The classification of non-local chiral CFT with $c < 1$

YASUYUKI KAWAHIGASHI[*]

Department of Mathematical Sciences

University of Tokyo, Komaba, Tokyo, 153-8914, Japan

e-mail: `yasuyuki@ms.u-tokyo.ac.jp`

ROBERTO LONGO[†]

Dipartimento di Matematica

Università di Roma "Tor Vergata"

Via della Ricerca Scientifica, 1, I-00133 Roma, Italy

e-mail: `longo@mat.uniroma2.it`

ULRICH PENNIG and KARL-HENNING REHREN

Institut für Theoretische Physik

Universität Göttingen

37077 Göttingen, Germany

e-mail: `pennig@theorie.physik.uni-goe.de`,
`rehren@theorie.physik.uni-goe.de`

October 18, 2005

Abstract

All non-local but relatively local irreducible extensions of Virasoro chiral CFTs with $c < 1$ are classified. The classification, which is a prerequisite for the classification of local $c < 1$ boundary CFTs on a two-dimensional half-space, turns out to be 1 to 1 with certain pairs of A-D-E graphs with distinguished vertices.

1

1 Introduction

Non-local chiral conformal quantum field theories have gained renewed interest because they give rise to local CFT on the two-dimensional Minkowski halfspace $x > 0$ (boundary CFT, BCFT), and vice versa [9].

More precisely, a BCFT contains *chiral* fields which generate a net A of local algebras on the circle, such that $A_+(O) = A(I) \vee A(J)$ are the chiral BCFT observables localized in the double cone $O = I \times J \equiv \{(t,x) : t + x \in I, t - x \in J\}$ where $I > J$ are two open intervals of the real axis (= the pointed circle). The two-dimensional *local* fields of the BCFT define a net of inclusions $A_+(O) \subset B_+(O)$ subject to locality, conformal covariance, and certain irreducibility requirements.

If A is assumed to be completely rational [7], then there is a 1 to 1 correspondence [9] between Haag dual BCFTs associated with a given chiral net A, and non-local chiral extensions B of A such that the net of inclusions $A(I) \subset B(I)$ is covariant, irreducible and relatively local, i.e., $A(I)$ commutes with $B(J)$ if I and J are disjoint. The correspondence is given by the simple relative commutant formula

$$B_+(O) = B(K)' \cap B(L)$$

where $O = I \times J$ as before, K is the open interval between I and J, and L is the interval spanned by I and J. Conversely,

$$B(L) = \bigvee_{I \subset L,\, J \subset L,\, I > J} B_+(O).$$

BCFTs which are not Haag dual are always intermediate between A_+ and a Haag dual BCFT.

The classification of local BCFTs on the two-dimensional halfspace is thus reduced to the classification of non-local chiral extensions, which in turn [8] amounts to the classification of Q-systems (Frobenius algebras) in the C^* tensor category of the superselection sectors [2] of A. The chiral nets $A = \mathrm{Vir}_c$ defined by the stress-energy tensor (Virasoro algebra) with $c < 1$ are known to be completely rational, so the classification program just outlined can be performed.

Local chiral extensions of Vir_c with $c < 1$ have a direct interpretation as local QFT models of their own. Their classification has been achieved previously ([5], see Remark 2.3) by imposing an additional condition [8] on the Q-system involving the braided structure (statistics [2]) of the tensor category. Of course, the present non-local classification contains the local one.

As in [5] we exploit the fact that the tensor subcategories of the "horizontal" and of the "vertical" superselection sectors of Vir_c with $c < 1$ are isomorphic with the tensor categories of the superselection sectors of $SU(2)$ current algebras. (The braiding is different, however.) We therefore first classify the Q-systems in the latter categories (Sect. 1), and then proceed from Q-systems in the subcategories to Q-systems in the tensor categories of all sectors of Vir_c (Sect. 2). Thanks to a cohomological triviality result [6], the classification problem simplifies considerably, and essentially
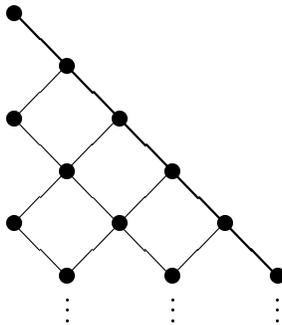
2

Figure 1: The Bratteli diagram of the first row

reduces to a combinatorial problem involving the Bratteli diagrams associated with the local subfactors $A(I) \subset B(I)$, combined with a "numerological" argument concerning Perron-Frobenius eigenvalues.

