On α-Induction, Chiral Generators and Modular Invariants for Subfactors

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Abstract

We consider a type III subfactor $N \subset M$ of finite index with a finite system of braided N-N morphisms which includes the irreducible constituents of the dual canonical endomorphism. We apply α-induction and, developing further some ideas of Ocneanu, we define chiral generators for the double triangle algebra. Using a new concept of intertwining braiding fusion relations, we show that the chiral generators can be naturally identified with the α-induced sectors. A matrix Z is defined and shown to commute with the S- and T-matrices arising from the braiding. If the braiding is non-degenerate, then Z is a "modular invariant mass matrix" in the usual sense of conformal field theory. We show that in that case the fusion rule algebra of the dual system of M-M morphisms is generated by the images of both kinds of α-induction, and that the structural information about its irreducible representations is encoded in the mass matrix Z. Our analysis sheds further light on the connection between (the classifications of) modular invariants and subfactors, and we will construct and analyze modular invariants from $SU(n)_k$ loop group subfactors in a forthcoming publication, including the treatment of all $SU(2)_k$ modular invariants.

Contents

1 Introduction

It is a surprising fact that a series of at first sight unrelated phenomena in mathematics and physics are governed by the scheme of A-D-E Dynkin diagrams, such as simple Lie algebras, finite subgroups of $SL(2;\mathbb{C})$, simple singularities of complex surfaces, quivers of finite type, modular invariant partition functions of $SU(2)$ WZW models and subfactors of Jones index less than four. Though a good understanding of the interrelations has not yet been achieved, this coincidence indicates that there are deep connections between these different fields which even seem to go beyond the A-D-E governed cases, e.g. finite subgroups of $SL(n;\mathbb{C})$, modular invariants of $SU(n)$ WZW models, or (certain) $SU(n)_k$ subfactors of larger index. This paper is addressed to the relation between the (classifications of) modular invariants in conformal field theory and subfactors in operator algebras.

In rational (chiral) conformal field theory one deals with a chiral algebra which possesses a certain finite spectrum of representations (or superselection sectors) π_λ acting on a Hilbert space $\mathcal{H}_\lambda$. Its characters $\chi_\lambda(\tau) = \mathrm{tr}_{\mathcal{H}_\lambda}(\mathrm{e}^{2\pi\mathrm{i}\tau(L_0-c/24)})$, $\mathrm{Im}(\tau) > 0$, L_0 being the conformal Hamiltonian and c the central charge, transform unitarily under "reparametrization of the torus", i.e. there are matrices S and T such that

$$\chi_\lambda(-1/\tau) = \sum_\mu S_{\lambda,\mu}\chi_\mu(\tau)\,, \qquad \chi_\lambda(\tau+1) = \sum_\mu T_{\lambda,\mu}\chi_\mu(\tau)\,,$$

which are the generators of a unitary representation of the (double cover of the) modular group $SL(2;\mathbb{Z})$ in which T is diagonal.[1] In order to classify conformal field theories, in particular extensions in a certain sense of a given theory, one searches for modular invariant partition functions $Z(\tau) = Z(-1/\tau) = Z(\tau+1)$ of the form

$$Z(\tau) = \sum_{\lambda,\mu} Z_{\lambda,\mu}\chi_\lambda(\tau)\chi_\mu(\tau)^*\,,$$

where

$$Z_{\lambda,\mu} = 0,1,2,\dots\,, \qquad\qquad Z_{0,0} = 1\,. \tag{1}$$

Here the label "0" refers to the "vacuum" representation, and the condition $Z_{0,0} = 1$ reflects the physical concept of uniqueness of the vacuum state. The matrix Z arising this way is called a modular invariant mass matrix. Mathematically speaking, the problem can be rephrased like this: Find all the matrices Z in the commutant of the unitary representation of $SL(2;\mathbb{Z})$ defined by S and T subject to the conditions in Eq. (1). In this paper we study this mathematical problem in the subfactor context. We start with a von Neumann algebra, more precisely a factor N endowed with a system of braided endomorphisms. Such a braiding defines matrices S and T which provide a unitary representation of $SL(2;\mathbb{Z})$ if it is non-degenerate. We then study embeddings $N \subset M$ in larger factors M which are in a certain sense compatible with the braided system of endomorphisms. We show that such an embedding $N \subset M$ determines a modular invariant mass matrix in exactly the sense specified above.

Longo and Rehren have studied nets of subfactors and defined a useful formula to extend a localized transportable endomorphism of the smaller to the larger observable algebra, realizing a suggestion in [43]. Xu [47, 48] has worked on essentially the same construction applied to subfactors arising from conformal inclusions with the loop group construction of A. Wassermann [45]. Two of us systematically analyzed the Longo-Rehren extension for nets of subfactors on S^1 [2, 4]. As sectors, a reciprocity between extension and restriction of localized transportable endomorphisms was established, analogous to the induction-restriction machinery of group representations, and therefore the extension was called α-induction in order to avoid confusion with the different sector induction. It was also noticed in [2] that the extended endomorphisms leave local algebras invariant and hence α-induction can also be considered as a map which takes certain endomorphisms of a local subfactor to endomorphisms of the embedding factor. This theory was applied to nets arising from conformal field theory models in [3, 4], and it was shown that for all type I modular invariants

[1] More precisely, for current algebras the characters depend also on other variables than τ, corresponding to Cartan subalgebra generators which are omitted here for simplicity. But these variables are responsible that one is in general dealing with the whole group $SL(2;\mathbb{Z})$ rather than $PSL(2;\mathbb{Z})$.

of $SU(2)$ respectively $SU(3)$ there are associated nets of subfactors and in turn α-induction gives rise to fusion graphs. In fact it was shown that that these graphs are the A-D-E Dynkin diagrams respectively their generalizations of [7, 8], and this is no accident: The homomorphism property of α-induction relates the spectrum of the fusion graphs to the non-zero diagonal entries of the modular invariant mass matrix.

A few months after the work of Longo-Rehren, Ocneanu presented his theory of "quantum symmetries" of Coxeter graphs and gave lectures [39] one year later. He introduced a notion of a "double triangle algebra" and defined elements $p_j^{\pm}$ which we refer to as "chiral generators" as they were not specifically named there. Ocneanu's analysis has much in common with work of Xu [47] and two of us [3, 4] about subfactors of type E_6, E_8 and $\mathrm{D}_{\mathrm{even}}$. The reason for this is that the same structures are studied from different viewpoints, as we will outline in this paper.

We start with a fairly general setting which admits both constructions, α-induction as well as Ocneanu's double triangle algebras and chiral generators. Namely, we consider a type III subfactor $N \subset M$ of finite index with a finite system of N-N morphisms which includes the irreducible constituents of the dual canonical endomorphism. (A "system of morphisms" means essentially that, as sectors, the morphisms form a closed algebra under the sector "fusion" product, see Definition 2.1 below.) Therefore the subfactor is in particular forced to have finite depth. The inclusion structure associates to the N-N system automatically N-M, M-N and M-M systems. The typical situation is that the system of M-M morphisms is the "unknown part" of the theory. As an easy reformulation of Ocneanu's idea from his work on Goodman-de la Harpe-Jones subfactors associated with Dynkin diagrams one can define the double triangle algebra for such a setting, and it provides a powerful tool to gain information about the "unknown part" from the "known part" of the theory. Namely, the double triangle algebra is a direct sum of intertwiner spaces equipped with two different product structures, and its center $\mathcal{Z}_h$ with respect to the "horizontal product" turns out to be isomorphic to the (in general non-commutative) fusion rule algebra of the M-M system when endowed with the "vertical product". This kind of duality is the subfactor analogue to the group algebra with its pointwise and convolution products.

Under the assumption that the N-N system is braided there is automatically the notion of α-induction, which extends N-N to (possibly reducible) M-M morphisms. (This notion does not even depend on the finite depth condition.) The braiding provides powerful tools to analyze the structure of the center $\mathcal{Z}_h$ at the same time, and the analysis is most conveniently carried out with a graphical intertwiner calculus which will be explained in detail in this paper. Besides the standard "braiding fusion symmetries" for wire diagrams representing intertwiners of the braided N-N morphisms, we show that the theory of α-induction gives rise naturally to an extended symmetry which we call "intertwining braiding fusion relations". This reduces all graphical manipulations representing the relations between intertwiners to easily visible purely topological moves, and it allows us to work without the "sliding moves along walls" involving "quantum $6j$-symbols for subfactors" which are the main technical tool in [39]. With a braiding on the N-N system we can define chiral generators $p^\pm_\lambda$ in the center $\mathscr{Z}_h$, and our notion essentially coincides with Ocneanu's definition of elements $p^\pm_j$ given graphically in his A-D-E setup. We show that the decomposition of the $p^\pm_\lambda$'s into minimal central projections in $\mathscr{Z}_h$ corresponds exactly to the sector decomposition of the α-induced sectors $[\alpha^\pm_\lambda]$, and therefore they can be naturally identified.

As shown by Rehren [40], a system of braided endomorphisms gives rise to S- and T-matrices which provide a unitary representation of the modular group $SL(2;\mathbb{Z})$ whenever the braiding is non-degenerate. (Relations between modular S- and T-matrices and braiding data are also discussed in [35, 14, 13].) In terms of α-induction we define a matrix Z with entries $Z_{\lambda,\mu} = \langle \alpha^+_\lambda, \alpha^-_\mu \rangle$ for N-N morphisms λ, μ, where the brackets denote the dimension of the intertwiner space $\mathrm{Hom}(\alpha^+_\lambda, \alpha^-_\mu)$. As it corresponds to the "vacuum" in physical applications, we use the label "0" for the identity morphism id_N, and hence our matrix Z satisfies the conditions in Eq. (1), where now $Z_{0,0} = 1$ is just the factor property of M. We show that Z commutes with S and T and therefore Z is a "modular invariant mass matrix" in the sense of conformal field theory if the braiding is non-degenerate. In fact, the non-degenerate case is the most interesting one, as in the $SU(n)_k$ examples in conformal field theory. We apply an argument of Ocneanu to our situation to show that in that case, due to the identification with chiral generators, both kinds of α-induction together generate the whole M-M fusion rule algebra. Moreover, the essential information about its representation theory (or equivalently, about the decomposition of the center $\mathscr{Z}_h$ with the vertical product into simple matrix algebras) is then encoded in the mass matrix Z: We show that the irreducible representations of the M-M fusion rule algebra are labelled by pairs λ, μ with $Z_{\lambda,\mu} \neq 0$, and that their dimensions are given exactly by the number $Z_{\lambda,\mu}$. Consequently, the M-M fusion rules are then commutative if and only if all $Z_{\lambda,\mu} \in \{0, 1\}$. An analogous result has been claimed by Ocneanu for his A-D-E setting related to the modular invariant mass matrices of the $SU(2)$ WZW models of [6, 23]. He has his own geometric construction of modular invariants sketched in the lectures but not included in the lecture notes [39]. Our construction is different and based on the results of [4], and it shows that the structural results do not depend on the very special properties of Dynkin diagrams and hold in a far more general context. We also analyze the representation of the M-M fusion rule algebra arising from its left action on M-N sectors. As corollaries of our analysis we find that the number of N-M (or M-N) morphisms is given by the trace $\mathrm{tr}(Z)$, whereas the number of M-M morphisms is given by $\mathrm{tr}(Z\,{}^{\mathrm{t}}Z)$.

In a forthcoming publication we will further analyze and apply our construction to subfactors constructed by means of the level k positive energy representations of the $SU(n)$ loop group theory. For these examples, the braiding is always non-degenerate and, moreover, the S- and T-matrices are the modular matrices performing the character transformations of the corresponding $SU(n)_k$ WZW theory. Therefore the construction of braided subfactors[2] for these models yields non-diagonal modular

[2]We remark that our short-hand notion of a "braided subfactor" meaning a subfactor for which

invariants Z. E.g. for $SU(2)_k$ one can construct the subfactors in terms of local loop groups which recover the A-D-E modular invariants of [6, 23]. In our setting also the "type II" or "non-blockdiagonal" invariants can be treated by dropping the chiral locality condition. (The chiral locality condition, expressing local commutativity of the extended chiral theory in the formulation of nets of subfactors [33], implies "$\alpha\sigma$-reciprocity" [2] which in turn forces the modular invariant to be of type I. Detailed explanation and non-local examples will be provided in [5].) Thus this paper extends the known results on conformal inclusions [47, 48, 3, 4] and simple current extensions [3, 4] of $SU(n)_k$, and it generally relates (the classification of) modular invariants to (non-degenerately) braided subfactors. Furthermore our results prove two conjectures by two of us [4, Conj. 7.1 & 7.2].

This paper is organized as follows. In Sect. 2 we review some basic facts about morphisms, intertwiners, sectors and braidings, and we reformulate Rehren's result about S- and T-matrices arising from superselection sectors in our context of braided factors. In Sect. 3 we establish the graphical methods for the intertwiner calculus we use in this paper. The abstract mathematical structure underlying the basic graphical calculus (Subsect. 3.1) is "strict monoidal C^*-categories" [9]. Graphical methods for calculations involving fusion and braiding have been used in various publications, see e.g. [34, 28, 46, 15, 14, 24, 22]. However, for our purposes it turns out to be extremely important to handle normalization factors with special care, and to the best of our knowledge, a comprehensive exposition which applies to our framework has not been published somewhere. So we work out a "rotation covariant" intertwiner calculus here, based on a formulation of Frobenius reciprocity by Izumi [19]. We then define α-induction for braided subfactors and use it to extend our graphical calculus conveniently. In Sect. 4 we present the double triangle algebra and analyze its properties. In Sect. 5 we present our version of Ocneanu's graphical notion of chiral generators, and we show that it can be naturally identified with the α-induced sectors. We then define the "mass matrix" Z and show that it commutes with the S- and T-matrices of the N-N system. Assuming now that the braiding is non-degenerate, we show that the M-M fusion rule algebra is generated by the images of the two kinds ($+$ and $-$) of α-induction. In Sect. 6 we decompose $\mathcal{Z}_h$ with the vertical product into simple matrix algebras which is equivalent to the determination of all the irreducible representations of the M-M fusion rule algebra, and we show that their dimensions are given by the entries of the modular invariant mass matrix. Then we analyze the representation arising from the left action on M-N sectors. In Sect. 7 we finally conclude this paper with general remarks and comments and an outlook to the applications to subfactors arising from conformal field theory which will be treated in [5].

Assumptions 4.1 and 5.1 below hold is different from the notion used in [31].

2 Preliminaries

2.1 Morphisms and sectors

For our purposes it turns out to be convenient to make use of the formulation of sectors between different factors. We follow here (up to minor notational changes) Izumi's presentation [19, 20] based on Longo's sector theory [30]. Let A, B be infinite factors. We denote by $\mathrm{Mor}(A,B)$ the set of unital $*$-homomorphisms from A to B. We also denote $\mathrm{End}(A) = \mathrm{Mor}(A,A)$, the set of unital $*$-endomorphisms. For $\rho \in \mathrm{Mor}(A,B)$ we define the statistical dimension $d_\rho = [B:\rho(A)]^{1/2}$, where $[B:\rho(A)]$ is the minimal index [21, 29]. A morphism $\rho \in \mathrm{Mor}(A,B)$ is called irreducible if the subfactor $\rho(A) \subset B$ is irreducible, i.e. if the relative commutant $\rho(A)' \cap B$ consists only of scalar multiples of the identity in B. Two morphisms $\rho, \rho' \in \mathrm{Mor}(A,B)$ are called equivalent if there exists a unitary $u \in B$ such that $\rho'(a) = u\rho(a)u^*$ for all $a \in A$. We denote by $\mathrm{Sect}(A,B)$ the quotient of $\mathrm{Mor}(A,B)$ by unitary equivalence, and we call its elements B-A sectors. Similar to the case $A = B$, $\mathrm{Sect}(A,B)$ has the natural operations, sums and products: For $\rho_1, \rho_2 \in \mathrm{Mor}(A,B)$ choose generators $t_1, t_2 \in B$ of a Cuntz algebra $\mathcal{O}_2$, i.e. such that $t_i^* t_j = \delta_{i,j}\mathbf{1}$ and $t_1 t_1^* + t_2 t_2^* = \mathbf{1}$. Define $\rho \in \mathrm{Mor}(A,B)$ by putting $\rho(a) = t_1\rho_1(a)t_1^* + t_2\rho_2(a)t_2^*$ for all $a \in A$, and then the sum of sectors is defined as $[\rho_1] \oplus [\rho_2] = [\rho]$. The product of sectors comes from the composition of endomorphisms, $[\rho_1][\rho_2] = [\rho_1 \circ \rho_2]$. We often omit the composition symbol "$\circ$", so $[\rho_1][\rho_2] = [\rho_1\rho_2]$. The statistical dimension is an invariant for sectors (i.e. equivalent morphisms have equal dimension) and is additive and multiplicative with respect to these operations. Moreover, for $[\rho] \in \mathrm{Sect}(A,B)$ there is a unique conjugate sector $\overline{[\rho]} \in \mathrm{Sect}(B,A)$ such that, if $[\rho]$ is irreducible, $\overline{[\rho]}$ is irreducible as well and $\overline{[\rho]} \times [\rho]$ contains the identity sector $[\mathrm{id}_A]$ and $[\rho] \times \overline{[\rho]}$ contains $[\mathrm{id}_B]$ precisely once. We choose a representative endomorphism of $\overline{[\rho]}$ and denote it naturally by $\bar{\rho}$, thus $[\bar{\rho}] = \overline{[\rho]}$. For conjugates we have $d_{\bar{\rho}} = d_\rho$. As for bimodules one may decorate B-A sectors $[\rho]$ with suffixes, ${}_B[\rho]_A$, and then we can multiply ${}_B[\rho]_A \times {}_A[\sigma]_B$ but not, for instance, ${}_B[\rho]_A$ with itself. For $\rho, \tau \in \mathrm{Mor}(A,B)$ we denote

$$\mathrm{Hom}(\rho,\tau) = \{t \in B : t\,\rho(a) = \tau(a)\,t\,,\ \ a \in A\}$$

and

$$\langle \rho, \tau \rangle = \dim \mathrm{Hom}(\rho,\tau)\,.$$

If $[\rho] = [\rho_1] \oplus [\rho_2]$ then

$$\langle \rho, \tau \rangle = \langle \rho_1, \tau \rangle + \langle \rho_2, \tau \rangle\,.$$

Note that if ρ is irreducible then for $t, t' \in \mathrm{Hom}(\rho,\tau)$ it follows that t^*t' is a scalar and then putting

$$t^*t' = \langle t, t' \rangle \mathbf{1}_B \tag{2}$$

defines an inner product on $\mathrm{Hom}(\rho,\tau)$. One often calls $\mathrm{Hom}(\rho,\tau)$ a "Hilbert space of isometries" in this case.

If $\rho \in \mathrm{Mor}(A,B)$ with $d_\rho < \infty$ then $\bar\rho \in \mathrm{Mor}(B,A)$ is a conjugate if there are isometries $r_\rho \in \mathrm{Hom}(\mathrm{id}_A, \bar\rho\rho)$ and $\bar r_\rho \in \mathrm{Hom}(\mathrm{id}_B, \rho\bar\rho)$ such that

$$\rho(r_\rho)^* \bar r_\rho = d_\rho^{-1} \mathbf{1}_B \qquad \text{and} \qquad \bar\rho(\bar r_\rho)^* r_\rho = d_\rho^{-1} \mathbf{1}_A \,,$$

and in the case that ρ is irreducible such isometries r_ρ and $\bar r_\rho$ are unique up to a common phase. If C is another factor and $\sigma \in \mathrm{Mor}(C,A)$ and $\tau \in \mathrm{Mor}(C,B)$ are morphisms with finite statistical dimensions $d_\sigma, d_\tau < \infty$, and conjugate morphisms $\bar\sigma \in \mathrm{Mor}(A,C)$, $\bar\tau \in \mathrm{Mor}(B,C)$, respectively, then the "left and right Frobenius reciprocity maps",

$$\begin{aligned}
\mathcal{L}_\rho : \mathrm{Hom}(\tau,\rho\sigma) &\longrightarrow \mathrm{Hom}(\sigma,\bar\rho\tau)\,, \qquad && t \longmapsto \sqrt{\frac{d_\rho d_\sigma}{d_\tau}}\, \bar\rho(t)^* r_\rho \,, \\
\mathcal{R}_\rho : \mathrm{Hom}(\bar\sigma,\bar\tau\rho) &\longrightarrow \mathrm{Hom}(\bar\tau,\bar\sigma\bar\rho)\,, \qquad && s \longmapsto \sqrt{\frac{d_\rho d_\tau}{d_\sigma}}\, s^* \bar\tau(\bar r_\rho) \,,
\end{aligned}$$

are anti-linear (vector space) isomorphisms with inverses

$$\begin{aligned}
\mathcal{L}_\rho^{-1} : \mathrm{Hom}(\sigma,\bar\rho\tau) &\longrightarrow \mathrm{Hom}(\tau,\rho\sigma)\,, \qquad && x \longmapsto \sqrt{\frac{d_\rho d_\tau}{d_\sigma}}\, \rho(x)^* \bar r_\rho \,, \\
\mathcal{R}_\rho^{-1} : \mathrm{Hom}(\bar\tau,\bar\sigma\bar\rho) &\longrightarrow \mathrm{Hom}(\bar\sigma,\bar\tau\rho)\,, \qquad && y \longmapsto \sqrt{\frac{d_\rho d_\sigma}{d_\tau}}\, y^* \bar\sigma(r_\rho) \,,
\end{aligned}$$

respectively [19]. (See also [14, Sect. 5] and [13, App. A] for such formulae arising from superselection sectors.) Hence we have in particular Frobenius reciprocity [19, 32],

$$\langle \tau,\rho\sigma\rangle = \langle \bar\rho\tau,\sigma\rangle = \langle \bar\rho,\sigma\bar\tau\rangle = \langle \bar\sigma\bar\rho,\bar\tau\rangle = \langle \bar\sigma,\bar\tau\rho\rangle = \langle \tau\bar\sigma,\rho\rangle \,.$$

If τ and σ are irreducible then the Frobenius reciprocity maps are even (anti-linearly) isometric: With the inner products as in Eq. (2) on the above intertwiner spaces we have $\langle t,t'\rangle = \langle \mathcal{L}_\rho(t'),\mathcal{L}_\rho(t)\rangle$ for $t,t' \in \mathrm{Hom}(\tau,\rho\sigma)$ and similarly $\langle s,s'\rangle = \langle \mathcal{R}_\rho(s'),\mathcal{R}_\rho(s)\rangle$ for $s,s' \in \mathrm{Hom}(\bar\sigma,\bar\tau\rho)$.

The map $\phi_\rho : B \to A$ defined by

$$\phi_\rho(b) = r_\rho^* \,\bar\rho(b)\, r_\rho \,, \qquad b \in B$$

is completely positive, normal, unital $\phi_\rho(\mathbf{1}_B) = \mathbf{1}_A$ and satisfies

$$\phi_\rho(\rho(a_1) b \rho(a_2)) = a_1 \phi_\rho(b) a_2 \,, \qquad a_1,a_2 \in A\,, \quad b \in B\,.$$

The map is called the (unique) standard left inverse. The minimal conditional expectation for the subfactor $\rho(A) \subset B$ is given by $E_\rho = \rho\circ\phi_\rho$. Let now ρ,σ,τ as above be irreducible with standard left inverses $\phi_\rho,\phi_\sigma,\phi_\tau$, respectively, and let $t \in \mathrm{Hom}(\tau,\rho\sigma)$ be non-zero. Then $\phi_\rho(tt^*) \in \mathrm{Hom}(\sigma,\sigma)$ is a positive scalar and $\tilde E_\tau : B \to \tau(C)$ given by $\rho\circ\phi_\rho(tt^*)\tilde E_\tau(b) = \tau\circ\phi_\sigma\circ\phi_\rho(tbt^*)$ for all $b \in B$ is a conditional expectation for the subfactor $\tau(C) \subset B$. Since conditional expectations for irreducible subfactors are unique we conclude that

$$\phi_\tau(b)\, E_\rho(tt^*) = \phi_\sigma \circ \phi_\rho(tbt^*)\,, \qquad b \in B$$

holds for any $t \in \mathrm{Hom}(\tau, \rho\sigma)$. Moreover, t^*t' is a scalar for any $t, t' \in \mathrm{Hom}(\tau, \rho\sigma)$, $t^*t' = \langle t, t' \rangle \mathbf{1}_B$, and so is $\mathcal{L}_\rho(t)^* \mathcal{L}_\rho(t')$, in fact

$$\langle t, t' \rangle \mathbf{1}_A = \langle \mathcal{L}_\rho(t'), \mathcal{L}_\rho(t) \rangle \mathbf{1}_A \equiv \mathcal{L}_\rho(t')^* \mathcal{L}_\rho(t) = \frac{d_\rho d_\sigma}{d_\tau}\, r_\rho^* \bar{\rho}(t't^*) r_\rho$$

and this is

$$\phi_\rho(t't^*) = \frac{d_\tau}{d_\rho d_\sigma}\, \langle t, t' \rangle \mathbf{1}_A\,. \tag{3}$$

Now let $N \subset M$ be an infinite subfactor of finite index. Let $\gamma \in \mathrm{End}(M)$ be a canonical endomorphism from M into N and $\theta = \gamma|_N \in \mathrm{End}(N)$. By $\iota \in \mathrm{Mor}(N, M)$ we denote the injection map, $\iota(n) = n \in M$, $n \in N$. Then $d_\iota = [M : N]^{1/2}$, and a conjugate $\bar{\iota} \in \mathrm{Mor}(M, N)$ is given by $\bar{\iota}(m) = \gamma(m) \in N$, $m \in M$. (These formulae could in fact be used to define the canonical and dual canonical endomorphism.) Note that $\gamma = \iota\bar{\iota}$ and $\theta = \bar{\iota}\iota$, and there are isometries $w \equiv r_\iota \in \mathrm{Hom}(\mathrm{id}_N, \theta)$ and $v \equiv \bar{r}_\iota \in \mathrm{Hom}(\mathrm{id}_M, \gamma)$ such that $w^*v = \gamma(v^*)w = [M : N]^{-1/2}\mathbf{1}$. Moreover, we have the pointwise equality $M = Nv$, and for each $m \in M$ the decomposition $m = nv$ yields a unique element $n \in N$. Explicitly, $n = [M : N]^{1/2} w^* \gamma(m)$.

Now let us consider a single factor A and its sectors. For a set of irreducible sectors which is closed under conjugation and irreducible decomposition of products (a "sector basis" in the notation of [2, 3, 4] in the case that the set is finite) it is often useful to choose one representative endomorphism for each sector.

Definition 2.1 We call a subset $\Delta \subset \mathrm{End}(A)$ a *system of endomorphisms* if it satisfies the following properties.

1. Each $\lambda \in \Delta$ is irreducible and has finite statistical dimension.
2. Different elements in Δ are inequivalent, i.e. different as sectors.
3. $\mathrm{id}_A \in \Delta$.
4. For any $\lambda \in \Delta$, we have a morphism $\bar{\lambda} \in \Delta$ such that $[\bar{\lambda}]$ is the conjugate sector of $[\lambda]$.
5. Δ is closed under composition and subsequent irreducible decomposition, i.e. for any $\lambda, \mu \in \Delta$ we have non-negative integers $N_{\lambda,\mu}^\nu$ with $[\lambda][\mu] = \sum_{\nu \in \Delta} N_{\lambda,\mu}^\nu [\nu]$ as sectors.

Note that we do not assume finiteness of Δ in this definition. The numbers $N_{\lambda\mu}^\nu = \langle \lambda\mu, \nu \rangle$ are called fusion coefficients. Frobenius reciprocity now reads $N_{\lambda,\mu}^\nu = N_{\bar{\lambda},\nu}^\mu = N_{\nu,\bar{\mu}}^\lambda$, and associativity of the sector product yields $\sum_{\mu \in \Delta} N_{\lambda,\mu}^\nu N_{\rho,\sigma}^\mu = \sum_{\tau \in \Delta} N_{\lambda,\rho}^\tau N_{\tau,\sigma}^\nu$.

The additivity and multiplicativity of the statistical dimension with respect to sector sums and products implies $\sum_{\nu\in\Delta} N_{\lambda,\mu}^{\nu} d_\nu = d_\lambda d_\mu$, $\lambda,\mu,\nu\in\Delta$. Defining matrices N_μ with entries $(N_\mu)_{\lambda,\nu} = N_{\lambda,\mu}^{\nu}$ gives $N_{\bar\mu}$ as the transpose of N_μ and defines the "regular representation" of the sector products, $N_\lambda N_\mu = \sum_{\nu\in\Delta} N_{\lambda,\mu}^{\nu} N_\nu$, and the statistical dimension can be regarded as a one-dimensional representation or as a simultaneous eigenvector of all matrices N_μ with eigenvalues d_μ ($\lambda,\mu,\nu\in\Delta$).

2.2 Braided endomorphisms

Let A again be an infinite factor and Δ a system of endomorphisms of A. In general the sector products are not commutative. If the sectors commute, then a "systematic choice of unitary intertwiners" in each space $\mathrm{Hom}(\lambda\mu,\mu\lambda)$, $\lambda,\mu\in\Delta$, is called a braiding (which need not exist in general). To be more precise, we give the following

Definition 2.2 We say that a system Δ of endomorphisms is *braided* if for any pair $\lambda,\mu\in\Delta$ there is a unitary operator $\varepsilon(\lambda,\mu)\in\mathrm{Hom}(\lambda\mu,\mu\lambda)$ subject to initial conditions

$$\varepsilon(\mathrm{id}_A,\mu) = \varepsilon(\lambda,\mathrm{id}_A) = \mathbf{1}\,, \tag{4}$$

and whenever $t\in\mathrm{Hom}(\lambda,\mu\nu)$ we have the braiding fusion equations (BFE's)

$$\begin{array}{rcl} \rho(t)\,\varepsilon(\lambda,\rho) & = & \varepsilon(\mu,\rho)\,\mu(\varepsilon(\nu,\rho))\,t\,, \\ t\,\varepsilon(\rho,\lambda) & = & \mu(\varepsilon(\rho,\nu))\,\varepsilon(\rho,\mu)\,\rho(t)\,, \\ \rho(t)^*\,\varepsilon(\mu,\rho)\,\mu(\varepsilon(\nu,\rho)) & = & \varepsilon(\lambda,\rho)\,t^*\,, \\ t^*\,\mu(\varepsilon(\rho,\nu))\,\varepsilon(\rho,\mu) & = & \varepsilon(\rho,\lambda)\,\rho(t)^*\,, \end{array} \tag{5}$$

for any $\lambda,\mu,\nu\in\Delta$.

