{\mathrm{opp}}$ such that $X = aV$.

Definition 2.3 For $\sigma\in\Sigma(\Phi)$ with a half-braiding $\mathcal{E}_\sigma = \{\mathcal{E}_\sigma(\beta)\}_{\beta\in\Delta}$, we define an extension $\eta(\sigma,\mathcal{E}_\sigma)\in\mathrm{End}(R)$ by putting

$$\begin{aligned} \eta(\sigma,\mathcal{E}_\sigma)(a) &= (\sigma\otimes\mathrm{id})(a), \quad a\in M\otimes M^{\mathrm{opp}}, \\ \eta(\sigma,\mathcal{E}_\sigma)(V) &= U(\sigma,\mathcal{E}_\sigma)^* V, \end{aligned} \tag{6}$$

where the unitary $U(\sigma,\mathcal{E}_\sigma)$ is defined as

$$U(\sigma,\mathcal{E}_\sigma) = \sum_{\beta\in\Delta} W_\beta(\mathcal{E}_\sigma(\beta)\otimes 1)(\sigma\otimes\mathrm{id}^{\mathrm{opp}})(W_\beta^*)\,. \tag{7}$$

Using

$$U^{\mathrm{opp}}(\sigma,\mathcal{E}_\sigma) = \sum_\beta W_\beta(1\otimes j(\mathcal{E}_\sigma(\beta)))(\mathrm{id}\otimes\sigma^{\mathrm{opp}})(W_\beta^*) \tag{8}$$

we similarly define an extension $\eta^{\mathrm{opp}}(\sigma,\mathcal{E}_\sigma)\in\mathrm{End}(R)$ of $\mathrm{id}\otimes\sigma^{\mathrm{opp}}$.

Let $\mathcal{D}(\Delta)$ be the system of irreducible endomorphisms of R arising from a choice of representative morphisms of irreducible subsectors of $\iota_{\mathrm{LR}}\circ\beta'\otimes\beta^{\mathrm{opp}}\circ\bar\iota_{\mathrm{LR}}$ for $\beta,\beta'\in\Delta$. Following [14, Def. 4.4], we call $\mathcal{D}(\Delta)$ the *quantum double system* of Δ. (Note that Izumi's notation $D(\Delta)$ for the quantum double includes reducible morphisms and thus corresponds to $\Sigma(\mathcal{D}(\Delta))$. Also note that the system $\mathcal{D}(\Delta)$ may be strictly larger than that arising from the Longo-Rehren subfactor $M\otimes M^{\mathrm{opp}}\subset R$ in the usual sense, i.e. arising from the decomposition of all powers of Γ. See Remark after [14, Thm. 4.6].) Izumi has proved in [14, Lemma 4.5, Thm. 4.6] that $\eta(\sigma,\mathcal{E})$ gives an endomorphism in $\Sigma(\mathcal{D}(\Delta))$ if we consider $\sigma\in\Sigma(\Delta)$ only, and then any endomorphism in $\Sigma(\mathcal{D}(\Delta))$ arises in this way. Note that this will no longer be true if we consider generic $\sigma\in\Sigma(\Phi)$.

The following is nothing but Izumi's [14, Thm. 4.6 (ii)]. We only provide a proof for the reader's convenience and in order to demonstrate that the arguments are the same though we work in a picture dual to Izumi's and extend $\sigma\in\Sigma(\Phi)\supset\Sigma(\Delta)$.

Theorem 2.4 *Let $\sigma,\sigma'\in\Sigma(\Phi)$ with half-braidings $\mathcal{E}_\sigma = \{\mathcal{E}_\sigma(\beta)\}_{\beta\in\Delta}$, $\mathcal{E}'_{\sigma'} = \{\mathcal{E}'_{\sigma'}(\beta)\}_{\beta\in\Delta}$. Then we have*

$$\begin{aligned} &\mathrm{Hom}(\eta(\sigma,\mathcal{E}_\sigma),\eta(\sigma',\mathcal{E}'_{\sigma'})) = \\ &\qquad = \{X\otimes\mathbf{1} \mid X\in\mathrm{Hom}(\sigma,\sigma')\,,\ \mathcal{E}'_{\sigma'}(\beta)X = \beta(X)\mathcal{E}_\sigma(\beta) \text{ for all } \beta\in\Delta\}\,. \end{aligned} \tag{9}$$

In particular, $\eta(\sigma,\mathcal{E}_\sigma)$ and $\eta(\sigma',\mathcal{E}'_{\sigma'})$ are unitarily equivalent as morphisms of R if and only if pairs $(\sigma,\mathcal{E}_\sigma)$ and $(\sigma',\mathcal{E}'_{\sigma'})$ are equivalent in the sense of Definition 2.2.

Proof. Let $T \in \operatorname{Hom}(\eta(\sigma,\mathcal{E}_\sigma),\eta(\sigma',\mathcal{E}'_{\sigma'}))$. Then it is decomposed as $T = aV$ with $a \in \operatorname{Hom}(\Theta\circ(\sigma\otimes\mathrm{id}^{\mathrm{opp}}),\sigma'\otimes\mathrm{id}^{\mathrm{opp}})$. Consequently $aW_\beta \in \operatorname{Hom}(\beta\sigma\otimes\beta^{\mathrm{opp}}),\sigma'\otimes\mathrm{id}^{\mathrm{opp}})$ can be non-zero only for $\beta = \mathrm{id}$. Hence $a = bW^*_{\mathrm{id}}$ with

$$b = aW_{\mathrm{id}} \in \operatorname{Hom}(\sigma\otimes\mathrm{id}^{\mathrm{opp}},\sigma'\otimes\mathrm{id}^{\mathrm{opp}}) = \{X\otimes\mathbf{1} \mid X \in \operatorname{Hom}(\sigma,\sigma')\}\,.$$

Since $W^*_{\mathrm{id}}V$ is a (non-zero) scalar we have found $T \in \{X\otimes\mathbf{1} \mid X \in \operatorname{Hom}(\sigma,\sigma')\}$. For such a $T = X\otimes\mathbf{1}$ the condition $TU(\sigma,\mathcal{E}_\sigma)^*V = U(\sigma',\mathcal{E}'_{\sigma'})^*VT$ is equivalent to $\Theta(X\otimes\mathbf{1})U(\sigma,\mathcal{E}_\sigma) = U(\sigma',\mathcal{E}'_{\sigma'})X\otimes\mathbf{1}$. Sandwiching with W^*_β and $(\sigma\otimes\mathrm{id}^{\mathrm{opp}})(W_\beta)$ gives the desired intertwining relations for all $\beta\in\Delta$. Conversely, any $T = X\otimes\mathbf{1}$ with $X \in \operatorname{Hom}(\sigma,\sigma)$ satisfying these relations intertwines $\eta(\sigma,\mathcal{E}_\sigma)$ and $\eta(\sigma',\mathcal{E}'_{\sigma'})$. $\square$

Since $\eta(\sigma,\mathcal{E}_\sigma)$ is an extension of $\sigma\otimes\mathrm{id}^{\mathrm{opp}}$ and since $[R : M\otimes M^{\mathrm{opp}}] < \infty$ we also find that its statistical dimension is d_σ, i.e. η preserves statistical dimensions. We have even more than that. Namely, for pairs $(\sigma,\mathcal{E}_\sigma)$ as above, we have natural notions of addition and multiplication extending those of the endomorphisms σ. Let $\sigma_i \in \Sigma(\Phi)$ with half-braidings $\mathcal{E}^i_{\sigma_i}$, $i = 1,2,...,n$. Let $\{t_i\}_{i=1}^n$ be a set of isometries in M satisfying the Cuntz relations and let $\sigma \in \Sigma(\Phi)$ be given by $\sigma(m) = \sum_i t_i\sigma_i(m)t_i^*$ for all $m \in M$. It is routine to show that putting

$$\mathcal{E}_\sigma(\beta) = \sum_{i=1}^n \beta(t_i)\mathcal{E}^i_{\sigma_i}(\beta)t_i^*\,, \qquad \beta\in\Delta\,,$$

defines a half-braiding for σ. Similarly, putting

$$\mathcal{E}'_{\sigma'}(\beta) = \mathcal{E}^1_{\sigma_1}(\beta)\sigma_1(\mathcal{E}^2_{\sigma_2}(\beta))\,, \qquad \beta\in\Delta\,,$$

defines a half-braiding $\{\mathcal{E}'_{\sigma'}(\beta)\}_{\beta\in\Delta}$ of products $\sigma' = \sigma_1\sigma_2$, as used [14]. It is straightforward to show that we have exact multiplicativity for the η-extensions,

$$\eta(\sigma',\mathcal{E}_{\sigma'}) = \eta(\sigma_1,\mathcal{E}^1_{\sigma_1})\eta(\sigma_2,\mathcal{E}^2_{\sigma_2})\,,$$

with this product half-braiding. Finally, conjugates were defined in [14, Thm. 4.6 (iv)] as follows. For a pair $(\sigma,\mathcal{E}_\sigma)$, operators

$$\bar{\mathcal{E}}_{\bar\sigma}(\beta) = d_\sigma R^*_\sigma\bar\sigma(\mathcal{E}_\sigma(\beta)^*\beta(\bar R_\sigma))\,, \qquad \beta\in\Delta\,,$$

where $R_\sigma \in \operatorname{Hom}(\mathrm{id},\bar\sigma\sigma)$, $\bar R_\sigma \in \operatorname{Hom}(\mathrm{id},\sigma\bar\sigma)$ are isometries with $\bar R^*_\sigma\sigma(R_\sigma) = R^*_\sigma\bar\sigma(\bar R_\sigma) = d_\sigma^{-1}$, give a half-braiding for the conjugate morphism $\bar\sigma$. The half-braiding $\{\bar{\mathcal{E}}_{\bar\sigma}(\beta)\}_{\beta\in\Delta}$ depends on the choices of $R_\sigma, \bar R_\sigma$ in general, however, its equivalence class does not [14]. Then Izumi's results give the following

Proposition 2.5 *The extension map $\eta : (\sigma,\mathcal{E}_\sigma) \to \eta(\sigma,\mathcal{E}_\sigma)$, regarded as a map from equivalence classes of pairs to sectors of R, preserves the operations of addition, multiplication, and conjugates.*

Proof. The preservation of addition and the multiplication is a straight-forward corollary of Theorem 2.4. The statement for the conjugates is derived in the same way as [14, Thm. 4.6 (iv)]. □

Next, [14, Prop. 6.4] gives the following

Proposition 2.6 *For a pair $\sigma \in \Sigma(\Phi)$ with a half-braiding $\{\mathcal{E}_\sigma(\beta)\}_{\beta\in\Delta}$, the extensions $\eta(\sigma,\mathcal{E}_\sigma)$ and $\eta^{\mathrm{opp}}(\bar\sigma,\bar{\mathcal{E}}_{\bar\sigma})$ are unitarily equivalent.*

Finally, [14, Thm. 4.1] and the remark at the end of [14, Sect. 4] give the following

Proposition 2.7 *Let $\mathcal{G}$ be the bipartite graph with odd vertices labelled by Δ and even vertices labelled $\mathcal{D}(\Delta)$, and the number of edges between a vertex labelled by $\beta \in \Delta$ and a vertex labelled by $\Omega \in \mathcal{D}(\Delta)$ such that $[\Omega] = [\eta(\sigma,\mathcal{E}_\sigma)]$ for $\sigma \in \Sigma(\Delta)$ with some half-braiding $\mathcal{E}_\sigma$ is given by $\langle\beta,\sigma\rangle$. Then the connected component $\mathcal{G}_0$ of $\mathcal{G}$ containing* id $\in \Delta$ *is the dual principal graph of the inclusion $M \otimes M^{\mathrm{opp}} \subset R$.*

This completes our review of [14].

3 Quantum doubles of full induced systems

In this section we study Longo-Rehren subfactors $M \otimes M^{\mathrm{opp}} \subset R(\Delta)$ arising from the system $\Delta = {}_M\mathcal{Y}_M^\alpha$, the full α-induced system associated to the subsystem ${}_N\mathcal{Y}_N \subset {}_N\mathcal{X}_N$. In order to proceed with η-extensions we first introduce some half-braidings.

For $\beta \in {}_M\mathcal{X}_M^\alpha$ choose an isometry $T \in \mathrm{Hom}(\beta, \alpha_\nu^+\alpha_{\nu'}^-)$ with some $\nu,\nu' \in {}_N\mathcal{X}_N$. (These exist by definition.) For any $\lambda \in \Sigma({}_N\mathcal{X}_N)$ we now put

$$\mathcal{E}_\lambda^\pm(\beta) = T^*\varepsilon^\pm(\lambda,\nu\nu')\alpha_\lambda^\pm(T) \tag{10}$$

We then have the following

Lemma 3.1 *The operators $\mathcal{E}_\lambda^\pm(\beta)$ are independent of the choice of T and ν,ν' in the sense that, if $\xi,\xi' \in {}_N\mathcal{X}_N$ and $S \in \mathrm{Hom}(\beta,\alpha_\xi^+\alpha_{\xi'}^-)$ is an isometry, then $\mathcal{E}_\lambda^\pm(\beta) = S^*\varepsilon^\pm(\lambda,\xi\xi')\alpha_\lambda^\pm(S)$. Moreover, for each $\lambda \in \Sigma({}_N\mathcal{X}_N)$, the family $\{\mathcal{E}_\lambda^\pm(\beta)\}_{\beta\in\Phi}$ is a half-braiding for the morphism $\alpha_\lambda^\pm$ with respect to the system $\Phi = {}_M\mathcal{X}_M^\alpha$.*

Proof. Note that if $\beta \in {}_M\mathcal{X}_M^\alpha$, $\nu,\nu' \in {}_N\mathcal{X}_N$ and $T \in \mathrm{Hom}(\beta,\alpha_\nu^+\alpha_{\nu'}^-)$ is an isometry, then $TT^* \in \mathrm{Hom}(\iota\nu\nu',\iota\nu\nu')$ since $\alpha_\nu^+\alpha_{\nu'}^-\iota = \iota\nu\nu'$. Hence $TT^*\varepsilon^\pm(\lambda,\nu\nu') = \varepsilon^\pm(\lambda,\nu\nu')\alpha_\lambda^\pm(TT^*)$ for any $\lambda \in \Sigma({}_N\mathcal{X}_N)$. With this it is easy to check that $\mathcal{E}_\lambda^\pm(\beta)$ is unitary. The first inclusion of Eq. (2) together with [1, Lemma 3.24] imply that $\varepsilon^\pm(\lambda,\nu\nu')$ is an intertwiner from $\alpha_\lambda^\pm\alpha_\nu^+\alpha_{\nu'}^-$ to $\alpha_\nu^+\alpha_{\nu'}^-\alpha_\lambda^\pm$. With that it is easy to check that $\mathcal{E}_\lambda^\pm(\beta) \in \mathrm{Hom}(\alpha_\lambda^\pm\beta,\beta\alpha_\lambda^\pm)$ (cf. the proof of [3, Lemma 3.20]). Next, for $\beta_j \in {}_M\mathcal{X}_M^\alpha$, $\nu_j,\nu_j' \in {}_N\mathcal{X}_N$ and $T_j \in \mathrm{Hom}(\beta_j,\alpha_{\nu_j}^+\alpha_{\nu_j'}^-)$ isometries, $j = 1,2,3$, and