In the last section, we determine the vacuum Hilbert spaces of the non-local extensions and of the associated BCFT's thus classified.

2 Classification of irreducible non-local extensions of the $SU(2)_k$-nets

As an easy preliminary, we first classify all irreducible, possibly non-local, extensions of the $SU(2)_k$-nets on the circle. Consider the representation category of the $SU(2)_k$-net and label the irreducible DHR sectors as $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_k$ as usual. (The vacuum sector is labeled as λ_0.) Label this net as A and an irreducible extension as B. Since A is completely rational in the sense of [7], the index $[B : A]$ is automatically finite by [5, Proposition 2.3]. We need to classify the irreducible B-A sectors $_B\iota_A$, where ι is the inclusion map. Note that the A-A sector $_A\bar{\iota}\iota_A$ gives the dual canonical endomorphism of the inclusion and this decomposes into a direct sum of λ_j by [8]. Suppose we have such a sector $_B\iota_A$, and consider the following sequence of commuting squares.

$$
\begin{array}{ccccccccc}
\mathrm{End}(_A\mathrm{id}_A) & \subset & \mathrm{End}(_A\lambda_{1A}) & \subset & \mathrm{End}(_A\lambda_{1A}^2) & \subset & \mathrm{End}(_A\lambda_{1A}^3) & \subset & \cdots \\
\cap & & \cap & & \cap & & \cap & & \\
\mathrm{End}(_B\iota_A) & \subset & \mathrm{End}(_B\iota\lambda_{1A}) & \subset & \mathrm{End}(_B\iota\lambda_{1A}^2) & \subset & \mathrm{End}(_B\iota\lambda_{1A}^3) & \subset & \cdots
\end{array}
$$

The Bratteli diagram of the first row arises from reflections of the Dynkin diagram A_{k+1} as in usual subfactor theory, that is, it looks like Fig. 1, where each vertex is labeled with irreducible sectors appearing in the irreducible decomposition of λ_1^k for $k = 0, 1, 2, \ldots$. (See [3, Section 9.6] for appearance of such graphs in subfactor theory.)
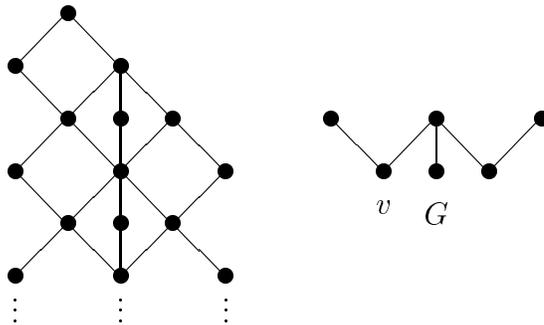
Figure 2: The Bratteli diagram of the second row

The Bratteli diagram of the second row also arises from reflections of some Dynkin diagrams having the same Coxeter number as A_{k+1} and starts with a single vertex because of the irreducibility of $_B\iota_A$. This gives a bipartite graph G, one of the A-D-E Dynkin diagram and its initial vertex v as an invariant of ι, but the vertex v is determined only up to the graph automorphism, so we denote the orbit of a vertex v under such automorphisms by $[v]$. (Note that the Dynkin diagrams A_n, D_n, and E_6 have non-trivial graph automorphisms of order 2.) Also note that the graph G is bipartite by definition. A B-A sector corresponding to an even vertex of G might be equivalent to another B-A sector corresponding to an odd vertex of G. Later it turns out that this case does not occur in the $SU(2)_k$-case, but it does occur in the Virasoro case below. Fig. 2 shows an example of G and v where G is the Dynkin diagram E_6.