The unitaries $\varepsilon(\lambda,\mu)$ are called *braiding operators* (or *statistics operators*). Note that a braiding $\varepsilon\equiv\varepsilon^+$ always comes along with another "opposite" braiding ε^-, namely operators $\varepsilon^-(\lambda,\mu) = (\varepsilon^+(\mu,\lambda))^*$, $\varepsilon^+(\mu,\lambda)\equiv\varepsilon(\mu,\lambda)$, satisfy the same relations. The unitaries $\varepsilon^+(\lambda,\mu)$ and $\varepsilon^-(\lambda,\mu)$ are different in general but may coincide for some λ, μ. Later we will also use the following notion of non-degeneracy of a braiding (cf. [40]).

Definition 2.3 We say that a braiding ε on a system of endomorphisms Δ is *non-degenerate*, if the following condition is satisfied: If some morphism $\lambda\in\Delta$ satisfies $\varepsilon^+(\lambda,\mu)=\varepsilon^-(\lambda,\mu)$ for all morphisms $\mu\in\Delta$, then we have $\lambda=\mathrm{id}_A$.

We may also extend a given braiding from Δ in a well defined manner to all equivalent and sum endomorphisms as follows. We denote by $\Sigma(\Delta)$ the set of all endomorphisms $\lambda,\rho\in\mathrm{End}(A)$ given as $\lambda(a)=\sum_{i=1}^n t_i\lambda_i(a)t_i^*$ and $\rho(a)=\sum_{j=1}^m s_j\rho_j(a)s_j^*$ for all $a\in A$, where $t_i\in A$, $i=1,2\ldots,n$, and $s_j\in A$, $j=1,2,\ldots,m$, are Cuntz algebra generators, i.e. $t_i^* t_k = \delta_{i,k}\mathbf{1}$ and $\sum_{i=1}^n t_i t_i^* = \mathbf{1}$, and similarly $s_j^* s_l = \delta_{j,l}\mathbf{1}$ and $\sum_{j=1}^m s_j s_j^* = \mathbf{1}$, and $\lambda_i, \rho_j \in \Delta$. (Here $n, m \geq 1$.) For λ, ρ as above we put

$$\varepsilon(\lambda, \rho) = \sum_{i=1}^{n} \sum_{j=1}^{m} s_j \rho_j(t_i)\, \varepsilon(\lambda_i, \rho_j)\, \lambda_i(s_j^*) t_i^* \,, \tag{6}$$

and one can check that this definition is independent of the ambiguities in the choice of isometries $t_i \in \mathrm{Hom}(\lambda_i, \lambda)$ and $s_j \in \mathrm{Hom}(\rho_j, \rho)$. Note that in the case $n = m = 1$ this reads

$$\varepsilon(\mathrm{Ad}(u) \circ \lambda, \mathrm{Ad}(q) \circ \rho) = q\rho(u)\, \varepsilon(\lambda, \rho)\, \lambda(q^*) u^* \tag{7}$$

with some unitaries $u, q \in A$. Then for any sum endomorphisms $\lambda, \mu, \rho \in \Sigma(\Delta)$ the BFE's (5) hold as well or, alternatively, we have the naturality equations

$$\rho(t)\, \varepsilon(\lambda, \rho) = \varepsilon(\mu, \rho)\, t \,, \qquad t\, \varepsilon(\rho, \lambda) = \varepsilon(\rho, \mu)\, \rho(t) \tag{8}$$

whenever $t \in \mathrm{Hom}(\lambda, \mu)$. Using decompositions of products $\lambda\mu$, $\lambda, \mu \in \Sigma(\Delta)$ one can then easily show by use of the BFE's that

$$\varepsilon(\lambda\mu, \rho) = \varepsilon(\lambda, \rho)\, \lambda(\varepsilon(\mu, \rho)) \,, \qquad \varepsilon(\lambda, \mu\rho) = \mu(\varepsilon(\lambda, \rho))\, \varepsilon(\lambda, \mu) \,. \tag{9}$$

By plugging this in Eq. (8) we find that BFE's hold for endomorphisms in $\Sigma(\Delta)$ as well and Eq. (8) yields for $\varepsilon(\lambda, \mu) \in \mathrm{Hom}(\lambda\mu, \mu\lambda)$ the braid relation (or "Yang-Baxter equation")

$$\rho(\varepsilon(\lambda, \mu))\, \varepsilon(\lambda, \rho)\, \lambda(\varepsilon(\mu, \rho)) = \varepsilon(\mu, \rho)\, \mu(\varepsilon(\lambda, \rho))\, \varepsilon(\lambda, \mu) \,. \tag{10}$$

Now let Δ be a braided system of endomorphisms and let $\rho, \bar{\rho} \in \Delta$ be conjugate morphisms. Denote by $r \equiv r_\rho \in \mathrm{Hom}(\mathrm{id}_A, \bar{\rho}\rho)$ and $\bar{r} \equiv \bar{r}_\rho \in \mathrm{Hom}(\mathrm{id}_A, \rho\bar{\rho})$ isometries such that

$$\rho(r)^* \bar{r} = \bar{\rho}(\bar{r})^* r = d_\rho^{-1} \mathbf{1} \,,$$

which are then unique up to a common phase.[3] Note that $\varepsilon(\bar{\rho}, \rho)^* \bar{r} \in \mathrm{Hom}(\mathrm{id}_A, \bar{\rho}\rho)$ is an isometry and hence $\varepsilon(\bar{\rho}, \rho)^* \bar{r} = \omega_\rho r$ for some phase $\omega_\rho \in \mathbb{T}$ which is called the *statistics phase* and is obviously independent of the common phase of r and $\bar{r}$. In fact ω_ρ is even independent of the choice of ρ and $\bar{\rho}$ within their sectors: If $\rho' = \mathrm{Ad}\ u \circ \rho$ and $\bar{\rho}' = \mathrm{Ad}\ \bar{u} \circ \bar{\rho}$ for some unitaries $u, \bar{u} \in A$, then it is easy to see that isometries $r' = \bar{u}\bar{\rho}(u) r \in \mathrm{Hom}(\mathrm{id}_A, \bar{\rho}'\rho')$ and $\bar{r}' = u\rho(\bar{u})\bar{r} \in \mathrm{Hom}(\mathrm{id}_A, \rho'\bar{\rho}')$ also fulfill $\rho(r')^* \bar{r}' = \bar{\rho}(\bar{r}')^* r' = d_\rho^{-1}\mathbf{1}$. Now the braiding operator transforms as $\varepsilon(\bar{\rho}', \rho') = u\rho(\bar{u})\varepsilon(\bar{\rho}, \rho)\bar{\rho}(u)^* \bar{u}^*$ and hence

$$\varepsilon(\bar{\rho}', \rho')^* \bar{r}' = \bar{u}\bar{\rho}(u)\varepsilon(\bar{\rho}, \rho)^* \bar{r} = \omega_\rho r' \,.$$

The statistics phase can also be obtained by

$$\phi_\rho(\varepsilon(\rho, \rho)) = r^* \bar{\rho}(\varepsilon(\rho, \rho)) r = \omega_\rho d_\rho^{-1} \mathbf{1} \,.$$

[3] If ρ is not self-conjugate then we may choose $r_{\bar{\rho}} = \bar{r}_\rho$ and $\bar{r}_{\bar{\rho}} = r_\rho$. However, if ρ is self-conjugate, $\rho = \bar{\rho}$, we do not have $r_\rho = \bar{r}_\rho$ in general. This is only true for so-called "real" sectors, and for "pseudo-real" sectors we have $r_\rho = -\bar{r}_\rho$.

(The number $\omega_\rho d_\rho^{-1}$ is usually called the *statistics parameter*.) This is obtained from the initial condition and the BFE:

$$\rho(r) = \rho(r)\varepsilon(\mathrm{id}_A,\rho) = \varepsilon(\bar\rho,\rho)\bar\rho(\varepsilon(\rho,\rho))r\,,$$

but since $r^*\varepsilon(\bar\rho,\rho)^* = \omega_\rho \bar r^*$ we obtain

$$r^*\bar\rho(\varepsilon(\rho,\rho))r = r^*\varepsilon(\bar\rho,\rho)^*\rho(r) = \omega_\rho\bar r^*\rho(r) = \omega_\rho d_\rho^{-1}\mathbf{1}\,.$$

Moreover we have $\omega_\rho = \omega_{\bar\rho}$. This can be seen as follows. We have

$$r = r\varepsilon(\rho,\mathrm{id}_A) = \bar\rho(\varepsilon(\rho,\rho))\varepsilon(\rho,\bar\rho)\rho(r)\,,$$

hence $r^*\bar\rho(\varepsilon(\rho,\rho)) = \rho(r)^*\varepsilon(\rho,\bar\rho)^*$, thus

$$\omega_\rho d_\rho^{-1}\mathbf{1} = \rho(r)^*\varepsilon(\rho,\bar\rho)^* r = \omega_{\bar\rho}\rho(r)^*\bar r = \omega_{\bar\rho} d_\rho^{-1}\,,$$

since $\varepsilon(\rho,\bar\rho)^* r = \omega_{\bar\rho}\bar r$ by definition. Therefore we have $\omega_\rho r^* = \bar r^*\varepsilon(\rho,\bar\rho)^*$. Another application of the BFE yields $\varepsilon(\rho,\rho)\rho(\bar r) = \rho(\varepsilon(\rho,\bar\rho))^*\bar r$, hence we have

$$\rho(\bar r)^*\varepsilon(\rho,\rho)\rho(\bar r) = \rho(\bar r)^*\rho(\varepsilon(\rho,\bar\rho))^*\bar r = \omega_\rho\rho(r)^*\bar r = \omega_\rho d_\rho^{-1}\mathbf{1}\,.$$

Now let $\lambda,\mu,\nu\in\Delta$. Let $r\equiv r_\lambda \in \mathrm{Hom}(\mathrm{id}_A,\bar\lambda\lambda)$ and $\bar r \equiv \bar r_\lambda \in \mathrm{Hom}(\mathrm{id}_A,\lambda\bar\lambda)$ be isometries such that $\lambda(r)^*\bar r = \bar\lambda(\bar r)^* r = d_\lambda^{-1}\mathbf{1}$. Let $t,t'\in\mathrm{Hom}(\lambda,\mu\nu)$. Recall that $\phi_\mu(t't^*) = d_\lambda d_\mu^{-1} d_\nu^{-1} t^*t' \in \mathrm{Hom}(\lambda,\lambda)$ is a scalar. We can now compute

$$\begin{aligned}
\omega_\lambda d_\mu^{-1} d_\nu^{-1}\, t^*t' &= \omega_\lambda d_\lambda^{-1}\,\phi_\nu\circ\phi_\mu(t't^*) = \phi_\nu\circ\phi_\mu(t'\lambda(\bar r)^*\varepsilon(\lambda,\lambda)\lambda(\bar r)t^*)\\
&= \bar r^*\,\phi_\nu\circ\phi_\mu(t'\varepsilon(\lambda,\lambda)t^*)\,\bar r = \bar r^*\,\phi_\nu\circ\phi_\mu(\varepsilon(\lambda,\mu\nu)\lambda(t')t^*)\,\bar r\\
&= \bar r^*\,\phi_\nu\circ\phi_\mu(\varepsilon(\lambda,\mu\nu)t^*)\,t'\bar r = \bar r^*t^*\,\phi_\nu\circ\phi_\mu(\varepsilon(\mu\nu,\mu\nu))\,t'\bar r\\
&= \bar r^*t^*\,\phi_\nu\circ\phi_\mu(\mu(\varepsilon(\mu,\nu))\mu^2(\varepsilon(\nu,\nu))\varepsilon(\mu,\mu)\mu(\varepsilon(\nu,\mu)))\,t'\bar r\\
&= \omega_\mu d_\mu^{-1}\,\bar r^*t^*\,\phi_\nu(\varepsilon(\mu,\nu)\mu(\varepsilon(\nu,\nu))\varepsilon(\nu,\mu))\,t'\bar r\\
&= \omega_\mu d_\mu^{-1}\,\bar r^*t^*\,\phi_\nu(\nu(\varepsilon(\nu,\mu)\varepsilon(\nu,\nu)\nu(\varepsilon(\mu,\nu))\,t'\bar r\\
&= \omega_\mu\omega_\nu d_\mu^{-1}d_\nu^{-1}\,\bar r^*t^*\,\varepsilon(\nu,\mu)\varepsilon(\mu,\nu)\,t'\bar r\\
&= \omega_\mu\omega_\nu d_\mu^{-1}d_\nu^{-1}\,t^*\varepsilon(\nu,\mu)\varepsilon(\mu,\nu)\,t'\,,
\end{aligned}$$

where we finally could omit the $\bar r$'s since $t^*\varepsilon(\nu,\mu)\varepsilon(\mu,\nu)t'\in\mathrm{Hom}(\lambda,\lambda)$ is a scalar. As $\varepsilon(\nu,\mu)\varepsilon(\mu,\nu)t'\in\mathrm{Hom}(\lambda,\mu\nu)$ we find $\omega_\lambda\langle t,t'\rangle = \omega_\mu\omega_\nu\langle t,\varepsilon(\nu,\mu)\varepsilon(\mu,\nu)t'\rangle$ for any $t,t'\in\mathrm{Hom}(\lambda,\mu\nu)$, and therefore we arrive at the important relation

$$\varepsilon(\nu,\mu)\varepsilon(\mu,\nu)\,t = \frac{\omega_\lambda}{\omega_\mu\omega_\nu}\,t \qquad \text{for all} \qquad t\in\mathrm{Hom}(\lambda,\mu\nu)\,. \tag{11}$$

Decomposing $[\mu\nu]$ in all irreducible sectors $[\lambda]$ and choosing for each $\lambda\in\Delta$ some orthonormal bases of intertwiners $t_{\lambda;i}\in\mathrm{Hom}(\lambda,\mu\nu)$, $i=1,2,\dots,N_{\mu,\nu}^\lambda$, where $N_{\mu,\nu}^\lambda =$

$\langle\lambda,\mu\nu\rangle$ as usual, we have $\sum_{\lambda\in\Delta}\sum_i t_{\lambda;i}t^*_{\lambda;i}=\mathbf{1}$, and therefore we find by Eqs. (3) and (11),

$$\phi_\mu(\varepsilon(\nu,\mu)\varepsilon(\mu,\nu))^*=\phi_\mu\left(\varepsilon(\nu,\mu)\varepsilon(\mu,\nu)\sum_{\lambda\in\Delta}\sum_i t_{\lambda;i}t^*_{\lambda;i}\right)^*=\sum_{\lambda\in\Delta}\frac{\omega_\mu\omega_\nu}{\omega_\lambda}N^\lambda_{\mu,\nu}\frac{d_\lambda}{d_\mu d_\nu}\mathbf{1}\,.$$

One then defines a matrix Y in terms of these numbers [40] (see also [14, 13]):

$$Y_{\mu,\nu}=\sum_{\lambda\in\Delta}\frac{\omega_\mu\omega_\nu}{\omega_\lambda}N^\lambda_{\mu,\nu}\,d_\lambda\,,\qquad \mu,\nu\in\Delta\,, \tag{12}$$

i.e. $d_\mu d_\nu\phi_\mu(\varepsilon(\nu,\mu)\varepsilon(\mu,\nu))^*=Y_{\mu,\nu}\mathbf{1}$. Then one has

$$Y_{\lambda,\mu}=Y_{\mu,\lambda}=Y^*_{\bar\lambda,\mu}=Y_{\bar\lambda,\bar\mu}\,.$$

The first equality is obvious from Eq. (12), so we only need to show $Y_{\lambda,\mu}=(Y_{\bar\lambda,\mu})^*$. In fact, applying the BFE again yields $\bar\lambda(\varepsilon(\lambda,\mu))r_\lambda=\varepsilon(\bar\lambda,\mu)^*\mu(r_\lambda)$ and $r^*_\lambda\bar\lambda(\varepsilon(\mu,\lambda))=\mu(r_\lambda)^*\varepsilon(\mu,\bar\lambda)^*$. Hence

$$\begin{aligned}Y_{\lambda,\mu}\mathbf{1}&=\phi_\mu(Y_{\lambda,\mu})=d_\lambda d_\mu(r^*_\mu\bar\mu(r^*_\lambda\bar\lambda(\varepsilon(\mu,\lambda)\varepsilon(\lambda,\mu))r_\lambda)r_\mu)^*\\&=d_\lambda d_\mu(r^*_\lambda r^*_\mu\bar\mu(\varepsilon(\mu,\bar\lambda)^*\varepsilon(\bar\lambda,\mu)^*)r_\mu r_\lambda)^*=(r^*_\lambda Y_{\bar\lambda,\mu}r_\lambda)^*=(Y_{\bar\lambda,\mu})^*\mathbf{1}\,.\end{aligned}$$

Moreover, we have

$$Y_{\nu,\rho}Y_{\mu,\rho}=d_\rho\sum_\lambda N^\lambda_{\mu,\nu}Y_{\rho,\lambda}\,,$$

since

$$\begin{aligned}Y_{\nu,\rho}Y_{\mu,\rho}\mathbf{1}&=d^2_\rho d_\mu d_\nu\,\phi_\nu(\varepsilon(\rho,\nu)\phi_\mu(\varepsilon(\rho,\mu)\varepsilon(\mu,\rho))\varepsilon(\nu,\rho))^*\\&=d^2_\rho d_\mu d_\nu\,\phi_\nu\circ\phi_\mu(\mu(\varepsilon(\rho,\nu))\varepsilon(\rho,\mu)\varepsilon(\mu,\rho)\mu(\varepsilon(\nu,\rho)))^*\\&=d^2_\rho d_\mu d_\nu\textstyle\sum_\lambda\sum_i\phi_\nu\circ\phi_\mu(\varepsilon(\rho,\mu\nu)\rho(t_{\lambda;i}t^*_{\lambda;i})\varepsilon(\mu\nu,\rho))^*\\&=d^2_\rho d_\mu d_\nu\textstyle\sum_\lambda\sum_i\phi_\nu\circ\phi_\mu(t_{\lambda;i}\varepsilon(\rho,\lambda)\varepsilon(\lambda,\rho)t^*_{\lambda;i})^*\\&=d^2_\rho d_\mu d_\nu\textstyle\sum_\lambda\sum_i\phi_\mu(t_{\lambda;i}t^*_{\lambda;i})^*\phi_\lambda(\varepsilon(\rho,\lambda)\varepsilon(\lambda,\rho))^*=d_\rho\sum_\lambda N^\lambda_{\mu,\nu}Y_{\rho,\lambda}\mathbf{1}\,.\end{aligned}$$

From now on we assume that the system Δ is finite. We define the complex number

$$z_\Delta=\sum_{\lambda\in\Delta}d^2_\lambda\omega_\lambda\,,$$

and if $z_\Delta\neq0$ we put $c=4\arg(z_\Delta)/\pi$. Note that the c is here only defined mod 8 and we may make a choice. Let C be the conjugation matrix with entries $C_{\lambda,\mu}=\delta_{\lambda,\bar\mu}$. Clearly, $C=C^*=C^{-1}$. We then have the following

Proposition 2.4 *Let Δ be finite system of endomorphisms with $z_\Delta\neq0$. Then S- and T-matrices defined by*

$$S_{\lambda,\mu}=|z_\Delta|^{-1}\,Y_{\lambda,\mu}\,,\qquad T_{\lambda,\mu}=\mathrm{e}^{-\pi\mathrm{i}c/12}\,\omega_\lambda\,\delta_{\lambda,\mu}\,,\qquad \lambda,\mu\in\Delta\,,$$

obey the partial Verlinde modular algebra $TSTST=S$, $CTC=T$, $CSC=S$ and $T^*T=\mathbf{1}$.

To prove the proposition, we simply compute

$$\begin{aligned}\sum_\mu \omega_\lambda Y_{\lambda,\mu}\omega_\mu Y_{\mu,\nu}\omega_\nu &= \omega_\lambda\omega_\nu \sum_\mu \omega_\mu Y^*_{\lambda,\bar\mu} Y^*_{\nu,\bar\mu} = \omega_\lambda\omega_\nu \sum_{\mu,\sigma} \omega_\mu d_\mu N^\sigma_{\lambda,\nu} Y^*_{\bar\mu,\sigma} \\ &= \omega_\lambda\omega_\nu \sum_{\mu,\rho,\sigma} \omega_\mu d_\mu N^\sigma_{\lambda,\nu} N^\rho_{\bar\mu,\sigma} \tfrac{\omega_\rho}{\omega_\mu\omega_\sigma} d_\rho = \omega_\lambda\omega_\nu \sum_{\rho,\sigma} d^2_\rho d_\sigma N^\sigma_{\lambda,\nu} \tfrac{\omega_\rho}{\omega_\sigma} \\ &= Y_{\lambda,\nu} \sum_\rho d^2_\rho \omega_\rho = Y_{\lambda,\nu} z_\Delta \,,\end{aligned}$$

hence $TSTST = \mathrm{e}^{-\pi \mathrm{i}c/4}|z_\Delta|^{-1} S z_\Delta = S$. The remaining relations $CTC = T$, $CSC = S$ and $T^*T = \mathbf{1}$ are obvious.

We define *weight vectors* y^λ with components $y^\lambda_\mu = Y_{\lambda,\mu}$ and *statistics characters* $\chi_\lambda : \Delta \to \mathbb{C}$ with evaluations $\chi_\lambda(\mu) = d_\lambda^{-1} Y_{\lambda,\mu}$, $\lambda,\mu \in \Delta$. We have seen that the weight vectors y^λ are simultaneous eigenvectors of the fusion matrices N_μ with eigenvalues $\chi_\lambda(\mu)$, $N_\mu y^\lambda = \chi_\lambda(\mu) y^\lambda$. Hence we obtain by computing inner products,

$$\chi_\mu(\rho)\langle y^\lambda, y^\mu\rangle = \langle y^\lambda, N_\rho y^\mu\rangle = \langle N_{\bar\rho} y^\lambda, y^\mu\rangle = \chi_\lambda(\bar\rho)^* \langle y^\lambda, y^\mu\rangle = \chi_\lambda(\rho)\langle y^\lambda, y^\mu\rangle \,.$$

Therefore the eigenvectors are either orthogonal, $\langle y^\lambda, y^\mu\rangle = 0$, or parallel, $d_\mu y^\lambda = d_\lambda y^\mu$ since then the characters are equal, $\chi_\lambda = \chi_\mu$. It is obvious that if some $\lambda \in \Delta$ is degenerate, i.e. has trivial monodromy with all other $\mu \in \Delta$, then y^λ is parallel to the vector y^0. (Here and later we use the label "0" for the identity $\mathrm{id}_A \in \Delta$.) Note that we have $y^0_\mu = d_\mu$, and then $Y_{\lambda,\mu} = d_\lambda d_\mu$. Conversely, if y^λ is parallel to y^0 we have seen that then necessarily $Y_{\lambda,\mu} = d_\lambda d_\mu$, hence

$$Y_{\lambda,\mu} = \sum_{\rho\in\Delta} \frac{\omega_\lambda\omega_\mu}{\omega_\rho} N^\rho_{\lambda,\mu}\, d_\rho = d_\lambda d_\mu = \sum_{\rho\in\Delta} N^\rho_{\lambda,\mu}\, d_\rho \,, \qquad \mu \in \Delta \,,$$

and this is clearly only possible if all the eigenvalues $\omega_\lambda\omega_\mu\omega_\rho^{-1}$ of the monodromy are trivial, i.e. if λ is degenerate. We conclude that a braiding on Δ is non-degenerate if and only if $\langle y^\lambda, y^0\rangle = \delta_{\lambda,0} w$, where $w = \sum_{\lambda\in\Delta} d^2_\lambda$ is the *global index*. We now arrive at Rehren's result [40].

Theorem 2.5 *The following conditions are equivalent for a finite braided system of endomorphisms Δ:*

1. *The braiding on Δ is non-degenerate.*

2. *We have $w = |z_\Delta|^2$ and the matrices S and T obey the full Verlinde modular algebra*
$$S^*S = T^*T = \mathbf{1} \,, \quad (ST)^3 = S^2 = C \,, \quad CTC = T \,,$$
moreover S diagonalizes the fusion rules (Verlinde formula):
$$N^\nu_{\lambda,\mu} = \sum_{\rho\in\Delta} \frac{S_{\lambda,\rho} S_{\mu,\rho} S^*_{\nu,\rho}}{S_{0,\rho}} \,.$$

Note that the implication *2.* ⇒ *1.* is trivial since invertibility of S implies that there is no vector y^λ parallel y^0. So let us assume that the braiding is non-degenerate: $\langle y^\lambda, y^0\rangle = \delta_{\lambda,0} w$ for all $\lambda \in \Delta$. Then we can first check

$$\begin{aligned} w &= \textstyle\sum_\mu \langle y^0, y^\mu\rangle d_\mu \omega_\mu^{-1} = \sum_{\mu,\nu} d_\nu Y_{\mu,\nu} d_\mu \omega_\mu^{-1} = \sum_{\mu,\nu,\lambda} d_\nu \frac{\omega_\mu\omega_\nu}{\omega_\lambda} N_{\mu,\nu}^\lambda d_\lambda d_\mu \omega_\mu^{-1} \\ &= \textstyle\sum_{\mu,\nu,\lambda} d_\lambda d_\nu \frac{\omega_\nu}{\omega_\lambda} N_{\bar\nu,\lambda}^\mu d_\mu = \sum_{\lambda,\nu} d_\lambda^2 \omega_\lambda^{-1} d_\nu^2 \omega_\nu \,, \end{aligned}$$

thus $w = \left|\sum_{\lambda\in\Delta} d_\lambda^2 \omega_\lambda\right|^2 \equiv |z_\Delta|^2$. Next we compute

$$\langle y^\lambda, y^\mu \rangle = \sum_\rho Y_{\lambda,\rho}^* Y_{\mu,\rho} = \sum_{\rho,\nu} N_{\bar\lambda,\mu}^\nu Y_{\rho,\nu} d_\rho = \sum_\nu N_{\bar\lambda,\mu}^\nu \langle y^0, y^\nu\rangle = N_{\bar\lambda,\mu}^0 w = \delta_{\lambda,\mu} w \,,$$

hence $S^* S = \mathbf{1}$. Similarly we observe that $\sum_\rho Y_{\lambda,\rho} Y_{\mu,\rho} = \sum_\rho Y_{\bar\lambda,\rho}^* Y_{\mu,\rho} = \delta_{\bar\lambda,\mu} w$, giving $S^2 = C$ which obviously commutes with T. Finally we check

$$\sum_\rho \frac{S_{\lambda,\rho} S_{\mu,\rho} S_{\nu,\rho}^*}{S_{0,\rho}} = w^{-1} \sum_\rho \frac{Y_{\lambda,\rho} Y_{\mu,\rho} Y_{\nu,\rho}^*}{d_\rho} = w^{-1} \sum_{\rho,\sigma} N_{\lambda,\mu}^\sigma Y_{\rho,\sigma} Y_{\nu,\rho}^* = \sum_\sigma N_{\lambda,\mu}^\sigma \delta_{\nu,\sigma} = N_{\lambda,\mu}^\nu \,,$$

proving the Verlinde identity.