$X \in \mathrm{Hom}(\beta_3, \beta_1\beta_2)$ one has $\alpha^+_{\nu_1}\alpha^-_{\nu'_1}(T_2)T_1XT_3^* \in \mathrm{Hom}(\iota\nu_3\nu'_3, \iota\nu_1\nu'_1\nu_2\nu'_2)$, and hence we can compute

$$\begin{aligned} X\mathcal{E}^\pm_\lambda(\beta_3) &= XT_3^*\varepsilon^\pm(\lambda,\nu_3\nu'_3)\alpha^\pm_\lambda(T_3) = T_1^*\alpha^+_{\nu_1}\alpha^-_{\nu'_1}(T_2^*T_2)T_1XT_3^*\varepsilon^\pm(\lambda,\nu_3\nu'_3)\alpha^\pm_\lambda(T_3) \\ &= T_1^*\alpha^+_{\nu_1}\alpha^-_{\nu'_1}(T_2)^*\nu_1\nu'_1(\varepsilon^\pm(\lambda,\nu_2\nu'_2))\varepsilon^\pm(\lambda,\nu_1\nu'_1)\alpha^\pm_\lambda(\alpha^+_{\nu_1}\alpha^-_{\nu'_1}(T_2)T_1X) \\ &= T_1^*\alpha^+_{\nu_1}\alpha^-_{\nu'_1}(T_2)^*\nu_1\nu'_1(\varepsilon^\pm(\lambda,\nu_2\nu'_2))\alpha^+_{\nu_1}\alpha^-_{\nu'_1}\alpha^\pm_\lambda(T_2)\varepsilon^\pm(\lambda,\nu_1\nu'_1)\alpha^\pm_\lambda(T_1X) \qquad , \\ &= \beta_1(T_2^*\varepsilon^\pm(\lambda,\nu_2\nu'_2)\alpha^\pm_\lambda(T_2)T_1^*\varepsilon^\pm(\lambda,\nu_1\nu'_1)\alpha^\pm_\lambda(T_1X) \\ &= \beta_1(\mathcal{E}^\pm_\lambda(\beta_2))\mathcal{E}^\pm_\lambda(\beta_1)\alpha^\pm_\lambda(X) \end{aligned}$$

establishing 2. of Definition 2.2. Finally, putting $\nu_2 = \nu_2 = \mathrm{id}$ so that consequently $\beta_2 = \mathrm{id}$ and $T_2 = \mathbf{1}$, and choosing $X = \mathbf{1}$ gives the desired invariance properties of $\mathcal{E}_\lambda(\beta)$ with $\beta = \beta_1 = \beta_2$. □

Restricting the half-braidings to $\Delta = {}_M\mathcal{Y}^\alpha_M \subset {}_M\mathcal{X}^\alpha_M = \Phi$, i.e. putting $\mathcal{E}^\pm_\lambda = \{\mathcal{E}^\pm_\lambda(\beta)\}_{\beta\in {}_M\mathcal{Y}^\alpha_M}$, we conclude that there are extensions $\eta(\alpha^\pm_\lambda, \mathcal{E}^\pm_\lambda)$ whenever $\lambda \in \Sigma({}_N\mathcal{X}_N)$. Note that

$$\mathcal{E}^\pm_\lambda(\beta)\alpha^\pm_\lambda(\mathcal{E}^\pm_\mu(\beta)) = T^*\varepsilon^\pm(\lambda,\nu\nu')\alpha^\pm_\lambda(TT^*)\lambda(\varepsilon^\pm(\mu,\nu\nu')\alpha^\pm_{\lambda\mu}(T)) = \mathcal{E}^\pm_{\lambda\mu}(\beta)$$

for all $\beta \in {}_M\mathcal{X}^\alpha_M$, and consequently

$$\eta(\alpha^\pm_{\lambda\mu}, \mathcal{E}^\pm_{\lambda\mu})) = \eta(\alpha^\pm_\lambda, \mathcal{E}^\pm_\lambda))\eta(\alpha^\pm_\mu, \mathcal{E}^\pm_\mu)) \tag{11}$$

for all $\lambda, \mu \in \Sigma({}_N\mathcal{X}_N)$.

We now state an inclusion of intertwiner spaces which is similar to the first inclusion in Eq. (2).

Lemma 3.2 *We have*

$$\mathrm{Hom}(\lambda,\mu)\otimes\mathbf{1} \subset \mathrm{Hom}(\eta(\alpha^\pm_\lambda, \mathcal{E}^\pm_\lambda), \eta(\alpha^\pm_\mu, \mathcal{E}^\pm_\mu)) \tag{12}$$

for any $\lambda, \mu \in \Sigma({}_N\mathcal{X}_N)$.

Proof. Thanks to Izumi's result, Theorem 2.4, and due to the first inclusion in Eq. (2), all what we have to verify is the relation $\beta(x)\mathcal{E}^\pm_\lambda(\beta) = \mathcal{E}^\pm_\mu(\beta)x$ for all $\beta \in {}_M\mathcal{Y}^\alpha_M$ whenever $x \in \mathrm{Hom}(\lambda,\mu)$. For $\beta \in {}_M\mathcal{Y}^\alpha_M$ there is some isometry $T \in \mathrm{Hom}(\beta, \alpha^+_\nu\alpha^-_{\nu'})$ with some $\nu, \nu' \in {}_N\mathcal{Y}_N$. Then this is just

$$\begin{aligned} \beta(x)\mathcal{E}^\pm_\lambda(\beta) &= \beta(x)T^*\varepsilon^\pm(\lambda,\nu\nu')\alpha^\pm_\lambda(T) = T^*\nu\nu'(x)\varepsilon^\pm(\lambda,\nu\nu')\alpha^\pm_\lambda(T) \\ &= T^*\varepsilon^\pm(\mu,\nu\nu')x\alpha^\pm_\lambda(T) = T^*\varepsilon^\pm(\mu,\nu\nu')\alpha^\pm_\lambda(T)x = \mathcal{E}^\pm_\mu(\beta)x\,, \end{aligned}$$

thanks to naturality. □

Immediately we obtain the following

Corollary 3.3 *The map* $\lambda \mapsto \eta(\alpha_\lambda^\pm, \mathcal{E}_\lambda^\pm)$*,* $\lambda \in \Sigma({}_N\mathcal{X}_N)$*, preserves sums, products, and conjugate sectors.*

Recall from [4, Sect. 4] that $\mathrm{Hom}(\mathrm{id}, \alpha_\rho^\pm) = \{w_\rho^* v : w_\rho \in \mathfrak{h}_\rho^\pm\}$ where $\mathfrak{h}_\rho^\pm \subset \mathrm{Hom}(\rho, \theta)$ is the Hilbert (sub-) space

$$\mathfrak{h}_\rho^\pm = \{w_\rho \in \mathrm{Hom}(\rho, \theta) : w_\rho^* \gamma(v) = w_\rho^* \varepsilon^\mp(\theta, \theta) \gamma(v)\}$$

for $\rho \in {}_N\mathcal{X}_N$. Note that by Assumption 2.1 the spaces $\mathrm{Hom}(\rho, \theta)$ and in turn $\mathfrak{h}_\rho^\pm$ can only be non-zero if $\rho \in {}_N\mathcal{Y}_N$. For any $\lambda, \mu \in {}_N\mathcal{X}_N$, $\rho \in {}_N\mathcal{Y}_N$, we may choose orthonormal basis of isometries $t(^\mu_{\rho,\lambda})_i \in \mathrm{Hom}(\mu, \rho\lambda)$, $i = 1, 2, ..., N_{\rho,\lambda}^\mu$ and $w_{\rho,r;\pm} \in \mathfrak{h}_\rho^\pm$, where $r = 1, 2, ..., Z_{\rho,0} = \langle \mathrm{id}, \alpha_\rho^+ \rangle$ respectively $r = 1, 2, ..., Z_{0,\rho} = \langle \mathrm{id}, \alpha_\rho^- \rangle$.

Lemma 3.4 *A basis of* $\mathrm{Hom}(\alpha_\lambda^\pm, \alpha_\mu^\pm)$ *is given by*

$$\{t(^\mu_{\rho,\lambda})_i^* w_{\rho,r;\pm}^* v : \ \rho \in {}_N\mathcal{Y}_N \,, \ \ i = 1, 2, \ldots, N_{\rho,\lambda}^\mu \,, \ \ r = 1, 2, \ldots, \langle \mathrm{id}, \alpha_\rho^\pm \rangle\} \tag{13}$$

for any $\lambda, \mu \in {}_N\mathcal{X}_N$*.*

Proof. It follows from $w_{\rho,r;\pm} \in \mathfrak{h}_\rho^\pm$ and the first inclusion in Eq. (2) that $t(^\mu_{\rho,\lambda})_i^* w_{\rho,r;\pm}^* v \in \mathrm{Hom}(\alpha_\lambda^\pm, \alpha_\mu^\pm)$. The elements are clearly linearly independent as $t(^\mu_{\rho,\lambda})_i^* w_{\rho,r;\pm}^*$ are orthonormal isometries in N. Now the statement follows since $\langle \alpha_\lambda^\pm, \alpha_\mu^\pm \rangle = \sum_\rho N_{\rho,\lambda}^\mu \langle \mathrm{id}, \alpha_\rho^\pm \rangle$ by Frobenius reciprocity. □

Next we define a subspace $\mathcal{L}(\lambda, \mu) \subset \mathrm{Hom}(\alpha_\lambda^\pm, \alpha_\mu^\pm)$ by putting

$$\mathcal{L}(\lambda, \mu) = \mathrm{span}\{t(^\mu_{\rho,\lambda})_i^* w_{\rho,r;\pm}^* v : \ \rho \in {}_N\mathcal{Y}_N^{\mathrm{deg}} \,, \ \ i = 1, 2, ..., N_{\rho,\lambda}^\mu \,, \ \ r = 1, 2, ..., Z_{\rho,0}\} \tag{14}$$

Note that there is no distinction between "+" and "−" anymore because $\alpha_\rho^+ = \alpha_\rho^-$ whenever $\rho \in {}_N\mathcal{Y}_N^{\mathrm{deg}}$.

Lemma 3.5 *We have*

$$\mathrm{Hom}(\eta(\alpha_\lambda^\pm, \mathcal{E}_\lambda^\pm), \eta(\alpha_\mu^\pm, \mathcal{E}_\mu^\pm)) = \mathcal{L}(\lambda, \mu) \otimes \mathbf{1} \tag{15}$$

and consequently $\langle \eta(\alpha_\lambda^\pm, \mathcal{E}_\lambda^\pm), \eta(\alpha_\mu^\pm, \mathcal{E}_\mu^\pm) \rangle = \sum_{\rho \in {}_N\mathcal{Y}_N^{\mathrm{deg}}} N_{\rho,\lambda}^\mu Z_{\rho,0}$ *for all* $\lambda, \mu \in {}_N\mathcal{X}_N$*.*

Proof. By Theorem 2.4 we have to show that

$$\mathcal{L}(\lambda, \mu) = \{X \in \mathrm{Hom}(\alpha_\lambda^\pm, \alpha_\mu^\pm) : \ \mathcal{E}_\mu^\pm(\beta) X = \beta(X) \mathcal{E}_\lambda^\pm(\beta) \text{ for all } \beta \in {}_M\mathcal{Y}_M^\alpha\} \,.$$

So first we assume that X is in the right-hand side, and such an $X \in \mathrm{Hom}(\alpha_\lambda^\pm, \alpha_\mu^\pm)$ satisfies $\mathcal{E}_\mu^\pm(\beta) X = \beta(X) \mathcal{E}_\lambda^\pm(\beta)$ in particular for all $\beta \in {}_M\mathcal{Y}_M^\pm \subset {}_M\mathcal{Y}_M^\alpha$. So choose an isometry $T \in \mathrm{Hom}(\beta, \alpha_\nu^\pm)$ with some $\nu \in {}_N\mathcal{Y}_N$. Then, by Eq. (3),

$$\mathcal{E}_\mu^\pm(\beta) X = T^* \varepsilon^\pm(\mu, \nu) \alpha_\mu^\pm(T) X = T^* \varepsilon^\pm(\mu, \nu) X \alpha_\lambda^\pm(T) = T^* \alpha_\nu^\mp(X) \varepsilon^\pm(\lambda, \nu) \alpha_\lambda^\pm(T)$$

whereas

$$\beta(X)\mathcal{E}^\pm_\lambda(\beta) = \beta(X)T^*\varepsilon^\pm(\lambda,\nu)\alpha^\pm_\lambda(T) = T^*\alpha^\pm_\nu(X)\varepsilon^\pm(\lambda,\nu)\alpha^\pm_\lambda(T)\,.$$

Equating these and multiplying by T from the left and T^* from the right we obtain, using again Eq. (3),

$$TT^*\alpha^\mp_\nu(X)\varepsilon^\pm(\lambda,\nu) = TT^*\alpha^\pm_\nu(X)\varepsilon^\pm(\lambda,\nu)\,.$$

Since this is supposed to hold for any $\beta \in {}_M\mathcal{Y}^\pm_M$ we may now take the sum over full orthonormal bases of $\mathrm{Hom}(\beta,\alpha^\pm_\nu)$ so that we find $\alpha^-_\nu(X) = \alpha^+_\nu(X)$ for all $\nu \in {}_N\mathcal{Y}_N$. Now recall that $X \in \mathrm{Hom}(\alpha^\pm_\lambda,\alpha^\pm_\mu)$ is a linear combination

$$X = \sum_{\rho\in {}_N\mathcal{Y}_N} \sum_{i=1}^{N^\mu_{\rho,\lambda}} \sum_{r=1}^{\langle \mathrm{id},\alpha^\pm_\rho\rangle} \zeta_{\rho,i,r} t({}^{\mu}_{\rho,\lambda})^*_i w^*_{\rho,r;\pm} v$$

with $\zeta_{\rho,i,r} \in \mathbb{C}$. But

$$\alpha^\pm_\nu(t({}^{\mu}_{\rho,\lambda})^*_i w^*_{\rho,r;\pm} v) = \nu(t({}^{\mu}_{\rho,\lambda})^*_i w^*_{\rho,r;\pm})\varepsilon^\mp(\theta,\nu) v = \nu(t({}^{\mu}_{\rho,\lambda})^*_i)\varepsilon^\mp(\rho,\nu) w^*_{\rho,r;\pm} v\,.$$