Theorem 2.1. *The pair $(G, [v])$ gives a complete invariant for irreducible extensions of nets $SU(2)_k$, and an arbitrary pair $(G, [v])$ arises as an invariant of some extension.*

Proof As in the proof of [1, Proposition A.3], we know that the paragroup generated by ι is uniquely determined by $(G, [v])$ and it is isomorphic to the paragroup of the Goodman-de la Harpe-Jones subfactor given by $(G, [v])$, which was defined in [4, Section 4.5]. Then we obtain uniqueness of the Q-system for the extension, up to unitary equivalence, as in [6, Theorem 5.3]. (We considered only a local extension of $SU(2)_{28}$ corresponding to E_8 and its vertex having the smallest Perron-Frobenius eigenvector entry there, but the same method works for any $(G, [v])$.)

Any combination of $(G, [v])$ is possible as in the proof of [1, Lemma A.1]. (We considered only the case of E_7 and its vertex having the smallest Perron-Frobenius eigenvector entry there, but the same method works for any $(G, [v])$. $\square$

Remark 2.2. Note that the pair $(G, [v])$ uniquely corresponds to an (isomorphism class of) irreducible Goodman-de la Harpe-Jones subfactor. So we may say that irreducible extensions of nets $SU(2)_k$ are labeled with irreducible Goodman-de la Harpe-Jones subfactors $N \subset M$ such that the inclusions $A(I) \subset B(I)$ of the localized algebras are isomorphic to $N \subset M$ tensored with a type III factor.

3   Classification of non-local extensions of the Virasoro nets Vir_c with $c < 1$

Let A be the Virasoro net with central charge $c < 1$. As studied in [5, Section 3], it is a completely rational net. We would like to classify all, possibly non-local, irreducible extensions of this net. For $c = 1 - 6/m(m + 1)$, $m = 3, 4, 5, \ldots$, we label the irreducible DHR sectors of the net A as follows. We have $\sigma_{j,k}$, $j = 0, 1, \ldots, m-2$, $k = 0, 1, \ldots, m - 1$, with identification of $\sigma_{j,k} = \sigma_{m-2-j,m-1-k}$. (Our notation $\sigma_{j,k}$ corresponds to $\lambda_{j+1,k+1}$ in [5, Section 3].) Our labeling gives that the identity sector is $\sigma_{0,0}$ and the statistical dimensions of $\sigma_{1,0}$ and $\sigma_{0,1}$ are $2\cos(\pi/m)$ and $2\cos(\pi/(m+1))$, respectively. We have $m(m-1)/2$ irreducible DHR sectors. We again need to classify the irreducible B-A sectors ${}_B\iota_A$, where ι is the inclusion map. Take such ${}_B\iota_A$ for a fixed Vir_c with $c = 1 - 6/m(m + 1)$ and we obtain an invariant as follows.

Consider the following sequence of commuting squares as in the Section 2.

$$
\begin{array}{ccccccccc}
\mathrm{End}({}_A\mathrm{id}_A) & \subset & \mathrm{End}({}_A\sigma_{1,0\,A}) & \subset & \mathrm{End}({}_A\sigma_{1,0\,A}^2) & \subset & \mathrm{End}({}_A\sigma_{1,0\,A}^3) & \subset & \cdots \\
\cap & & \cap & & \cap & & \cap & & \\
\mathrm{End}({}_B\iota_A) & \subset & \mathrm{End}({}_B\iota\sigma_{1,0\,A}) & \subset & \mathrm{End}({}_B\iota\sigma_{1,0\,A}^2) & \subset & \mathrm{End}({}_B\iota\sigma_{1,0\,A}^3) & \subset & \cdots
\end{array}
$$

From the Bratteli diagram of the second row, we obtain a graph G_1 and its vertex v_1 as in Section 2. The graph G_1 is one of the A-D-E Dynkin diagrams and has the Coxeter number m. We also use $\sigma_{0,1}$ instead of $\sigma_{1,0}$ in this procedure and obtain a graph G_2 and its vertex v_2. The graph G_2 is one of the A-D-E Dynkin diagrams and has the Coxeter number $m + 1$. The quadruple $(G_1, [v_1], G_2, [v_2])$ is an invariant for ι. (The notation $[\cdot]$ means the orbit under the graph automorphisms as in Section 2.) Note that one of the graphs G_1, G_2 must be of type A because the D and E diagrams have even Coxeter numbers. We then prove the following classification theorem.