Corollary 2.6 *If the braiding on Δ is non-degenerate, then the matrix S and the diagonal matrix T are the images $S = U(\mathcal{S})$ and $T = U(\mathcal{T})$ of canonical generators*

$$\mathcal{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

in a unitary representation U of the modular group[4] *$SL(2;\mathbb{Z})$ with dimension $|\Delta|$, the cardinality of Δ.*

3 Graphical Intertwiner Calculus

3.1 Basic graphical intertwiner calculus

We now introduce our conventions to represent and manipulate intertwiners graphically. We consider a braided system of endomorphisms $\Delta \subset \mathrm{End}(A)$ with A a type III factor. Essentially we represent intertwiners by "wire diagrams" where the (oriented) wires represent endomorphisms $\lambda \in \Delta$. This works as follows. For an intertwiner $x \in \mathrm{Hom}(\lambda_1\lambda_2\cdots\lambda_n, \mu_1\mu_2\cdots\mu_m)$ we draw a (dashed) box with n (downward) incoming wires labelled by $\lambda_1,\ldots,\lambda_n$ and m (downward) outgoing wires $\mu_1,\ldots,\mu_m$ as in Fig. 1, $\lambda_i, \mu_j \in \Delta$. Therefore the diagrammatic representation of x does not only specify it as an operator, it even specifies the intertwiner space it is considered to belong to. (Note that the same operator can belong to different intertwiner spaces as e.g. the identity operator belongs to any $\mathrm{Hom}(\lambda,\lambda)$ with λ varying.) If a morphism $\rho \in \Delta$ is applied to x, then $\rho(x) \in \mathrm{Hom}(\rho\lambda_1\lambda_2\cdots\lambda_n, \rho\mu_1\mu_2\cdots\mu_m)$ is represented graphically

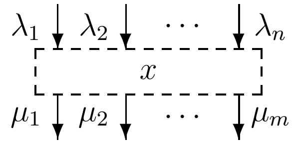


Figure 1: An intertwiner x

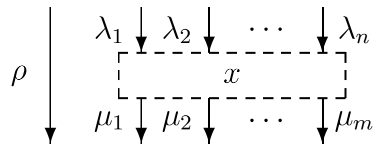


Figure 2: The intertwiner $\rho(x)$

by adding a straight wire on the left as in Fig. 2. Reflecting the fact that x can also be considered as an intertwiner in $\mathrm{Hom}(\lambda_1\lambda_2\cdots\lambda_n\rho, \mu_1\mu_2\cdots\mu_m\rho)$ we can always add (or remove) a straight wire on the right as in Fig. 3 without changing the inter-

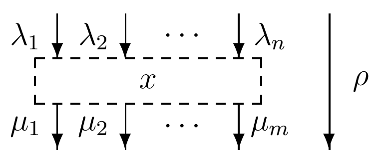


Figure 3: The intertwiner x

twiner as an operator. We say that intertwiners $x \in \mathrm{Hom}(\lambda_1\lambda_2\cdots\lambda_n, \mu_1\mu_2\cdots\mu_m)$ and $y \in \mathrm{Hom}(\nu_1\nu_2\cdots\nu_k, \rho_1\rho_2\cdots\rho_l)$, $\rho_j \in \Delta$, are *diagrammatically composable* if $m = k$ and $\mu_i = \nu_i$ for all $i = 1, 2, \ldots, m$. Then the composed intertwiner $yx \in \mathrm{Hom}(\lambda_1\lambda_2\cdots\lambda_n, \rho_1\rho_2\cdots\rho_l)$ is represented graphically by putting the wire diagram for x on top of that for y as in Fig. 4. We also call this graphical procedure composition of wire diagrams. Sometimes diagrammatic composability may be achieved by adding or removing straight wires on the right. Now let also $x' \in \mathrm{Hom}(\lambda'_1\lambda'_2\cdots\lambda'_{n'}, \mu'_1\mu'_2\cdots\mu'_{m'})$ with $\lambda'_i, \mu'_j \in \Delta$. The intertwining property of x yields the identity $\mu_1\mu_2\cdots\mu_m\rho_1\rho_2\cdots\rho_l(x')x = x\lambda_1\lambda_2\cdots\lambda_n\rho_1\rho_2\cdots\rho_l(x')$, and this is diagrammatically given in Fig. 5. Thus we have some freedom in translating intertwiner boxes vertically without actually changing the represented intertwiner.

[4]In the literature the name "modular group" is often reserved for $PSL(2;\mathbb{Z}) = SL(2;\mathbb{Z})/\mathbb{Z}_2$ rather than $SL(2;\mathbb{Z})$. Clearly, we obtain a representation of $PSL(2;\mathbb{Z})$ whenever the charge conjugation is trivial, $C = \mathbf{1}$.

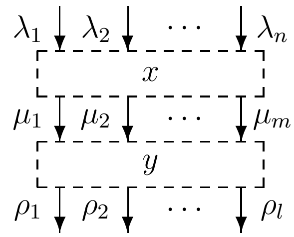


Figure 4: Product yx of diagrammatically composable intertwiners x and y

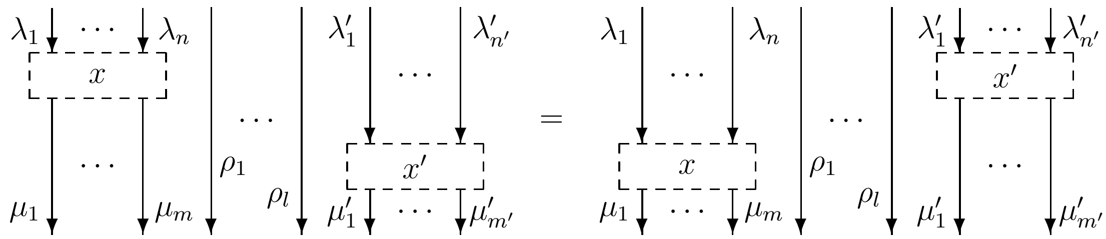


Figure 5: Vertical translation intertwiners x and x'

The intertwiners we consider are (sums over) compositions of *elementary intertwiners* arising from the unitary braiding operators $\varepsilon(\lambda,\mu)\in\mathrm{Hom}(\lambda\mu,\mu\lambda)$ and isometries $t\in\mathrm{Hom}(\lambda,\mu\nu)$. The wire diagrams and boxes we are dealing with are therefore compositions of "elementary boxes" representing the elementary intertwiners. We now have to introduce some normalization convention. First, the identity intertwiner $\mathbf{1}\equiv\mathbf{1}_A$ is naturally given by the "trivial box" with only straight wires of arbitrary labels. The next elementary intertwiner is $\rho_1\rho_2\cdots\rho_n(\varepsilon(\lambda,\mu))$ for which we draw a box as in Fig. 6 where the arbitrary labels $\nu_1,\ldots,\nu_m$ are irrelevant and

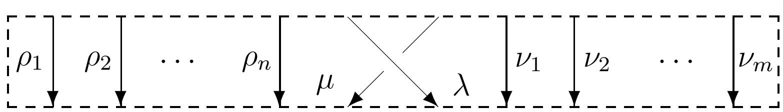


Figure 6: $\rho_1\rho_2\cdots\rho_n(\varepsilon(\lambda,\mu))$

may be omitted. Similarly, the box of Fig. 7 represents the elementary intertwiner $d_\mu^{1/4}d_\nu^{1/4}d_\lambda^{-1/4}\rho_1\rho_2\cdots\rho_n(t)$, where $t\in\mathrm{Hom}(\lambda,\mu\nu)$ is an isometry. We label the trivalent vertex in the box by t since $\mathrm{Hom}(\lambda,\mu\nu)$ may be more than one-dimensional and so we have to specify the intertwiner. (Note that there would still be an ambiguity of a phase for the choice of an isometry even if $\mathrm{Hom}(\lambda,\mu\nu)$ is only one-dimensional.) Finally, the elementary intertwiners $\varepsilon(\lambda,\mu)^*=\varepsilon^-(\mu,\lambda)$ and $d_\mu^{1/4}d_\nu^{1/4}d_\lambda^{-1/4}\rho_1\rho_2\cdots\rho_n(t)^*$ are represented by Figs. 8 and 9, i.e. they are obtained from the original boxes in Figs.

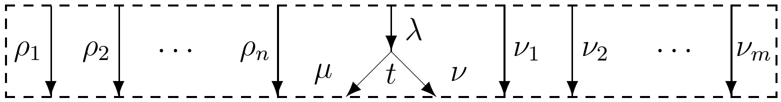


Figure 7: $\sqrt[4]{\frac{d_\mu d_\nu}{d_\lambda}}\rho_1\rho_2\cdots\rho_n(t)$ where $t \in \mathrm{Hom}(\lambda, \mu\nu)$ is an isometry

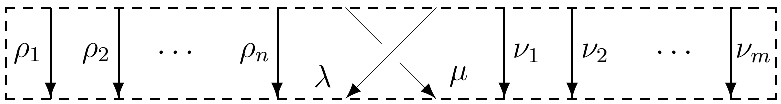


Figure 8: $\rho_1\rho_2\cdots\rho_n(\varepsilon(\lambda,\mu))^* = \rho_1\rho_2\cdots\rho_n(\varepsilon^-(\mu,\lambda))$

6 and 7 by vertical reflection and inversion of all the arrows. Note that $\varepsilon \equiv \varepsilon^+$ repre-

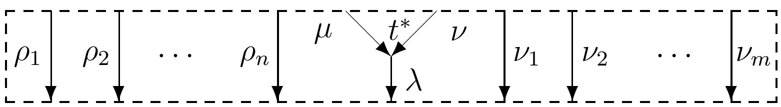


Figure 9: $\sqrt[4]{\frac{d_\mu d_\nu}{d_\lambda}}\rho_1\rho_2\cdots\rho_n(t)^*$ where $t \in \mathrm{Hom}(\lambda, \mu\nu)$ is an isometry

sents overcrossing and ε^- undercrossing of wires. We will consider intertwiners which are products of diagrammatically composable elementary intertwiners. In terms of wire diagrams we are correspondingly dealing with compositions of elementary boxes of Figs. 6, 7, 8, 9 so that the wires with the same labels (and orientations) can and will be glued together in parallel and then we finally forget about the boundaries of the (dashed) boxes. Therefore, if a wire diagram represents some intertwiner x then x^* is represented by the diagram obtained by vertical reflection and reversing all the arrows. Note that our resulting wire diagrams are then composed only from straight lines, over- and undercrossings (in X-shape) and trivalent vertices (in Y-shape or inverted Y-shape).

So far, we have considered only wires with downward orientation. We now introduce also the reversed orientation in terms of conjugation as follows: Reversing the orientation of an arrow on a wire changes its label λ to $\bar{\lambda}$. Also we will usually omit drawing a wire labelled by $\mathrm{id} \equiv \mathrm{id}_A$. For each $\lambda \in \Delta$ we fix (the common phase of) isometries $r_\lambda \in \mathrm{Hom}(\mathrm{id}, \bar{\lambda}\lambda)$ and $\bar{r}_\lambda \in \mathrm{Hom}(\mathrm{id}, \lambda\bar{\lambda})$ such that $\lambda(r_\lambda)^*\bar{r}_\lambda = \bar{\lambda}(\bar{r}_\lambda)^* r_\lambda = d_\lambda^{-1}\mathbf{1}$ and in turn for $\sqrt{d_\lambda}r_\lambda$ we draw one of the equivalent diagrams in Fig. 10. So the normalized isometries and their adjoints appear in wire diagrams as "caps" and "cups", respectively. The point is that with our normalization convention, the relation $\lambda(r_\lambda)^*\bar{r}_\lambda = d_\lambda^{-1}\mathbf{1}$ (and its adjoint) gives a *topological invariance* for intertwiners represented by wire diagrams, displayed in Fig. 11. Note

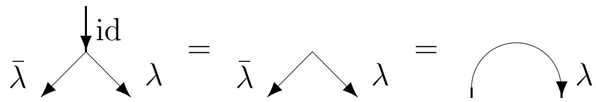


Figure 10: Wire diagrams for $\sqrt{d_\lambda} r_\lambda$

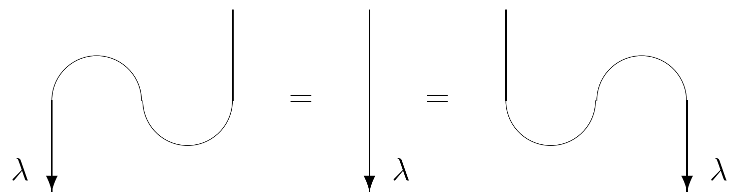


Figure 11: A topological invariance for intertwiners represented by wire diagrams

that then the wire diagrams in Fig. 12 represent the scalar d_λ (i.e. the intertwiner

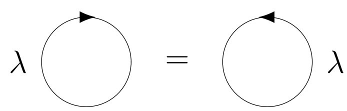


Figure 12: Wire diagrams for the statistical dimension d_λ

$d_\lambda \mathbf{1} \in \mathrm{Hom}(\mathrm{id}, \mathrm{id})$). Also note the "vertical Reidemeister move of type II" in Fig. 13 is just the unitarity condition $\varepsilon(\lambda,\mu)^*\varepsilon(\lambda,\mu) = \mathbf{1} = \varepsilon(\mu,\lambda)\varepsilon(\mu,\lambda)^*$. The BFE's yield another topological invariance, see Fig. 14 for the first equation and Fig. 15 for the second equation. The third and fourth equations are obtained similarly by use of the co-isometry t^*; we leave it as an exercise to the reader to draw the corresponding wire diagrams. Up to conjugation they can also be obtained by changing over- to undercrossings in Figs. 14 and 15. Finally, the braid relation, Eq. (10), represents graphically a vertical Reidemeister move of type III, presented in Fig. 16. The topological invariance gives us the freedom to write down the intertwiner algebraically from a given wire diagram: We can deform the wire diagram by finite sequences of the above moves and then split it in elementary wire diagrams — in whatever way we decompose the wire diagrams into horizontal slices of elementary intertwiners, we always obtain the same intertwiner due to our topological invariance identities.

Next we recall that we can write the statistics phase ω_λ as the intertwiner $d_\lambda r_\lambda^* \bar\lambda(\varepsilon(\lambda,\lambda)) r_\lambda$. Therefore we obtain for ω_λ the wire diagram on the left-hand side of Fig. 17. The diagram on the right-hand side expresses that ω_λ can also be obtained as $d_\lambda \bar\lambda(r_\lambda)^* \varepsilon(\bar\lambda,\bar\lambda)\bar\lambda(r_\lambda)$. Note that we obtain the complex conjugate ω_λ^* by exchanging over- and undercrossings. Similarly, we recall that we can write

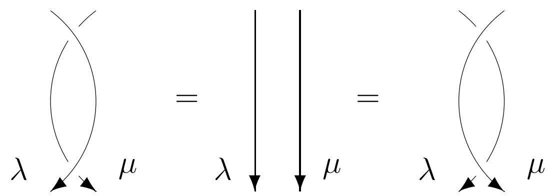


Figure 13: Unitarity of braiding operators as a vertical Reidemeister move of type II

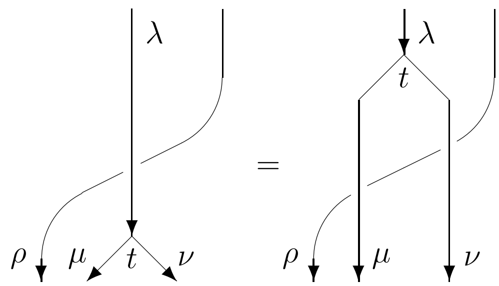


Figure 14: The first braiding fusion equation

a matrix element $Y_{\lambda,\mu} = Y_{\mu,\lambda}$ of Rehren's Y-matrix as $d_\lambda d_\mu \phi_\mu(\varepsilon(\lambda,\mu)\varepsilon(\mu,\lambda))^* = d_\lambda d_\mu r_\mu^* \bar{\mu}(\varepsilon^-(\lambda,\mu)\varepsilon^-(\mu,\lambda))r_\mu$. Dividing by d_λ we obtain $\chi_\lambda(\mu)$, the statistics character χ_λ evaluated on μ, represented graphically by the wire diagram in Fig. 18. We have drawn the circle μ symmetrically relative to the straight wire λ because it does not make a difference whether we put the "caps" and "cups" for the isometry r_μ and its conjugate r_μ^* on the left or on the right due to the braiding fusion relations. As it is a scalar, we can write $Y_{\lambda,\mu} = \bar{r}_\mu^* Y_{\lambda,\mu} \bar{r}_\mu$ and therefore its expression $d_\lambda d_\mu \bar{r}_\mu^* r_\lambda^* \bar{\lambda}(\varepsilon^-(\mu,\lambda)\varepsilon^-(\lambda,\mu))r_\lambda \bar{r}_\mu$ yields exactly the "Hopf link" as the wire diagram for the matrix element $Y_{\lambda,\mu}$, given by the left-hand side of Fig. 19. The equality to the right-hand side is just the relation $Y_{\lambda,\mu} = Y_{\lambda,\bar{\mu}}^*$ together with the prescription of

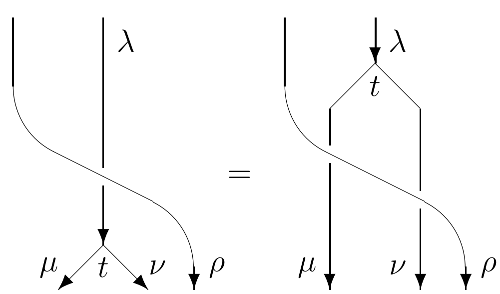


Figure 15: The second braiding fusion equation

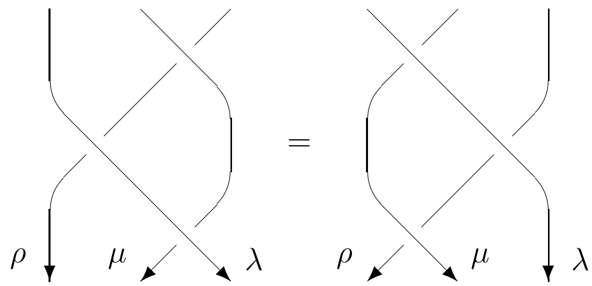


Figure 16: The braid relation as a vertical Reidemeister move of type III

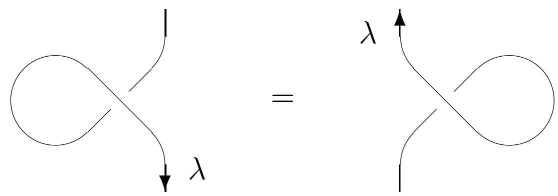


Figure 17: Statistics phase ω_λ as a "twist"

representing conjugates. Recall that if Δ is finite then the Y-matrix differs from the S-matrix just by an overall normalization factor $\sqrt{w}$, where w is the global index.

Often we consider intertwiners which are sums over intertwiners represented by the same wire diagram but the sum runs over one or more of the labels. Then we simply write the sum symbol in front of the diagram, we may similarly insert scalar factors. Now recall that for finite Δ the non-degeneracy of the braiding is encoded in the orthogonality relation $\langle y^0, y^\lambda \rangle = \delta_{\lambda,0} w$. In terms of the statistics characters this reads $\sum_\mu d_\mu \chi_\lambda(\mu) = d_\lambda^{-1} \delta_{\lambda,0} w = \delta_{\lambda,0} w$. Graphically this can be represented as in Fig. 20. This kind of (graphical) relation has also been used more recently in [44, 38, 25] and was called a "killing ring" in [38].

Wire diagrams can also be used for intertwiners of morphisms between different factors. Let A, B, C infinite factors, $\rho \in \mathrm{Mor}(A, B)$, $\sigma \in \mathrm{Mor}(C, B)$, $\tau \in \mathrm{Mor}(A, C)$

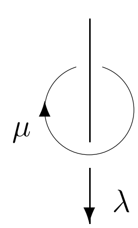


Figure 18: Rehren's statistics character χ_λ evaluated on μ: $\chi_\lambda(\mu)$

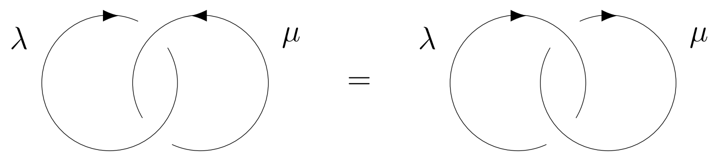


Figure 19: Matrix element $Y_{\lambda,\mu}$ of Rehren's Y-matrix as a "Hopf link"

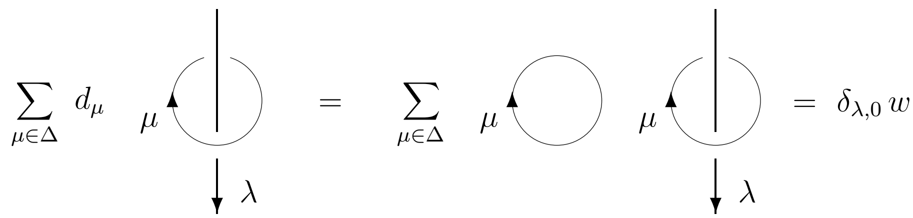


Figure 20: Orthogonality relation for a non-degenerate braiding ("killing ring")

irreducible morphisms and $t \in \mathrm{Hom}(\rho, \sigma\tau)$ an isometry. Then Fig. 21 represents the

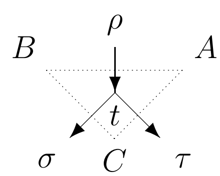


Figure 21: The intertwiner $\sqrt[4]{\frac{d_\sigma d_\tau}{d_\rho}}t$ as a triangle

intertwiner $d_\sigma^{1/4} d_\tau^{1/4} d_\rho^{-1/4} t$. Similarly we can draw a picture using a co-isometry. Along the lines of the previous paragraphs, we can similarly build up larger wire diagrams out of trivalent vertices involving different factors. We do not need the triangles with corners labelled by factors as we can also label the regions between the wires. So far we do not have a meaningful way to cross wires with differently labelled regions left and right, but all the arguments listed above which do not involve braidings can be used for intertwiners of morphisms between different factors exactly as proceeded above. Moreover, the diagrams may also involve wires where left and right regions are labelled by the same factor, i.e. these wires correspond to *endo*morphisms of some factor which may well form a braided system, and then one may have crossings for those wires.

3.2 Frobenius reciprocity and rotations

Let A, B, C be infinite factors, $\rho \in \mathrm{Mor}(A,B)$, $\tau \in \mathrm{Mor}(C,B)$, $\sigma \in \mathrm{Mor}(C,A)$ morphisms with finite statistical dimensions $d_\rho, d_\tau, d_\sigma < \infty$, respectively, and let $t \in \mathrm{Hom}(\tau, \rho\sigma)$. Then

$$\mathcal{L}_\rho(t) = \sqrt{\frac{d_\rho d_\sigma}{d_\tau}}\bar{\rho}(t)^* r_\rho \in \mathrm{Hom}(\sigma, \bar{\rho}\tau)$$

and

$$\mathcal{R}_\sigma(t) = \sqrt{\frac{d_\rho d_\sigma}{d_\tau}} t^* \rho(\bar{r}_\sigma) \in \mathrm{Hom}(\rho, \tau\bar{\sigma})$$

are the images under left and right Frobenius maps. Displaying the intertwiners $d_\rho^{1/2} r_\rho^* \bar{\rho}(t)$ and $d_\sigma^{1/2} \rho(\bar{r}_\sigma)^* t$ graphically yields the identities in Figs. 22 and 23, re-

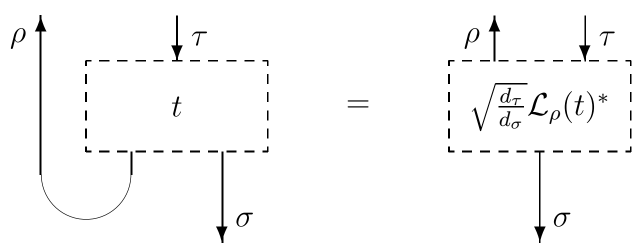


Figure 22: Left Frobenius reciprocity for an intertwiner $t \in \mathrm{Hom}(\tau, \rho\sigma)$

spectively. These morphisms need not be irreducible. Taking them as products,

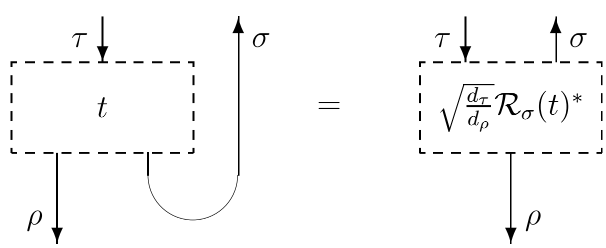


Figure 23: Right Frobenius reciprocity for an intertwiner $t \in \mathrm{Hom}(\tau, \rho\sigma)$

we may replace any of them by bundles of wires. We call the linear isomorphisms $t \mapsto d_\rho^{1/2} r_\rho^* \bar{\rho}(t)$ and $t \mapsto d_\sigma^{1/2} \rho(\bar{r}_\sigma)^* t$ the left and right Frobenius rotations.

Now let us assume that t is isometric and labels a trivalent vertex of wires corresponding to irreducible morphisms ρ, τ, σ. With the above "transformation law" we then have the identity of Fig. 24, where the first equality is just a definition which gives us some prescription of "tightening" wires at trivalent vertices. In fact, the

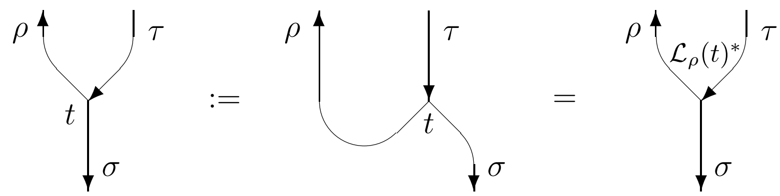


Figure 24: Left Frobenius reciprocity for a trivalent vertex labelled by an isometry

label $\mathcal{L}_\rho(t)^*$ of the trivalent vertex makes sense since it is a co-isometry: Due to irreducibility of τ and σ, the map $t \mapsto \mathcal{L}_\rho(t)^*$ is isometric. Similarly, we get Fig. 25 (using irreducibility of τ and ρ). Hence the prefactor in Figs. 22 and 23 is just such

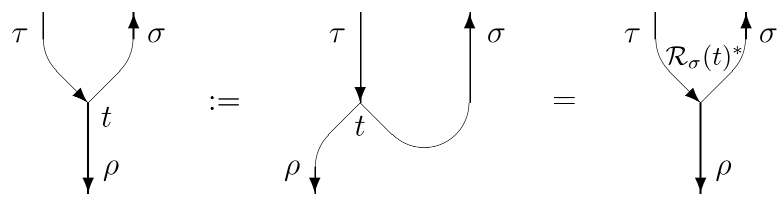


Figure 25: Right Frobenius reciprocity for a trivalent vertex labelled by an isometry

that it transforms isometries with natural normalization prefactors into co-isometries with natural normalization prefactors and, by taking adjoints, the other way round which gives the graphical identities given in Fig. 26. We may now use the replacement

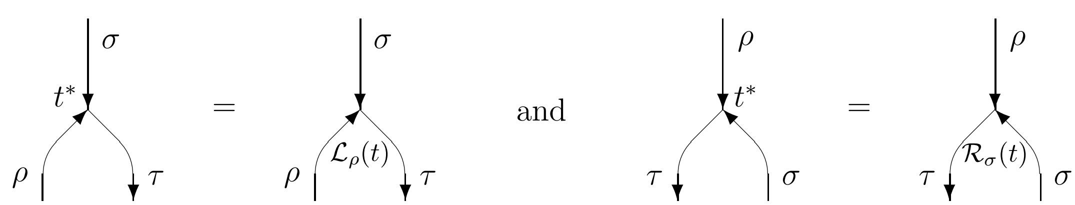


Figure 26: Frobenius reciprocity for a trivalent vertex labelled by a co-isometry

prescription three times, beginning with a trivalent vertex labelled by an isometry $t \in \mathrm{Hom}(\tau, \rho\sigma)$ and proceeding in a clockwise direction. Then we end up with a co-isometry $\Theta(t)^* \in \mathrm{Hom}(\bar\sigma\bar\rho, \bar\tau)$ in the corner where we originally had the label t. In fact,

$$\Theta(t) = \mathcal{R}_\rho(\mathcal{L}_\tau(\mathcal{R}_\sigma(t))) = \sqrt{d_\rho d_\sigma d_\tau}\, r_\tau^* \bar\tau(t^* \rho(\bar r_\sigma)\bar r_\rho)\,.$$

Similarly we can go in the counter-clockwise direction and then we obtain $\tilde\Theta(t)^* \in$

$\mathrm{Hom}(\bar\sigma\bar\rho,\bar\tau)$, where

$$\tilde\Theta(t) = \mathcal{L}_\sigma(\mathcal{R}_\tau(\mathcal{L}_\rho(t))) = \sqrt{d_\rho d_\sigma d_\tau}\,\bar\sigma\bar\rho(\bar r_\tau^* t^*)\bar\sigma(r_\rho)r_\sigma\,,$$

and in order to establish a well-defined rotation procedure we have to show that $\Theta(t)=\tilde\Theta(t)$. Now

$$\begin{aligned}\Theta(t)^*\tilde\Theta(t) &= \sqrt{d_\rho d_\sigma d_\tau}\,\bar\tau(\bar r_\tau^* t^*)\Theta(t)^*\bar\sigma(r_\rho)r_\sigma\\ &= d_\rho d_\sigma d_\tau\,\bar\tau(\bar r_\tau^* t^*)\bar\tau(\bar r_\rho^*\rho(\bar r_\sigma^*)t)\bar\tau\tau(\bar\sigma(r_\rho)r_\sigma)r_\tau\\ &= d_\rho d_\sigma d_\tau\,\bar\tau(\bar r_\tau^* t^*\bar r_\rho^*\rho(\bar r_\sigma^*)\rho\sigma(\bar\sigma(r_\rho)r_\sigma)t)r_\tau\\ &= d_\rho d_\tau\,\bar\tau(\bar r_\tau^* t^*\bar r_\rho^*\rho(r_\rho)t)r_\tau = d_\tau\,\bar\tau(\bar r_\tau^*)r_\tau = \mathbf{1}\,,\end{aligned}$$

hence $(\Theta(t)-\tilde\Theta(t))^*(\Theta(t)-\tilde\Theta(t))=0$, i.e. $\Theta(t)=\tilde\Theta(t)$. Thus a trivalent vertex labelled with an isometry $t\in\mathrm{Hom}(\tau,\rho\sigma)$ can equivalently be labelled with a co-isometry $\Theta(t)^*\in\mathrm{Hom}(\bar\sigma\bar\rho,\bar\tau)$. So here we have established some "rotation invariance" of trivalent vertices (in standard inverted Y-shape or Y-shape) with a replacement prescription for the rotated labelling (co-) isometries.