Therefore, using $nv = 0$ implies $n = 0$ as well as orthonormality of the $w_{\rho,r;\pm}$'s, we find that $\alpha^-_\nu(X) = \alpha^+_\nu(X)$ for all ν implies

$$\sum_{i=1}^{N^\mu_{\rho,\lambda}} \zeta_{\rho,i,r}\nu(t({}^{\mu}_{\rho,\lambda})^*_i)(\varepsilon^+(\rho,\nu)\varepsilon^+(\nu,\rho) - \mathbf{1}) = 0$$

for all $\nu,\rho \in {}_N\mathcal{Y}_N$ and all $r = 1,...,\langle \mathrm{id},\alpha^\pm_\rho\rangle$. Taking the adjoint and applying the left inverse ϕ_ν yields

$$\sum_{i=1}^{N^\mu_{\rho,\lambda}} \zeta^*_{\rho,i,r}\left(\frac{Y_{\nu,\rho}}{d_\nu d_\rho} - 1\right) t({}^{\mu}_{\rho,\lambda})_i = 0\,, \qquad \nu,\rho \in {}_N\mathcal{Y}_N\,, \quad r = 1,...,\langle \mathrm{id},\alpha^\pm_\rho\rangle\,,$$

as the monodromy matrix Y is obtained [27, 12, 11] from $d_\nu d_\rho \phi_\nu(\varepsilon^+(\rho,\nu)\varepsilon^+(\nu,\rho))^* = Y_{\nu,\rho}\mathbf{1}$. Hence we have $\zeta^*_{\rho,i,r}(Y_{\nu,\rho} - d_\nu d_\rho) = 0$ for all ν,ρ,i,r. But $Y_{\nu,\rho} = d_\nu d_\rho$ for all $\nu \in {}_N\mathcal{Y}_N$ if and only if $\rho \in {}_N\mathcal{Y}^{\mathrm{deg}}_N$ by [27]. Consequently $\zeta_{\rho,i,r} = 0$ whenever $\rho \notin {}_N\mathcal{Y}^{\mathrm{deg}}_N$, so that indeed $X \in \mathcal{L}(\lambda,\mu)$.

Conversely, if we start with $X \in \mathcal{L}(\lambda,\mu)$, i.e.

$$X = \sum_{\rho\in {}_N\mathcal{Y}^{\mathrm{deg}}_N} \sum_{i=1}^{N^\mu_{\rho,\lambda}} \sum_{r=1}^{\langle \mathrm{id},\alpha^\pm_\rho\rangle} \zeta_{\rho,i,r} t({}^{\mu}_{\rho,\lambda})^*_i w^*_{\rho,r;\pm} v\,, \qquad \zeta_{\rho,i,r} \in \mathbb{C}\,,$$

then we find $\alpha^-_\nu(X) = \alpha^+_\nu(X)$ for all $\nu \in {}_N\mathcal{Y}_N$. Hence, if $\beta \in {}_M\mathcal{Y}^\alpha_M$ and $T \in \mathrm{Hom}(\beta,\alpha^+_\nu\alpha^-_{\nu'})$ is an isometry with some $\nu,\nu' \in {}_N\mathcal{Y}_N$, then

$$\begin{aligned}
\mathcal{E}^\pm_\mu(\beta)X &= T^*\varepsilon^\pm(\mu,\nu\nu')\alpha^\pm_\mu(T)X = T^*\varepsilon^\pm(\mu,\nu\nu')X\alpha^\pm_\lambda(T)\\
&= T^*\alpha^\mp_{\nu\nu'}(X)\varepsilon^\pm(\lambda,\nu\nu')\alpha^\pm_\lambda(T) = T^*\alpha^+_\nu\alpha^-_{\nu'}(X)\varepsilon^\pm(\lambda,\nu\nu')\alpha^\pm_\lambda(T)\\
&= \beta(X)T^*\varepsilon^\pm(\lambda,\nu\nu')\alpha^\pm_\lambda(T) = \beta(X)\mathcal{E}^\pm_\lambda(\beta)
\end{aligned}$$

by Eq. (3). Thus X satisfies the desired intertwining relations. $\square$

Next we compare η-extensions with different signature.

Lemma 3.6 *We have*

$$\mathrm{Hom}(\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+),\eta(\alpha_\mu^-,\mathcal{E}_\mu^-)) = \begin{cases} \mathcal{L}(\lambda,\mu)\otimes\mathbf{1} & : \quad \lambda,\mu\in{}_N\mathcal{Y}_N^{\mathrm{per}} \\ \{0\} & : \quad \textit{otherwise} \end{cases} \tag{16}$$

for all $\lambda,\mu\in{}_N\mathcal{X}_N$.

Proof. Again by Theorem 2.4, we only need to show that for $\lambda,\mu\in{}_N\mathcal{X}_N$ the linear space of intertwiners $X\in\mathrm{Hom}(\alpha_\lambda^+,\alpha_\mu^-)$ satisfying $\mathcal{E}_\mu^-(\beta)X=\beta(X)\mathcal{E}_\lambda^+(\beta)$ for all $\beta\in{}_M\mathcal{Y}_M^\alpha$ is given by $\mathcal{L}(\lambda,\mu)$ whenever $\lambda,\mu\in{}_N\mathcal{Y}_N^{\mathrm{per}}$ and vanishes otherwise. Thus suppose that $X\in\mathrm{Hom}(\alpha_\lambda^\pm,\alpha_\mu^\mp)$ satisfies $\mathcal{E}_\mu^\mp(\beta)X=\beta(X)\mathcal{E}_\lambda^\pm(\beta)$ for all $\beta\in{}_M\mathcal{Y}_M^\alpha$. Then in particular

$$T^*\varepsilon^\mp(\mu,\nu)\alpha_\mu^\mp(T)X=\beta(X)T^*\varepsilon^\pm(\lambda,\nu)\alpha_\lambda^\pm(T)$$

whenever $T\in\mathrm{Hom}(\beta,\alpha_\nu^\mp)$ is an isometry and $\nu\in{}_N\mathcal{Y}_N$. Sandwiching this with T and $\alpha_\lambda^\pm(T)^*$ yields by use of Eq. (3)

$$\varepsilon^\mp(\mu,\nu)\alpha_\mu^\mp(TT^*)X=\alpha_\nu^\mp(X)\varepsilon^\pm(\lambda,\mu)\alpha_\lambda^\pm(TT^*)\,,$$

and since this is supposed for any subsector $[\beta]$ of any $[\alpha_\nu^\mp]$ we can sum over orthonormal bases of $\mathrm{Hom}(\beta,\alpha_\nu^\mp)$ so that we arrive at

$$\varepsilon^\mp(\mu,\nu)X=\alpha_\nu^\mp(X)\varepsilon^\pm(\lambda,\nu)=\varepsilon^\pm(\mu,\nu)X \qquad \text{for all } \nu\in{}_N\mathcal{Y}_N\,.$$

If $X\neq0$ then $X=t^*v$ with $t\in\mathrm{Hom}(\mu,\theta\lambda)$ some necessarily non-zero multiple of an isometry. Therefore we have found that $\varepsilon^+(\mu,\nu)t^*=\varepsilon^-(\mu,\nu)t^*$ for all $\nu\in{}_N\mathcal{Y}_N$ implying $\mu\in{}_N\mathcal{Y}_N^{\mathrm{per}}$. Now note that if $X\otimes\mathbf{1}\in\mathrm{Hom}(\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+),\eta(\alpha_\mu^-,\mathcal{E}_\mu^-))$ then $X^*\otimes\mathbf{1}\in\mathrm{Hom}(\eta(\alpha_\mu^-,\mathcal{E}_\mu^-),\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+))$ so that our calculation also yields $\lambda\in{}_N\mathcal{Y}_N^{\mathrm{per}}$. We conclude that the intertwiner space on the left-hand side of Eq. (16) is zero unless $\lambda,\mu\in{}_N\mathcal{Y}_N^{\mathrm{per}}$. But if $\mu\in{}_N\mathcal{Y}_N^{\mathrm{per}}$, then $\alpha_\mu^+=\alpha_\mu^-$ as well as $\mathcal{E}_\mu^+=\mathcal{E}_\mu^-$, so that clearly $\mathrm{Hom}(\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+),\eta(\alpha_\mu^-,\mathcal{E}_\mu^-))=\mathrm{Hom}(\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+),\eta(\alpha_\mu^+,\mathcal{E}_\mu^+))$. Then the conclusion follows from Lemma 3.5. $\square$

For $\lambda\in{}_N\mathcal{X}_N$ conjugate half-braiding operators are given by

$$\bar{\mathcal{E}}_\lambda^\pm(\beta)=d_\lambda\bar{r}_\lambda^*\alpha_\lambda^\pm(\mathcal{E}_{\bar\lambda}^\pm(\beta)^*\beta(r_\lambda))\,,\qquad \beta\in{}_M\mathcal{X}_M^\alpha\,,$$

where $r_\lambda\in\mathrm{Hom}(\mathrm{id},\bar\lambda\lambda)$ and $\bar r_\lambda\in\mathrm{Hom}(\mathrm{id},\lambda\bar\lambda)$ are the R-isometries, i.e. satisfying $\lambda(r_\lambda)^*\bar r_\lambda=\bar\lambda(\bar r_\lambda)^*r_\lambda=d_\lambda^{-1}\mathbf{1}$. (Recall that these isometries also serve as R-isometries for the α-induced morphisms due to the first inclusion in Eq. (2).)

Lemma 3.7 *We have $\bar{\mathcal{E}}^{\pm}_{\lambda}(\beta) = \mathcal{E}^{\pm}_{\lambda}(\beta)$ for all $\beta \in {}_M\mathcal{X}^{\alpha}_M$ and all $\lambda \in {}_N\mathcal{X}_N$.*

Proof. Let $T \in \mathrm{Hom}(\beta, \alpha^+_\nu \alpha^-_{\nu'})$ be an isometry, $\nu, \nu' \in {}_N\mathcal{X}_N$. Then

$$\begin{aligned} \bar{\mathcal{E}}^{\pm}_{\lambda}(\beta) &= d_\lambda \bar{r}^*_\lambda \alpha^{\pm}_\lambda (\mathcal{E}^{\pm}_{\bar\lambda}(\beta)^* \beta(r_\lambda)) = d_\lambda \bar{r}^*_\lambda \alpha^{\pm}_\lambda (\alpha^{\pm}_{\bar\lambda}(T)^* \varepsilon^{\pm}(\bar\lambda, \nu\nu')^* T \beta(r_\lambda)) \\ &= d_\lambda T^* \bar{r}^*_\lambda \lambda(\varepsilon^{\mp}(\nu\nu', \bar\lambda)\nu\nu'(r_\lambda)) \alpha^{\pm}_\lambda(T) = d_\lambda T^* \bar{r}^*_\lambda \lambda(\bar\lambda(\varepsilon^{\mp}(\nu\nu', \lambda)^*) r_\lambda) \alpha^{\pm}_\lambda(T) \\ &= d_\lambda T^* \varepsilon^{\pm}(\lambda, \nu\nu') \bar{r}^*_\lambda \lambda(r_\lambda) \alpha^{\pm}_\lambda(T) = \mathcal{E}^{\pm}_\lambda(\beta)\,, \end{aligned}$$

where we used the BFE $r_\lambda = \bar\lambda(\varepsilon^{\mp}(\nu\nu', \lambda)\varepsilon^{\mp}(\nu\nu', \bar\lambda)\nu\nu'(r_\lambda)$. $\square$

Considering only $\beta \in {}_M\mathcal{Y}^{\alpha}_M$, Lemma 3.7 yields with Proposition 2.6 the following

Corollary 3.8 *We have $[\eta^{\mathrm{opp}}(\alpha^{\pm}_{\lambda}, \mathcal{E}^{\pm}_{\lambda})] = [\eta(\alpha^{\pm}_{\bar\lambda}, \mathcal{E}^{\pm}_{\bar\lambda})]$ for all $\lambda \in {}_N\mathcal{X}_N$.*

We are now ready to state the main result of this section in the following

Theorem 3.9 *We have*

$$\langle \eta(\alpha^+_\lambda, \mathcal{E}^+_\lambda)\eta(\alpha^-_\mu, \mathcal{E}^-_\mu), \eta(\alpha^+_{\lambda'}, \mathcal{E}^+_{\lambda'})\eta(\alpha^-_{\mu'}, \mathcal{E}^-_{\mu'})\rangle = \sum_{\nu,\xi \in {}_N\mathcal{Y}^{\mathrm{per}}_N} \sum_{\rho \in {}_N\mathcal{Y}^{\mathrm{deg}}_N} N^{\nu}_{\bar\lambda',\lambda} N^{\xi}_{\mu',\bar\mu} N^{\rho}_{\nu,\xi} Z_{\rho,0}\,, \tag{17}$$

for all $\lambda, \lambda', \mu, \mu' \in {}_N\mathcal{X}_N$.

Proof. Using Proposition 2.5 and Lemma 3.7, we can compute

$$\begin{aligned} &\langle \eta(\alpha^+_\lambda, \mathcal{E}^+_\lambda)\eta(\alpha^-_\mu, \mathcal{E}^-_\mu), \eta(\alpha^+_{\lambda'}, \mathcal{E}^+_{\lambda'})\eta(\alpha^-_{\mu'}, \mathcal{E}^-_{\mu'})\rangle = \\ &\quad = \langle \eta(\alpha^+_{\bar\lambda'}, \mathcal{E}^+_{\bar\lambda'})\eta(\alpha^+_\lambda, \mathcal{E}^+_\lambda), \eta(\alpha^-_{\mu'}, \mathcal{E}^-_{\mu'})\eta(\alpha^-_{\bar\mu}, \mathcal{E}^-_{\bar\mu})\rangle \\ &\quad = \textstyle\sum_{\nu,\xi \in {}_N\mathcal{X}_N} N^{\nu}_{\bar\lambda',\lambda} N^{\xi}_{\mu',\bar\mu} \langle \eta(\alpha^+_\nu, \mathcal{E}^+_\nu), \eta(\alpha^-_\xi, \mathcal{E}^-_\xi)\rangle\,, \end{aligned}$$

and now the result follows by Lemma 3.6 and since $\dim\mathcal{L}(\nu,\xi) = \sum_{\rho \in {}_N\mathcal{Y}^{\mathrm{deg}}_N} N^{\rho}_{\nu,\xi} Z_{\rho,0}$. $\square$

Theorem 3.9 has some simple consequences in the non-degenerate case. Let us review a bit of category language first. A system $\mathcal{S} \subset \mathrm{End}(Q)$ for some type III factor Q gives a strict C^*-tensor category (with conjugates, subobjects, and direct sums) in the sense of [7, 21], whose objects are in $\Sigma(\mathcal{S})$. There is a natural notion of equivalence of such categories, and two such categories $\mathcal{C}$ and $\mathcal{C}'$ are equivalent [13, Prop. 1.1] if and only if there is a C^*-tensor functor $F : \mathcal{C} \to \mathcal{C}'$ such that any object in $\mathcal{C}'$ is isomorphic (unitarily equivalent) to an object in the image of F, and the arrow functions $F_{\rho,\sigma} : \mathrm{Hom}(\rho,\sigma) \to \mathrm{Hom}(F(\rho), F(\sigma))$ are isomorphisms for any ρ and σ in $\mathcal{C}$. We also have a notion of direct product for two such strict C^*-tensor categories. That is, if we have two systems of irreducible endomorphisms of two (type III) factors Q and R, we have a system of irreducible endomorphisms arising as tensor products of pairs of irreducible endomorphisms on $Q \otimes R$. Moreover we can pass from one system of such endomorphisms on R to another "opposite" system on the opposite algebra R^{opp} naturally.