Theorem 3.1. *The quadruple $(G_1, [v_1], G_2, [v_2])$ gives a complete invariant for irreducible extensions of nets Vir_c, and an arbitrary quadruple, subject to the conditions on the Coxeter numbers as above, arises as an invariant of some extension.*

We will distinguish certain sectors by their dimensions. For this purpose, we need the following technical lemma on the values of dimensions, which we prove before the proof of the above theorem.

Lemma 3.2. *Let m be a positive odd integer and G one of the A-D-E Dynkin diagrams having a Coxeter number n with $|n-m| = 1$. Take a Perron-Frobenius eigenvector $(\mu_a)_a$ for the graph G, where a denotes a vertex of G. Set $d_j = \sin(j\pi/m)/\sin(\pi/m)$ for $j = 1, 2, \ldots, m - 1$. Then the sets*

$$\{\mu_a/\mu_b \mid a, b \text{ are vertices of } G\}$$

and

$$\{d_2, d_3, \ldots, d_{m-2}\}$$

are disjoint.

Proof If $m = 1, 3$, then the latter set is empty, so we may assume $m \geq 5$. Note that the value 1 is not in the latter set.

Suppose a number ω is in the intersection and we will derive a contradiction. Then ω is in the intersection of the cyclotomic fields $\mathbb{Q}(\exp(\pi i/m))$ and $\mathbb{Q}(\exp(\pi i/n))$, which is $\mathbb{Q}$ since $(2m, 2n) = 2$ and $\mathbb{Q}(\exp(2\pi i/2)) = \mathbb{Q}$. Suppose d_j is equal to this ω. We may and do assume $2 \leq j \leq (m-1)/2$. We have

$$\omega = \frac{\zeta^j - \zeta^{-j}}{\zeta - \zeta^{-1}} = \zeta^{j-1} + \zeta^{j-3} + \zeta^{j-5} + \cdots + \zeta^{-j+1},$$

where $\zeta = \exp(2\pi i/(2m))$.

First assume that j is even. We note $(m-2, 2m) = 1$ since m is odd. Then the map $\sigma : \zeta^k \mapsto \zeta^{k(m-2)}$ for $k = 0, 1, \ldots, 2m-1$ gives an element of the Galois group for the cyclotomic extension $\mathbb{Q} \subset \mathbb{Q}(\zeta)$. We have

$$\sigma(\omega) = \zeta^{(j-1)(m-2)} + \zeta^{(j-3)(m-2)} + \zeta^{(j-5)(m-2)} + \cdots + \zeta^{(-j+1)(m-2)}.$$

Here the set

$$\{\zeta^{(j-1)(m-2)}, \zeta^{(j-3)(m-2)}, \zeta^{(j-5)(m-2)}, \ldots, \zeta^{(-j+1)(m-2)}\}$$

has j distinct roots of unity containing ζ^{m-2} and it is a subset of

$$Z = \{\zeta^k \mid k = 1, 3, 5, \ldots, 2m-1\}.$$

The set

$$\{\zeta^{j-1}, \zeta^{j-3}, \zeta^{j-5}, \ldots, \zeta^{-j+1}\}$$

is the unique subset having j distinct elements of Z that attains the maximum of $\text{Re} \sum_{k=1}^{j} \alpha_k$ among all subsets $\{\alpha_1, \alpha_2, \ldots, \alpha_j\}$ having j distinct elements of Z. However, we have $m > 3$, which implies $j \leq (m-1)/2 < m-2$, thus the complex number ζ^{m-2} is not in the above unique set, and thus the sum

$$\zeta^{(j-1)(m-2)} + \zeta^{(j-3)(m-2)} + \zeta^{(j-5)(m-2)} + \cdots + \zeta^{(-j+1)(m-2)}$$

cannot be equal to

$$\zeta^{j-1} + \zeta^{j-3} + \zeta^{j-5} + \cdots + \zeta^{-j+1},$$

which shows that ω is not fixed by σ, so ω is not an element of $\mathbb{Q}$, which is a contradiction.