Next we turn to the rotation of crossings when we have a braiding. Assume we have a braided system of endomorphisms $\Delta\ni\lambda,\mu,\nu$ of some factor A. From the BFE we obtain $r_\lambda=\bar\lambda(\varepsilon^\mp(\mu,\lambda))\varepsilon^\mp(\mu,\bar\lambda)\mu(r_\lambda)$. Applying λ and multiplying by $d_\lambda\varepsilon^\pm(\lambda,\mu)\bar r_\lambda^*$ from the left yields

$$\varepsilon^\pm(\lambda,\mu) = d_\lambda\bar r_\lambda^*\lambda(\varepsilon^\mp(\mu,\bar\lambda))\lambda\mu(r_\lambda)\,. \tag{13}$$

The BFE yields similarly $\lambda(\bar r_\mu)=\varepsilon^\pm(\mu,\lambda)\mu(\varepsilon^\mp(\bar\mu,\lambda))\bar r_\mu$, and by multiplying with $d_\mu\mu\lambda(r_\mu^*)\varepsilon^\pm(\lambda,\mu)$ from the left we obtain

$$\varepsilon^\pm(\lambda,\mu) = d_\mu\mu\lambda(r_\mu^*)\mu(\varepsilon^-(\bar\mu,\lambda))\bar r_\mu\,,$$

and therefore we have the graphical identity given in Fig. 27, here displayed only for overcrossings. Then this procedure can even be iterated so that we obtain arbi-

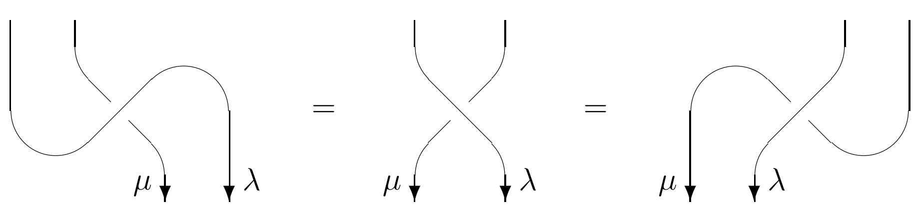


Figure 27: Rotation of crossings

trarily twisted crossings. Note that for the rotation of crossings we do not need any relabelling prescription as this is encoded in the BFE's.

We now turn to the discussion of "abstract pictures" which admit different intertwiner interpretations according to Frobenius rotations. Let $A_1, A_2, ..., A_\ell$ be factors equipped with sets $\Delta_{i,j} \subset \mathrm{Mor}(A_i, A_j)$, $i, j = 1, 2, ..., \ell$, of irreducible, pairwise inequivalent morphisms with finite index such that $\bigsqcup_{i,j} \Delta_{i,j}$ is closed under conjugation and irreducible decomposition of products (whenever composable) as sectors, and in particular each $\Delta_{i,i}$ is a system of endomorphisms. Some of the systems $\Delta_{i,i}$ may be braided.

We now consider "labelled knotted graphs" of the following form. On a finite connected and simply connected region in the plane we have a finite number of wires (i.e. images of piecewise C^∞ maps from the unit interval into the region). Within the region there is a finite number of trivalent vertices (i.e. common endpoints of three wires) and crossings of two wires, and for the latter there is a notion of over- and undercrossing (i.e. for each crossing there is one wire "on top of the other"). If wires are not closed (i.e. if their two endpoints do not coincide) then they are only allowed to have trivalent vertices or distinguished points on the boundary of the region as their endpoints. The wires meet each other only at the trivalent vertices and crossings, and they are directed and labelled by the morphisms in $\bigsqcup_{i,j} \Delta_{i,j}$ subject to the following rules. Crossings are only possible for wires with labelling morphisms in some $\Delta_{i,i}$ with braiding. Furthermore it must be possible to associate the factors A_i to the free regions between the wires such that any wire labelled by some $\rho \in \Delta_{i,j}$ has the "source" factor A_i on its left and the "range" factor A_j on its right relative to the orientation (composition compatibility). We identify graphs which are transformed into each other by inversion of the orientation of a wire and simultaneous replacement of its label, say $\rho \in \Delta_{i,j}$, by the representative conjugate morphism $\bar{\rho} \in \Delta_{j,i}$. Finally, the trivalent vertices are labelled either by isometric or co-isometric intertwiners which are associated locally to one corner region of the trivalent vertex as follows. If $\tau \in \Delta_{i,j}$, $\rho \in \Delta_{k,j}$, $\sigma \in \Delta_{i,k}$ label the three wires of a trivalent vertex, τ is entering and, following counter-clockwise, ρ and σ are outgoing (as e.g. the trivalent vertex in Figs. 24 and 25, possibly up to isotopy and rotation), then in the local corner region opposite to τ the label must either be an isometry $t \in \mathrm{Hom}(\tau, \rho\sigma)$ or a co-isometry $s^* \in \mathrm{Hom}(\bar{\sigma}\bar{\rho}, \bar{\tau})$. If the wires at a trivalent vertex have orientation different from this, the rule can be derived from the previous case by reversing orientations and simultaneous relabelling by conjugate morphisms.

Now let $\mathcal{G}$ be such a labelled knotted graph as above. To interpret $\mathcal{G}$ as an intertwiner, we may put it in some "Frobenius annulus" as shown in Fig. 28 for an example.[5] A Frobenius annulus has labelled wires inside such that each of them meets an open end of a wire of $\mathcal{G}$ at one endpoint (labelled by ρ_1,...,ρ_{12} in our example), matching the label and orientation of this wire, and this way all the open ends of the wires of $\mathcal{G}$ are either connected to the top or bottom of the outside square boundary of the annulus. No crossings or trivalent vertices are allowed in the annulus, but it may contain cups or caps. Gluing the wires together and forgetting about the boundary

[5] Our notion of a Frobenius annulus is inspired by the annular invariance used in Jones' definition of a "general planar algebra" [22].

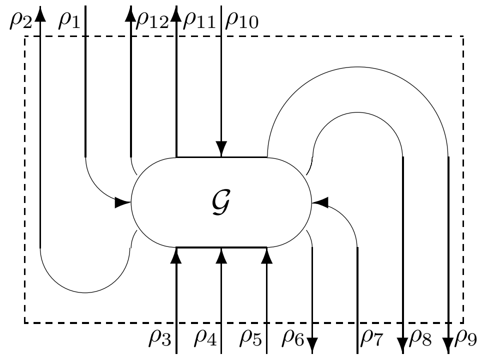


Figure 28: A Frobenius annulus surrounding $\mathcal{G}$

of $\mathcal{G}$ and the annulus, we will read the result as a wire diagram and therefore the annulus corresponds to a “Frobenius choice”, deciding whether we will get a certain intertwiner or its image by certain Frobenius rotations, cf. Figs. 22 and 23 (and their adjoints).

Reading vertically downwards, we may now have the problem that on a finite number of horizontal levels a finite number of singular points of crossings, trivalent vertices, cups and caps are exactly on the same level (or “height”) so that we cannot time slice the diagram into stripes containing only one elementary intertwiner. Also some wires may have pieces going exactly horizontally. We now allow to make small vertical translations such that these crossings and trivalent vertices are put on slightly different levels and all wires obtain piecewise slopes, without letting wires touch or producing new crossings, but we may possibly produce some new cups or caps. In the latter case we can always arrange it so that even each new cup or cap appears on a distinct level. The trivalent vertices and crossings may not be in “standard form”, i.e. in Y- or inverted Y shape respectively X-shape. In an “ϵ-neighborhood” of a trivalent vertex, we now bend the wires so that the angles are arranged in standard form. Similarly we modify the crossings to bend them into an X-shape. Using for labels at trivalent vertices our replacement prescription by Frobenius reciprocity, we can obtain isometries as labels for trivalent vertices in inverted Y-shape, located on the bottom corner region, and co-isometries as labels for trivalent vertices in Y-shape, located on the top corner region.

Again, these topological moves are allowed to produce at most new cups or caps, all on different levels so that the resulting diagram can be time sliced into stripes of elementary diagrams. Clearly, this procedure of deforming a labelled knotted graph in a Frobenius annulus into a regular wire diagram is highly ambiguous. However, the ambiguities in the above procedures are irrelevant: The ambiguities arising from the production of slopes of wires and different levels of certain elementary intertwiners are irrelevant due to the topological invariance of Fig. 11 and the freedom of translating intertwiners vertically as shown in Fig. 5, and the ambiguities arising from rotations of the elementary intertwiners are irrelevant due to the rotation invariance of trivalent vertices and crossings, as we have established in Figs. 24, 25, 26 and 27.

Now let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two labelled knotted graphs as above which are defined on the same (connected, simply connected) region in the plane and have the same entering and outgoing wires at the same points with the same orientation, i.e. they have coinciding open ends so that they fit in the same Frobenius annuli. When embedded in some Frobenius annulus it may now happen that the corresponding intertwiners are the same, even if $\mathcal{G}_1$ and $\mathcal{G}_2$ are different. Because of the isomorphism property of Frobenius rotations it is clear that then $\mathcal{G}_1$ and $\mathcal{G}_2$ yield the same intertwiner through embedding in any Frobenius annulus. We can write down sufficient conditions for such equality in terms of some "regular isotopy": For given $\mathcal{G}_1$ and $\mathcal{G}_2$ as above choose a Frobenius annulus and regularize the pictures into two wire diagrams $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively. We call $\mathcal{G}_1$ and $\mathcal{G}_2$ regularly isotopic if $\mathcal{W}_1$ can be transformed into $\mathcal{W}_2$ by the following list of moves:

1. Reversing orientation of some wires with simultaneous relabelling by conjugate morphisms,
2. any horizontal translations of elementary intertwiners which may change slopes of wires but which do not let the wires meet or involve cups or caps,
3. vertical translations of elementary intertwiners as in Fig. 5,
4. topological moves as in Fig. 11,
5. rotations of trivalent vertices and their labels as in Figs. 24, 25, 26,
6. and for wires corresponding to a braided system $\Delta_{i,i}$ we additionally admit
   (a) vertical Reidemeister moves of type II as in Fig. 13,
   (b) moving crossings over and under trivalent vertices, cups and caps according to the BFE's (cf. Figs. 14 and 15 for the first two relations),
   (c) vertical Reidemeister moves of type III for crossings (cf. Fig. 16 for overcrossings),
   (d) rotations of crossings (cf. Fig. 27 for overcrossings).

Thus the ambiguity in the regularization procedure means in particular that from one graph we can only obtain wire diagrams that can be transformed into each other by these moves. It is easy to see that regular isotopy is an equivalence relation for knotted labelled graphs. Moreover, for closed labelled knotted graphs (i.e. without open ends) which are then embedded in a trivial annulus, the local rotation invariance of the elementary intertwiners ends up in a total rotation invariance: We can rotate the picture freely, the rotated graph is always regularly isotopic to the original one and we will always end up with the same scalar (times $\mathbf{1}_{A_i}$, where A_i is the factor associated to the outside region).[6]

Let us finally consider an intertwiner $x \in \mathrm{Hom}(\rho, \rho)$ with $\rho \in \mathrm{Mor}(A, B)$ irreducible. Then clearly x is a scalar: $x = \xi \mathbf{1}_B$, $\xi \in \mathbb{C}$. Hence we have the identity $d_\rho \xi \mathbf{1}_B \equiv d_\rho x = d_\rho \bar{r}_\rho^* x \bar{r}_\rho$, and this is graphically the left-hand side in Fig. 29. On the

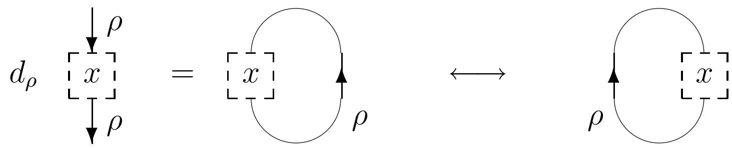


Figure 29: Two intertwiners of the same scalar value

other hand, application of the left inverse yields $d_\rho \phi_\rho(x) = d_\rho r_\rho^* \bar{\rho}(x) r_\rho = d_\rho \xi \mathbf{1}_A$, which is a different intertwiner of the same scalar value, and it is represented graphically by the right-hand side in Fig. 29. Thus the left- and right-hand side in Fig. 29 represent the same scalar. If we consider closed wire diagrams and are only interested in the scalars they represent, then we therefore have a "regular isotopy on the 2-sphere".

3.3 α-Induction for braided subfactors

We now consider α-induction of [2, 3, 4] in the setting of braided subfactors. Here we work with a type III subfactor $N \subset M$, equipped with a braided system $\Delta \subset \mathrm{End}(N)$ in the sense of Definition 2.1 such that for the injection map $\iota : N \to M$, the sector $[\bar{\iota}\iota]$ decomposes into a finite sum of sectors of morphisms in Δ. (Here $\bar{\iota}$ denotes any choice of a representative morphism for the conjugate sector of $[\iota]$.) Note that since elements in Δ have by definition finite statistical dimension, it follows that the injection map has finite statistical dimension and thus the subfactor $N \subset M$ has finite index. But also note that we did neither assume the finite depth condition on $N \subset M$ (we did not assume finiteness of Δ) nor non-degeneracy of the braiding at this point. As usual, we denote the canonical endomorphism $\iota\bar{\iota} \in \mathrm{End}(M)$ by $\gamma = \iota\bar{\iota}$, the dual canonical endomorphism $\bar{\iota}\iota \in \mathrm{End}(N)$ by $\theta = \bar{\iota}\iota$ and "canonical" isometries by $v \in M$ and $w \in N$, more precisely, we have $v \in \mathrm{Hom}(\mathrm{id}_M, \gamma)$ and $w \in \mathrm{Hom}(\mathrm{id}_N, \theta)$ such that $w^* v = \gamma(v^*) w = [M : N]^{-1/2} \mathbf{1}$. Recall that we have pointwise equality $M = Nv$.

With a braiding ε on Δ and its extension to $\Sigma(\Delta)$ as in Subsection 2.2 we can define the α-induced $\alpha_\lambda^\pm$ for $\lambda \in \Sigma(\Delta)$ exactly as in [33, 2], namely we define

$$\alpha_\lambda^\pm = \bar{\iota}^{-1} \circ \mathrm{Ad}(\varepsilon^\pm(\lambda, \theta)) \circ \lambda \circ \bar{\iota} \,.$$

Then α_λ^+ and α_λ^- are morphisms in $\mathrm{Mor}(M, M)$ with the properties $\alpha_\lambda^\pm \circ \iota = \iota \circ \lambda$, $\alpha_\lambda^\pm(v) = \varepsilon^\pm(\lambda, \theta)^* v$, $\alpha_{\lambda\mu}^\pm = \alpha_\lambda^\pm \alpha_\mu^\pm$ if also $\mu \in \Sigma(\Delta)$, and clearly $\alpha_{\mathrm{id}_N}^\pm = \mathrm{id}_M$. Note

[6] For a single kind of wire corresponding to a braided system, this invariance is similar to the complex number-valued regular isotopy invariant of knotted graphs obtained in [36].

that the first property yields immediately $d_{\alpha_\lambda^\pm} = d_\lambda$ by the multiplicativity of the minimal index [31]. We also obtain easily that $\overline{\alpha_\lambda^\pm} = \alpha_{\bar\lambda}^\pm$, since we obtain $r_\lambda = \varepsilon^\pm(\theta, \bar\lambda\lambda)\theta(r_\lambda)$, and similarly $\bar r_\lambda = \varepsilon^\pm(\theta, \lambda\bar\lambda)\theta(\bar r_\lambda)$ easily from Eq. (8). Multiplying both relations by v from the right yields $r_\lambda v = \alpha_{\bar\lambda}^\pm\alpha_\lambda^\pm(v)r_\lambda$ and $\bar r_\lambda v = \alpha_\lambda^\pm\alpha_{\bar\lambda}^\pm(v)\bar r_\lambda$, hence $r_\lambda \in \mathrm{Hom}(\mathrm{id}_M, \alpha_{\bar\lambda}^\pm\alpha_\lambda^\pm)$, $\bar r_\lambda \in \mathrm{Hom}(\mathrm{id}_M, \alpha_\lambda^\pm\alpha_{\bar\lambda}^\pm)$ as $M = Nv$, thus we can put $R_{\alpha_\lambda^\pm} = \iota(r_\lambda)$, $\bar R_{\alpha_\lambda^\pm} = \iota(\bar r_\lambda)$ as R-isometries for the α-induced morphisms, i.e. $\overline{\alpha_\lambda^\pm} = \alpha_{\bar\lambda}^\pm$. Note also that the definition of $\alpha_\lambda^\pm$ does not depend on the choice of the representative morphism $\bar\iota$ for the conjugate sector of $[\iota]$ due to the transformation properties of the braiding operators, Eq. (7).

Though the local net structure for $N(I) \subset M(I)$ is assumed in [33, 2], we need only an assumption of a braiding for the definition of $\alpha_\lambda^\pm$. We, however, have to be careful, because we do not assume the chiral locality condition $\varepsilon(\theta,\theta)\gamma(v) = \gamma(v)$ in this paper. (The name "chiral locality" is motivated from the treatment of extensions of chiral observables in conformal field theory in the setting of nets of subfactors [33], where the extended net is shown to satisfy local commutativity if and only if the condition $\varepsilon(\theta,\theta)\gamma(v) = \gamma(v)$ is met [33, Thm. 4.9].) Some theorems in [2, 3, 4] do depend on the chiral locality condition and are *not* true in this more general setting of α-induction. Namely, with $\varepsilon(\theta,\theta)\gamma(v) = \gamma(v)$ it was easily derived [2, Lemma 3.5] by using the BFE that then $\mathrm{Hom}(\alpha_\lambda^\pm, \alpha_\mu^\pm) = \mathrm{Hom}(\iota\lambda, \iota\mu)$ for $\lambda, \mu \in \Sigma(\Delta)$. As a surprising corollary (cf. [2, Cor. 3.6]) one found by putting $\lambda = \mu = \mathrm{id}_N$ that ι, thus the subfactor $N \subset M$, was irreducible which had not been assumed. Another corollary was then the "main formula" [2, Thm. 3.9], giving $\langle \alpha_\lambda^\pm, \alpha_\mu^\pm \rangle = \langle \iota\lambda, \iota\mu \rangle = \langle \theta\lambda, \mu \rangle$ by Frobenius reciprocity. (Moreover, in the framework of *nets of* subfactors $\mathcal{N} \subset \mathcal{M}$, where the braidings arise from the transportability of localized endomorphisms, a certain reciprocity formula $\langle \alpha_\lambda^\pm, \beta \rangle = \langle \lambda, \sigma_\beta \rangle$, called "$\alpha\sigma$-reciprocity", between localized transportable endomorphisms λ and β of the smaller respectively the larger net was established; here σ-restriction is essentially $\sigma_\beta = \bar\iota\beta\iota$.) Without chiral locality, these results are in general not true: The subfactor $N \subset M$ is neither forced to be irreducible, nor does the main formula hold, however, we always have the inequality $\langle \alpha_\lambda^\pm, \alpha_\mu^\pm \rangle \le \langle \theta\lambda, \mu \rangle$, since only the "$\ge$" part of the proof of [2, Thm. 3.9] uses chiral locality.

It is a simple application of the braiding fusion equation and does not involve chiral locality that for $\lambda, \mu, \nu \in \Sigma(\Delta)$ we have the (equivalent) relations [2, Lemma 3.25]

$$\alpha_\rho^\mp(Q)\varepsilon^\pm(\lambda,\rho) = \varepsilon^\pm(\mu,\rho)Q\,, \qquad Q\varepsilon^\pm(\rho,\lambda) = \varepsilon^\pm(\rho,\mu)\alpha_\rho^\pm(Q) \tag{14}$$

whenever $Q \in \mathrm{Hom}(\iota\lambda, \iota\mu)$.

Let $a \in \mathrm{Mor}(M,N)$ be such that $[a]$ is a subsector of $[\mu\bar\iota]$ for some $\mu \in \Sigma(\Delta)$. Hence $a\iota \in \Sigma(\Delta)$. Similarly, let $\bar b \in \mathrm{Mor}(N,M)$ be such that $[\bar b]$ is a subsector of $[\iota\bar\nu]$ for some $\bar\nu \in \Sigma(\Delta)$. If $T \in \mathrm{Hom}(\bar b, \iota\bar\nu)$ is an isometry we put

$$\mathcal{E}^\pm(\lambda, \bar b) = T^*\varepsilon^\pm(\lambda, \bar\nu)\alpha_\lambda^\pm(T)\,, \qquad \mathcal{E}^\pm(\bar b, \lambda) = (\mathcal{E}^\mp(\lambda, \bar b))^*\,.$$

Note that the definition is independent of the choice of T and $\bar\nu$ in the following sense: If also $S \in \mathrm{Hom}(\bar b, \iota\bar\tau)$ is an isometry for some $\bar\tau \in \Sigma(\Delta)$ then $ST^* \in \mathrm{Hom}(\iota\bar\nu, \iota\bar\tau)$ and therefore

$$\mathcal{E}^{\pm}(\lambda,\bar{b}) = S^*ST^*\varepsilon^{\pm}(\lambda,\bar{\nu})\alpha^{\pm}_{\lambda}(T) = S^*\varepsilon^{\pm}(\lambda,\bar{\tau})\alpha^{\pm}_{\lambda}(ST^*T) = S^*\varepsilon^{\pm}(\lambda,\bar{\tau})\alpha^{\pm}_{\lambda}(S)\,.$$

Similarly one easily checks that $\mathcal{E}^{\pm}(\lambda,\bar{b})$ is unitary.

Proposition 3.1 *Let $\lambda\in\Sigma(\Delta)$, let $a\in\mathrm{Mor}(M,N)$ be such that $[a]$ is a subsector of $[\mu\bar{\iota}]$ for some $\mu\in\Sigma(\Delta)$ and let $\bar{b}\in\mathrm{Mor}(N,M)$ be such that $[\bar{b}]$ is a subsector of $[\iota\bar{\nu}]$ for some $\bar{\nu}\in\Sigma(\Delta)$. Then we have*

$$\varepsilon^{\pm}(\lambda,a\iota)\in\mathrm{Hom}(\lambda a,a\alpha^{\pm}_{\lambda})\,,\qquad \mathcal{E}^{\pm}(\lambda,\bar{b})\in\mathrm{Hom}(\alpha^{\pm}_{\lambda}\bar{b},\bar{b}\lambda)\,. \tag{15}$$

Proof. The first relation in Eq. (15) is trivial on N, so we only need to show it for v since $M=Nv$. Note that $a(v)\in\mathrm{Hom}(a\iota,a\iota\theta)$, therefore Eq. (5) yields

$$a(v)\varepsilon^{\pm}(\lambda,a\iota) = a\iota(\varepsilon^{\pm}(\lambda,\theta))\varepsilon^{\pm}(\lambda,a\iota)\lambda(a(v))\,,$$

hence

$$a\circ\alpha^{\pm}_{\lambda}(v) = a\iota(\varepsilon^{\pm}(\lambda,\theta)^*)a(v) = \varepsilon^{\pm}(\lambda,a\iota)\lambda(a(v))\varepsilon^{\pm}(\lambda,a\iota)^* = \mathrm{Ad}\,\varepsilon^{\pm}(\lambda,a\iota)\circ\lambda\circ a(v)\,.$$

For the second relation we use the fact that $TT^*\in\mathrm{Hom}(\iota\bar{\nu},\iota\bar{\nu})$ for $T\in\mathrm{Hom}(\bar{b},\iota\bar{\nu})$:

$$\begin{aligned}\mathcal{E}^{\pm}(\lambda,\bar{b})\alpha^{\pm}_{\lambda}\bar{b}(n) &= T^*\varepsilon^{\pm}(\lambda,\bar{\nu})\alpha^{\pm}_{\lambda}(TT^*\bar{\nu}(n)T) = T^*\varepsilon^{\pm}(\lambda,\bar{\nu})\lambda\bar{\nu}(n)\alpha^{\pm}_{\lambda}(T)\\ &= T^*\bar{\nu}\lambda(n)\varepsilon^{\pm}(\lambda,\bar{\nu})\alpha^{\pm}_{\lambda}(T) = \bar{b}\lambda(n)\mathcal{E}^{\pm}(\lambda,\bar{b})\end{aligned}$$

for all $n\in N$. □

Due to Prop. 3.1 we can now draw the pictures in Fig. 30 for the operators $\varepsilon^{\pm}(\lambda,a\iota)$ and $\mathcal{E}^{\pm}(\lambda,\bar{b})$. The pictures for their conjugates $\varepsilon^{\mp}(a\iota,\lambda)$ and $\mathcal{E}^{\mp}(\bar{b},\lambda)$ are as usual

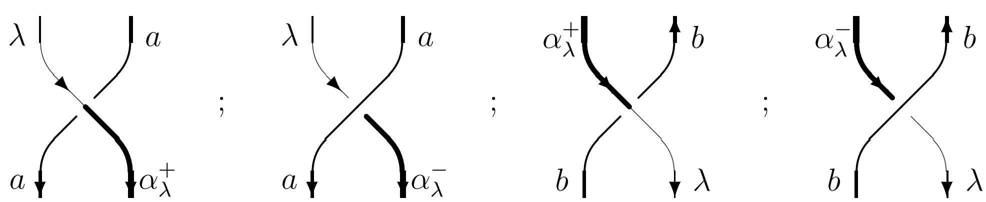


Figure 30: Wire diagrams for $\varepsilon^+(\lambda,a\iota)$, $\varepsilon^-(\lambda,a\iota)$, $\mathcal{E}^+(\lambda,\bar{b})$, $\mathcal{E}^-(\lambda,\bar{b})$, respectively

obtained by horizontal reflection and inversion of arrows of the pictures in Fig. 30.

Lemma 3.2 *Let $\bar{a},\bar{b}\in\mathrm{Mor}(M,N)$ be such that $[\bar{a}]$ and $[\bar{b}]$ are subsectors of $[\iota\bar{\mu}]$ and $[\iota\bar{\nu}]$ for some $\bar{\mu},\bar{\nu}\in\Sigma(\Delta)$, respectively. Whenever $Y\in\mathrm{Hom}(\bar{a},\bar{b})$ we have*

$$\alpha^{\mp}_{\rho}(Y)\,\mathcal{E}^{\pm}(\bar{a},\rho) = \mathcal{E}^{\pm}(\bar{b},\rho)\,Y\,,\qquad Y\,\mathcal{E}^{\pm}(\rho,\bar{a}) = \mathcal{E}^{\pm}(\rho,\bar{b})\,\alpha^{\pm}_{\rho}(Y)\,.$$

Proof. Let $S \in \mathrm{Hom}(\bar{a}, \iota\bar{\mu})$ and $T \in \mathrm{Hom}(\bar{b}, \iota\bar{\nu})$ be isometries. Then $\mathcal{E}^\pm(\bar{a}, \rho) = \alpha_\rho^\mp(S)^* \varepsilon^\pm(\bar{\mu}, \rho) S$ and $\mathcal{E}^\pm(\rho, \bar{b}) = T^* \varepsilon^\pm(\rho, \bar{\nu}) \alpha_\rho^\pm(T)$. Now $TYS^* \in \mathrm{Hom}(\iota\bar{\mu}, \iota\bar{\nu})$. Inserting this in Eq. (14) yields the statement. □

In order to establish a symmetry for "moving crossings over trivalent vertices" we can now state the following

Proposition 3.3 *Let $\lambda, \rho \in \Sigma(\Delta)$, let $a, b \in \mathrm{Mor}(M, N)$ be such that $[a]$ and $[b]$ are subsectors of $[\mu\bar{\iota}]$ and $[\nu\bar{\iota}]$ for some $\mu, \nu \in \Sigma(\Delta)$ and let $\bar{a}, \bar{b} \in \mathrm{Mor}(N, M)$ be conjugates, respectively. Whenever $t \in \mathrm{Hom}(\lambda, a\bar{b})$, $x \in \mathrm{Hom}(a, \lambda b)$ and $Y \in \mathrm{Hom}(\bar{a}, \bar{b}\lambda)$, we have the intertwining braiding fusion equations (IBFE's):*

$$\begin{aligned}
\rho(t)\, \varepsilon^\pm(\lambda, \rho) &= \varepsilon^\pm(a\iota, \rho)\, a(\mathcal{E}^\pm(\bar{b}, \rho))\, t\,, & (16)\\
t\, \varepsilon^\pm(\rho, \lambda) &= a(\mathcal{E}^\pm(\rho, \bar{b}))\, \varepsilon^\pm(\rho, a\iota)\, \rho(t)\,, & (17)\\
\rho(x)\, \varepsilon^\pm(a\iota, \rho) &= \varepsilon^\pm(\lambda, \rho)\, \lambda(\varepsilon^\pm(b\iota, \rho))\, x\,, & (18)\\
x\, \varepsilon^\pm(\rho, a\iota) &= \lambda(\varepsilon^\pm(\rho, b\iota))\, \varepsilon^\pm(\rho, \lambda)\, \rho(x)\,, & (19)\\
\alpha_\rho^\mp(Y)\, \mathcal{E}^\pm(\bar{a}, \rho) &= \mathcal{E}^\pm(\bar{b}, \rho)\, \bar{b}(\varepsilon^\pm(\lambda, \rho))\, Y\,, & (20)\\
Y\, \mathcal{E}^\pm(\rho, \bar{a}) &= \bar{b}(\varepsilon^\pm(\rho, \lambda))\, \mathcal{E}^\pm(\rho, \bar{b})\, \alpha_\rho^\pm(Y)\,. & (21)
\end{aligned}$$