Note that the right-hand side of Eq. (17) collapses dramatically in case that ${}_N\mathcal{Y}_N = {}_N\mathcal{X}_N$ and if the original braiding is non-degenerate, i.e. if ${}_N\mathcal{Y}_N^{\mathrm{deg}} = \{\mathrm{id}\}$: Then we are simply left with Kronecker symbols $\delta_{\lambda,\lambda'}\delta_{\mu,\mu'}$. As shown in [5], non-degeneracy of the braiding implies ${}_M\mathcal{X}_M^\alpha = {}_M\mathcal{X}_M$, and then the global index of $\Delta = {}_M\mathcal{X}_M^\alpha$ is equal to the global index ${}_N\mathcal{X}_N$, $[[\Delta]] = [[{}_N\mathcal{X}_N]]$. Theorem 3.9 and Corollary 3.8 imply that $\{\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta^{\mathrm{opp}}(\alpha_\mu^-,\mathcal{E}_\mu^-)\}_{\lambda,\mu\in {}_N\mathcal{X}_N}$ gives a system of irreducible R-R morphisms. Note that each of the morphisms in this system gives a sector arising from $\mathcal{D}(\Delta)$, and with a suitable choice of representatives in $\mathcal{D}(\Delta)$ we may assume that this system is a subsystem of $\mathcal{D}(\Delta)$. As the statistical dimension of $\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta^{\mathrm{opp}}(\alpha_\mu^-,\mathcal{E}_\mu^-)$ is $d_\lambda d_\mu$, we know that its global index is equal to $[[{}_N\mathcal{X}_N]]^2$. But since $[[\mathcal{D}(\Delta)]] = [[\Delta]]^2$, it follows that our system is in fact the entire $\mathcal{D}(\Delta)$. With non-degeneracy, Theorem 3.9 implies that $\langle \eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+), \eta(\alpha_\mu^+,\mathcal{E}_\mu^+)\rangle = \langle\lambda,\mu\rangle$ for any $\lambda,\mu\in\Sigma({}_N\mathcal{X}_N)$ and consequently Lemma 3.2 gives equalities

$$\mathrm{Hom}(\lambda,\mu)\otimes\mathbf{1} = \mathrm{Hom}(\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+),\eta(\alpha_\mu^+,\mathcal{E}_\mu^+)) \qquad \text{for all} \quad \lambda,\mu\in\Sigma({}_N\mathcal{X}_N)\,.$$

A similar statement holds for $\mathrm{Hom}(\eta^{\mathrm{opp}}(\alpha_\lambda^-,\mathcal{E}_\lambda^-),\eta^{\mathrm{opp}}(\alpha_\mu^-,\mathcal{E}_\mu^-))$ and we thus have

$$\begin{aligned}&\mathrm{Hom}(\lambda',\mu')\otimes\mathrm{Hom}(\lambda^{\mathrm{opp}},\mu^{\mathrm{opp}})\\ &\qquad = \mathrm{Hom}(\eta(\alpha_{\lambda'}^+,\mathcal{E}_{\lambda'}^+)\eta^{\mathrm{opp}}(\alpha_\lambda^-,\mathcal{E}_\lambda^-),\eta(\alpha_{\mu'}^+,\mathcal{E}_{\mu'}^+)\eta^{\mathrm{opp}}(\alpha_\mu^-,\mathcal{E}_\mu^-))\end{aligned}$$

for all $\lambda',\mu',\lambda,\mu\in\Sigma({}_N\mathcal{X}_N)$. Let now $\mathcal{C}$ be the strict C^*-tensor category arising from the direct product of ${}_N\mathcal{X}_N$ and $({}_N\mathcal{X}_N)^{\mathrm{opp}}$ and $\mathcal{C}'$ be the one arising from $\mathcal{D}(\Delta)$. We may now introduce a functor $F:\mathcal{C}\to\mathcal{C}'$ which maps any pair $(\lambda',\lambda^{\mathrm{opp}})$ to the R-R morphism $\eta(\alpha_{\lambda'}^+,\mathcal{E}_{\lambda'}^+)\eta^{\mathrm{opp}}(\alpha_\lambda^-,\mathcal{E}_\lambda^-)$, and with arrow functions $F_{(\lambda',\lambda^{\mathrm{opp}}),(\mu',\mu^{\mathrm{opp}})}$ mapping $x\otimes y\in\mathrm{Hom}(\lambda',\mu')\otimes\mathrm{Hom}(\lambda^{\mathrm{opp}},\mu^{\mathrm{opp}})$ to $x\otimes y\in \mathrm{Hom}(\eta(\alpha_{\lambda'}^+,\mathcal{E}_{\lambda'}^+)\eta^{\mathrm{opp}}(\alpha_\lambda^-,\mathcal{E}_\lambda^-),\eta(\alpha_{\mu'}^+,\mathcal{E}_{\mu'}^+)\eta^{\mathrm{opp}}(\alpha_\mu^-,\mathcal{E}_\mu^-))$, which is obviously a (rather trivial) C^*-tensor functor. It is similarly clear that any object in $\mathcal{C}'$ is unitarily equivalent to some object in the image of F, and that the arrow functions are isomorphisms. Therefore we have the following

Corollary 3.10 *If the braiding on ${}_N\mathcal{X}_N$ is non-degenerate, then the strict C^*-tensor category given by the system of irreducible R-R morphisms for the Longo-Rehren subfactor $M\otimes M^{\mathrm{opp}}\subset R$ arising from the system ${}_M\mathcal{X}_M$ and that given as a direct product of those arising from the systems ${}_N\mathcal{X}_N$ and $({}_N\mathcal{X}_N)^{\mathrm{opp}}$ are equivalent.*

By Izumi's result, Proposition 2.7, we find that the irreducible $M\otimes M^{\mathrm{opp}}$-$R$ morphisms arising in our system are labelled with $\beta\in\Delta={}_M\mathcal{X}_M^\alpha$ and the multiplicity of the edges between this morphism and $\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta(\alpha_{\bar\mu}^-,\mathcal{E}_{\bar\mu}^-)$ is given as $\langle\beta,\alpha_\lambda^+\alpha_{\bar\mu}^-\rangle$, $\lambda,\mu\in{}_N\mathcal{X}_N$. Consequently the canonical endomorphism $\Gamma\in\mathrm{End}(R)$ for the subfactor $M\otimes M^{\mathrm{opp}}\subset R$ decomposes as

$$[\Gamma] = \bigoplus_{\lambda,\mu\in{}_N\mathcal{X}_N} Z_{\lambda,\mu}\,[\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta^{\mathrm{opp}}(\alpha_\mu^-,\mathcal{E}_\mu^-)]\,,$$

as $Z_{\lambda,\mu} = \langle \mathrm{id}, \alpha^+_\lambda \alpha^-_{\bar\mu} \rangle$ and by Corollary 3.8. Using the isomorphism of the tensor categories of Corollary 3.10 gives another

Corollary 3.11 *Assume that the braiding on ${}_N\mathcal{X}_N$ is non-degenerate. Then $\bigoplus_{\lambda,\mu\in {}_N\mathcal{X}_N} Z_{\lambda,\mu}[\lambda \otimes \mu^{\mathrm{opp}}]$ is the sector of a canonical endomorphism for some subfactor of $B \subset N \otimes N^{\mathrm{opp}}$.*

This is a special case of a recent result of Rehren [28, Cor. 1.6], and our method gives a new proof of this statement by looking at the dual of the usual Longo-Rehren subfactor arising from ${}_M\mathcal{X}^\alpha_M$. (Note that a canonical endomorphism does not determine a subfactor uniquely. So our construction and Rehren's might produce non-isomorphic subfactors, while they give the same canonical endomorphism. We expect that these two subfactors are related by an "$\mathcal{E}$-twist" in the sense of Izumi as in the remark after [14, Prop. 7.3].)

4 Quantum doubles of chiral systems

In this section we study the Longo-Rehren subfactors arising from chiral subsystems $\Delta = {}_M\mathcal{Y}^\pm_M \subset {}_M\mathcal{X}^\pm_M = \Phi$. Recall from [3, Subsect. 3.3] that for $\beta_\pm \in {}_M\mathcal{X}^\pm_M$ the operators

$$\mathcal{E}_\mathrm{r}(\beta_+,\beta_-) = S^*\alpha^-_\mu(T)^*\varepsilon^+(\lambda,\mu)\alpha^+_\lambda(S)T$$

are unitaries in $\mathrm{Hom}(\beta_+\beta_-,\beta_-\beta_+)$ whenever $\lambda,\mu \in {}_N\mathcal{X}_N$ and $T \in \mathrm{Hom}(\beta_+,\alpha^+_\lambda)$ and $S \in \mathrm{Hom}(\beta_-,\alpha^-_\mu)$ are isometries, and they do not depend on the special choices of λ,μ and S,T which realize $\beta_\pm$. Moreover, they constitute a "relative braiding" between the chiral systems ${}_M\mathcal{X}^+_M$ and ${}_M\mathcal{X}^-_M$. Recall that the ambichiral system is defined as ${}_M\mathcal{X}^0_M = {}_M\mathcal{X}^+_M \cap {}_M\mathcal{X}^-_M$. For any $\tau \in \Sigma({}_M\mathcal{X}^0_M)$ we now put

$$\begin{aligned} \mathcal{E}^-_\tau(\beta) &= \mathcal{E}_\mathrm{r}(\beta,\tau)^* \qquad \text{for all} \quad \beta \in {}_M\mathcal{X}^+_M\,, \\ \mathcal{E}^+_\tau(\beta) &= \mathcal{E}_\mathrm{r}(\tau,\beta) \qquad \text{for all} \quad \beta \in {}_M\mathcal{X}^-_M\,. \end{aligned}$$

Then the following lemma plays the role of Lemma 3.1.

Lemma 4.1 *For each $\tau \in \Sigma({}_M\mathcal{X}^0_M)$, the family $\{\mathcal{E}^\mp_\tau(\beta)\}_{\beta\in\Phi}$ is a half-braiding with respect to the system $\Phi = {}_M\mathcal{X}^\pm_M$.*

Proof. Immediate from [3, Prop. 3.12]. □

The restricted half-braidings $\mathcal{E}^\mp_\tau = \{\mathcal{E}^\mp_\tau(\beta)\}_{\beta\in\Delta}$ with $\Delta = {}_M\mathcal{Y}^\pm_M \subset {}_M\mathcal{X}^\pm_M = \Phi$ will provide η-extensions. Thanks to the composition rules of the relative braiding operators we have multiplicativity for the η-extensions,

$$\eta(\tau\tau', \mathcal{E}^\pm_{\tau\tau'})) = \eta(\tau, \mathcal{E}^\pm_\tau)\eta(\tau', \mathcal{E}^\pm_{\tau'})) \tag{18}$$

for all $\tau, \tau' \in \Sigma({}_M\mathcal{X}_M^0)$. Let us now consider such η-extensions using only $\tau \in {}_M\mathcal{X}_M^0$ (rather than in $\Sigma({}_M\mathcal{X}_M^0)$). Then it is trivial by Theorem 2.4 that

$$\mathrm{Hom}(\eta(\tau, \mathcal{E}_\tau^\mp), \eta(\tau', \mathcal{E}_{\tau'}^\mp)) = \delta_{\tau,\tau'}\mathbb{C}\,,$$

so that all such η-extensions are irreducible and disjoint. Note that for $R = R(\Delta)$ with $\Delta = {}_M\mathcal{X}_M^\pm$ we also have extensions for $\alpha_\lambda^\pm$, $\lambda \in {}_N\mathcal{X}_N$, using the restrictions of their half-braidings $\{\mathcal{E}_\lambda^\pm(\beta)\}_{\beta\in {}_M\mathcal{X}_M^\alpha}$ of Section 3 to the subsystems ${}_M\mathcal{X}_M^\pm$. By a slight abuse of notation, we also denote them by $\eta(\alpha_\lambda^\pm, \mathcal{E}_\lambda^\pm)$. In order to avoid too confusing $\pm$-indices, we now better focus on the case $\Delta = {}_M\mathcal{X}_M^+$. The other case, $\Delta = {}_M\mathcal{X}_M^-$, is of course completely analogous. The following lemma is the analogue of Lemma 3.5, now addressing the extensions $\eta(\alpha_\lambda^+, \mathcal{E}_\lambda^+)$ for $\Delta = {}_M\mathcal{X}_M^+$.

Lemma 4.2 *We have*

$$\mathrm{Hom}(\eta(\alpha_\lambda^+, \mathcal{E}_\lambda^+), \eta(\alpha_\mu^+, \mathcal{E}_\mu^+)) = \mathcal{L}(\lambda, \mu) \otimes \mathbf{1} \tag{19}$$

and consequently $\langle \eta(\alpha_\lambda^+, \mathcal{E}_\lambda^+), \eta(\alpha_\mu^+, \mathcal{E}_\mu^+)\rangle = \sum_{\rho\in {}_N\mathcal{Y}_N^{\mathrm{deg}}} N_{\rho,\lambda}^\mu Z_{\rho,0}$ *for all* $\lambda, \mu \in {}_N\mathcal{X}_N$.

Proof. Literally the same as the proof of Lemma 3.5, apart from the simplification in the second half that we now only have to consider $\nu' = \mathrm{id}$. ☐

Next we compare our different kinds of η-extensions.