Next we assume that j is odd. We now have that

$$\zeta^{2(j-1)/2} + \zeta^{2(j-3)/2} + \cdots + \zeta^{2(1-j)/2} \in \mathbb{Q}.$$

Since $(m, (m-1)/2) = 1$, the map $\sigma : \zeta^{2k} \mapsto \zeta^{k(m-1)}$ for $k = 0, 1, \ldots, m-1$ gives an element of the Galois group for the cyclotomic extension $\mathbb{Q} \subset \mathbb{Q}(\zeta^2)$. Since $m - 1 > j - 1$, $\sigma(\omega)$ contains a term ζ^{m-1} which does not appear in ω. Then by an argument similar to the above case of even j, we obtain a contradiction. $\square$
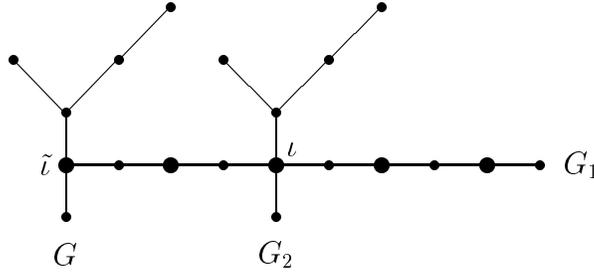
Figure 3: The graphs G_1, G_2, G

We now start the proof of Theorem 3.1.

Proof Without loss of generality, we may assume that m is odd and hence that the graph G_1 is A_{m-1}. (Otherwise, the graph G_2 is A_m, and we can switch the symmetric roles of G_1 and G_2.) Note that a sector corresponding to an even vertex of A_{m-1} can be equivalent to another sector corresponding to an odd vertex of A_{m-1}.

The tensor category having the irreducible objects $\{\sigma_{0,0}, \sigma_{1,0}, \ldots, \sigma_{m-2,0}\}$ is isomorphic to the representation category of $SU(2)_{m-2}$, thus each of the irreducible objects is labeled with a vertex of the Dynkin diagram A_{m-1}. Let $\sigma_{j,0}$ be one of the two sectors corresponding to $[v_1]$. We choose j to be even, and then j is uniquely determined. Let Δ be the set of the irreducible B-A sectors arising from the decomposition of $_B\iota\sigma_{1,0}^{2k}{}_A$ for all k. Note that Δ is a subset of the vertices of G_1. Let $\tilde{\iota}$ be one of the B-A sectors in Δ having the smallest dimension. By the Perron-Frobenius theory and the definition of the graph G_1, which is now A_{m-1}, we know that such $\tilde{\iota}$ is uniquely determined and that the set

$$\{d(\lambda)/d(\tilde{\iota}) \mid \lambda \in \Delta\}$$

is equal to

$$\{\sin(k\pi/m)/\sin(\pi/m) \mid k = 1, 2, \ldots, (m-1)/2\}.$$

The situation is illustrated in Fig. 3 where we also have the graph G which will be defined below. The vertices corresponding to the elements in Δ are represented as larger circles.

Now we consider the Bratteli diagram for $\text{End}(\tilde{\iota}\sigma_{0,1}^n)$ for $n = 0, 1, 2, \ldots$, and obtain a graph G and an orbit $[v]$ whose reflection gives this Bratteli diagram. The graph G is one of the A-D-E Dynkin diagrams and its Coxeter number differs from m by 1.

We want to show that the pairs $(G_2, [v_2])$ and $(G, [v])$ are equal as follows. We first claim that the irreducible decomposition of $\bar{\tilde{\iota}}\tilde{\iota}$ contains only sectors among $\sigma_{0,k}$, $k = 0, 1, \ldots, m-1$. Suppose that $\bar{\tilde{\iota}}\tilde{\iota}$ contains $\sigma_{2l,k}$ with $l > 0$ on the contrary. By the Frobenius reciprocity, we have

$$0 < \langle \bar{\tilde{\iota}}\tilde{\iota}, \sigma_{2l,k} \rangle = \langle \tilde{\iota}\sigma_{0,k}, \tilde{\iota}\sigma_{2l,0} \rangle.$$

By the above description of the graph G_1, we know that $\tilde{\iota}\sigma_{2l,0}$ is irreducible and distinct from $\tilde{\iota}$. The assumption that $\tilde{\iota}\sigma_{2l,0}$ appears in the irreducible decomposition of $\tilde{\iota}\sigma_{0,k}$ means that the graphs G and G_1 have a common vertex other than $\tilde{\iota}$ and this is impossible by Lemma 3.2. We have thus proved that the irreducible decomposition of $\bar{\tilde{\iota}}\tilde{\iota}$ contains only sectors among $\sigma_{0,k}$, $k = 0, 1, \ldots, m - 1$.