Proof. Since $[\bar{b}]$ must be a subsector of $[\iota\bar{\nu}]$ for $\bar{\nu} \in \Sigma(\Delta)$ a conjugate of ν, there is an isometry $T \in \mathrm{Hom}(\bar{b}, \iota\bar{\nu})$. Note that then $a(T) \in \mathrm{Hom}(a\bar{b}, a\iota\bar{\nu})$. Hence by naturality and Proposition 3.1 we compute

$$\begin{aligned}
\varepsilon^\pm(\rho, a\bar{b}) &= a(T^*)\varepsilon^\pm(\rho, a\iota\bar{\nu})\rho a(T) = a(T^*)a(\varepsilon^\pm(\rho, \bar{\nu}))\varepsilon^\pm(\rho, a\iota)\rho a(T)\\
&= a(T^*)a(\varepsilon^\pm(\rho, \bar{\nu}))a\alpha_\rho^\pm(T)\varepsilon^\pm(\rho, a\iota) = a(\mathcal{E}^\pm(\rho, \bar{b}))\varepsilon^\pm(\rho, a\iota)\,,
\end{aligned}$$

and hence also $\varepsilon^\pm(a\bar{b}, \rho) = \varepsilon^\pm(a\iota, \rho)a(\mathcal{E}^\pm(\bar{b}, \rho)$. We also obtain $\varepsilon^\pm(\lambda b\iota, \rho) = \varepsilon^\pm(\lambda, \rho)\lambda(\varepsilon^\pm(b\iota, \rho))$ and $\varepsilon^\pm(\rho, \lambda b\iota) = \lambda(\varepsilon^\pm(\rho, b\iota))\varepsilon^\pm(\rho, \lambda)$ by Eq. (9). Note that $x \in \mathrm{Hom}(a\iota, \lambda b\iota)$ by restriction. Eqs. (16)-(19) follow now by naturality, Eq. (8). Next, we note that $T \in \mathrm{Hom}(\bar{b}\lambda, \iota\bar{\nu}\lambda)$, and hence $\mathcal{E}^\pm(\rho, \bar{b}\lambda) = T^*\varepsilon^\pm(\rho, \bar{\nu}\lambda)\alpha_\rho^\pm(T)$. Therefore

$$\begin{aligned}
\mathcal{E}^\pm(\rho, \bar{b}\lambda) &= T^*\bar{\nu}(\varepsilon^\pm(\rho, \lambda))\varepsilon^\pm(\rho, \bar{\nu})\alpha_\rho^\pm(T) = \bar{b}(\varepsilon^\pm(\rho, \lambda))T^*\varepsilon^\pm(\rho, \bar{\nu})\alpha_\rho^\pm(T)\\
&= \bar{b}(\varepsilon^\pm(\rho, \lambda))\mathcal{E}^\pm(\rho, \bar{b})\,,
\end{aligned}$$

and hence also $\mathcal{E}^\pm(\bar{b}\lambda, \rho) = \mathcal{E}^\pm(\bar{b}, \rho)\bar{b}(\varepsilon^\pm(\lambda, \rho))$. Now Eqs. (20) and (21) follow from Lemma 3.2. □

These IBFE's can be nicely visualized in diagrams. We display Eq. (16) in Fig. 31 and Eq. (21) in Fig. 32, both for overcrossings. We leave the remaining diagrams as a straightforward exercise to the reader. Note that the IBFE's give us the freedom to move wires with label ρ and $\alpha_\rho^\pm$ freely over trivalent vertices which involve one N-N wire and two N-M wires. Unitarity of operators $\mathcal{E}^\pm(\lambda, \bar{b})$ yields a "vertical

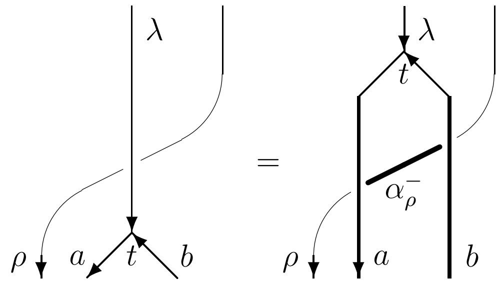


Figure 31: The first intertwining braiding fusion equation (overcrossings)

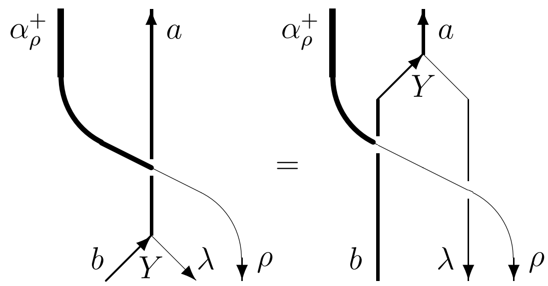


Figure 32: The sixth intertwining braiding fusion equation (overcrossings)

Reidemeister move of type II" similar to Fig. 13. We can now also easily elaborate the rotation behavior of mixed crossings displayed in Fig. 30 (and consequently their conjugates). Crucial for this is the fact that $R_{\alpha^\pm_\lambda} = \iota(r_\lambda) \equiv r_\lambda$ and $\bar{R}_{\alpha^\pm_\lambda} = \iota(\bar{r}_\lambda) \equiv \bar{r}_\lambda$ can be used as R-isometries for the α-induced morphisms as $R_{\alpha^\pm_\lambda} \in \mathrm{Hom}(\mathrm{id}_M, \overline{\alpha^\pm_\lambda}\alpha^\pm_\lambda)$ and $\bar{R}_{\alpha^\pm_\lambda} \in \mathrm{Hom}(\mathrm{id}_M, \alpha^\pm_\lambda\overline{\alpha^\pm_\lambda})$ satisfy $\alpha^\pm_\lambda(R_{\alpha^\pm_\lambda})^*\bar{R}_{\alpha^\pm_\lambda} = d_\lambda^{-1}\mathbf{1}_M$ and $\overline{\alpha^\pm_\lambda}(\bar{R}_{\alpha^\pm_\lambda})^*\bar{R}_{\alpha^\pm_\lambda} = d_\lambda^{-1}\mathbf{1}_M$ and $d_{\alpha^\pm_\lambda} = d_\lambda$. First we notice that we have

$$\varepsilon^\pm(\lambda, a\iota) = d_\lambda\, \bar{r}_\lambda\, \lambda(\varepsilon^\mp(a\iota, \bar{\lambda}))\, \lambda a(r_\lambda)$$

by Eq. (13). Now let $R_a \in \mathrm{Hom}(\mathrm{id}_M, \bar{a}a)$ and $\bar{r}_a \in \mathrm{Hom}(\mathrm{id}_N, a\bar{a})$ be isometries such that $a(R_a)^*\bar{r}_a = d_a^{-1}\mathbf{1}_N$ and $\bar{a}(\bar{r}_a)^*R_a = d_a^{-1}$, and otherwise we keep the notations as in Prop. 3.3. From Eq. (17) we obtain $a(\mathcal{E}^\mp(\bar{a},\lambda))\bar{r}_a = \varepsilon^\pm(\lambda, a\iota)\lambda(\bar{r}_a)$. Hence we have

$$\begin{aligned} \varepsilon^\pm(\lambda, a\iota) &= d_a\, \varepsilon^\pm(\lambda, a\iota)\, \lambda a(R_a)^*\lambda(\bar{r}_a) = d_a\, a\alpha^\pm_\lambda(R_a)^*\, \varepsilon^\pm(\lambda, a\iota)\, \lambda(\bar{r}_a) \\ &= d_a\, a\alpha^\pm_\lambda(R_a)^*\, a(\mathcal{E}^\mp(\bar{a},\lambda))\, \bar{r}_a\,. \end{aligned}$$

Next we compute, using again Eq. (13),

$$\begin{aligned} \mathcal{E}^\pm(\lambda, \bar{b}) &= T^*\, \varepsilon^\pm(\lambda, \bar{\nu})\, \alpha^\pm_\lambda(T) = d_\lambda\, T^*\, \bar{r}_\lambda\lambda(\varepsilon^\mp(\bar{\nu}, \bar{\lambda}))\lambda\bar{\nu}(r_\lambda)\, \alpha^\pm_\lambda(T) \\ &= d_\lambda\, \bar{r}^*_\lambda\alpha^\pm_\lambda(\alpha^\pm_{\bar{\lambda}}(T)^*\varepsilon^\mp(\bar{\nu},\bar{\lambda})T)\alpha^\pm_\lambda\bar{b}(r_\lambda) = d_\lambda\, \bar{r}^*_\lambda\alpha^\pm_\lambda(\mathcal{E}^\mp(\bar{b},\bar{\lambda}))\alpha^\pm_\lambda\bar{b}(r_\lambda)\,. \end{aligned}$$

Finally, as Eq. (17) yields $\bar{r}_a^* a(\mathcal{E}^\pm(\lambda,\bar{a}) = \lambda(\bar{r}_a)^*\varepsilon^\mp(a\iota,\lambda)$, we obtain

$$\mathcal{E}^\pm(\lambda,\bar{a}) = d_a\,\bar{a}(\bar{r}_a)^*\,\bar{a}a(\mathcal{E}^\pm(\lambda,\bar{a}))R_a = d_a\,\bar{a}\lambda(\bar{r}_a)^*\,\bar{a}(\varepsilon^\mp(a\iota,\lambda))R_a\,.$$

Drawing for $R_{\alpha_\lambda^\pm} = \iota(r_\lambda)$ and $\bar{R}_{\alpha_\lambda^\pm} = \iota(\bar{r}_\lambda)$ caps of the wires $\alpha_\lambda^\pm$, these relations yield graphically the analogues of Fig. 27. We conclude that we can include the crossings of Fig. 30 consistently in our "rotation covariant" graphical framework.

4 Double Triangle Algebras for Subfactors

We now formulate Ocneanu's construction [39] for a subfactor with finite index and finite depth rather than for bi-unitary connections and bimodules arising from Goodman-de la Harpe-Jones subfactors associated to A-D-E Dynkin diagrams in order to apply it in a more general context. From now on we work with $N\subset M$ satisfying the following

Assumption 4.1 Let $N\subset M$ be a type III subfactor with finite index. We assume that we have a system of endomorphisms ${}_N\mathcal{X}_N \subset \mathrm{Mor}(N,N)\equiv\mathrm{End}(N)$ in the sense of Definition 2.1 such that for the injection map $\iota: N\to M$, the sector $[\theta]=[\bar{\iota}\iota]$ decomposes into a sum of sectors of morphisms in ${}_N\mathcal{X}_N$. We choose sets of morphisms ${}_N\mathcal{X}_M\subset\mathrm{Mor}(M,N)$, ${}_M\mathcal{X}_N\subset\mathrm{Mor}(N,M)$ and ${}_M\mathcal{X}_M\subset\mathrm{Mor}(M,M)\equiv\mathrm{End}(M)$ consisting of representative endomorphisms of irreducible subsectors of sectors of the form $[\lambda\bar{\iota}]$, $[\iota\lambda]$ and $[\iota\lambda\bar{\iota}]$, $\lambda\in{}_N\mathcal{X}_N$, respectively. (We may and do choose id_M in ${}_M\mathcal{X}_M$ as the endomorphism representing the trivial sector.) We also assume that ${}_N\mathcal{X}_N$ is finite. Consequently, the set $\mathcal{X}={}_N\mathcal{X}_N\sqcup{}_N\mathcal{X}_M\sqcup{}_M\mathcal{X}_N\sqcup{}_M\mathcal{X}_M$ is finite.

Note that Assumption 4.1 implies that representative morphisms for all irreducible sectors appearing in decompositions of powers $[\gamma^k]$ ($[\theta^k]$) of Longo's (dual) canonical endomorphism are contained in ${}_M\mathcal{X}_M$ (${}_N\mathcal{X}_N$). In other words, the set $\mathcal{X}$ contains at least the morphisms corresponding to the (equivalence classes of) bimodules arising from this subfactor through the Jones tower, and therefore we may call an $\mathcal{X}$ which does not contain any other morphisms a *minimal choice*. We conclude that finiteness of ${}_N\mathcal{X}_N$ in Assumption in 4.1 automatically implies that the subfactor $N\subset M$ has finite depth. We used sectors instead of bimodules in view of our "identification" of chiral generators with α-induced sectors below. Therefore we need a sector approach in order to define α-induction since its definition involves $\bar{\iota}^{-1}$, and hence we work with factors of type III. (We do not need hyperfiniteness of M for our purposes.)

We now use the graphical calculus presented in Section 3. In the graphical method of [37] (and [11, Chapter 12]), factors, bimodules (morphisms), and intertwiners are represented with trivalent vertices, edges, and triangles, respectively, and this is where the name "double triangle algebra" comes from. However, here (as in [38, 39]) these three kinds of objects are represented by regions, wires, and trivalent vertices, respectively, though the labels for regions are omitted for notational simplicity.

For $\mathcal{X}$ in Assumption 4.1, we define the *double triangle algebra* $\diamondminus$ with two multiplications $*_h$ and $*_v$ as follows. As a linear space, we set

$$\diamondminus = \bigoplus_{a,b,c,d \in {}_N\mathcal{X}_M} \mathrm{Hom}(a\bar{b}, c\bar{d})\,.$$

This is a finite dimensional complex linear space. An element in $\diamondminus$ is presented graphically as in Fig. 33 under the interpretation in Section 3 with the convention of reading the diagram from the top to the bottom. (A general element in $\diamondminus$ is

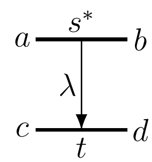


Figure 33: An element in $\diamondminus$

a linear combination of this type of element.) We can interpret the same diagram with the convention of reading the diagram from the left to the right or, equivalently, keeping the top-to-bottom convention but putting the diagram in a suitable Frobenius annulus. Then the resulting intertwiner is in

$$\diamondvert = \bigoplus_{a,b,c,d \in {}_N\mathcal{X}_M} \mathrm{Hom}(\bar{c}a, \bar{d}b)\,.$$

The isomorphism of these two spaces is given by application of two Frobenius rotations, and we can use this isomorphism to identify $\diamondminus$ and $\diamondvert$. By our convention of the normalization in Section 3, the diagram of Fig. 33 represents an element $d_a^{1/4} d_b^{1/4} d_c^{1/4} d_d^{1/4} d_\lambda^{-1/2} t s^*$ in the block $\mathrm{Hom}(a\bar{b}, c\bar{d})$, where $s \in \mathrm{Hom}(\lambda, a\bar{b})$ and $t \in \mathrm{Hom}(\lambda, c\bar{d})$ are isometries and $\lambda \in {}_N\mathcal{X}_N$. Similarly we may use elements in $\diamondminus$ which are graphically represented as in Fig. 34 with isometries $S \in \mathrm{Hom}(\beta, \bar{c}a)$, $T \in \mathrm{Hom}(\beta, \bar{d}b)$ and $\beta \in {}_M\mathcal{X}_M$. Note that elements of the form in Fig. 33, or

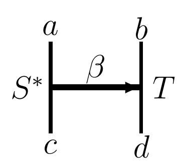


Figure 34: An element in $\diamondminus$

equivalently of the form in Fig. 34, span $\diamondminus$ linearly.

Our graphical convention is as follows. We use thin, thick, and very thick wires for N-N morphisms, N-M morphisms, and M-M morphisms, respectively, analogous to the convention [39]. We call them N-N wires, and so on. We label N-N morphisms with Greek letters $\lambda, \mu, \nu, \ldots$, N-M morphisms with Roman letters $a, b, c, d, \ldots$, and M-M morphisms with Greek letters $\beta, \beta', \beta'', \ldots$. We orient N-N or M-M wires but we put no orientations on N-M wires since it is clear from the context whether we mean an N-M morphism a or an M-N morphism $\bar{a}$. We simply put a label a for an unoriented thick wire for both. Note that, whatever we consider, ⟠ or ⟡, the same intertwiner (as an operator) may appear in different blocks of the double triangle algebra, e.g. the identity id_N is an element in any $\mathrm{Hom}(a\bar{b}, a\bar{b})$, $a, b \in {}_N\mathcal{X}_M$. The graphical notation is particularly useful in order to avoid this kind of confusion because diagrams as in Figs. 33 and 34 always specify also the associated block in addition to the intertwiner as an operator.

The *horizontal product* $*_h$ on ⟠ is defined as in Fig. 35. The meaning of the right-

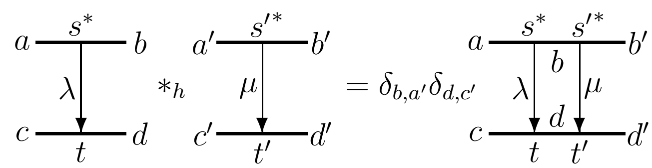


Figure 35: The horizontal product $*_h$ on ⟠

hand side is as follows. The product is by definition zero if the labels of the open ends of the wires facing each other do not match. If they match, we glue the wires of the two diagrams together as in Fig. 35 and interpret it as an intertwiner. It belongs to the block of the double triangle algebra which is specified by the four remaining open ends of the new diagram. This is a horizontal version of the composition of intertwiners described in Section 3.

We also can represent this horizontal product in terms of elements in Fig. 34. This is described in Fig. 36, because the convention of Section 3 means that this product is just the composition of the intertwiners in ⟡, and this composition is realized by taking the inner product of the two intertwiners in the right-hand side in Fig. 36.

We similarly define the *vertical product* $*_v$ on ⟠ by composing two diagrams vertically, but with extra coefficients as in Fig. 37. The meaning of the right-hand side is as before. Note that the definitions of horizontal and vertical products are not completely symmetric due to the extra coefficients we chose. This choice is somewhat arbitrary but it just turns out to be useful for our purposes. Namely, with this definition of the products, the minimal central projections of (⟠, $*_h$) have simple and useful composition rules with respect to the vertical product $*_v$, see Theorem 4.4 below. We clearly also have a $*$-structure for the horizontal product obtained by vertical reflection of the diagram, adjoining labels for trivalent vertices and reversing orientations of wires. Analogously, a $*$-structure for the vertical product comes from

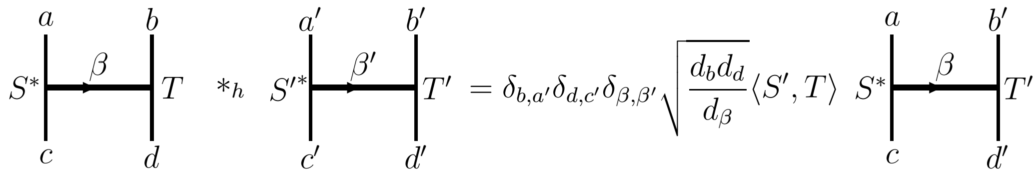


Figure 36: The horizontal product presented in another way

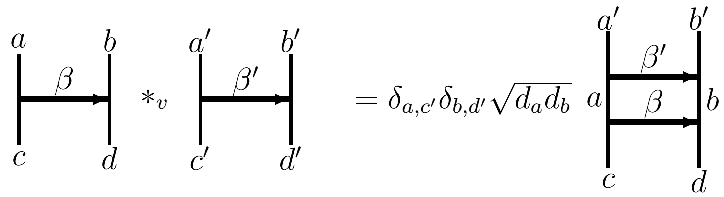


Figure 37: The vertical product $*_v$ in $\diamondminus$

horizontal reflection. The basic idea is that the 90-degree rotation is something like a "Fourier transform" which transforms the two products into each other, similar to the situation of the group algebra of a finite or compact group.

For each β, λ, a, b we choose orthonormal bases of isometries $T^{\beta;i}_{\bar{b},a} \in \mathrm{Hom}(\beta, \bar{b}a)$, $i = 1, 2, ..., N^{\beta}_{\bar{b},a}$, and $t^{\lambda;j}_{a,\bar{b}} \in \mathrm{Hom}(\lambda, a\bar{b})$ $j = 1, 2, ..., N^{\lambda}_{a\bar{b}}$, so that

$$\sum_{\beta \in {}_M\mathcal{X}_M} \sum_{i=1}^{N^{\beta}_{\bar{b},a}} T^{\beta;i}_{\bar{b},a} (T^{\beta;i}_{\bar{b},a})^* = \mathbf{1}_M \qquad \text{and} \qquad \sum_{\lambda \in {}_N\mathcal{X}_N} \sum_{j=1}^{N^{\lambda}_{a,\bar{b}}} t^{\lambda;j}_{a,\bar{b}} (t^{\lambda;j}_{a,\bar{b}})^* = \mathbf{1}_N \tag{22}$$

for all $a, b \in {}_N\mathcal{X}_M$. Then it is easy to see that the elements in Fig. 38 form bases of

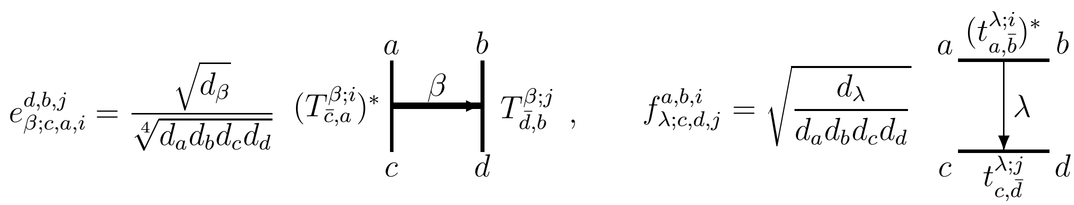


Figure 38: Matrix units $e^{d,b,j}_{\beta;c,a,i}$ for $(\diamondminus, *_h)$ and $f^{a,b,i}_{\lambda;c,d,j}$ for $(\diamondminus, *_v)$

$\diamondminus$ which constitute complete systems of matrix units $(\diamondminus, *_h)$ respectively $(\diamondminus, *_v)$. Thus for each of the two multiplications the double triangle algebra is a direct sum of full matrix algebras. The two different bases are transformed into each other by a unitary transformation with coefficients given by the $6j$-symbols for subfactors of [37] (see [11, Chapter 12] for the basic properties of "quantum $6j$-symbols"), but this will not be exploited here.

Definition 4.2 For each $\beta \in {}_M\mathcal{X}_M$ we define an element $e_\beta = \sum_{a,b,i} e^{b,a,i}_{\beta;b,a,i} \in \Diamond$. Graphically, this element is given by the left-hand side in Fig. 39. We use the convention shown on the right-hand side in Fig. 39 to represent this element.

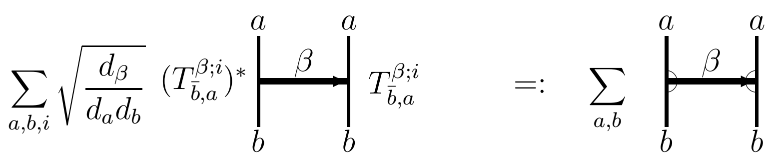


Figure 39: The minimal central projection e_β

Due to the summation over $i = 1, 2, ..., N^{\beta}_{\bar{b},a}$, the definition is independent of the choice of the intertwiner bases as different orthonormal bases are related by a unitary matrix. We will use such a graphical convention whenever we have a sum over internal "fusion channels" of two corresponding trivalent vertices together with prefactors which renormalize the trivalent vertices to isometries. Note that we obtain a prefactor, as displayed in Fig. 40 for an example, when we turn around the small arcs at trivalent vertices. Here the dotted parts mean that there might be expansions

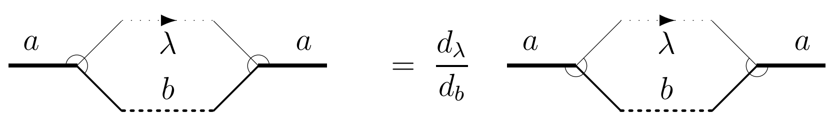


Figure 40: Turning around small arcs yields a prefactor

as given in the following lemma or later even be braiding operators in between; it is just important that the small arcs at corresponding trivalent vertices denote the same summation over internal fusion channels.

Lemma 4.3 *The identity of Fig. 41 holds. Analogous identities hold if a, b, β are replaced by wires of other type (in a compatible way).*

Proof. With the normalization convention as in Fig. 39, this is just the expansion of the identity in Eq. (22), and this certainly holds as well using similar expansions with other intertwiner bases. □

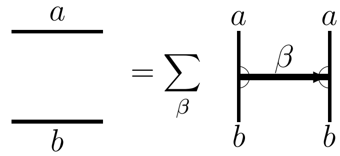


Figure 41: The identity with expansion using β

Note that the identity in Fig. 41 may, for example, also appear rotated by 90 degrees as we can put the left- and right-hand sides in some Frobenius annulus as described in Subsect. 3.2.

As we have already indicated, the horizontal product is essentially the composition of intertwiners in $\Diamond$. The main point of the double triangle algebra is the following. Suppose we have complete information on the fusion rules of N-N, N-M, M-N morphisms in $\mathcal{X}$ and their $6j$-symbols. We can define the algebra $\Diamond$ in terms of matrix elements $f^{\lambda;a,b,i}_{c,d,j}$ and determine their composition with respect to the horizontal product without any information of the M-M morphisms. Then we can *find* M-M sectors and determine their fusion rules by the following theorem which generalizes a result for Goodman-de la Harpe-Jones subfactors in [39] in a straightforward manner.

Theorem 4.4 *For any $\beta \in {}_M\mathcal{X}_M$ the element $e_\beta \in \Diamond$ of Definition 4.2 is a minimal central projection with respect to the horizontal product, and all minimal central projections arise in this way in a bijective correspondence. Furthermore, we have*[7]

$$e_\beta *_v e_{\beta'} = \sum_{\beta'' \in {}_M\mathcal{X}_M} \frac{d_\beta d_{\beta'}}{d_{\beta''}} N^{\beta''}_{\beta,\beta'} \, e_{\beta''} \tag{23}$$

for all $\beta, \beta' \in {}_M\mathcal{X}_M$. In particular, the center $\mathcal{Z}_h$ of $\Diamond$ with respect to the horizontal product is closed under the vertical product.

Proof. That each e_β is a minimal central projection and that all minimal central projections arise in this way is obvious from the description of the matrix units. The vertical product $e_\beta *_v e_{\beta'}$ is given graphically by the left-hand side of Fig 42. We can use the expansion of Lemma 4.3 for the two parallel wires β and β' in the middle. Now note that the horizontal unit is given by $\mathbf{1}_h = \sum_\beta e_\beta$. Therefore, by multiplying $\mathbf{1}_h$ from the left and from the right, we obtain the diagram on the right-hand side of Fig. 42. Reading the diagram from left to right, we observe that intertwiners in $\mathrm{Hom}(\beta''', \beta'')$ and $\mathrm{Hom}(\beta'', \beta'''')$ are involved here. Hence we first obtain a factor $\delta_{\beta''',\beta''}\delta_{\beta'',\beta''''}$. Next, we can use the trick of Fig. 40 to turn around the small arcs at the trivalent vertices involving a, b, β'. This yields a factor d'_β/d_b. This way we see that the diagram on the right-hand side of Fig. 42 represents the same element of the $\Diamond$ as the diagram in. Fig. 43. Now let us look at the part of this picture inside

[7]Note that the fusion coefficients with dimension prefactors as in Eq. (23) coincide with the structure constants used for C-algebras [1].

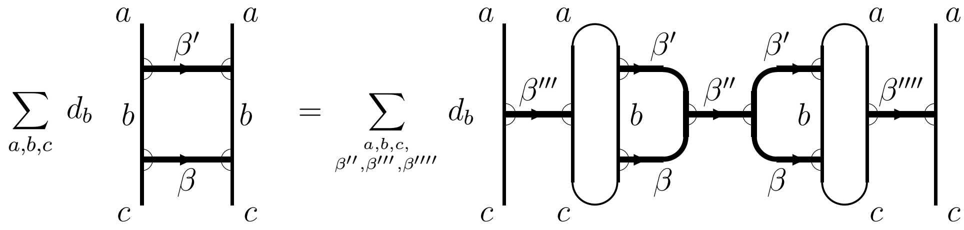


Figure 42: The vertical product $e_\beta *_v e_{\beta'}$

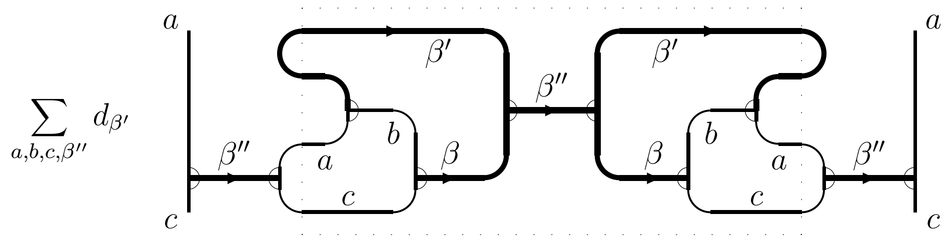


Figure 43: The vertical product $e_\beta *_v e_{\beta'}$

the dotted box. Reading it from the left, this part can be read for fixed a and c as $\sum_{i,k} T_i T^{\beta'';k}_{\beta,\beta'} (T^{\beta'';k}_{\beta,\beta'})^* T_i^*$, and the sum over i runs over a full orthonormal basis of isometries T_i in the Hilbert space $\mathrm{Hom}(\beta, \bar{c}a\bar{\beta}')$ since we have the summation over b. Next we look at the part inside the dotted box of the diagram in Fig. 44. Here,

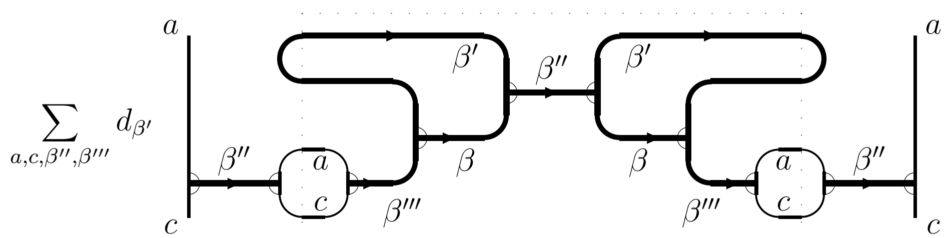


Figure 44: The vertical product $e_\beta *_v e_{\beta'}$

since we introduced the sum over β''', the part can be similarly read for fixed a and c as $\sum_{j,k} S_j T^{\beta'';k}_{\beta,\beta'} (T^{\beta'';k}_{\beta,\beta'})^* S_j^*$, where the sum over j runs over another orthonormal basis of isometries S_i in the Hilbert space $\mathrm{Hom}(\beta, \bar{c}a\bar{\beta}')$. Since such bases $\{T_i\}$ and $\{S_j\}$ are related by a unitary matrix transformation (this is essentially "unitarity of $6j$-symbols"), we conclude that the diagrams in Figs. 43 and 44 represent the same element in $\diamondsuit$. We now see that we first obtain a factor $\delta_{\beta'',\beta'''}$. Next we can turn around the small arcs at the outer two trivalent vertices involving β, β' and $\beta''' = \beta''$

so that we obtain a factor $d_\beta/d_{\beta''}$. Then, by "stretching" the diagram a bit, we can read the diagram for fixed a, c, β'' as

$$\sum_{i,j,m=1}^{N_{\bar{c},a}^{\beta''}} \sum_{k,l=1}^{N_{\beta,\beta'}^{\beta''}} \frac{d_\beta d_{\beta'}}{d_{\beta''}} T_{\bar{c},a}^{\beta'';i} (T_{\bar{c},a}^{\beta'';i})^* T_{\bar{c},a}^{\beta'';j} (T_{\beta,\beta'}^{\beta'';l})^* T_{\beta,\beta'}^{\beta'';k} (T_{\beta,\beta'}^{\beta'';k})^* T_{\beta,\beta'}^{\beta'';l} (T_{\bar{c},a}^{\beta'';j})^* T_{\bar{c},a}^{\beta'';m} (T_{\bar{c},a}^{\beta'';m})^*$$
$$= \sum_{i=1}^{N_{\bar{c},a}^{\beta''}} \frac{d_\beta d_{\beta'}}{d_{\beta''}} N_{\beta,\beta'}^{\beta''}\, T_{\bar{c},a}^{\beta'';i} (T_{\bar{c},a}^{\beta'';i})^* \,.$$

Now proceeding with the summations over a, c, β'' yields the statement. $\square$

Now consider the vector space with basis elements $[\beta]$, $\beta \in {}_M\mathcal{X}_M$ which we can endow with a product through $[\beta][\beta'] = \sum_{\beta''} N_{\beta,\beta'}^{\beta''}[\beta'']$. We call the algebra defined this way the M-M fusion rule algebra. Similarly we define the N-N fusion rule algebra using morphisms in ${}_N\mathcal{X}_N$.