Lemma 4.3 *We have*

$$\mathrm{Hom}(\eta(\tau, \mathcal{E}_\tau^-)), \eta(\alpha_\lambda^+, \mathcal{E}_\lambda^+)) = \begin{cases} \mathrm{Hom}(\tau, \alpha_\lambda^+) \otimes \mathbf{1} & : \quad \lambda \in {}_N\mathcal{Y}_N^{\mathrm{per}} \\ \{0\} & : \quad \textit{otherwise} \end{cases} \tag{20}$$

for all $\tau \in {}_M\mathcal{X}_M^0$ *and all* $\lambda \in {}_N\mathcal{X}_N$.

Proof. Using once again Theorem 2.4, we first show that if a non-zero $X \in \mathrm{Hom}(\tau, \alpha_\lambda^+)$ satisfies $\mathcal{E}_\lambda^+(\beta)X = \beta(X)\mathcal{E}_\tau^-(\beta)$ for all $\beta \in {}_M\mathcal{Y}_M^+$, then this implies $\lambda \in {}_N\mathcal{Y}_N^{\mathrm{per}}$. So suppose we have such an $X \neq 0$. Since $\tau \in {}_M\mathcal{X}_M^0$ there will also be some $\mu \in {}_N\mathcal{X}_N$ and an isometry $Q \in \mathrm{Hom}(\tau, \alpha_\mu^-)$. Then the intertwining condition reads

$$T^*\varepsilon^+(\lambda, \nu)\alpha_\lambda^+(T)X = \beta(X)T^*\alpha_\nu^+(Q)^*\varepsilon^-(\mu, \nu)\alpha_\mu^-(T)Q$$

whenever $\nu \in {}_N\mathcal{Y}_N$ and $T \in \mathrm{Hom}(\beta, \alpha_\nu^+)$ is an isometry. Multiplication with T from the right yields

$$\varepsilon^+(\lambda, \nu)\alpha_\lambda^+(T)X = \alpha_\nu^+(XQ^*)\varepsilon^-(\mu, \nu)\alpha_\mu^-(T)Q = \varepsilon^-(\lambda, \nu)\alpha_\lambda^+(T)X\,,$$

where we exploited $XQ^* \in \mathrm{Hom}(\alpha_\mu^-, \alpha_\lambda^+) \subset \mathrm{Hom}(\iota\mu, \iota\lambda)$ to apply Eq. (3). Now we can multiply by $\tau(T)^*$ from the right, and then we may use a summation over full orthonormal bases of $\mathrm{Hom}(\beta, \alpha_\nu^+)$ to obtain $\varepsilon^+(\lambda, \nu)X = \varepsilon^-(\lambda, \nu)X$ for all $\nu \in {}_N\mathcal{Y}_N$.

Now note that a non-zero $X \in \mathrm{Hom}(\tau, \alpha^+_\lambda)$ is necessarily of the from $X = t^* v$ with $t \in \mathrm{Hom}(\lambda, \bar\iota\tau\iota)$ a non-zero multiple of an isometry. Hence we find $\varepsilon^+(\lambda,\nu) = \varepsilon^-(\lambda,\nu)$ for all $\nu \in {}_N\mathcal{Y}_N$, proving that the left-hand side of Eq. (20) is zero unless $\lambda \in {}_N\mathcal{Y}_N^{\mathrm{per}}$.

On the other hand, if $\lambda \in {}_N\mathcal{Y}_N^{\mathrm{per}}$ then $\alpha^+_\lambda = \alpha^-_\lambda$. Hence, for an arbitrary $X \in \mathrm{Hom}(\tau, \alpha^+_\lambda)$ we find $X^* \in \mathrm{Hom}(\alpha^-_\lambda, \tau)$, and therefore the naturality of the relative braiding of [3, Prop. 3.12] gives us $\mathcal{E}_{\mathrm{r}}(\beta,\tau)\beta(X)^* = X^*\mathcal{E}_{\mathrm{r}}(\beta,\alpha^-_\lambda)$ for any $\beta \in {}_M\mathcal{X}_M^+$. By taking adjoints this reads

$$\beta(X)\mathcal{E}^-_\tau(\beta) = T^*\varepsilon^-(\lambda,\nu)\alpha^-_\lambda(T)X = T^*\varepsilon^+(\lambda,\nu)\alpha^+_\lambda(T)X = \mathcal{E}^+_\lambda(\beta)X\,,$$

so that the desired intertwining relation is automatically fulfilled in particular for $\beta \in {}_M\mathcal{Y}_M^+$. This completes the proof. $\square$

Conjugate half-braiding operators are given for $\tau \in {}_M\mathcal{X}_M^0$ by

$$\bar{\mathcal{E}}^-_\tau(\beta) = d_\tau \bar R^*_\tau \tau(\mathcal{E}^-_{\bar\tau}(\beta)^*\beta(R_\tau))\,, \qquad \beta \in {}_M\mathcal{X}_M^+\,,$$

with R-isometries $R_\tau \in \mathrm{Hom}(\mathrm{id}, \bar\tau\tau)$ and $\bar R_\tau \in \mathrm{Hom}(\mathrm{id}, \tau\bar\tau)$ satisfying $\tau(R_\tau)^*\bar R_\tau = \bar\tau(\bar R_\tau)^* R_\tau = d_\tau^{-1}\mathbf{1}$.

Lemma 4.4 *We have $\bar{\mathcal{E}}^-_\tau(\beta) = \mathcal{E}^-_\tau(\beta)$ for all $\beta \in {}_M\mathcal{X}_M^+$ and all $\tau \in {}_M\mathcal{X}_M^0$.*

Proof. We compute

$$\bar{\mathcal{E}}^-_\tau(\beta) = d_\tau \bar R^*_\tau \tau(\mathcal{E}_{\mathrm{r}}(\beta,\bar\tau)\beta(R_\tau)) = d_\tau \bar R^*_\tau \tau(\bar\tau(\mathcal{E}_{\mathrm{r}}(\beta,\tau)^* R_\tau) = \mathcal{E}^-_\tau(\beta)\,,$$

where we used the BFE for the relative braiding [3, Prop. 3.12], $R_\tau = \bar\tau(\mathcal{E}_{\mathrm{r}}(\beta,\tau)\mathcal{E}_{\mathrm{r}}(\beta,\bar\tau)\beta(R_\tau)$. $\square$

Considering only $\beta \in {}_M\mathcal{Y}_M^+$, Lemma 4.4 yields with Proposition 2.6 the following

Corollary 4.5 *We have $[\eta^{\mathrm{opp}}(\tau, \mathcal{E}^-_\tau)] = [\eta(\bar\tau, \mathcal{E}^-_{\bar\tau})]$ for all $\tau \in {}_M\mathcal{X}_M^0$.*

Recall from [6] that $b^\pm_{\tau,\lambda} = \langle \tau, \alpha^\pm_\lambda\rangle$ denote the chiral branching coefficients for ambichiral τ and $\lambda \in {}_N\mathcal{X}_N$.

Theorem 4.6 *We have*

$$\langle \eta(\alpha^+_\lambda, \mathcal{E}^+_\lambda)\eta(\tau, \mathcal{E}^-_\tau), \eta(\alpha^+_\mu, \mathcal{E}^+_\mu)\eta(\tau', \mathcal{E}^-_{\tau'})\rangle = \sum_{\tau'' \in {}_M\mathcal{X}_M^0} \sum_{\rho \in {}_N\mathcal{Y}_N^{\mathrm{per}}} N_{\bar\tau,\tau'}^{\tau''} N_{\lambda,\bar\mu}^{\rho} b^+_{\tau'',\rho} \tag{21}$$

for all $\lambda, \mu \in {}_N\mathcal{X}_N$ and all $\tau, \tau' \in {}_M\mathcal{X}_M^0$.

Proof. Analogous to the proof of Theorem 3.9, this is reduced to Lemma 4.3 by use of Proposition 2.5. $\square$

Let $\Upsilon \subset \mathcal{D}({}_M\mathcal{Y}_M^+)$ denote the subset of morphisms $\Omega \in \mathcal{D}({}_M\mathcal{Y}_M^+)$ which correspond to subsectors of $[\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta(\tau,\mathcal{E}_\tau^-)]$ considering $\lambda \in {}_N\mathcal{Y}_N$ and $\tau \in {}_M\mathcal{Y}_M^0$ only. Now the question arises whether Υ is a proper subsystem or whether it may exhaust the entire quantum double system and therefore we would like to measure its size. For this purpose we compare the global indices $[[\Upsilon]]$ and $[[\mathcal{D}({}_M\mathcal{Y}_M^+)]] = [[{}_M\mathcal{Y}_M^+]]^2$.

Proposition 4.7 *The global index of Υ is given by*

$$[[\Upsilon]] = \frac{\sum_{\rho\in{}_N\mathcal{Y}_N^{\mathrm{deg}}} d_\rho Z_{\rho,0}}{[[{}_N\mathcal{Y}_N^{\mathrm{deg}}]]}\,[[\mathcal{D}({}_M\mathcal{Y}_M^+)]]\,. \tag{22}$$

Proof. Let $R_{\tau,\lambda}$, $\tau\in{}_M\mathcal{Y}_M^0$, $\lambda\in{}_N\mathcal{Y}_N$, denote matrices with entries

$$R_{\tau,\lambda;\Omega}^{\Omega'} = \langle \Omega\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta(\tau,\mathcal{E}_\tau^-),\Omega'\rangle\,,\qquad \Omega,\Omega'\in\Upsilon\,.$$

Further let $\vec{d}$ denote the column vector with entries d_Ω, $\Omega\in\Upsilon$. Then $\vec{d}$ is a simultaneous eigenvector of the matrices $R_{\tau,\lambda}$ with respective eigenvalues $d_\tau d_\lambda$. We define another vector $\vec{v}$ by putting

$$v_\Omega = \sum_{\tau\in{}_M\mathcal{Y}_M^0}\sum_{\lambda\in{}_N\mathcal{Y}_N} d_\tau d_\lambda \langle\Omega,\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta(\tau,\mathcal{E}_\tau^-)\rangle\,,\qquad \Omega\in\Upsilon\,.$$

Then we have $R_{\tau,\lambda}\vec{v} = d_\tau d_\lambda\vec{v}$, as we can compute

$$\begin{aligned}(R_{\tau,\lambda}\vec{v})_\Omega &= \textstyle\sum_{\Omega'\in\Upsilon}\sum_{\tau'\in{}_M\mathcal{Y}_M^0}\sum_{\mu\in{}_N\mathcal{Y}_N}\langle\Omega\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta(\tau,\mathcal{E}_\tau^-),\Omega'\rangle\\ &\qquad\qquad\qquad\times d_{\tau'}d_\mu\langle\Omega',\eta(\alpha_\mu^+,\mathcal{E}_\mu^+)\eta(\tau',\mathcal{E}_{\tau'}^-)\rangle\\ &= \textstyle\sum_{\tau'\in{}_M\mathcal{Y}_M^0}\sum_{\mu\in{}_N\mathcal{Y}_N} d_{\tau'}d_\mu\langle\Omega,\eta(\alpha_{\bar\lambda}^+\mathcal{E}_{\bar\lambda}^+)\eta(\alpha_\mu^+,\mathcal{E}_\mu^+)\eta(\tau',\mathcal{E}_{\tau'}^-)\eta(\bar\tau,\mathcal{E}_{\bar\tau}^-)\rangle\\ &= \textstyle\sum_{\tau',\tau''\in{}_M\mathcal{Y}_M^0}\sum_{\mu,\nu\in{}_N\mathcal{X}_N} d_{\tau'}d_\mu N_{\bar\lambda,\mu}^\nu N_{\tau',\bar\tau}^{\tau''}\langle\Omega,\eta(\alpha_\nu^+,\mathcal{E}_\nu^+)\eta(\tau'',\mathcal{E}_{\tau''}^-)\rangle\\ &= \textstyle\sum_{\tau''\in{}_M\mathcal{Y}_M^0}\sum_{\nu\in{}_N\mathcal{X}_N} d_\tau d_\lambda d_{\tau''}d_\nu\langle\Omega,\eta(\alpha_\nu^+,\mathcal{E}_\nu^+)\eta(\tau'',\mathcal{E}_{\tau''}^-)\rangle = d_\tau d_\lambda v_\Omega\,.\end{aligned}$$

Because the sum matrix $\sum_{\tau,\lambda}R_{\tau,\lambda}$ is irreducible it follows $\vec{v}=\zeta\vec{d}$, $\zeta\in\mathbb{R}$, by the uniqueness of the Perron-Frobenius eigenvector. Note that

$$d_\tau d_\lambda = \sum_{\Omega\in\Upsilon}\langle\Omega,\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta(\tau,\mathcal{E}_\tau^-)\rangle d_\Omega\,,$$

and hence $[[{}_N\mathcal{Y}_N]][[{}_M\mathcal{Y}_M^0]] = \sum_\Omega v_\Omega d_\Omega = \zeta[[\Upsilon]]$. We next notice that $\zeta = v_{\mathrm{id}}$ as $d_{\mathrm{id}}=1$. But v_{id} can be computed as

$$v_{\mathrm{id}} = \sum_{\tau\in{}_M\mathcal{Y}_M^0}\sum_{\lambda\in{}_N\mathcal{Y}_N} d_\tau d_\lambda\langle\eta(\bar\tau,\mathcal{E}_{\bar\tau}^-),\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\rangle = \sum_{\lambda\in{}_N\mathcal{Y}_N^{\mathrm{deg}}} d_\lambda\sum_{\tau\in{}_M\mathcal{Y}_M^0} b_{\tau,\lambda}^+ d_\tau = \sum_{\lambda\in{}_N\mathcal{Y}_N^{\mathrm{deg}}} d_\lambda^2\,,$$

where we used Lemma 4.3. Hence $[[\Upsilon]] = [[{}_N\mathcal{Y}_N]][[{}_M\mathcal{Y}_M^0]]/[[{}_N\mathcal{Y}_N^{\mathrm{deg}}]]$, and now the claim follows since $[[{}_M\mathcal{Y}_M^0]] = (\sum_{\rho\in{}_N\mathcal{Y}_N^{\mathrm{deg}}} d_\rho Z_{\rho,0})[[{}_M\mathcal{Y}_M^+]]^2/[[{}_N\mathcal{Y}_N]]$ by [4, Prop. 3.1]. □