Definition 4.5 We define a linear map Φ from the M-M fusion rule algebra to $\mathcal{Z}_h$ by linear extension of $\Phi([\beta]) = e_\beta/d_\beta$.

Theorem 4.4 now says that this map Φ is an isomorphism from the M-M fusion rule algebra onto $(\mathcal{Z}_h, *_v)$. Note that $(\mathcal{Z}_h, *_v)$ is a non-unital subalgebra of $(\Diamond, *_v)$. The unit $\mathbf{1}_v$ of $(\Diamond, *_v)$ is given by $\mathbf{1}_v = \sum_\lambda f_\lambda$, where $f_\lambda = \sum_{a,b,j} f_{\lambda;a,b,j}^{a,b,j}$ whereas the unit of $(\mathcal{Z}_h, *_v)$ is given by e_0.

Definition 4.6 We define two linear functionals φ_h and τ_v on $\Diamond$ corresponding to the two product structures $*_h$ and $*_v$ by linear extension of

$$\begin{aligned} \varphi_h(e_{\beta;c,a,i}^{d,b,j}) &= \delta_{a,b}\delta_{c,d}\delta_{i,j}\, d_a d_c d_\beta / w^2 \,, \\ \tau_v(f_{\lambda;c,d,j}^{a,b,i}) &= \delta_{a,c}\delta_{c,d}\delta_{i,j}\, d_\lambda \,. \end{aligned} \tag{24}$$

Applied to an element in Fig. 33 (Fig. 34) the functional φ_h (τ_v) can be characterized graphically as in Fig. 45 (Fig. 46). Therefore these functionals correspond to closing

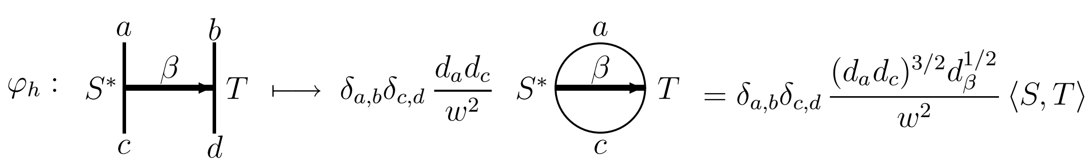
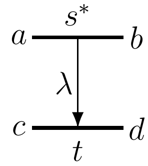


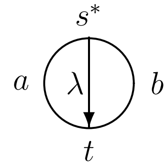
Figure 45: The horizontal functional φ_h

the open ends of a diagram with prefactors as in the middle part of Figs. 45 and 46.

$\tau_v :$



$\longmapsto \delta_{a,c}\delta_{b,d}\sqrt{d_a d_b}$



$= \delta_{a,b}\delta_{c,d}\, d_a d_b d_\lambda^{1/2}\, \langle s, t\rangle$

Figure 46: The vertical functional τ_v

Recall that the global index of ${}_N\mathcal{X}_N$ is given by $w = \sum_{\lambda\in {}_N\mathcal{X}_N} d_\lambda^2$. Note that we have sector decompositions $[a\iota] = \sum_\lambda \langle\lambda, a\iota\rangle[\lambda]$ and hence $d_a d_\iota = \sum_\lambda\langle\lambda, a\iota\rangle d_\lambda$ for any $a \in {}_N\mathcal{X}_M$. Using Frobenius reciprocity $\langle\lambda, a\iota\rangle = \langle\lambda\bar\iota, a\rangle$ we obtain similarly $d_\lambda d_\iota = \sum_a\langle\lambda, a\iota\rangle d_a$. Hence $w = \sum_\lambda d_\lambda^2 = \sum_{\lambda,a}\langle\lambda, a\iota\rangle d_\lambda d_a/d_\iota = \sum_a d_a^2$. Similarly we obtain $w = \sum_\beta d_\beta^2$ (cf. [37]).

Lemma 4.7 *We have $\varphi_h(e_\beta) = d_\beta^2/w$. In particular, the functional φ_h is a faithful state on $(\diamondsuit, *_h)$. The functional τ_v is a (un-normalized) faithful trace on $(\diamondsuit, *_v)$.*

Proof. By Definition 4.6 and Fig. 39, we compute

$$\varphi_h(e_\beta) = \sum_{a,b\in {}_N\mathcal{X}_M} N_{\bar a,b}^\beta d_a d_b d_\beta w^{-2} = \sum_{a\in {}_N\mathcal{X}_M}\left(\sum_{b\in {}_N\mathcal{X}_M} N_{a,\beta}^b d_b\right) d_\beta d_a w^{-2} = d_\beta^2 w^{-1}\,.$$

Since the horizontal unit $\mathbf{1}_h$ is given by $\mathbf{1}_h = \sum_\beta e_\beta$ we find that $\varphi(\mathbf{1}_h) = 1$. As φ_h sends off-diagonal matrix units to zero and the diagonal ones to strictly positive numbers, this proves that φ_h is a faithful state. Obviously also τ_v sends off-diagonal matrix units (with respect to $*_v$) to zero and the diagonal ones to strictly positive numbers, and hence it is a strictly positive functional but it is not normalized. The trace property $\tau_v(xy) = \tau_v(yx)$ is clear from the definition of τ_v using matrix units for x and y. $\square$

For τ_v we could have gained analogous properties as for φ_h by replacing the scalar d_λ in Eq. (24) by $d_a d_b d_\lambda/w^2$ (and by multiplying the scalars in Fig. 46 also by $d_a d_b/w^2$). However, we chose a different normalization on each matrix unit in order to turn τ_v into a trace on $(\diamondsuit, *_v)$. Later we want to study the center $(\mathcal{Z}_h, *_v)$ which is, as we have seen, a subalgebra of $(\diamondsuit, *_v)$. Therefore τ_v provides a faithful trace on $(\mathcal{Z}_h, *_v)$ but it has in general different weightings on its simple summands. To construct from τ_v a trace which sends one-dimensional projections to one will in particular be possible in the case that ${}_N\mathcal{X}_N$ is non-degenerately braided, see Subsect. 6.1 below.

This is also the case in the following most basic example of the double triangle algebra. Let N be a type III factor and G a finite group acting freely on N. Consider the subfactor $N \subset N \rtimes G = M$. Then (with the minimal choice for $\mathcal{X}$) the double triangle algebra $\diamondsuit$ for this subfactor is just the group algebra of G. That is, the double triangle algebra is spanned by the group elements linearly. The horizontal product is given by the group multiplication. By Proposition 4.4 we conclude that the minimal central projections in $\diamondsuit$ and thus irreducible M-M sectors are labelled by the irreducible representations of G. (Of course, this identification of the M-M sectors is well-known for that example.) The functional τ_v gives the standard trace on the group algebra, and the vertical product corresponds to the ordinary tensor product of group representations.

5 α-Induction, Chiral Generators and Modular Invariants

5.1 Relating α-induction to chiral generators

We will now define chiral generators for braided subfactors and prove that the concepts of α-induction and chiral generators are essentially the same. For the rest of this paper deal with the following

Assumption 5.1 In addition to Assumption 4.1 we now assume that the system ${}_N\mathcal{X}_N$ is braided.

With the braiding we have now the notion of α-induction in the sense of Subsect. 3.3. From now on we are also dealing with crossings of N-N wires and mixed crossings introduced in Subsect. 3.3. We now present chiral generators as our version of a definition Ocneanu originally introduced for systems of bimodules arising from A-D-E Dynkin diagrams in [39]. The construction of the chiral generator is similar to the "Ocneanu projection" in the tube algebra [38] (see also [12]) and also related to Izumi's analysis [20] of the tube algebra in terms of sectors for the Longo-Rehren inclusion [33].

Definition 5.2 For any $\lambda \in {}_N\mathcal{X}_N$, we define an element $p_\lambda^+ \in \diamondsuit$ by the diagram on the left-hand side of Fig. 47 and call it a *chiral generator*. Similarly, we also define p_λ^- by exchanging over- and undercrossings.

Note that we do *not* assume the non-degeneracy of the braiding for the definition p_λ^+.

We obtain the diagram in the middle from the one on the left-hand side in Fig. 47 by applying two IBFE's. This way we obtain two twists in the semi-circular thin wires which correspond to the label λ but they give complex conjugate phases so that their effects cancel out. The diagram on the right-hand side is obtained by Lemma 4.3 and application of the IBFE, and this shows that our definition coincides with Ocneanu's notion given in his setting.

Since $\alpha_\lambda^\pm \iota = \iota\lambda$ we find that each irreducible subsector $[\beta]$ of $[\alpha_\lambda^\pm]$ is the equivalence class of some $\beta \in {}_M\mathcal{X}_M$ if $\lambda \in {}_N\mathcal{X}_N$. Therefore we have the sector decomposition

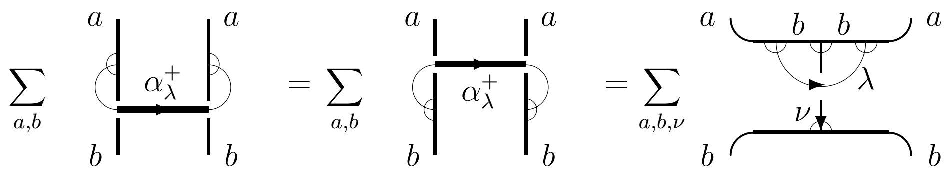


Figure 47: A chiral generator p_λ^+

$[\alpha_\lambda^\pm] = \sum_{\beta\in {}_M\mathcal{X}_M} \langle\beta,\alpha_\lambda^\pm\rangle[\beta]$, and we can consider $[\alpha_\lambda^\pm]$ as an element of the M-M fusion algebra. The relation between the sector decomposition of $[\alpha_\lambda^\pm]$ and the chiral generator is clarified by the following result.

Theorem 5.3 *For any $\lambda \in {}_N\mathcal{X}_N$, we have $d_\lambda^{-1} p_\lambda^\pm = \sum_{\beta\in {}_M\mathcal{X}_M} d_\beta^{-1}\langle\beta,\alpha_\lambda^+\rangle e_\beta$, and consequently $p_\lambda^\pm = d_\lambda \Phi([\alpha_\lambda^\pm])$. In particular, $p_\lambda^\pm$ is in the center $\mathcal{Z}_h$.*

Proof. We only show the statement for the $+$-sign; the other case is analogous. First we fix $a,b \in {}_N\mathcal{X}_M$ and $\lambda \in {}_N\mathcal{X}_N$. For each $\beta \in {}_M\mathcal{X}_M$ we choose orthonormal bases of isometries $T_{\bar{b}a}^{\beta;i} \in \mathrm{Hom}(\beta,\bar{b}a)$, $i=1,2,...,N_{\bar{b},a}^\beta$, so that $\sum_{\beta,i} T_{\bar{b}a}^{\beta;i}(T_{\bar{b}a}^{\beta;i})^* = \mathbf{1}_M$. Using Frobenius reciprocity, we obtain an orthonormal basis of isometries $\mathcal{L}_b^{-1}(T_{\bar{b}a}^{\beta;i}) = d_a^{1/2} d_b^{1/2} d_\beta^{-1/2} b(T_{\bar{b}a}^{\beta;i})^* \bar{r}_b \in \mathrm{Hom}(a,b\beta)$. Here we chose an isometry $\bar{r}_b \in \mathrm{Hom}(\mathrm{id}_N, b\bar{b})$ such that there is an isometry $R_b \in \mathrm{Hom}(\mathrm{id}_M,\bar{b}b)$ subject to relations $b(R_b)^*\bar{r}_b = d_b^{-1}\mathbf{1}_N$ and $\bar{b}(\bar{r}_b)^* R_b = d_b^{-1}\mathbf{1}_M$, as usual. Choosing also orthonormal bases of isometries $V_{\beta;\ell} \in \mathrm{Hom}(\beta,\alpha_\lambda^+)$, $\ell = 1,2,...,\langle\beta,\alpha_\lambda^+\rangle$, for each $\beta \in {}_M\mathcal{X}_M$ (so that $\sum_{\beta,\ell} V_{\beta;\ell}V_{\beta;\ell}^* = \mathbf{1}_M$) we find that $\{b(V_{\beta;\ell})\mathcal{L}_b^{-1}(T_{\bar{b}a}^{\beta;i})\}_{\beta,i,\ell}$ gives an orthonormal basis of isometries of $\mathrm{Hom}(a,b\alpha_\lambda^+)$. Finally, using Proposition 3.1, we find that putting

$$s_{\beta;\ell,i} = \varepsilon^+(\lambda,b\iota)^* b(V_{\beta;\ell})\mathcal{L}_b^{-1}(T_{\bar{b}a}^{\beta;i}) = \sqrt{\frac{d_a d_b}{d_\beta}}\,\varepsilon^+(\lambda,b\iota)^* b(V_{\beta;\ell}(T_{\bar{b}a}^{\beta;i})^*)\bar{r}_b$$

defines an orthonormal basis of isometries $\{s_{\beta;\ell,i}\}_{\beta,i,\ell}$ of $\mathrm{Hom}(a,\lambda b)$. Then we have for any $\ell = 1,2,...,\langle\beta,\alpha_\lambda^+\rangle$ by the elementary relations for the intertwiners $R_b,\bar{r}_b$ the following identity:

$$\begin{aligned} T_{\bar{b}a}^{\beta;i}(T_{\bar{b}a}^{\beta;i})^* &= d_b^2\,\bar{b}(\bar{r}_b)^*\,\bar{b}b(T_{\bar{b}a}^{\beta;i}V_{\beta;\ell}^*)\,R_bR_b^*\,\bar{b}b(V_{\beta;\ell}(T_{\bar{b}a}^{\beta;i})^*)\,\bar{b}(\bar{r}_b) \\ &= \frac{d_\beta d_b}{d_a}\,\bar{b}(s_{\beta;\ell,i}\varepsilon^+(\lambda,b\iota)^*)\,R_bR_b^*\,\bar{b}(\varepsilon^+(\lambda,b\iota)s_{\beta;\ell,i}^*)\,. \end{aligned}$$

The second line yields graphically exactly the diagram in Fig. 48 where we read the diagram from the left to the right in order to interpret it as an intertwiner in Φ. Now let us take on both sides first the summation over $i=1,2,...,N_{\bar{b},a}^\beta$. Then the left-hand side gives exactly the $\mathrm{Hom}(\bar{b}a,\bar{b}a)$ part of e_β (in Φ) as defined in Definition 4.2. Next we divide by d_β and we proceed with the summation over $\ell = 1,2,...,\langle\beta,\alpha_\lambda^+\rangle$ and

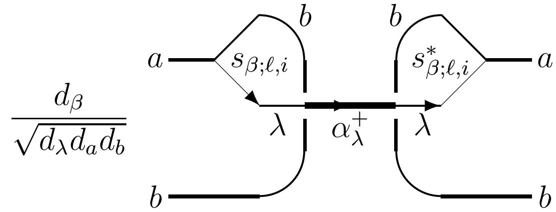


Figure 48: Diagram for $T^{\beta;i}_{\bar b a}(T^{\beta;i}_{\bar b a})^*$

$\beta \in {}_M\mathcal{X}_M$. On the left-hand side we obtain the $\mathrm{Hom}(\bar b a, \bar b a)$ part of $\sum_\beta d_\beta^{-1}\langle\beta, \alpha^+_\lambda\rangle e_\beta$ this way, and this is exactly the $\mathrm{Hom}(\bar b a, \bar b a)$ part of $\Phi([\alpha^+_\lambda])$. On the right-hand side we now have a summation over the full basis $\{s_{\beta;\ell,i}\}_{\beta,i,\ell}$ of $\mathrm{Hom}(a, \lambda b)$. Therefore we can use the graphical convention of Fig. 39 to put a small semi-circle around the wire labelled by λ at the two trivalent vertices. This gives us a factor $\sqrt{d_a d_b / d_\lambda}$ so that only a factor d_λ^{-1} remains from the original prefactor in Fig. 48. Thus, by repeating the above procedure for all $a, b \in {}_N\mathcal{X}_M$ and making finally the summation over $a, b \in {}_N\mathcal{X}_M$, we obtain on the left the full $\Phi([\alpha^+_\lambda])$ whereas the right-hand side gives graphically the diagram in Fig. 49. The diagram on the left-hand side in Fig.

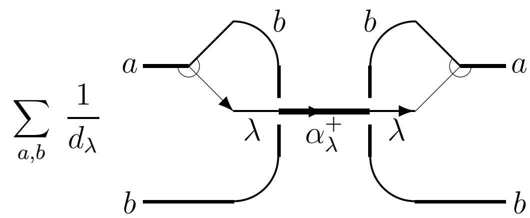


Figure 49: The image $\Phi([\alpha^+_\lambda]) = \sum_\beta d_\beta^{-1}\langle\beta, \alpha^+_\lambda\rangle e_\beta$

47 is obtained from Fig. 49, up to the factor d_λ, by a topological move. $\square$

Note that it was not clear from the definition that the chiral generators are in the center $\mathcal{Z}_h$, but Theorem 5.3 proves this centrality as it states that $p^\pm_\lambda$ is a linear combination of e_β's. Also note that if $\alpha^\pm_\lambda$ is irreducible then $p^\pm_\lambda$ is a (horizontal) projection, however, if $\alpha^\pm_\lambda$ is not irreducible, then $p^\pm_\lambda$ is a sum over projections with weight coefficients arising from the nature of the isomorphism Φ in Definition 4.5.

Two of us [4, Subsection 3.3] established a relative braiding between the two kinds of α-induction, which holds in a fairly general context. (It does neither depend on chiral locality nor even on finite depth.) Theorem 5.3 now shows that Ocneanu's relative braiding [39] is a special case of the analysis in [4, Subsection 3.3].

From Theorem 5.3 and the homomorphism property of α-induction [2, Lemma 3.10], we obtain immediately the following

Corollary 5.4 *The chiral generators $p_\lambda^\pm$ are in $\mathcal{Z}_h$. For $\lambda, \mu \in {}_N\mathcal{X}_N$, we have*

$$p_\lambda^\pm *_v p_\mu^\pm = \sum_{\nu \in {}_N\mathcal{X}_N} \frac{d_\lambda d_\mu}{d_\nu} N_{\lambda,\mu}^\nu \, p_\nu^\pm \, .$$

Note that this corollary shows that the M-M fusion rule algebra contains two representations of the N-N fusion rule algebra.

5.2 Modular invariants for braided subfactors

We will now show that a notion of "modular invariant" arises naturally for a braided subfactor. We first note that under Assumption 5.1, we have matrices $Y = (Y_{\lambda,\mu})$ and $T = (T_{\lambda,\mu})$ for the system $\Delta = {}_N\mathcal{X}_N$ as in Subsection 2.2. We recall that in the case that the braiding is non-degenerate, the matrix $S = w^{-1/2}Y$ is unitary and the matrices S and (the diagonal) T obey the Verlinde modular algebra by Theorem 2.5. Motivated by the results of [4] we now construct a certain matrix Z commuting with Y and T such that it is a "modular invariant mass matrix" in the usual sense of conformal field theory whenever the braiding is non-degenerate.

Definition 5.5 For a system $\mathcal{X}$ satisfying Assumption 5.1, we define a matrix Z with entries $Z_{\lambda,\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$, $\lambda, \mu \in {}_N\mathcal{X}_N$.

As $Z_{\lambda,\mu}$ is by definition a dimension and since $\alpha_{\mathrm{id}_N}^\pm = \mathrm{id}_M$ is irreducible by virtue of the factor property of M, the matrix elements obviously satisfy the conditions in Eq. (1) for $\lambda, \mu \in {}_N\mathcal{X}_N$, where the label "0" refers as usual to the identity morphism $\mathrm{id}_N \in {}_N\mathcal{X}_N$. We relate the definition of Z to the chiral generators by the following

Theorem 5.6 *We have the identity*

$$Z_{\lambda,\mu} = \frac{w}{d_\lambda d_\mu} \, \varphi_h(p_\lambda^+ *_h p_\mu^-) \, , \qquad \lambda, \mu \in {}_N\mathcal{X}_N \, . \tag{25}$$

Therefore the number $Z_{\lambda,\mu}$ is graphically represented as in Fig. 50.

Proof. From Theorem 5.3 we obtain

$$\sum_{\beta \in {}_M\mathcal{X}_M} \frac{1}{d_\beta} \langle \alpha_\lambda^+, \beta \rangle e_\beta = \frac{1}{d_\lambda} p_\lambda^+ \, .$$

Hence

$$\sum_{\beta \in {}_M\mathcal{X}_M} \frac{1}{d_\beta^2} \langle \alpha_\lambda^+, \beta \rangle \langle \alpha_\mu^-, \beta \rangle e_\beta = \frac{1}{d_\lambda d_\mu} \, p_\lambda^+ *_h p_\mu^- \, .$$

$$Z_{\lambda,\mu} = \sum_{b,c} \frac{d_b d_c}{w d_\lambda d_\mu}$$

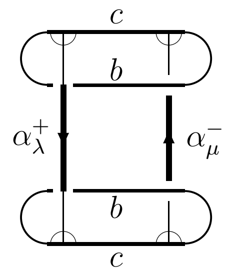


Figure 50: Graphical representation of $Z_{\lambda,\mu}$

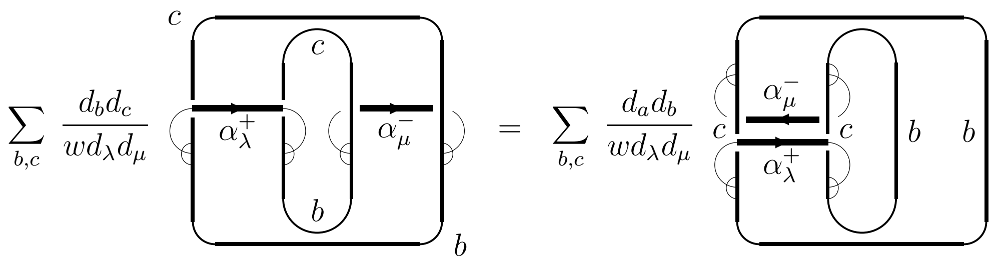


Figure 51: The scalar $wd_\lambda^{-1}d_\mu^{-1}\varphi_h(p_\lambda^+ *_h p_\mu^-)$

Application of the horizontal state φ_h of Definition 4.6 and multiplication by w yields Eq. (25) since $[\alpha_\lambda^+]$ and $[\alpha_\mu^-]$ decompose into sectors $[\beta]$ with $\beta \in {}_M\mathcal{X}_M$, and by Lemma 4.7. Now the right-hand side of Eq. (25) is given graphically by the diagram on the left in Fig. 51, and we can slide around the trivalent vertices to obtain the diagram on the right-hand side. Without changing the scalar value we can now open the outer wire labelled by b and close it on the other side, as in Fig. 29. This way we obtain the picture in Fig. 50 up to a 90 degree rotation, but a rotation is irrelevant for the scalar values. □

We remark that we can apply Lemma 4.3 to replace the two horizontal wires labelled by b by a summation over a thin wire ν, and this way we obtain an equivalent diagram from Fig. 50 for the matrix elements $Z_{\lambda,\mu}$, which only consists of thin (N-N) wires λ, μ, ν and thick (N-M) wires b, c but which does not involve very thick (M-M) wires labelled by α-induced morphisms $\alpha_\lambda^+, \alpha_\mu^-$.

Theorem 5.7 *The matrix Z of Definition 5.5 commutes with the matrices Y and T of the system ${}_N\mathcal{X}_N$.*

Proof. Using the diagram for the matrix elements $Y_{\nu,\lambda}$ in Fig. 19, the sum $\sum_\lambda Y_{\nu,\lambda} Z_{\lambda,\mu}$ can be represented by the diagram on the left-hand side of Fig. 52. Using

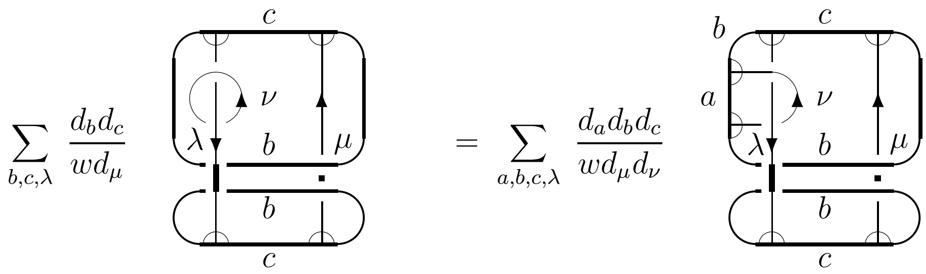


Figure 52: Commutation of Y and Z

Lemma 4.3 and also the trick to turn around the small arcs given in Fig. 40, we obtain the right-hand side of Fig. 52. We can now slide around the lower trivalent vertex of the wire ν to obtain the left-hand side of Fig. 53. Next, we can use Lemma

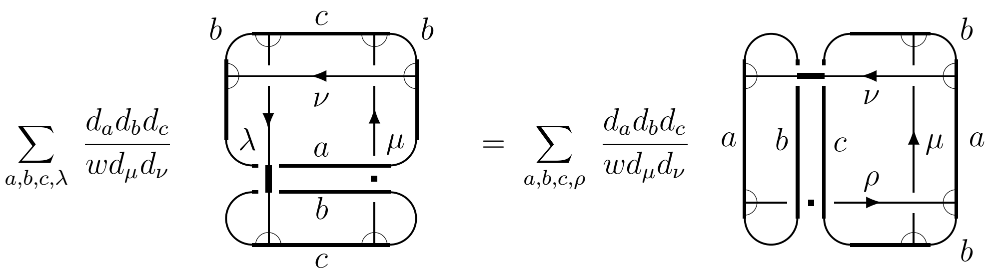
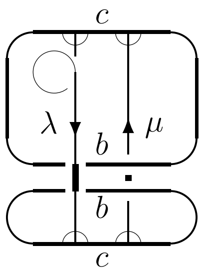


Figure 53: Commutation of Y and Z

4.3 to replace the two parallel horizontal wires with labels a and b by a summation over a thin wire ρ. Similarly, but the other way round, we can then use Lemma 4.3 to replace the summation over the wire with label λ by two straight horizontal wires with labels b and c. This way we obtain the right-hand side of Fig. 53. Now it should be clear how to proceed: We slide around the upper trivalent vertex of the wire μ counter-clockwise. Then we see that the result gives us the diagram for $\sum_\rho Z_{\nu,\rho} Y_{\rho,\mu}$, rotated by 90 degrees. This proves $YZ = ZY$. Next we show commutativity of Z with T. We have to show $\omega_\lambda Z_{\lambda,\mu} = Z_{\lambda,\mu}\omega_\mu$. Using the graphical expression for the statistics phase ω_λ on the left-hand side of Fig. 17, we can represent $\omega_\lambda Z_{\lambda,\mu}$ by the left-hand side of Fig. 54. We now start to rotate the upper oval consisting of the thick wires b and c in a clockwise direction. This way we obtain the right-hand side of Fig. 54. It should now be clear that, if we continue rotating to a full rotation by 360 degrees, then we remove the twist from the wire λ whereas we obtain a twist in the wire μ which is of the type displayed on the right-hand side of Fig. 17, thus representing ω_μ. Hence $TZ = ZT$. $\square$

$$\sum_{b,c} \frac{d_b d_c}{w d_\lambda d_\mu}$$



$$= \sum_{b,c} \frac{d_b d_c}{w d_\lambda d_\mu}$$

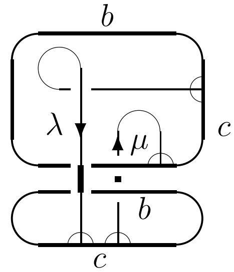


Figure 54: Commutation of T and Z

The following is now immediate by Thm. 2.5, which states that in the non-degenerate case matrices $S = w^{-1/2}Y$ and T provide a unitary representation of the modular group $SL(2;\mathbb{Z})$.

Corollary 5.8 *If the braiding on ${}_N\mathcal{X}_N$ is non-degenerate, then the matrix Z defined in Definition 5.5 is a modular invariant mass matrix.*

In conformal field theory the $SL(2;\mathbb{Z})$ action arises from a "reparametrization of the torus", and in the parameter space S corresponds to a 90 degree rotation and T to twisting the torus. Note that this action is nicely reflected in the proof of Thm. 5.7.