Similar to Theorem 3.9, the degenerate morphisms ρ appearing in Eq. (21) are responsible that some of the $\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta(\tau,\mathcal{E}_\tau^-)$'s will be equivalent or are reducible, and this will cause some kind of orbifolding as we will show in Section 5 by examples. Note, however, that also the right-hand side of Eq. (21) simplifies considerably if the original braiding is non-degenerate and if we have ${}_N\mathcal{X}_N = {}_N\mathcal{Y}_N$: We are just left with Kronecker symbols $\delta_{\lambda,\mu}\delta_{\tau,\tau'}$. Since the statistical dimension of $\eta(\tau,\mathcal{E}_\tau^-)$ is d_τ and as $[[{}_N\mathcal{X}_N]][[{}_M\mathcal{X}_M^0]] = [[{}_M\mathcal{X}_M^+]]^2$ thanks to [6, Thm. 4.2], we conclude that the family of morphisms $\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta^{\mathrm{opp}}(\tau,\mathcal{E}_\tau^-)$ serves as a system $\mathcal{D}(\Delta)$. In the non-degenerate case, it is derived similarly from Theorem 4.6 and Corollary 4.3 that then

$$\mathrm{Hom}(\lambda,\mu)\otimes\mathbf{1} = \mathrm{Hom}(\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+),\eta(\alpha_\mu^+,\mathcal{E}_\mu^+)) \qquad \text{for all} \quad \lambda,\mu\in\Sigma({}_N\mathcal{X}_N)\,,$$

as well as

$$\mathbf{1}\otimes\mathrm{Hom}(\tau^{\mathrm{opp}},\tau'^{\mathrm{opp}}) = \mathrm{Hom}(\eta^{\mathrm{opp}}(\tau,\mathcal{E}_\tau^-),\eta^{\mathrm{opp}}(\tau',\mathcal{E}_{\tau'}^-)) \qquad \text{for all} \quad \tau,\tau'\in\Sigma({}_M\mathcal{X}_M^0)\,.$$

By the same arguments which lead to Corollary 3.10 this gives the following

Corollary 4.8 *If the braiding on ${}_N\mathcal{X}_N$ is non-degenerate, then the strict C^*-tensor category given by the system of irreducible R-R morphisms for the Longo-Rehren subfactor $M\otimes M^{\mathrm{opp}}\subset R$ arising from the system ${}_M\mathcal{X}_M^\pm$ and that given as a direct product of those arising from the systems ${}_N\mathcal{X}_N$ and $({}_M\mathcal{X}_M^0)^{\mathrm{opp}}$ are equivalent.*

This corollary seems to be a precise statement of an announcement by Ocneanu. Namely, at the Taniguchi Conference in Nara, Japan, in December 1998, he announced as a part of his "big sandwich of theorems" that "the quantum double of a quantum subgroup$^+$ of a non-degenerately braided quantum group is equal to the quantum group $\times$ $\overline{\text{ambichirals}}$" (in whatever sense).

Note that the braiding on the "quantum double" system of R-R morphisms is given by the direct product of the original one on ${}_N\mathcal{X}_N$ and the one on ${}_M\mathcal{X}_M^0$ in the above theorem. Since [14, Thm. 5.5] implies that this braiding is non-degenerate and since we assumed non-degeneracy of the original braiding on ${}_N\mathcal{X}_N$ here, Corollary 4.8 implies that also the braiding on the ambichiral system ${}_M\mathcal{X}_M^0$ is non-degenerate, in perfect agreement with our result [6, Thm. 4.2].

The A-D-E cases studied in [25, 30, 2, 6] provide the following examples.

Corollary 4.9 *As strict C^*-tensor categories, the quantum double systems of the chiral systems* E_6, E_8, *and* D_{2n} *are equivalent to* $\mathrm{A}_{11}\times(\mathrm{A}_3)^{\mathrm{opp}}$, $\mathrm{A}_{29}\times(\mathrm{A}_4^{\mathrm{even}})^{\mathrm{opp}}$, *and* $\mathrm{A}_{4n-3}\times(\mathrm{D}_{2n}^{\mathrm{even}})^{\mathrm{opp}}$, *respectively.*

By the same arguments used in Section 3 we now find for the non-degenerate case and ${}_N\mathcal{X}_N = {}_N\mathcal{Y}_N$ that

$$[\Gamma] = \bigoplus_{\lambda\in{}_N\mathcal{X}_N}\bigoplus_{\tau\in{}_M\mathcal{X}_M^0} b_{\tau,\lambda}^+\,[\eta(\alpha_\lambda^+,\mathcal{E}_\lambda^+)\eta^{\mathrm{opp}}(\tau,\mathcal{E}_\tau^-)]$$

is the canonical endomorphisms sector of $M \otimes M^{\mathrm{opp}} \subset R$ arising from $\Delta = {}_M\mathcal{X}_M^+$. However, if one considers the Longo-Rehren subfactor arising from $\Delta = {}_M\mathcal{Y}_M^+$ where ${}_N\mathcal{Y}_N$ is now a proper and degenerate subsystem of ${}_N\mathcal{X}_N$, then the computations for the structure of $\mathcal{D}(\Delta)$ and the dual principal graph become more involved. In that case one needs the whole general machinery of this section which takes care of possible degeneracies. Such situations will be handled in Section 5.

5 Quantum doubles of color zero subsystems of chiral systems

Subfactors with principal graphs E_6, E_8 are basic and important examples of subfactors arising from α-induction [30, 2]. The Longo-Rehren subfactor arising from the subfactor with principal graph E_6 has been studied and the principal and the dual principal graphs have been computed, as well as other information, by Izumi [15]. Note that this subfactor is *different* from the Longo-Rehren subfactor arising from the chiral system for the conformal inclusion $SU(2)_{10} \subset SO(5)_1$ as studied in Section 4. The reason is that Izumi considers in [15] the quantum double system of the endomorphisms corresponding to the *even* three vertices rather than all nodes of the graph E_6. This is more natural from the viewpoint of the usual theory of type II_1 subfactors, since we obtain this quantum double system of the system of the three M-M bimodules, if we apply the construction of the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ to the hyperfinite II_1 subfactor $N \subset M$ with principal graph E_6 and compute the system of M_∞-M_∞ bimodules. So we will study the Longo-Rehren subfactors arising from α-induction corresponding to this type of asymptotic inclusions in this section. That is, from the view point of the α-induction, the chiral system ${}_M\mathcal{X}_M^+$ for the subfactor $N \subset M$ arising from the conformal inclusion $SU(2)_{10} \subset SO(5)_1$ has a natural "coloring" for irreducible objects with colors 0 and 1, inherited from the coloring of the $SU(2)_{10}$ system coming from the even-odd parity of the spins. More precisely and generally, thanks to Wassermann's work [29], we know that there are (non-degenerately) braided systems $\mathcal{X}_{n,k} = \{\lambda_\Lambda : \Lambda \in \mathcal{A}_{n,k}\}$, where $\mathcal{A}_{n,k}$ denotes the $SU(n)$ level k Weyl alcove, such that the morphisms $\lambda_\Lambda \in \mathrm{End}(N)$ satisfy the $SU(n)_k$ fusion rules and have statistics phases $\omega_\Lambda = \mathrm{e}^{2\pi \mathrm{i} h_\Lambda}$, where h_Λ are the conformal dimensions, for any $n, k = 1, 2, \dots$. The Weyl alcove has a natural coloring ("n-ality") $t : \mathcal{A}_{n,k} \to \mathbb{Z}_n$, and the color zero subsystems $\mathcal{Y}_{n,k} \subset \mathcal{X}_{n,k}$ are given by $\mathcal{Y}_{n,k} = \{\lambda_\Lambda : t(\Lambda) = 0\}$. Now let $N \subset M$ be a subfactor arising from a conformal inclusion $SU(n)_k \subset G_1$ for some Lie group G, as treated in [6]. We put ${}_N\mathcal{X}_N = \mathcal{X}_{n,k}$. Note that then the ambichiral system ${}_M\mathcal{X}_M^0$ corresponds to the positive energy representations π_ℓ of G_1, and the chiral branching coefficients are the well-known branching coefficients of the conformal inclusion at hand $b^{\pm}_{\tau_\ell,\lambda_\Lambda} = b_{\ell,\Lambda}$ and the modular invariant matrix is given by

$$Z_{\Lambda,\Lambda'} = \sum_\ell b_{\ell,\Lambda} b_{\ell,\Lambda'} \,.$$

We now set $\Phi = {}_M\mathcal{X}_M^+$ and $\Delta = {}_M\mathcal{Y}_M^+$, where ${}_M\mathcal{Y}_M^+$ arises from the color zero subsystem ${}_N\mathcal{Y}_N = \mathcal{Y}_{n,k}$. Note that ${}_N\mathcal{Y}_N$ will in general be degenerate though we have always non-degeneracy for ${}_N\mathcal{X}_N$ here. Here we will study Longo-Rehren subfactors $M \otimes M^{\mathrm{opp}} \subset R$ with $R = R(\Delta)$ and illustrate that the degeneracy of ${}_N\mathcal{Y}_N$ causes naturally a certain orbifold procedure by means of Theorem 4.6. The examples we cover correspond to subfactors with principal graph E_6 and E_8, and all the three analogues arising from conformal inclusions of $SU(3)_k$.

Example 5.1 We start with the subfactor $N \subset M$ arising from the conformal inclusion $SU(2)_{10} \subset SO(5)_1$. The irreducible endomorphisms in $\lambda_j \in {}_N\mathcal{X}_N$ are labelled with $j \in \{0, 1, \dots, 10\}$ as usual and those in ${}_M\mathcal{X}_M^0$ are labelled with τ_ℓ, $\ell = 0, 1, 2$ as the vertices of A_3. (Such that $\tau_0 = \mathrm{id}$.) The morphisms τ_ℓ obey Ising fusion rules, $[\tau_1\tau_1] = [\tau_0] \oplus [\tau_2]$, the non-vanishing branching coefficients $b^+_{\tau_\ell,\lambda_j} = b_{\ell,j}$ are given by

$$b_{0,0} = b_{0,6} = b_{1,3} = b_{1,7} = b_{2,4} = b_{2,10} = 1\,,$$

and the E_6 modular matrix is given by $Z_{j,j'} = \sum_{\ell=0}^{2} b_{\ell,j} b_{\ell,j'}$ (cf. [2, Example 2.2]). With ${}_N\mathcal{Y}_N = \{\lambda_j : j = 0, 2, 4, 6, 8, 10\}$, we study the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R$ arising from the system $\Delta = {}_M\mathcal{Y}_M^+$ whose irreducible morphisms correspond to the three even vertices of the graph E_6. Note that λ_{10} is degenerate in ${}_N\mathcal{Y}_N$, in fact we have ${}_N\mathcal{Y}_N^{\mathrm{deg}} = {}_N\mathcal{Y}_N^{\mathrm{per}} = \{\lambda_0, \lambda_{10}\}$. Since $Z_{10,0} = 0$ we find by Proposition 4.7 that the set Υ is only provides half of the quantum double system $\mathcal{D}({}_M\mathcal{Y}_M^+)$. Thus, as considering even j and even ℓ only will not exhaust $\mathcal{D}({}_M\mathcal{Y}_M^+)$, we now consider $\eta(\alpha_j^+, \mathcal{E}_j^+)$, $\eta(\tau_\ell, \mathcal{E}_\ell^-)$ with $j \in \{0, 1, \dots, 10\}$ and $\ell \in \{0, 1, 2\}$, where we write α_j for α_{λ_j} as in [2, Example 2.2]. These extended endomorphisms of R may not decompose into direct sums of irreducible morphisms in $\mathcal{D}(\Delta)$ any more, but $\eta(\alpha_j^+, \mathcal{E}_j^+)\eta(\tau_\ell, \mathcal{E}_\ell^-)$ do decompose into direct sums of irreducible morphisms in $\mathcal{D}(\Delta)$ if $j + \ell$ is even, since then $\alpha_j^+\tau_\ell$ decompose into a direct sum of morphisms in ${}_M\mathcal{Y}_M^+$. Now Lemma 4.2 yields easily irreducibility of $\eta(\alpha_j^+, \mathcal{E}_j^+)$ for $j \in \{0, 2, 4, 6, 8, 10\}$. Lemma 4.3 yields similarly

$$\langle \eta(\alpha_{10}^+, \mathcal{E}_{10}^+), \eta(\tau_2, \mathcal{E}_2^-) \rangle = 1\,,$$

and by Theorem 4.6 we find more generally

$$\langle \eta(\alpha_j^+, \mathcal{E}_j^+)\eta(\tau_0, \mathcal{E}_0^-), \eta(\alpha_{10-j}^+, \mathcal{E}_{10-j}^+)\eta(\tau_2, \mathcal{E}_2^-) \rangle = 1$$

for $j = 0, 2, 4, ..., 10$. We similarly compute

$$\begin{array}{rcl}
\langle \eta(\alpha_1^+, \mathcal{E}_1^+)\eta(\tau_1, \mathcal{E}_1^-), \eta(\alpha_1^+, \mathcal{E}_1^+)\eta(\tau_1, \mathcal{E}_1^-) \rangle &=& 1, \\
\langle \eta(\alpha_3^+, \mathcal{E}_3^+)\eta(\tau_1, \mathcal{E}_1^-), \eta(\alpha_3^+, \mathcal{E}_3^+)\eta(\tau_1, \mathcal{E}_1^-) \rangle &=& 1, \\
\langle \eta(\alpha_5^+, \mathcal{E}_5^+)\eta(\tau_1, \mathcal{E}_1^-), \eta(\alpha_5^+, \mathcal{E}_5^+)\eta(\tau_1, \mathcal{E}_1^-) \rangle &=& 2, \\
\langle \eta(\alpha_7^+, \mathcal{E}_7^+)\eta(\tau_1, \mathcal{E}_1^-), \eta(\alpha_7^+, \mathcal{E}_7^+)\eta(\tau_1, \mathcal{E}_1^-) \rangle &=& 1, \\
\langle \eta(\alpha_9^+, \mathcal{E}_9^+)\eta(\tau_1, \mathcal{E}_1^-), \eta(\alpha_9^+, \mathcal{E}_9^+)\eta(\tau_1, \mathcal{E}_1^-) \rangle &=& 1, \\
\langle \eta(\alpha_1^+, \mathcal{E}_1^+)\eta(\tau_1, \mathcal{E}_1^-), \eta(\alpha_9^+, \mathcal{E}_9^+)\eta(\tau_1, \mathcal{E}_1^-) \rangle &=& 1, \\
\langle \eta(\alpha_3^+, \mathcal{E}_3^+)\eta(\tau_1, \mathcal{E}_1^-), \eta(\alpha_7^+, \mathcal{E}_7^+)\eta(\tau_1, \mathcal{E}_1^-) \rangle &=& 1.
\end{array}$$