5.3 Generating property of α-induction

We now show that both kinds of α-induction generate the whole M-M fusion rule algebra (or the sector algebra in our terminology of [2, 3, 4]) in the case that the N-N system is non-degenerately braided. That is, from now on we work with the following

Assumption 5.9 In addition to Assumption 5.1, we now assume that the braiding on ${}_N\mathcal{X}_N$ is non-degenerate in the sense of Definition 2.3.

With Assumption 5.9 we can now use the "killing ring", the orthogonality relation of Fig. 20, and this turns out to be a powerful tool in the graphical framework.

The following theorem states in particular that any minimal central projection e_β of $(\diamondminus, *_h)$ appears in the linear decomposition of some $p^+_\lambda *_v p^-_\mu$. Such a generating property of $p^\pm_j$'s has also been noticed by Ocneanu in the setting of the lectures [39]. We can apply his idea of the proof (which is not included in the notes [39]) to our situation without essential change.

Theorem 5.10 *Under Assumption 5.9, we have $\sum_{\lambda,\mu\in{}_N\mathcal{X}_N} p^+_\lambda *_v p^-_\mu = w\mathbf{1}_h$ in $\diamondminus$, and consequently*

$$\sum_{\lambda,\mu\in{}_N\mathcal{X}_N} d_\lambda d_\mu [\alpha^+_\lambda][\alpha^-_\mu] = w \sum_{\beta\in{}_M\mathcal{X}_M} d_\beta [\beta] \tag{26}$$

in the M-M fusion rule algebra. In particular, for any $\beta \in {}_M\mathcal{X}_M$ the sector $[\beta]$ is a subsector of $[\alpha_\lambda^+][\alpha_\mu^-]$ for some $\lambda, \mu \in {}_N\mathcal{X}_N$.

Proof. The sum $\sum_{\lambda,\mu} p_\lambda^+ *_v p_\mu^-$ is given graphically by the left-hand side of Fig. 55. By using Lemma 4.3 for the two parallel vertical wires c on the bottom and the IBFE

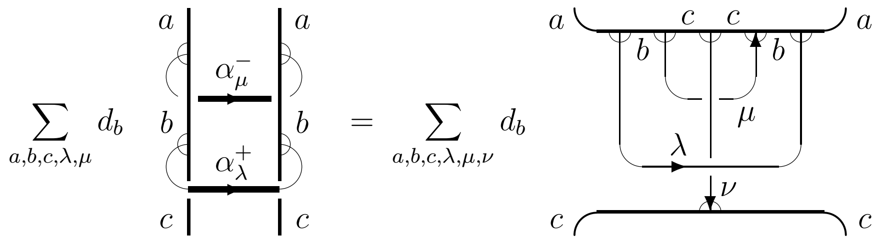


Figure 55: The sum $\sum_{\lambda,\mu} p_\lambda^+ *_v p_\mu^-$

moves we obtain the right-hand side of Fig. 55. For the summation over the thin wire λ we can use Lemma 4.3 again to obtain the left-hand side of Fig. 56. Now we can

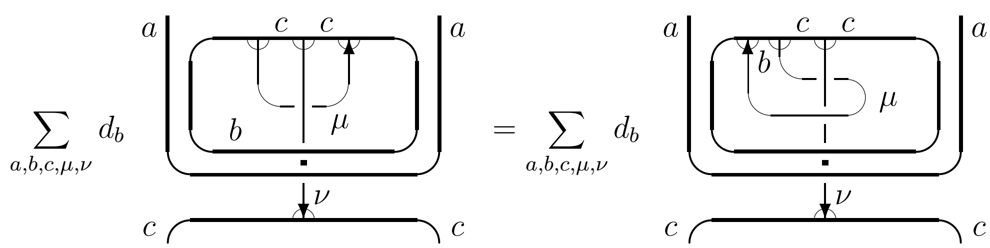
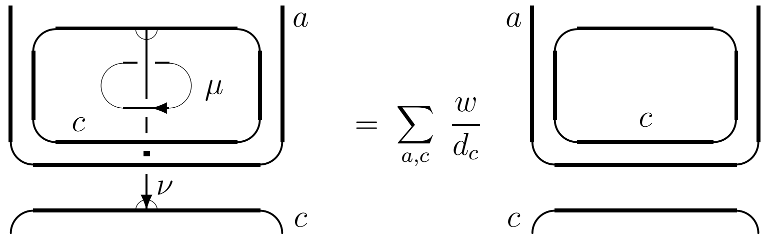


Figure 56: The sum $\sum_{\lambda,\mu} p_\lambda^+ *_v p_\mu^-$

slide around the right trivalent vertex of the wire μ, and this yields the right-hand side of Fig. 56. Next we can use the trick of Fig. 40 to turn around the small arcs from the wire μ to the wire b. This yields a factor d_μ/d_b. Then we can proceed with the summation over b, using Lemma 4.3 once more, and this gives us the left-hand side of Fig. 57. Now we observe that the summation over μ provides a killing ring, and hence we obtain a factor $w\delta_{\nu,0}$. The normalization convention for the small arcs yields another factor $1/d_c$, and hence we get exactly the right-hand side of Fig. 57. The circular wire c cancels the factor $1/d_c$, and thus we are left exactly with the global index w times a summation over two straight horizontal wires, and the latter is exactly the horizontal unit $\mathbf{1}_h = \sum_\beta e_\beta$. The rest is application of the isomorphism Φ. $\square$

We remark that the non-degeneracy of the braiding played an essential role in the proof. In fact there are counter-examples showing that the generating property does



$$\sum_{a,c,\mu,\nu} d_\mu \qquad = \sum_{a,c} \frac{w}{d_c}$$

Figure 57: The sum $\sum_{\lambda,\mu} p^+_\lambda *_v p^-_\mu$

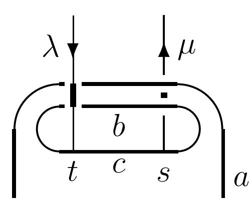
not hold in general if the braiding is degenerate (e.g. the finite group case discussed in Section 4.2 of [2] serves as such an example).

6 Representations of the M-M Fusion Rule Algebra

6.1 Irreducible representations of the M-M fusion rules

We next study in detail the algebra $(\mathcal{Z}_h, *_v)$ or, equivalently, the M-M fusion rule algebra in the case that the N-N system is non-degenerately braided. Note that the Assumption 5.1 implies in particular that the N-N fusion rules algebra is Abelian. However, the M-M fusion rules are in general non-commutative, and therefore so is the center $(\mathcal{Z}_h, *_v)$. We are now going to decompose $(\mathcal{Z}_h, *_v)$ in simple matrix algebras. Note that such a decomposition of $(\mathcal{Z}_h, *_v)$ is equivalent to the determination of the irreducible representations of the M-M fusion rule algebra.

We need some preparation. As in the graphical setting for the double triangle algebra, we can consider the diagram in Fig. 58 as a vector $\Omega^{\lambda,\mu}_{b,c,t,s} \in \mathcal{H}_{\lambda,\mu}$, where $\mathcal{H}_{\lambda,\mu}$



$$\sum_a$$

Figure 58: The vector $\Omega^{\lambda,\mu}_{b,c,t,s} \in \mathcal{H}_{\lambda,\mu}$

is the vector space $\mathcal{H}_{\lambda,\mu} = \bigoplus_{a\in {}_N\mathcal{X}_M} \mathrm{Hom}(\lambda\bar{\mu}, a\bar{a})$, $\lambda,\mu \in {}_N\mathcal{X}_N$. Here $b,c \in {}_N\mathcal{X}_M$, and $t \in \mathrm{Hom}(\lambda, b\bar{c})$ and $s \in \mathrm{Hom}(\bar{\mu}, c\bar{b})$ are isometries labelling the two trivalent vertices in Fig. 58. It is important to notice that we do not allow coefficients depending on a: The same isometries t, s are used in each block $\mathrm{Hom}(\lambda\bar{\mu}, a\bar{a})$ of $\mathcal{H}_{\lambda,\mu}$. We next define the subspace $H_{\lambda,\mu} \subset \mathcal{H}_{\lambda,\mu}$ spanned by such vectors:

$$H_{\lambda,\mu} = \mathrm{span}\{\Omega^{\lambda,\mu}_{b,c,t,s} \mid b,c \in {}_N\mathcal{X}_M\,,\ t \in \mathrm{Hom}(\lambda, b\bar{c})\,,\ s \in \mathrm{Hom}(\bar{\mu}, c\bar{b})\}\,.$$

Take two such vectors $\Omega^{\lambda,\mu}_{b,c,t,s}$ and $\Omega^{\lambda,\mu}_{b',c',t',s'}$. We define an element $|\Omega^{\lambda,\mu}_{b',c',t',s'}\rangle\langle\Omega^{\lambda,\mu}_{b,c,t,s}| \in \Diamond$ by the diagram in Fig. 59. (This notation will be justified by Lemma 6.1 below.)

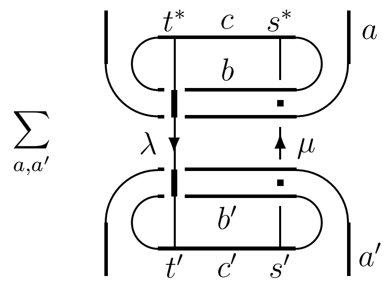


Figure 59: The element $|\Omega^{\lambda,\mu}_{b',c',t',s'}\rangle\langle\Omega^{\lambda,\mu}_{b,c,t,s}| \in \Diamond$

We now choose orthonormal bases of isometries $t^{\lambda;i}_{b,\bar{c}} \in \mathrm{Hom}(\lambda, b\bar{c})$, $i = 1,2,...,N^{\lambda}_{b,\bar{c}}$, for each λ, b, c and put $\Omega^{\lambda,\mu}_{\xi} = \Omega^{\lambda,\mu}_{b,c,t^{\lambda;i}_{b,\bar{c}},t^{\bar{\mu};j}_{c,\bar{b}}}$ with some multi-index $\xi = (b,c,i,j)$. Varying ξ, we obtain a generating set of $H_{\lambda,\mu}$ which will, however, in general not be a basis as the vectors $\Omega^{\lambda,\mu}_{\xi}$ may be linearly dependent in $H_{\lambda,\mu}$. Let $\Phi^{\lambda,\mu}_j \in H_{\lambda,\mu}$, $j = 1,2$, any two vectors. We can expand them as $\Phi^{\lambda,\mu}_j = \sum_\xi c^\xi_j \Omega^{\lambda,\mu}_\xi$ with $c^\xi_j \in \mathbb{C}$, but note that this expansion is not unique. We now define an element $|\Phi^{\lambda,\mu}_1\rangle\langle\Phi^{\lambda,\mu}_2| \in \Diamond$ by

$$|\Phi^{\lambda,\mu}_1\rangle\langle\Phi^{\lambda,\mu}_2| = \sum_{\xi,\xi'} c^\xi_1 (c^{\xi'}_2)^* |\Omega^{\lambda,\mu}_\xi\rangle\langle\Omega^{\lambda,\mu}_{\xi'}|\,, \tag{27}$$

and a scalar $\langle\Phi^{\lambda,\mu}_2, \Phi^{\lambda,\mu}_1\rangle \in \mathbb{C}$,

$$\langle\Phi^{\lambda,\mu}_2, \Phi^{\lambda,\mu}_1\rangle = \frac{1}{d_\lambda d_\mu}\,\tau_v(|\Phi^{\lambda,\mu}_1\rangle\langle\Phi^{\lambda,\mu}_2|)\,. \tag{28}$$

Lemma 6.1 *Eq. (27) extends to a sesqui-linear map $H_{\lambda,\mu} \times H_{\lambda,\mu} \to \mathcal{Z}_h$ which is positive definite: If $|\Phi^{\lambda,\mu}\rangle\langle\Phi^{\lambda,\mu}| = 0$ for some $\Phi^{\lambda,\mu} \in H_{\lambda,\mu}$ then $\Phi^{\lambda,\mu} = 0$. Consequently, Eq. (28) defines a scalar product turning $H_{\lambda,\mu}$ into a Hilbert space.*

Proof. As in particular $\Phi_j \in \mathcal{H}_{\lambda,\mu}$, we can write $\Phi_j = \bigoplus_a (\Phi_j)_a$ with $(\Phi_j)_a \in \mathrm{Hom}(\lambda\bar{\mu}, a\bar{a})$ according to the direct sum structure of $\mathcal{H}_{\lambda,\mu}$, $j = 1,2$. Assume $\Phi_1 = 0$. Then clearly $(\Phi_1)_a = 0$ for all a. Now the $\mathrm{Hom}(a\bar{a}, a'\bar{a'})$ part of $|\Phi^{\lambda,\mu}_1\rangle\langle\Phi^{\lambda,\mu}_2| \in \Diamond$ is given by $(\Phi_1)_{a'}(\Phi_2)^*_a$, hence $|\Phi^{\lambda,\mu}_1\rangle\langle\Phi^{\lambda,\mu}_2| = 0$. A similar argument applies to Φ_2, and hence the element $|\Phi^{\lambda,\mu}_1\rangle\langle\Phi^{\lambda,\mu}_2| \in \Diamond$ is independent of the linear expansions of the Φ_j's. Therefore Eq. (27) defines a sesqui-linear map $H_{\lambda,\mu} \times H_{\lambda,\mu} \to \Diamond$. Now assume $|\Phi_1^{\lambda,\mu}\rangle\langle\Phi_1^{\lambda,\mu}| = 0$. Then in particular $(\Phi_1)_a(\Phi_1)_a^* = 0$ for all $a \in {}_N\mathcal{X}_M$, and hence $\Phi_1 = 0$, proving strict positivity. That the sesqui-linear form $\langle\cdot,\cdot\rangle$ on $H_{\lambda,\mu}$ is non-degenerate follows now from positive definiteness of τ_v. It remains to show that $|\Phi_1^{\lambda,\mu}\rangle\langle\Phi_2^{\lambda,\mu}| \in \mathcal{Z}_h$. But this is clear since any element of the form in Fig. 33 can be "pulled through" the diagram in Fig. 59 by using the IBFE's. $\square$

Lemma 6.2 *We have the identity in Fig. 60 for intertwiners in* $\mathrm{Hom}(\lambda'\bar{\mu}', \lambda\bar{\mu})$, $\lambda, \mu, \lambda', \mu' \in {}_N\mathcal{X}_N$.

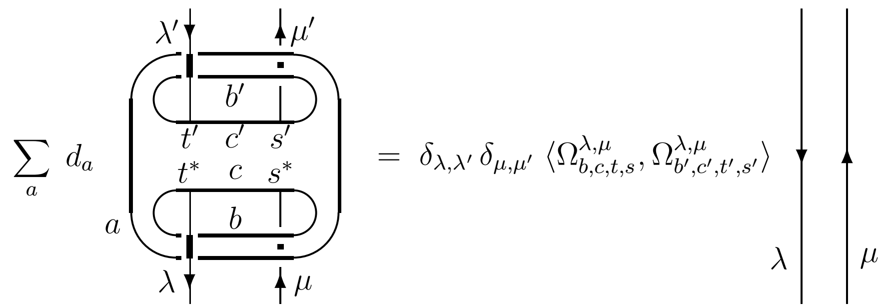


Figure 60: An identity in $\mathrm{Hom}(\lambda'\bar{\mu}', \lambda\bar{\mu})$

Proof. Using Lemma 4.3 we can replace the left-hand side of Fig. 60 by the left-hand side of Fig. 61. Next we can slide one of the trivalent vertices of the wire ν

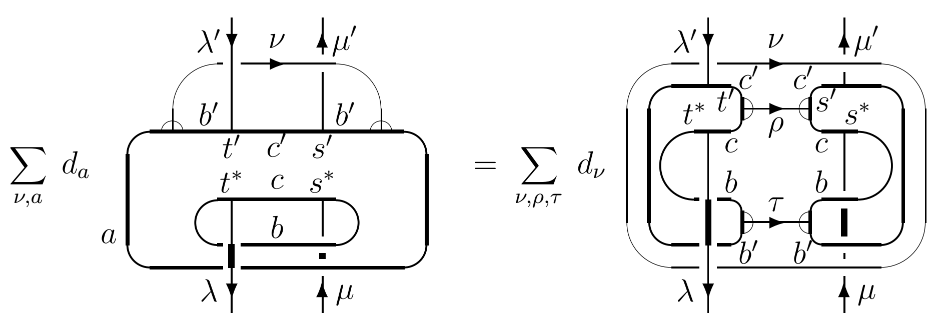


Figure 61: The identity in $\mathrm{Hom}(\lambda'\bar{\mu}', \lambda\bar{\mu})$

around the wire a. Using the identity of Fig. 40, we obtain a factor d_ν/d_a, and we can now proceed with the summation over a, again using Lemma 4.3. Using also Lemma 4.3 for the parallel wires c, c' as well as b and b', we obtain the right-hand side of Fig. 61. Using now Lemma 4.3 once again for the wires ρ, τ, we can pull the wire ν over the middle expansion. The summation over ν yields a killing ring which disconnects the picture into two halves, one is an intertwiner in $\mathrm{Hom}(\lambda',\lambda)$ and the other in $\mathrm{Hom}(\bar{\mu'},\bar{\mu})$. Hence we obtain a factor $\delta_{\lambda,\lambda'}\delta_{\mu,\mu'}$, and we conclude that the left-hand side in Fig. 60 represents a scalar intertwiner $\delta_{\lambda,\lambda'}\delta_{\mu,\mu'}\zeta\mathbf{1}_N \in \mathrm{Hom}(\lambda\bar{\mu},\lambda\bar{\mu})$, $\zeta \in \mathbb{C}$. To compute that scalar, we can start again on the left-hand side of Fig. 60, now putting $\lambda' = \lambda$ and $\mu' = \mu$. The diagram on the left-hand side of Fig. 62 clearly represents an intertwiner of the same scalar value ζ. We can now use the move

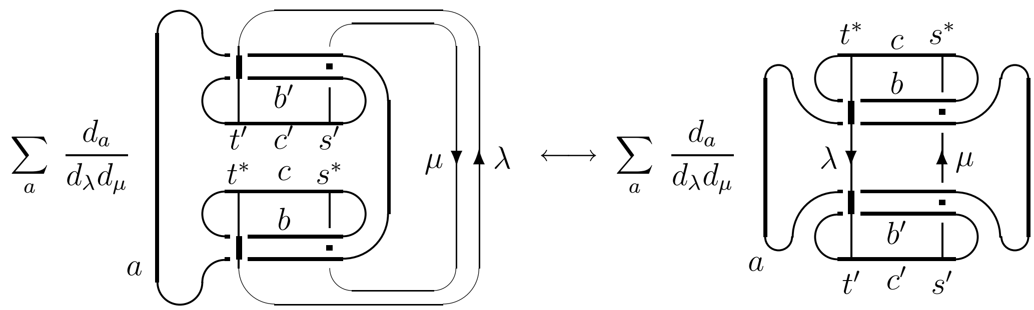


Figure 62: Computation of the scalar ζ

of Fig. 29 which does not change the scalar value: We open the wire a on the left and close it on the right. The resulting diagram is regularly isotopic to the diagram on the right-hand side of Fig. 62. Thus we are left with exactly the diagram for $d_\lambda^{-1}d_\mu^{-1}\tau_v(|\Omega^{\lambda,\mu}_{b',c',t',s'}\rangle\langle\Omega^{\lambda,\mu}_{b,c,t,s}|)$. This proves the lemma. $\square$

The following is now immediate by the definition of the vertical product.

Corollary 6.3 *Let $\Phi_j^{\lambda,\mu} \in H_{\lambda,\mu}$ and $\Psi_j^{\lambda',\mu'} \in H_{\lambda',\mu'}$, $j=1,2$. Then we have*

$$|\Phi_1^{\lambda,\mu}\rangle\langle\Phi_2^{\lambda,\mu}| *_v |\Psi_1^{\lambda',\mu'}\rangle\langle\Psi_2^{\lambda',\mu'}| \;=\; \delta_{\lambda,\lambda'}\,\delta_{\mu,\mu'}\,\langle\Phi_2^{\lambda,\mu},\Psi_1^{\lambda,\mu}\rangle\,|\Phi_1^{\lambda,\mu}\rangle\langle\Psi_2^{\lambda,\mu}| \tag{29}$$

in the double triangle algebra.

Whenever $H_{\lambda,\mu} \neq \{0\}$ we can choose an orthonormal basis $\{E_i^{\lambda,\mu}\}_{i=1}^{\dim H_{\lambda,\mu}}$. Then Lemma 6.1 and Corollary 6.3 tell us that $\{\,|E_i^{\lambda,\mu}\rangle\langle E_j^{\lambda,\mu}|\,\}_{\lambda,\mu,i,j}$ forms a set of non-zero matrix units in $(\mathcal{Z}_h,*_v)$. However, we do not know yet whether this is a complete set.

Lemma 6.4 *Let $\pi_{\lambda,\mu}(e_\beta)\Omega^{\lambda,\mu}_{b,c,t,s} \in \mathcal{H}_{\lambda,\mu}$ denote the vector which is given graphically by the diagram in Fig. 63, where $\lambda,\mu \in {}_N\mathcal{X}_N$, $b,c \in {}_N\mathcal{X}_M$, and $t \in \mathrm{Hom}(\lambda,b\bar{c})$, $s \in \mathrm{Hom}(\bar{\mu},c\bar{b})$ are isometries. Then in fact $\pi_{\lambda,\mu}(e_\beta)\Omega^{\lambda,\mu}_{b,c,t,s} \in H_{\lambda,\mu}$.*

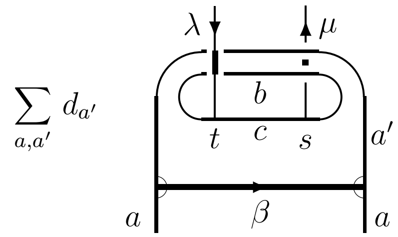


Figure 63: The vector $\pi_{\lambda,\mu}(e_\beta)\Omega^{\lambda,\mu}_{b,c,t,s} \in \mathcal{H}_{\lambda,\mu}$

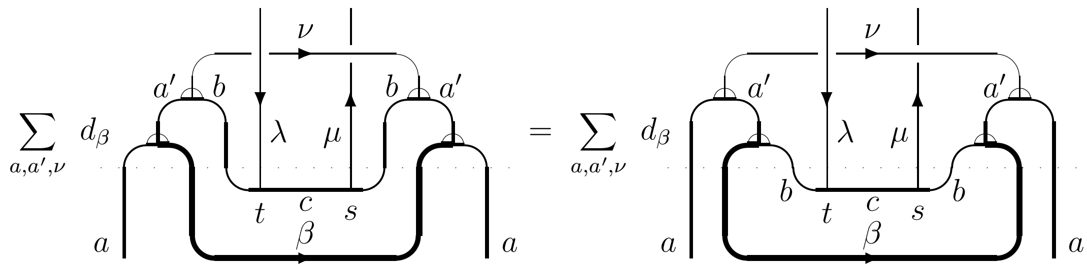


Figure 64: The vector $\pi_{\lambda,\mu}(e_\beta)\Omega^{\lambda,\mu}_{b,c,t,s} \in \mathcal{H}_{\lambda,\mu}$

Proof. Using Lemma 4.3 and also the trick of Fig. 40, we can draw the diagram on the left-hand side in Fig. 64 for $\pi_{\lambda,\mu}(e_\beta)\Omega^{\lambda,\mu}_{b,c,t,s}$. Now let us look at the part of this picture above the dotted line. In a suitable Frobenius annulus, this part can be read for fixed ν and a as $\sum_i \lambda\bar{\mu}(t_i)\varepsilon^-(\nu,\lambda\bar{\mu})t_i^*$, and the sum runs over a full orthonormal basis of isometries t_i in the Hilbert space $\mathrm{Hom}(\nu, b\bar{\beta}\bar{a})$ since we have the summation over a'. Next we look at the part above the dotted line on the right-hand side of Fig. 64. This can be similarly read for fixed ν and a as $\sum_j \lambda\bar{\mu}(s_j)\varepsilon^-(\nu,\lambda\bar{\mu})s_j^*$, where the sum runs over another full orthonormal basis of isometries $s_j \in \mathrm{Hom}(\nu, b\bar{\beta}\bar{a})$. Since such bases $\{t_i\}$ and $\{s_j\}$ are related by a unitary matrix transformation (this is again just "unitarity of $6j$-symbols"), the left- and right-hand side represent the same vector in $\mathcal{H}_{\lambda,\mu}$. Then, using again Lemma 4.3 and also the trick of Fig. 40, we conclude that the vector $\pi_{\lambda,\mu}(e_\beta)\Omega^{\lambda,\mu}_{b,c,t,s}$ can be represented by the diagram on the left-hand side of Fig. 65. Now let us look at the part of the diagram inside the dotted box. In a suitable Frobenius annulus, this can be interpreted as an intertwiner in $\mathrm{Hom}(\lambda\bar{\mu}, a'\bar{a}')$. But any element in this space can be written as a linear combination of elements constructed from basis isometries $t^{\lambda;i}_{a',\bar{c}'}$, $t^{\bar{\mu};j}_{c',\bar{a}'}$, as indicated in the dotted box on the right-hand side of Fig. 65. The coefficients in its linear expansion depend only on c', i, j for fixed a', β, b, c, t, s, but certainly not on a. This shows that $\pi_{\lambda,\mu}(e_\beta)\Omega^{\lambda,\mu}_{b,c,t,s}$ is a linear combination of $\Omega^{\lambda,\mu}_\xi$'s, thus $\pi_{\lambda,\mu}(e_\beta)\Omega^{\lambda,\mu}_{b,c,t,s} \in H_{\lambda,\mu}$. □

The map $\Omega^{\lambda,\mu}_{b,c,t,s} \mapsto \pi_{\lambda,\mu}(e_\beta)\Omega^{\lambda,\mu}_{b,c,t,s}$ defines clearly a linear map $\pi_{\lambda,\mu}(e_\beta) : H_{\lambda,\mu} \to$

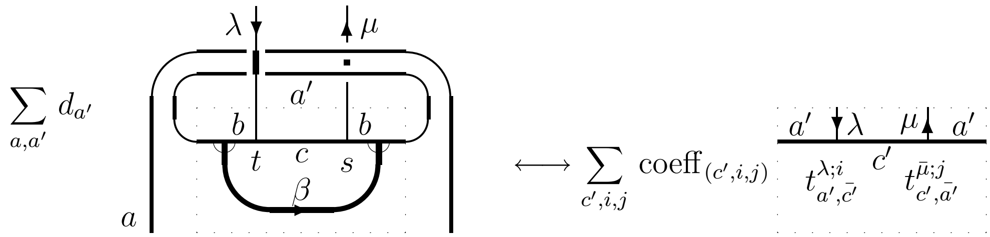


Figure 65: The vector $\pi_{\lambda,\mu}(e_\beta)\Omega^{\lambda,\mu}_{b,c,t,s} \in \mathcal{H}_{\lambda,\mu}$

$\mathcal{H}_{\lambda,\mu}$ since it is just a linear intertwiner multiplication on each $\mathrm{Hom}(\lambda\bar{\mu}, a\bar{a})$ block. From Lemma 6.4 we now learn that $\pi_{\lambda,\mu}(e_\beta)$ is in fact a linear operator on $H_{\lambda,\mu}$. With the definition of the vertical product we now immediately obtain the following

Corollary 6.5 *With orthonormal bases* $\{E^{\lambda,\mu}_i\}_{i=1}^{\dim H_{\lambda,\mu}}$ *of each* $H_{\lambda,\mu}$ *we have*

$$|E^{\lambda,\mu}_i\rangle\langle E^{\lambda,\mu}_j| *_v e_\beta *_v |E^{\lambda',\mu'}_k\rangle\langle E^{\lambda',\mu'}_l| \;=\; \delta_{\lambda,\lambda'}\,\delta_{\mu,\mu'}\,\langle E^{\lambda,\mu}_j, \pi_{\lambda,\mu}(e_\beta)E^{\lambda,\mu}_k\rangle\;|E^{\lambda,\mu}_i\rangle\langle\; E^{\lambda,\mu}_l|\,. \tag{30}$$

Since $\mathcal{Z}_h$ is spanned by the e_β's, we obtain a map $\pi_{\lambda,\mu} : \mathcal{Z}_h \to B(H_{\lambda,\mu})$ by linear extension, and we obtain similarly the following

Corollary 6.6 *The map* $\pi_{\lambda,\mu} : \mathcal{Z}_h \to B(H_{\lambda,\mu})$ *is a representation of* $(\mathcal{Z}_h, *_v)$.

We now tackle the problem of completeness of the system of matrix units.

Definition 6.7 For $\lambda, \mu \in {}_N\mathcal{X}_N$ we define the *vertical projector* $q_{\lambda,\mu} \in \Diamond$ by

$$q_{\lambda,\mu} = \frac{\sqrt{d_\lambda d_\mu}}{w^2} \sum_\xi \,|\Omega^{\lambda,\mu}_\xi\rangle\langle\Omega^{\lambda,\mu}_\xi|\,. \tag{31}$$

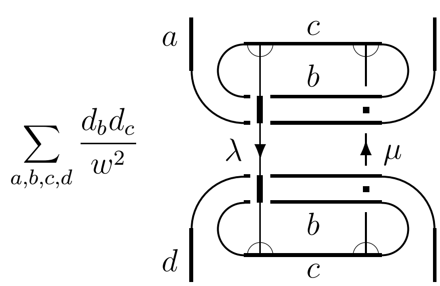


Figure 66: A vertical projector $q_{\lambda,\mu}$

This is given graphically in Fig. 66. (Clearly, we can use Lemma 4.3 twice to obtain an equivalent picture which does not involve pieces of very thick wires corresponding to α^+_λ and α^-_μ.) We are now ready to prove the main result of this section.