We have a decomposition of $\eta(\alpha_5^+, \mathcal{E}_5^+)\eta(\tau_1, \mathcal{E}_1^-)$ into two irreducible, mutually inequivalent endomorphisms $\Omega, \Omega' \in \mathrm{End}(R)$ which must belong (up to equivalence) to $\mathcal{D}(\Delta)$. By Izumi's result [14, Lemma 4.5] we conclude that that there are morphisms $\sigma, \sigma' \in \Sigma({}_M\mathcal{Y}_M^+)$ with half-braidings $\mathcal{E}_\sigma, \mathcal{E}'_{\sigma'}$ such that $\Omega = \eta(\sigma, \mathcal{E}_\sigma)$ and $\Omega' = \eta(\sigma', \mathcal{E}'_{\sigma'})$. But since we have $[\alpha_5^+][\tau_1] = 2[\alpha_2^+]$ and since η-extension preserves the statistical dimension we must have $d_\Omega + d_{\Omega'} = d_5 d_{\tau_1} = 2d_2$. (Recall $d_{\alpha_j^+} = d_j$.) Moreover, Theorem 2.4 implies that $[\alpha_2^+]$ is a subsector of both, $[\sigma]$ and $[\sigma']$, and in turn $d_\Omega = d_\sigma \geq d_2$, $d_{\Omega'} = d_{\sigma'} \geq d_2$. This forces $d_\Omega = d_{\Omega'} = d_2$ and consequently $[\alpha_2^+] = [\sigma] = [\sigma']$, i.e. the product $\eta(\alpha_5^+, \mathcal{E}_5^+)\eta(\tau_1, \mathcal{E}_1^-)$ decomposes into two irreducibles of equal statistical dimension. So we have (at least) the following 8 irreducible, mutually inequivalent endomorphisms $\eta(\alpha_0^+, \mathcal{E}_0^+)$, $\eta(\alpha_2^+, \mathcal{E}_2^+)$, $\eta(\alpha_4^+, \mathcal{E}_4^+)$, $\eta(\alpha_6^+, \mathcal{E}_6^+)$, $\eta(\alpha_8^+, \mathcal{E}_8^+)$, $\eta(\alpha_{10}^+, \mathcal{E}_{10}^+)$, $\eta(\alpha_1^+, \mathcal{E}_1^+)\eta(\tau_1, \mathcal{E}_1^-)$, $\eta(\alpha_3^+, \mathcal{E}_3^+)\eta(\tau_1, \mathcal{E}_1^-)$ and two more irreducible endomorphisms Ω, Ω' of R arising from $\eta(\alpha_5^+, \mathcal{E}_5^+)\eta(\tau_1, \mathcal{E}_1^-)$. Counting the global index, we conclude that these are all the R-R morphisms in the system $\mathcal{D}(\Delta)$. Thus, with Proposition 2.7 and recalling that $\eta(\alpha_j^+, \mathcal{E}_j^+)\eta(\tau_\ell, \mathcal{E}_\ell^-)$ is the η-extension of $\alpha_j^+ \circ \tau_\ell$ with the composed half-braiding, we can compute the dual principal graph of the subfactor $M \otimes M^{\mathrm{opp}} \subset R$ which we display in Fig. 1. Of course it is the same as the

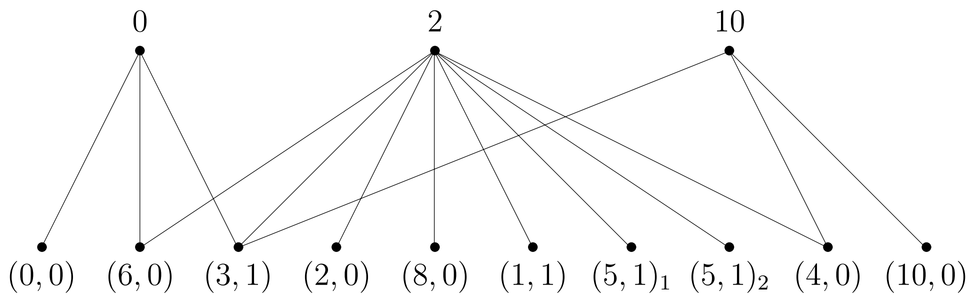

Figure 1: The dual principal graph for the Longo-Rehren subfactor arising from E_6

one first computed by Izumi [15] by direct computations of the tube algebra involving $6j$-symbols. In the graph, we used an obvious notation for the vertices labelled by morphisms in ${}_M\mathcal{Y}_M^+$, and we simply wrote (j, ℓ) for the pair $\eta(\alpha_j^+, \mathcal{E}_j^+)\eta(\tau_\ell, \mathcal{E}_\ell^-)$ with $j \in \{0, 1, \ldots, 10\}$ and $\ell \in \{0, 1, 2\}$. The labels $(5, 1)_1$ and $(5, 1)_2$ stand for the two irreducible endomorphisms Ω, Ω' corresponding to the subsectors of $\eta(\alpha_5^+, \mathcal{E}_5^+)\eta(\tau_1, \mathcal{E}_1^-)$. The procedure yielding the R-R morphisms here is an orbifold procedure of order 2 for the (j, ℓ) with $j + \ell \in 2\mathbb{N}$ with symmetry $(j, \ell) \leftrightarrow (10 - j, 2 - \ell)$.

Example 5.2 We next study the Longo-Rehren subfactor arising from the four even vertices of the graph E_8 and compute the dual principal graph, which is new. The subfactor $N \subset M$ now arises from the conformal inclusion $SU(2)_{28} \subset (\mathrm{G}_2)_1$ as in [2, Example 2.3]. Analogously to Example 5.1, the full system is given ${}_N\mathcal{X}_N = \{\lambda_j \mid$

$j = 0, 1, 2, ..., 28\}$. We label the ambichiral morphisms in ${}_M\mathcal{X}_M^0$ with τ_k, $k = 0, 2$, corresponding to the extremal vertices of the two long legs of E_8. They τ_ℓ's obey Lee-Yang fusion rules, $[\tau_2\tau_2] = [\tau_0] \oplus [\tau_2]$ which is the fusion of the even vertices of A_4. The non-vanishing branching coefficients $b^+_{\tau_\ell,\lambda_j} = b_{\ell,j}$ are given by

$$b_{0,0} = b_{0,10} = b_{0,18} = b_{0,28} = b_{2,6} = b_{2,12} = b_{2,16} = b_{2,22} = 1\,,$$

determining the modular invariant Z as before. The color zero subsystem is ${}_N\mathcal{Y}_N = \{\lambda_j \mid j = 0, 2, 4, ..., 28\}$, and then ${}_N\mathcal{Y}_N^{\mathrm{deg}} = \{\lambda_0, \lambda_{28}\}$. We will study the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R$ arising from the chiral induced system $\Delta = {}_M\mathcal{Y}_M^+$. The system $\Phi = {}_M\mathcal{X}_M^+$ corresponds to the labels of vertices of E_8 in [2, Fig. 8], and the subsystem $\Delta = {}_M\mathcal{Y}_M^+$ to the even ones. As the ambichiral vertices are both even we find ${}_M\mathcal{X}_M^0 = {}_M\mathcal{Y}_M^0$ here. Note that the degenerate morphism λ_{28} appears in the vacuum column and row of Z this time. Therefore Proposition 4.7 tells us that $\Upsilon = \mathcal{D}(\Delta)$, i.e. in contrast to Example 5.1 we do not need to consider the odd spins at all in order to produce the entire $\mathcal{D}(\Delta)$ by our method. Lemma 4.2 gives $[\eta(\alpha_j^+, \mathcal{E}_j^+)] = [\eta(\alpha_{28-j}^+, \mathcal{E}_{28-j}^+)]$ for $j = 0, 2, 4, \ldots, 12$ and it similarly implies that $\eta(\alpha_j^+, \mathcal{E}_j^+)$'s are irreducible and mutually inequivalent for $j = 0, 2, 4, \ldots, 12$, due to the fusion rules $N_{j,j'}^{28} = 0$ for $j, j' = 0, 2, 4, \ldots, 12$. But we obtain

$$\langle \eta(\alpha_{14}^+, \mathcal{E}_{14}^+), \eta(\alpha_{14}^+, \mathcal{E}_{14}^+) \rangle = 2$$

since $N_{14,14}^{28} = 1$ Since $[\alpha_{14}^+] = 2[\alpha_4^+]$, we conclude by the same argument as used in Example 5.1 that the endomorphism $\eta(\alpha_{14}^+, \mathcal{E}_{14}^+)$ decomposes into two mutually inequivalent irreducible endomorphisms with equal statistical dimensions. We conclude that the system

$$\begin{aligned} &\{\eta(\alpha_j^+, \mathcal{E}_j^+) \mid j = 0, 2, 4, \ldots 14\} \cup \{\eta(\alpha_j^+, \mathcal{E}_j^+)\eta(\tau_2, \mathcal{E}_2^-) \mid j = 0, 2, 4, \ldots 14\} \\ &\cup \{\eta(\alpha_{14}^+, \mathcal{E}_{14}^+)_1, \eta(\alpha_{14}^+, \mathcal{E}_{14}^+)_2, (\eta(\alpha_{14}^+, \mathcal{E}_{14}^+)\eta(\tau_2, \mathcal{E}_2^-))_1, (\eta(\alpha_{14}^+, \mathcal{E}_{14}^+)\eta(\tau_2, \mathcal{E}_2^-))_2\} \end{aligned} \tag{23}$$

gives the entire $\mathcal{D}(\Delta)$, where $\eta(\alpha_{14}^+, \mathcal{E}_{14}^+)_1$ and $\eta(\alpha_{14}^+, \mathcal{E}_{14}^+)_2$ are irreducible endomorphisms arising from decomposition of $\eta(\alpha_{14}^+, \mathcal{E}_{14}^+)$, and $(\eta(\alpha_{14}^+, \mathcal{E}_{14}^+)\eta(\tau_2, \mathcal{E}_2^-))_1$ and $(\eta(\alpha_{14}^+, \mathcal{E}_{14}^+)\eta(\tau_2, \mathcal{E}_2^-))_2$ are irreducible endomorphisms arising from decomposition of $\eta(\alpha_{14}^+, \mathcal{E}_{14}^+)\eta(\tau_2, \mathcal{E}_2^-)$. We can then draw the dual principal graph of the subfactor $M \otimes M^{\mathrm{opp}} \subset R$ as in Fig. 2, where we use a similar convention for labeling vertices to the one in Fig. 1. The procedure to get the labels for the R-R morphisms here is again an orbifold procedure of order 2 for the labels (j, ℓ) with $= 0, 2, 4, \ldots, 28$, $\ell = 0, 2$ with symmetry $(j, \ell) \leftrightarrow (28 - j, \ell)$.

Example 5.3 We next study the subfactor $N \subset M$ arising from the conformal inclusion $SU(3)_5 \subset SU(6)_1$, as treated in [2, Sect. 2.3 (iv)]. The 21 irreducible endomorphisms in the full $SU(3)_5$ system are labelled as ${}_N\mathcal{X}_N = \{\lambda_{(p,q)} \mid 0 \le q \le p \le 5\}$ as usual. Those in the ambichiral system ${}_M\mathcal{X}_M^0$ are labelled with the six circled vertices of the graph $\mathcal{E}^{(8)}$ in [2, Fig. 11] and obey $\mathbb{Z}_6$ fusion rules. We label

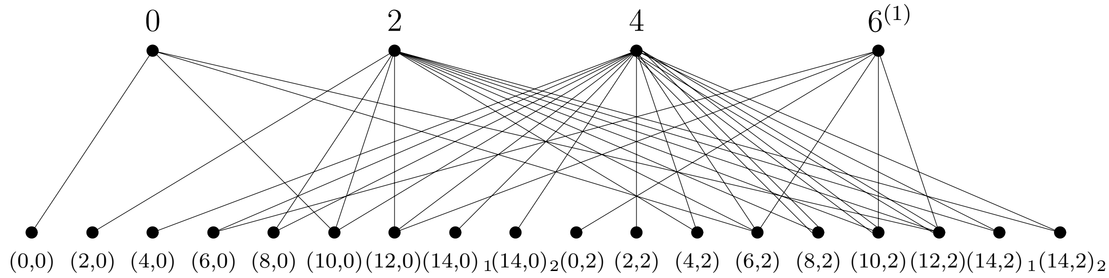

Figure 2: The dual principal graph for the Longo-Rehren subfactor arising from E_8

them as τ_ℓ, $\ell = 0, 1, \ldots, 5$ such that the fusion rules read $[\tau_\ell][\tau_{\ell'}] = [\tau_{\ell+\ell' (\mathrm{mod}\ 6)}]$. The non-vanishing branching coefficients $b^+_{\tau_\ell, \lambda_{(p,q)}} = b_{\ell,(p,q)}$ are

$$b_{0,(0,0)} = b_{0,(4,2)} = b_{1,(2,0)} = b_{1,(5,3)} = b_{2,(3,1)} = b_{2,(5,5)} = \\ = b_{3,(3,0)} = b_{3,(3,3)} = b_{4,(3,2)} = b_{4,(5,0)} = b_{5,(2,2)} = b_{5,(5,2)} = 1$$

The colour zero subsystem is given by

$${}_N\mathcal{Y}_N = \{\lambda_{(0,0)}, \lambda_{(3,0)}, \lambda_{(2,1)}, \lambda_{(5,1)}, \lambda_{(4,2)}, \lambda_{(3,3)}, \lambda_{(5,4)}\}\,.$$

The situation is particularly simple as this system is still non-degenerate, i.e. ${}_N\mathcal{Y}_N^{\mathrm{deg}} = \{\mathrm{id}\}$. (Note that a degenerate subsystem must be the dual of a group.) Then ${}_M\mathcal{Y}_M^+$ consists of four endomorphisms $\alpha^+_{(0,0)}$, $\alpha^+_{(5,1)}$, $\alpha^+_{(5,4)}$, $\alpha^{+,(1)}_{(3,0)}$ labelled as in [2, Fig. 11], where we write $\alpha^+_{(p,q)}$ for $\alpha^+_{\lambda_{(p,q)}}$. We study the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R(\Delta)$ arising from this system $\Delta = {}_M\mathcal{Y}_M^+$. Again, Proposition 4.7 implies that we only need to consider η-extensions of color zero morphisms to obtain the entire $\mathcal{D}(\Delta)$. But in fact, due to the non-degeneracy Corollary 4.8 applies and yields that $\mathcal{D}(\Delta)$ is equivalent (as C^*-tensor categories) to ${}_N\mathcal{Y}_N \times \mathbb{Z}_2$ as the subsystem of ${}_M\mathcal{Y}_M^0 \subset {}_M\mathcal{X}_M^0$ of color zero ambichirals consists of τ_0, τ_3, obeying $\mathbb{Z}_2$ fusion rules. In particular, here is no orbifold procedure. Alternatively, one checks by Theorem 4.6 easily that $\{\eta(\alpha^+_\lambda, \mathcal{E}^+_\lambda)\eta(\tau_k, \mathcal{E}^-_k) \mid \lambda \in {}_N\mathcal{Y}_N\,,\ k = 0, 3\}$ constitutes as set of 14 irreducible, mutually inequivalent endomorphisms, hence yielding the entire quantum double system $\mathcal{D}(\Delta)$. The subfactor $\alpha^+_{(1,0)}(M) \subset M$ is a natural analogue of the subfactors with principal graphs E_6 or E_8, and our Longo-Rehren subfactor corresponds to the asymptotic inclusion of (the corresponding hyperfinite II_1 subfactor of) this inclusion. From [2, Fig. 11], it is easy to extract the dual principal graph of the subfactor $\alpha^+_{(1,0)}(M) \subset M$, drawn in Fig. 3. Using Proposition 2.7 it is now a straight-forward calculation yielding the dual principal graph of the associated Longo-Rehren inclusion, displayed in Fig. 4. Here we used the short-hand notation $(p, q; \ell)$ for $\eta(\alpha^+_{(p,q)}, \mathcal{E}^+_{(p,q)})\eta(\tau_\ell, \mathcal{E}^-_\ell)$ and an obvious notation for the vertices labelled by morphisms in ${}_M\mathcal{Y}_M^+$. It seems that the system $\mathcal{D}(\Delta)$ is equivalent to the system of Q_∞-Q_∞ bimodules arising from the asymptotic inclusion $Q \vee (Q' \cap Q_\infty)$ of the