Theorem 6.8 *Under Assumption 5.9, the vertical projector $q_{\lambda,\mu}$ is either zero or a minimal central projection in $(\mathcal{Z}_h, *_v)$. We have mutual orthogonality $q_{\lambda,\mu} *_v q_{\lambda',\mu'} = \delta_{\lambda,\lambda'}\delta_{\mu,\mu'} q_{\lambda,\mu}$ and the vertical projectors sum up to the multiplicative identity of $(\mathcal{Z}_h, *_v)$: $\sum_{\lambda,\mu\in {}_N\mathcal{X}_N} q_{\lambda,\mu} = e_0$. Moreover, $q_{\lambda,\mu} = 0$ whenever $Z_{\lambda,\mu} = 0$ and otherwise the simple summand $q_{\lambda,\mu} *_v \mathcal{Z}_h$ is a full $Z_{\lambda,\mu} \times Z_{\lambda,\mu}$ matrix algebra, where $Z_{\lambda,\mu}$ is the (λ, μ)-entry of the modular invariant mass matrix of Definition 5.5.*

Proof. It follows from Corollary 6.3 that $q_{\lambda,\mu} *_v q_{\lambda',\mu'} = 0$ unless $\lambda = \lambda'$ and $\mu = \mu'$. We now show that $\sum_{\lambda,\mu} q_{\lambda,\mu} = e_0$. (We denote $e_0 \equiv e_{\mathrm{id}_M}$.) The sum is given graphically by the left-hand side in Fig. 67. A twofold application of Lemma 4.3

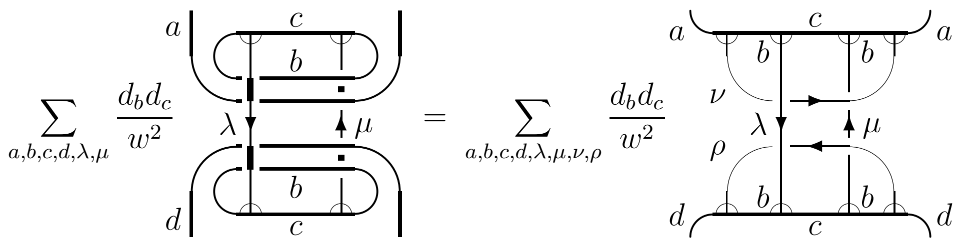


Figure 67: The sum $\sum_{\lambda,\mu} q_{\lambda,\mu}$

yields the right-hand side in Fig. 67. Applying Lemma 4.3 twice again, we obtain the left-hand side of Fig. 68. We can now slide the upper trivalent vertex of the wire

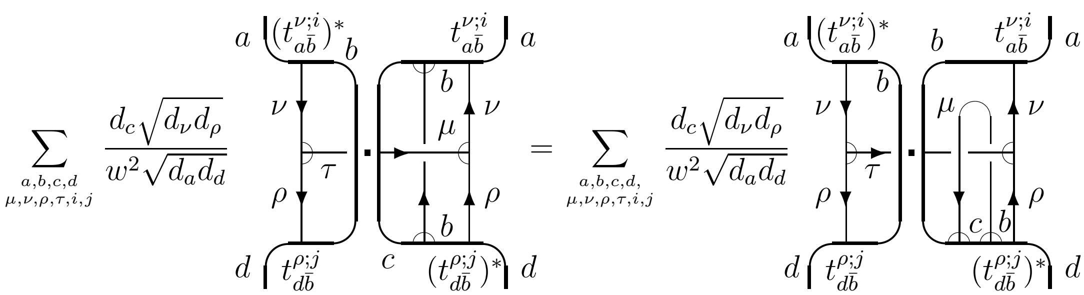


Figure 68: The sum $\sum_{\lambda,\mu} q_{\lambda,\mu}$

μ around to obtain the right-hand side of Fig. 68. Next we can use the trick of Fig. 40 to turn around the small arcs at the trivalent vertices of the wire μ, yielding a factor d_μ/d_c. This gives the right-hand side of Fig. 68. Since we have a summation over c, we can again use Lemma 4.3, and this gives us the left-hand side of Fig. 69. As we have a prefactor d_μ, the summation over μ provides a killing ring, and only $\tau = \mathrm{id}_N$ survives it: We obtain a factor $w\delta_{\tau,0}$. Now our picture starts to collapse. The factor $\delta_{\tau,0}$ yields, with the normalization convention as in Fig. 39, a factor $d_\nu^{-1}\delta_{\nu,\rho}$. Since our picture is now disconnected into two parts which represent intertwiners in

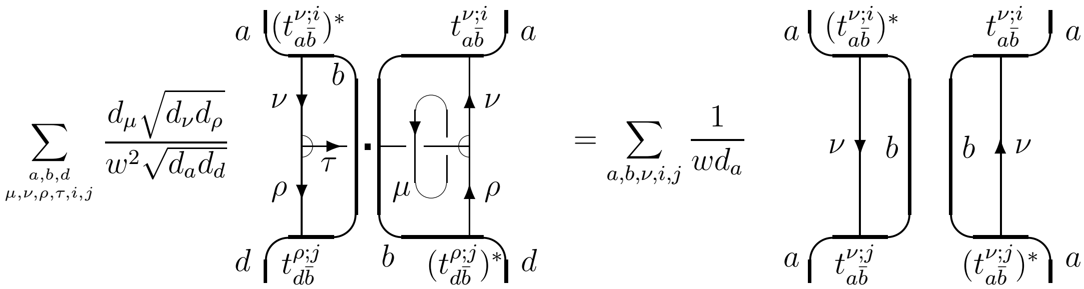


Figure 69: The sum $\sum_{\lambda,\mu} q_{\lambda,\mu}$

$\mathrm{Hom}(a,d)$, they are scalars and we obtain a factor $\delta_{a,d}$. This gives us the right-hand side of Fig. 69. Therefore we are now left with a sum over scalars times two straight vertical wires labelled by a, representing a scalar intertwiner in $\mathrm{Hom}(a\bar{a},a\bar{a})$. The scalar value of each connected part of the picture is $\delta_{i,j}\sqrt{d_\nu d_b/d_a}$, therefore we can compute the prefactor as

$$\frac{1}{wd_a}\sum_{b,\nu}\sum_{i,j=1}^{N^{\nu}_{a\bar{b}}}\left(\sqrt{\frac{d_\nu d_b}{d_a}}\delta_{i,j}\right)^2 = \frac{1}{wd_a^2}\sum_{b,\nu} d_b N^{\nu}_{a,\bar{b}} d_\nu = \frac{1}{wd_a}\sum_b d_b^2 = \frac{1}{d_a}\,.$$

Thus we are left with a sum over two vertical straight wires with label a and prefactor d_a^{-1}. This is e_0.

Next, we can expand each vector $\Omega_\xi^{\lambda,\mu} \in H_{\lambda,\mu}$, in an orthonormal basis as

$$\Omega_\xi^{\lambda,\mu} = \sum_{i=1}^{\dim H_{\lambda,\mu}} \langle E_i^{\lambda,\mu}, \Omega_\xi^{\lambda,\mu}\rangle E_i^{\lambda,\mu}\,.$$

Inserting this in Eq. (31) yields

$$q_{\lambda,\mu} = \frac{\sqrt{d_\lambda d_\mu}}{w^2}\sum_{i,j}^{\dim H_{\lambda,\mu}}\sum_\xi \langle E_i^{\lambda,\mu}, \Omega_\xi^{\lambda,\mu}\rangle\langle \Omega_\xi^{\lambda,\mu}, E_j^{\lambda,\mu}\rangle\, |E_i^{\lambda,\mu}\rangle\langle E_j^{\lambda,\mu}|\,.$$

Now using $\sum_{\lambda,\mu} q_{\lambda,\mu} = e_0$ and Corollary 6.3 we compute

$$\begin{aligned}\delta_{i,j}\,|E_i^{\lambda,\mu}\rangle\langle E_j^{\lambda,\mu}| &= \textstyle\sum_{\lambda',\mu'} |E_i^{\lambda,\mu}\rangle\langle E_i^{\lambda,\mu}| *_v q_{\lambda',\mu'} *_v |E_j^{\lambda,\mu}\rangle\langle E_j^{\lambda,\mu}| \\ &= \frac{\sqrt{d_\lambda d_\mu}}{w^2}\sum_\xi \langle E_i^{\lambda,\mu}, \Omega_\xi^{\lambda,\mu}\rangle\langle \Omega_\xi^{\lambda,\mu}, E_j^{\lambda,\mu}\rangle\, |E_i^{\lambda,\mu}\rangle\langle E_j^{\lambda,\mu}|\,,\end{aligned}$$

hence

$$q_{\lambda,\mu} = \sum_{i=1}^{\dim H_{\lambda,\mu}} |E_i^{\lambda,\mu}\rangle\langle E_i^{\lambda,\mu}|\,.$$

Thus $q_{\lambda,\mu}$ is a projection and we also have $e_0 = \sum_{\lambda,\mu} \sum_{i=1}^{\dim H_{\lambda,\mu}} |E_i^{\lambda,\mu}\rangle\langle E_i^{\lambda,\mu}|$. Hence for any $\beta \in {}_M\mathcal{X}_M$ we find

$$e_\beta = e_0 *_v e_\beta *_v e_0 = \sum_{\lambda,\mu} \sum_{i,j=1}^{\dim H_{\lambda,\mu}} \langle E_i^{\lambda,\mu}, \pi_{\lambda,\mu}(e_\beta) E_j^{\lambda,\mu}\rangle \, |E_i^{\lambda,\mu}\rangle\langle\, E_j^{\lambda,\mu}|$$

by Corollary 6.5. Thus each e_β can be expanded in our matrix units, and since $\mathcal{Z}_h$ is spanned by the e_β's we conclude that $\{|E_i^{\lambda,\mu}\rangle\langle\, E_j^{\lambda,\mu}|\}_{\lambda,\mu,i,j}$ is a complete system of matrix units. It follows that the non-zero vertical projectors are minimal central projections in $(\mathcal{Z}_h, *_v)$, and that the simple summand $q_{\lambda,\mu} *_v \mathcal{Z}_h$ is a full $\dim H_{\lambda,\mu} \times \dim H_{\lambda,\mu}$ matrix algebra. It remains to show $\dim H_{\lambda,\mu} = Z_{\lambda,\mu}$. The dimension of $H_{\lambda,\mu}$ can be counted as

$$\dim H_{\lambda,\mu} = \sum_{i=1}^{\dim H_{\lambda,\mu}} \langle E_i^{\lambda,\mu}, E_i^{\lambda,\mu}\rangle = \sum_{i=1}^{\dim H_{\lambda,\mu}} \frac{1}{d_\lambda d_\mu} \tau_v(|E_i^{\lambda,\mu}\rangle\langle E_i^{\lambda,\mu}|) = \frac{1}{d_\lambda d_\mu} \tau_v(q_{\lambda,\mu}) \,.$$

Now $d_\lambda^{-1} d_\mu^{-1} \tau_v(q_{\lambda,\mu})$ is given graphically in Fig. 70. By the IBFE's we can pull out

$$\sum_{a,b,c} \frac{d_a d_b d_c}{w^2 d_\lambda d_\mu}$$

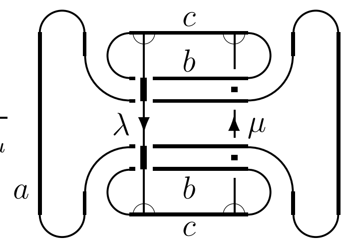


Figure 70: The number $d_\lambda^{-1} d_\mu^{-1} \tau_v(q_{\lambda,\mu})$

the circle with label a which gives us another factor d_a. We can therefore proceed with the summation over a, and this yields a factor w, the global index, and then we are left exactly with the picture in Fig. 50. $\square$

Note that we learn from the proof that putting $\mathrm{Tr}_v(z) = \sum_{\lambda,\mu} d_\lambda^{-1} d_\mu^{-1} \tau_v(q_{\lambda,\mu} *_v z)$ for $z \in \mathcal{Z}_h$ gives a matrix trace Tr_v on $(\mathcal{Z}_h, *_v)$ which sends the minimal projections to one. Next we have learnt that for all λ, μ with $Z_{\lambda,\mu} \neq 0$, the $\pi_{\lambda,\mu}$'s are the irreducible representations of $(\mathcal{Z}_h, *_v)$ and hence the $\pi_{\lambda,\mu} \circ \Phi$'s are the irreducible representations of the M-M fusion rule algebra.

Corollary 6.9 *Under Assumption 5.9, the M-M fusion rule algebra is commutative if and only if $Z_{\lambda,\mu} \in \{0,1\}$ for all $\lambda, \mu \in {}_N\mathcal{X}_N$.*

Corollary 6.10 *Under Assumption 5.9, the total number of morphisms in ${}_M\mathcal{X}_M$ is equal to* $\mathrm{tr}(Z\,{}^{\mathrm{t}}Z) = \sum_{\lambda,\mu \in {}_N\mathcal{X}_N} Z_{\lambda,\mu}^2$.

6.2 The left action on M-N sectors

The decomposition of $(\mathcal{Z}_h, *_v)$ into simple matrix algebras is equivalent to the irreducible decomposition of the "regular representation" (up to multiplicities given as the dimensions) of the M-M fusion rule algebra, i.e. the representation obtained by its action on itself as a vector space. There is another representation of the M-M fusion rule algebra, namely the one obtained by its (left) action on the M-N sectors. This is what we study in the following.

We define the vector space K by $K = \bigoplus_{a \in {}_N\mathcal{X}_M} \mathrm{Hom}(\mathrm{id}_N, a\bar{a})$. Note that each block consists just of scalar multiples of the isometries $\bar{r}_a$ but we need the explicit form of K. We define basis vectors $v_{\bar{a}} \in K$ corresponding to $d_a^{-1/2}\bar{r}_a$ in each block $\mathrm{Hom}(\mathrm{id}_N, a\bar{a})$. We can display each $v_{\bar{a}}$ graphically by a thick wire "cap" with label $a \in {}_N\mathcal{X}_M$ together with a prefactor $1/d_a$. We furnish K with a Hilbert space structure by putting $\langle v_{\bar{a}}, v_{\bar{b}} \rangle = \delta_{a,b}$. For each $a \in {}_N\mathcal{X}_M$ we define a vector $\varrho(e_\beta)v_{\bar{a}}$ by putting

$$\varrho(e_\beta)v_{\bar{a}} = d_\beta \sum_b N_{\beta,\bar{a}}^{\bar{b}}\, v_{\bar{b}}\,. \tag{32}$$

We can display the right-hand side graphically as in Fig. 71. The left- and right-hand

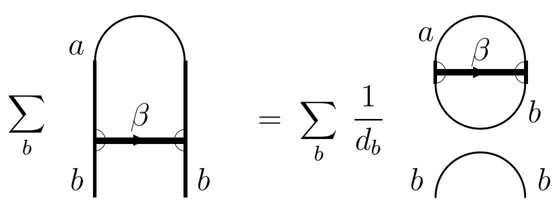


Figure 71: The element $\varrho(e_\beta)v_{\bar{a}} \in K$

side in Fig. 71 are the same because both sides are scalar multiples of the isometry $\bar{r}_a$ in each block $\mathrm{Hom}(\mathrm{id}_N, a\bar{a})$. The map $\varrho(e_\beta) : v_{\bar{a}} \mapsto \varrho(e_\beta)v_{\bar{a}}$ clearly defines a linear operator on K for each $\beta \in {}_M\mathcal{X}_M$, and we can extend the map $e_\beta \mapsto \varrho(e_\beta)$ linearly to $\mathcal{Z}_h$. Graphically, this action of $\mathcal{Z}_h$ is quite similar to the vertical product. (Note that there also appears a factor d_a cancelling the d_a^{-1} in the definition of $v_{\bar{a}}$ when gluing the picture for $v_{\bar{a}}$ on top of that for e_β.)

We observe that the map $\varrho : e_\beta \mapsto \varrho(e_\beta)$ extends linearly to a representation of $(\mathcal{Z}_h, *_v)$ as we can compute for $\beta, \beta' \in {}_M\mathcal{X}_M$ as follows:

$$\begin{aligned}
\varrho(e_\beta)(\varrho(e_{\beta'})v_{\bar{a}}) &= \varrho(e_\beta)\left(d_\beta \textstyle\sum_b N_{\beta',\bar{a}}^{\bar{b}}\, v_{\bar{b}}\right) = d_\beta d_{\beta'} \textstyle\sum_{b,c} N_{\beta,\bar{b}}^{\bar{c}} N_{\beta',\bar{a}}^{\bar{b}} v_{\bar{c}} \\
&= d_\beta d_{\beta'} \textstyle\sum_{\beta'',c} N_{\beta,\beta'}^{\beta''} N_{\beta'',\bar{a}}^{\bar{c}} v_{\bar{c}} = d_\beta d_{\beta'} \textstyle\sum_{\beta'',c} d_{\beta''}^{-1} N_{\beta,\beta'}^{\beta''} \varrho(e_{\beta''})v_{\bar{a}} \\
&= \varrho(e_\beta *_v e_{\beta'})v_{\bar{a}}\,,
\end{aligned}$$

where we used associativity of the sector product in the third equality. Consequently, $\varrho(q_{\lambda,\mu})$ is a projection onto a subspace, and $\varrho|_{\varrho(q_{\lambda,\mu})K}$ is a subrepresentation.

Lemma 6.11 *We have $K = \bigoplus_{\lambda \in {}_N\mathcal{X}_N} K_\lambda$, where $K_\lambda = \varrho(q_{\lambda,\lambda})K$.*

Proof. The vector $\varrho(q_{\lambda,\mu})v_{\bar{a}} \in K$ is given graphically by the left-hand side of Fig. 72. Now note that the upper part of the diagram represents an intertwiner

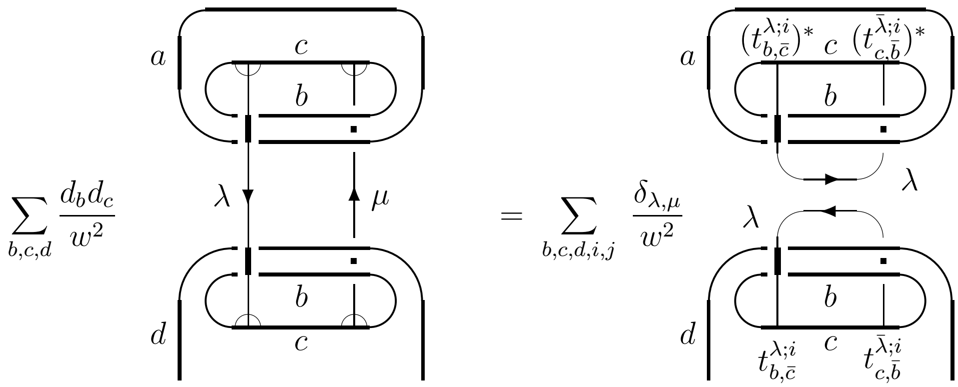


Figure 72: The vector $\varrho(q_{\lambda,\mu})v_{\bar{a}} \in K$

in $\mathrm{Hom}(\mathrm{id}_N, \lambda\bar{\mu})$. Therefore it vanishes unless $\lambda = \mu$ and then it must be a scalar multiple of $\bar{r}_\lambda$. Hence we can insert a term $\bar{r}_\lambda \bar{r}_\lambda^*$ which corresponds graphically to the disconnection of the wires as on the right-hand side in Fig. 72 and multiplication by d_λ^{-1}. Then the factor $d_b d_c / d_\lambda$ disappears because of the normalization convention for trivalent vertices with small arcs, and we are left exactly with the right-hand side of Fig. 72. It follows in particular that $\varrho(q_{\lambda,\mu})K = 0$ unless $\lambda = \mu$. The claim follows now since the vertical projectors sum up to e_0 and $\varrho(e_0)$ is the identity on K. □

We are now ready to prove the following

Theorem 6.12 *The representation ϱ of $(\mathcal{Z}_h, *_v)$ on K obtained by Eq. (32) is unitarily equivalent to the direct sum over the irreducible representations $\pi_{\lambda,\lambda}$:*

$$\varrho \simeq \bigoplus_{\lambda \in {}_N\mathcal{X}_N} \pi_{\lambda,\lambda} \,. \tag{33}$$

Consequently, the representation $\varrho \circ \Phi$ of the M-M fusion rule algebra which is obtained by the action on the M-N sectors arising from ${}_M\mathcal{X}_N$ decomposes into irreducibles as $\varrho \circ \Phi \simeq \bigoplus_\lambda \pi_{\lambda,\lambda} \circ \Phi$.

Proof. For $b, c \in {}_N\mathcal{X}_M$ and isometries $t \in \mathrm{Hom}(\lambda, b\bar{c})$ and $s \in \mathrm{Hom}(\bar{\lambda}, c\bar{b})$ we define a vector $k^\lambda_{b,c,t,s} \in K$ by the diagram in Fig. 73. Using again intertwiner bases, we also put $k^\lambda_\xi = k_{b,c,t^{\lambda;i}_{b,\bar{c}},t^{\bar{\lambda};j}_{c,\bar{b}}}$ with some multi-index $\xi = (b,c,i,j)$. It follows from the right-hand side in Fig. 72 that $K_\lambda \subset \mathrm{span}\{k^\lambda_\xi \,|\, \xi = (b,c,i,j)\}$. Conversely, we obtain by Lemma 6.2 that $\varrho(q_{\mu,\mu})k^\lambda_\xi = 0$ unless $\lambda = \mu$, hence $K_\lambda = \mathrm{span}\{k^\lambda_\xi \,|\, \xi = (b,c,i,j)\}$.

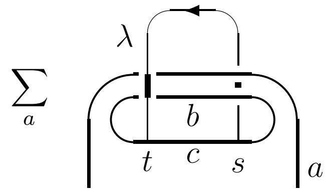


Figure 73: The vector $k^\lambda_{b,c,t,s} \in K$

With $\lambda = \mu$, closing the wires on the bottom and on the top on both sides of Fig. 60 yields

$$\langle k^\lambda_\xi, k^\lambda_{\xi'} \rangle = d_\lambda \langle \Omega^{\lambda,\lambda}_\xi, \Omega^{\lambda,\lambda}_{\xi'} \rangle \,.$$

Hence linear extension of $\Omega^{\lambda,\lambda}_\xi \mapsto d_\lambda^{-1/2} k^\lambda_\xi$ defines a unitary operator $U_\lambda : H_{\lambda,\lambda} \to K_\lambda$. Note that U means multiplication by $\bar{r}_\lambda$ from the right in each block $\mathrm{Hom}(\lambda\bar{\lambda}, a\bar{a})$ and this corresponds graphically to closing the open ends of the wires λ in Fig. 58 and multiplying by $d_\lambda^{-1/2}$. Therefore we find

$$U\left[\pi_{\lambda,\lambda}(e_\beta)\Omega^{\lambda,\lambda}_\xi\right] = d_\lambda^{-1/2} \varrho_\lambda(e_\beta) k^\lambda_\xi = \varrho_\lambda(e_\beta) U\left[\Omega^{\lambda,\lambda}_\xi\right] ,$$

where $\varrho_\lambda = \varrho|_{K_\lambda}$. Thus $\varrho_\lambda \simeq \pi_{\lambda,\lambda}$. $\square$

Since the dimension of K is the cardinality of ${}_N\mathcal{X}_M$ we immediately obtain the following

Corollary 6.13 *Under Assumption 5.9, the total number of morphisms in ${}_N\mathcal{X}_M$ (or, equivalently, in ${}_M\mathcal{X}_N$) is equal to* $\mathrm{tr}(Z) = \sum_{\lambda \in {}_N\mathcal{X}_N} Z_{\lambda,\lambda}$.

7 Conclusions and Outlook

We have analyzed braided type III subfactors and shown that in the non-degenerate case the system of M-M system is entirely generated by α-induction, including in particular the subsectors of Longo's canonical endomorphism γ. We established that in that case the essential structural information about the M-M fusion rules is encoded in the modular invariant mass matrix Z. Our setting applies in particular to $SU(n)$ loop group subfactors $\pi^0(L_I SU(n))'' \subset \pi^0(L_I G)''$ of conformal inclusions $SU(n)_k \subset G_1$ and $\pi_0(L_I SU(n))'' \subset \pi_0(L_I SU(n))'' \rtimes_\sigma \mathbb{Z}_m$ which were analyzed by α-induction in [3, 4]. Here π^0 denotes the level 1 vacuum representation of the loop group LG, π_0 the level k representation of $LSU(n)$, $I \subset S^1$ is an interval, and σ is a "simple current". The braiding here arises from the localized transportable endomorphisms of the net of local algebras $A(I) = \pi_0(L_I SU(n))''$. Since it follows from Wassermann's work [45] that these endomorphisms obey the $SU(n)_k$ fusion rules and from the conformal spin-statistics theorem [18] that the statistics phases are given by $\omega_\lambda = \mathrm{e}^{2\pi\mathrm{i}h_\lambda}$ with h_λ denoting the $SU(n)_k$ conformal dimensions, it follows that the S- and T-matrices from the braiding coincide with the well-known S- and T-matrices which transform the conformal characters. Therefore Theorem 5.10 shows in particular that Condition 4 in Proposition 5.1 in [4] holds in the setting of conformal inclusions, and in turn it proves Conjecture 7.1 in [4]. It also follows that in the setting of Proposition 5.1 in [4], the sum of e_β for "marked vertices" $[\beta]$ (the M-M sectors arising from the positive energy representations of the ambient theory) correspond to the projections appearing in the decomposition of $\sum_{\lambda,\mu} p^+_\lambda *_h p^-_\mu$, the "ambichiral projector" in Ocneanu's language. Similarly, the results of this paper also prove Conjecture 7.2 in [4]. Theorem 5.10 shows in particular that there are *no* counter-examples for conformal inclusions where the M-M sectors arising from the conformal inclusion subfactor are not generated by the mixed α-induction (cf. [48]). Xu made some computation in [47] (see also [3]) to find an example with non-commutative fusion rules of (M-M) sectors generated by the image of only one "positive" induction for subfactors arising from conformal inclusions. By Corollary 6.9, it is at least very easy to find examples of a non-commutative entire M-M fusion rule algebra. The D_4 case mentioned in [4, Subsection 6.1] is one such example. In fact, the whole D_{2n} series arising from simple current extension of $SU(2)_{4n-4}$ also give examples of non-commutative M-M fusion rule algebras. Such non-commutativity for $\mathrm{D}_{\mathrm{even}}$ has been also pointed out in the setting of [39] (though not in the context of conformal inclusions or simple current extensions).

We will present the details and more analysis about $SU(n)_k$ loop group subfactors, including the treatment of all $SU(2)$ modular invariants, in a forthcoming publication [5]. Our treatment can now also incorporate the type II invariants which were not considered in [3, 4], because we dropped the chiral locality condition which automatically forces the mass matrix Z to be type I, i.e. block-diagonal.

Let us remark that we could also have defined $Z_{\lambda,\mu}$ with exchanged $\pm$-signs in Def. 5.5, and this would correspond to replacing Z by the transposed mass matrix ${}^{\mathrm{t}}Z$. It is not hard to see that all our calculations go through with ${}^{\mathrm{t}}Z$ as well. That means α-induction for a (non-degenerately) braided subfactor determines actually two modular invariant mass matrices Z and ${}^{\mathrm{t}}Z$, and it is not clear to us at present whether they can in fact be different in our general setting. (We have $Z = {}^{\mathrm{t}}Z$ for all $SU(2)$ and $SU(3)$ modular invariants).

A notion of subequivalent paragroups was introduced in [27]. Since ${}_N\mathcal{X}_N$ and ${}_M\mathcal{X}_M$ are equivalent systems of endomorphisms by definition, α-induction produces an example of a subequivalent paragroup. That is, for $\lambda \in {}_N\mathcal{X}_N$, the subfactors $\alpha^\pm_\lambda(M) \subset M$ are subequivalent to $\lambda(N) \subset N$. Various examples in [27] arise from this construction. Indeed, the most fundamental example in [27] comes from the Goodman-de la Harpe-Jones subfactor [17, Section 4.5] with index $3+\sqrt{3}$. In our current setting, this example comes from the conformal inclusion $SU(2)_{10} \subset SO(5)_1$ and shows that the two paragroups with principal graph E_6 are subequivalent to the paragroup with principal graph A_{11}.

As a corollary of a rigidity theorem presented by Ocneanu in Madras in January

1997, there are only finitely many paragroups with global index below a given upper bound. This implies that for a given paragroup we have only finitely many subequivalent paragroups since their global indices are less than or equal to the global index of the given paragroup. In the context of modular invariants, a simple argument of Gannon [16] shows $\sum_{\lambda,\mu} Z_{\lambda,\mu} \leq 1/S_{0,0}^2$, which in turn implies that there are only finitely many modular invariant mass matrices Z for a given unitary representation of $SL(2;\mathbb{Z})$, where the S-matrix satisfies the standard relations $S_{0,\lambda} \geq S_{0,0} > 0$. As for a non-degenerately braided system of morphisms this bound coincides with the global index, $w = 1/S_{0,0}^2$, and in view of the relations between modular invariants and subfactors elaborated in this paper, it is natural to expect that these two finiteness arguments are not completely unrelated. We consider a good understanding of the connections between these two arguments to be highly desirable.

Let us finally remark that in a recent paper of Rehren [42] the embedding of left and right chiral observables in a $2D$ conformal field theory are studied. Such embeddings give rise to subfactors and in turn to coupling matrices which are invariant mass matrices if the Fourier transform matrix of the chiral fusion rules is modular. As these subfactors are quite different from ours which appear in a framework considering chiral observables only, the relation between the two approaches also calls for a coherent understanding.

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