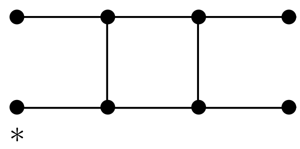

Figure 3: The (dual) principal graph for the subfactor $\alpha^+_{(1,0)}(M) \subset M$

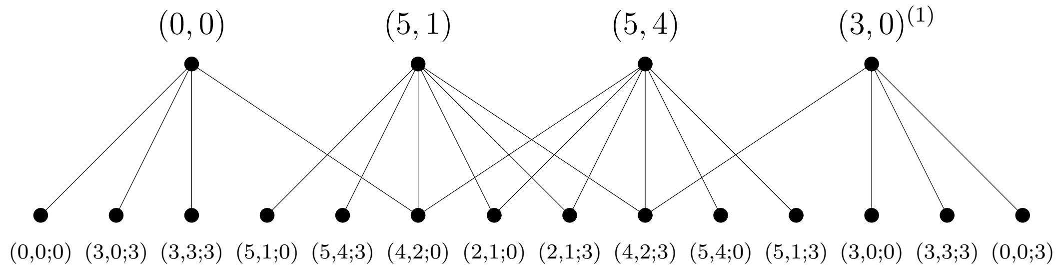

Figure 4: The dual principal graph for the Longo-Rehren subfactor arising from $\mathcal{E}^{(8)}$

hyperfinite II_1 subfactor $P \subset Q$ with principal graph A_7 of Jones [16], but we have no proof.

Example 5.4 We next study the subfactor $N \subset M$ arising from the conformal inclusion $SU(3)_9 \subset (\mathrm{E}_6)_1$, as treated in [3, Sect. 6.4]. The 55 irreducible endomorphisms in ${}_N\mathcal{X}_N$ are labelled with $\lambda_{(p,q)}$, $0 \le q \le p \le 9$ as usual. The chiral system ${}_M\mathcal{X}_M^+$ corresponds to the vertices of the graph $\mathcal{E}^{(12)}$, and the ambichiral system ${}_M\mathcal{X}_M^0$ to the three vertices marked with circles in [3, Fig. 12], obeying the $\mathbb{Z}_3$ fusion rules. We label them as τ_ℓ, $\ell = 0,1,2$, so that $[\tau_\ell][\tau_{\ell'}] = [\tau_{\ell+\ell'(\mathrm{mod}\ 3)}]$.

The non-vanishing branching coefficients $b^+_{\tau_\ell,\lambda_{(p,q)}} = b_{\ell,(p,q)}$ are

$$\begin{aligned} & b_{0,(0,0)} = b_{0,(5,1)} = b_{0,(5,4)} = b_{0,(8,4)} = b_{0,(9,0)} = b_{0,(9,9)} = \\ & = b_{1,(4,2)} = b_{1,(7,1)} = b_{1,(7,7)} = b_{2,(4,2)} = b_{2,(7,1)} = b_{2,(7,7)} = 1 \end{aligned}$$

The color zero subsystem ${}_N\mathcal{Y}_N$ is given by those 19 morphisms $\lambda_{(p,q)} \in {}_N\mathcal{X}_N$ subject to $p+q \in 3\mathbb{Z}$. It now contains the simple currents, and as a consequence [3, Lemma 6.11] we have ${}_N\mathcal{Y}_N^{\mathrm{deg}} = \{\lambda_{(0,0)}, \lambda_{(9,0)}, \lambda_{(9,9)}\}$. Then the system ${}_M\mathcal{Y}_M^+$ consists of four endomorphisms $\alpha^+_{(0,0)} = \tau_0$, $\alpha^+_{(2,1)}$, τ_1, τ_2, so that in particular ${}_M\mathcal{X}_M^0 = {}_M\mathcal{Y}_M^0$. (We use labels as in [3, Fig. 12], apart from denoting the ambichiral η_j by τ_j here, $j = 1,2$, as η_j is obviously no suitable notation when considering η-extensions.) As usual, we study the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R(\Delta)$ arising from $\Delta = {}_M\mathcal{Y}_M^+$. Since the degenerate morphisms appear in the vacuum column of the modular invariant, we only need to consider η-extensions of α^+_λ and τ with $\lambda \in {}_N\mathcal{Y}_N$

and $\tau \in {}_M\mathcal{Y}_M^0$ only, thanks to Proposition 4.7. The subfactor $\alpha^+_{(1,0)}(M) \subset M$ is again a natural analogue of the subfactors with principal graphs E_6 and E_8. From [3, Fig. 12] it is easy extract the dual principal graph of the subfactor $\alpha^+_{(1,0)}(M) \subset M$, drawn in Fig. 5. By Lemma 4.2 we find that $\eta(\alpha^+_{(0,0)}, \mathcal{E}^+_{(0,0)})$, $\eta(\alpha^+_{(2,1)}, \mathcal{E}^+_{(2,1)})$,

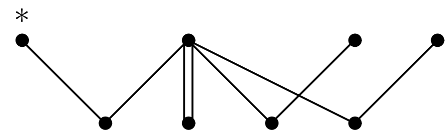

Figure 5: The (dual) principal graph for the subfactor $\alpha^+_{(1,0)}(M) \subset M$

$\eta(\alpha^+_{(3,0)}, \mathcal{E}^+_{(3,0)})$, $\eta(\alpha^+_{(3,3)}, \mathcal{E}^+_{(3,3)})$, $\eta(\alpha^+_{(4,2)}, \mathcal{E}^+_{(4,2)})$, and $\eta(\alpha^+_{(5,1)}, \mathcal{E}^+_{(5,1)})$ are irreducible and mutually inequivalent endomorphisms of R. We similarly obtain

$$\langle \eta(\alpha^+_{(6,3)}, \mathcal{E}^+_{(6,3)}), \eta(\alpha^+_{(6,3)}, \mathcal{E}^+_{(6,3)}) \rangle = 3\,,$$

and find that $\eta(\alpha^+_{(6,3)}, \mathcal{E}^+_{(6,3)})$ is disjoint from the others since $\lambda_{(6,3)}$ is a fixed point of the simple currents. Consequently, the decomposition of $[\eta(\alpha^+_{(6,3)}, \mathcal{E}^+_{(6,3)})]$ yields three new irreducible sectors. Next, Theorem 4.6 yields $\langle \eta(\alpha^+_\lambda, \mathcal{E}^+_\lambda), \eta(\tau_\ell, \mathcal{E}^-_\ell) \rangle = 0$ for $\lambda \in {}_N\mathcal{Y}_N$ and $\ell = 1, 2$, because $[\tau_1]$ and $[\tau_2]$ do not appear as subsectors of $[\alpha^+_\lambda]$ for $\lambda \in {}_N\mathcal{Y}_N^{\mathrm{deg}}$. Similarly we find that $\eta(\alpha^+_\lambda, \mathcal{E}^+_\lambda)\eta(\tau_\ell, \mathcal{E}^-_\ell)$ are disjoint for different $\ell = 0, 1, 2$. We now have a set of irreducible, mutually inequivalent endomorphisms of R consisting of 27 endomorphisms, which can be considered as ${}_N\mathcal{Y}_N/\mathbb{Z}_3 \times \mathbb{Z}_3$. Here for the orbifold ${}_N\mathcal{Y}_N/\mathbb{Z}_3$, 18 objects collapse into 6 objects by identification arising from a $\mathbb{Z}_3$ symmetry, and the fixed point of the symmetry splits into 3 objects. The total number of the irreducible objects is therefore $(6 + 3) \times 3 = 27$. Let the statistical dimensions of the irreducible morphisms appearing in the decomposition of $\eta(\alpha^+_{(6,3)}, \mathcal{E}^+_{(6,3)})$ be d_1, d_2, d_3 respectively. We then have $d_1 + d_2 + d_3 = d_{(6,3)}$. The square sum $d_1^2 + d_2^2 + d_3^2$ attains the minimum $d_{(6,3)}^2/3$ with $d_1 = d_2 = d_3 = d_{(6,3)}/3$ under the constraint $d_1 + d_2 + d_3 = d_{(6,3)}$. Assume for contradiction that we are off the minimum. Then $d_1^2 + d_2^2 + d_3^2 > d_{(6,3)}^2/3$, and in turn the global index of the system ${}_N\mathcal{Y}_N/\mathbb{Z}_3 \times \mathbb{Z}_3$ is strictly bigger than $[[{}_N\mathcal{Y}_N]]$. But then it exceeds $[[\mathcal{D}(\Delta)]] = [[{}_M\mathcal{Y}_M^+]]^2$ because [4, Prop. 3.1] tells us that $[[{}_M\mathcal{Y}_M^+]]^2 = [[{}_N\mathcal{Y}_N]][[{}_M\mathcal{Y}_M^0]]/\sum_{\lambda \in {}_N\mathcal{Y}_N^{\mathrm{deg}}} d_\lambda Z_{\lambda,0}$ which obviously yields $[[{}_M\mathcal{Y}_M^+]]^2 = [[{}_N\mathcal{Y}_N]]$ here; contradiction. We conclude that $d_1 = d_2 = d_3 = d_{(6,3)}/3$ and that the above set of morphisms gives the entire system $\mathcal{D}(\Delta)$.

Example 5.5 We finally study the subfactor $N \subset M$ arising from the conformal inclusion $SU(3)_{21} \subset (\mathrm{E}_7)_1$, as treated in [3, Sect. 6.4]. The 253 irreducible endomorphisms in ${}_N\mathcal{X}_N$ are labelled with $\lambda_{(p,q)}$, $0 \le q \le p \le 21$, as usual. The morphisms in ${}_M\mathcal{X}_M^+$ correspond to the vertices of the graph $\mathcal{E}^{(24)}$, and those in the ambichiral system ${}_M\mathcal{X}_M^0$ to the two encircled vertices in [3, Fig. 13], the latter obeying the $\mathbb{Z}_2$

fusion rules. We label them as τ_ℓ here, $\ell = 0, 1$, so that $[\tau_1][\tau_1] = [\tau_0]$. (Note that our τ_1 is denoted by ϵ in[3, Fig. 13].) The color zero subsystem ${}_N\mathcal{Y}_N$ is given by those 85 morphisms $\lambda_{(p,q)} \in {}_N\mathcal{X}_N$ subject to $p + q \in 3\mathbb{Z}$. It now contains the simple currents, and as a consequence we have ${}_N\mathcal{Y}_N^{\mathrm{deg}} = \{\lambda_{(0,0)}, \lambda_{(21,0)}, \lambda_{(21,21)}\}$. The system ${}_M\mathcal{Y}_M^+$ consists of eight endomorphisms $\alpha^+_{(0,0)} = \tau_0$, $\alpha^+_{(2,1)}$, $\alpha^{+,(1)}_{(4,2)}$, $\alpha^{+,(2)}_{(4,2)}$, $\alpha^+_{(3,0)}$, $\alpha^+_{(3,3)}$, $\alpha^{+,(1)}_{(5,1)}$, and τ_1. We study the Longo-Rehren subfactor $M \otimes M^{\mathrm{opp}} \subset R(\Delta)$ arising from this system $\Delta = {}_M\mathcal{Y}_M^+$, which corresponds to the asymptotic inclusion of the subfactor $\alpha^+_{(1,0)}(M) \subset M$ as the natural analogue of the subfactors with principal graphs E_6, E_8. From [3, Fig. 13] we extract the dual principal graph of the subfactor $\alpha^+_{(1,0)}(M) \subset M$, displayed in Fig. 6. Since the degenerate morphisms appear in the vacuum column

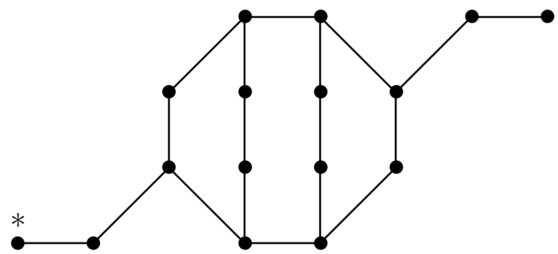

Figure 6: The (dual) principal graph for the subfactor $\alpha^+_{(1,0)}(M) \subset M$

of the modular invariant, the situation is similar to Examples 5.2 and 5.4. Their $\mathbb{Z}_3$ symmetry has $\lambda_{(14,7)} \in {}_N\mathcal{Y}_N$ as a fixed point, and the other 84 endomorphisms give 28 orbits under this symmetry. We then have $\langle \eta(\alpha^+_{(14,7)}, \mathcal{E}^+_{(14,7)}), \eta(\alpha^+_{(14,7)}, \mathcal{E}^+_{(14,7)}) \rangle = 3$. Along the same lines as in Example 5.4, we conclude that the system $\mathcal{D}(\Delta)$ contains $(28+3) \times 2 = 62$ irreducible endomorphisms, corresponding to ${}_N\mathcal{Y}_N/\mathbb{Z}_3 \times \mathbb{Z}_2$. Namely, the 28 irreducible endomorphisms $\eta(\alpha^+_\lambda, \mathcal{E}^+_\lambda)$ where we select one $\lambda \in {}_N\mathcal{Y}_N$ of each $\mathbb{Z}_3$ orbit together with the three irreducible endomorphisms of equal statistical dimensions arising from decomposition of $\eta(\alpha^+_{(14,7)}, \mathcal{E}^+_{(14,7)})$ correspond to ${}_N\mathcal{Y}_N/\mathbb{Z}_3$, and the blowing up by $\mathbb{Z}_2$ arises from multiplication with $\eta(\tau_\ell, \mathcal{E}^-_\ell)$, $\ell = 0, 1$.

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