We know $\tilde{\iota}\sigma_{j,0}$ and ι are equivalent sectors since they both are irreducible. To show $(G_2, [v_2]) = (G, [v])$, we therefore need to compare the irreducible decompositions of $\tilde{\iota}\sigma_{j,0}\sigma_{0,1}^k$ and $\tilde{\iota}\sigma_{0,1}^k$ for $k = 0, 1, 2, \ldots$. Suppose that λ is an irreducible sector appearing in the decomposition of $\tilde{\iota}\sigma_{0,1}^k$ for some k. We have

$$\langle \lambda\sigma_{j,0}, \lambda\sigma_{j,0} \rangle = \langle \bar{\lambda}\lambda, \sigma_{j,0}^2 \rangle.$$

Now the decomposition of $\bar{\lambda}\lambda$ contains only sectors among $\sigma_{0,l}$, $l = 0, 1, \ldots, m - 1$ as above. Thus the only irreducible sector appearing in decompositions of both $\bar{\lambda}\lambda$ and $\sigma_{j,0}^2$ is the identity sector, which appears exactly once in the both. We conclude that $\lambda\sigma_{j,0}$ is also irreducible. Thus the irreducible decompositions of $\tilde{\iota}\sigma_{j,0}\sigma_{0,1}^k$ and $\tilde{\iota}\sigma_{0,1}^k$ for $k = 0, 1, 2, \ldots$ are described by the same Bratteli diagram and $(G, [v])$ and $(G_2, [v_2])$ are equal.

Then ${}_A\bar{\tilde{\iota}}\tilde{\iota}_A$ decomposes into some irreducible sectors among $\sigma_{0,0}, \sigma_{0,1}, \ldots, \sigma_{0,m-1}$. Since the tensor category having the irreducible objects

$$\{\sigma_{0,0}, \sigma_{0,1}, \ldots, \sigma_{0,m-1}\}$$

is isomorphic to the representation category of $SU(2)_{m-1}$, we obtain uniqueness of $\tilde{\iota}$ for a given $(G, [v])$ hence $(G_2, [v_2])$. Then $\iota = \tilde{\iota}\sigma_{j,0}$ determines a Q-system uniquely, up to unitary equivalence.

We next prove a realization of a given $(G_1, [v_1], G_2, [v_2])$. We continue to assume that G_1 is A_{m-1}. Let $\sigma_{j,0}$ be one of the two sectors corresponding to $[v_1]$ as above. Using the tensor category having the irreducible objects $\{\sigma_{0,0}, \sigma_{0,1}, \ldots, \sigma_{0,m-1}\}$, we have $\tilde{\iota}$ corresponding to $(G_2, [v_2])$ as in the proof of Theorem 2.1. Set $\iota = \tilde{\iota}\sigma_{j,0}$. Then one can verify that this ι produces the quadruple $(G_1, [v_1], G_2, [v_2])$ by the same argument as in the above one showing $G = G_2$. $\square$

Remark 3.3. By [5, Theorem 4.1], we already know that the local extensions among the above classification are labeled with (A_{n-1}, A_n), (A_{4n}, D_{2n+2}), (D_{2n+2}, A_{4n+2}), (A_{10}, E_6), (E_6, A_{12}), (A_{28}, E_8) and (E_8, A_{30}) for (G_1, G_2) and the vertices v_1, v_2 are those having the smallest Perron-Frobenius eigenvector entries.

Remark 3.4. As in Remark 2.2, we may say that irreducible extensions of the Virasoro nets with $c < 1$ are labeled with pairs of irreducible Goodman-de la Harpe-Jones subfactors having the Coxeter numbers differing by 1.

Remark 3.5. The graphs G_1 and G_2 are by definition bipartite, thus excluding the tadpole graphs which also have Frobenius norm < 2. Tadpole diagrams arise by pairwise identification of the vertices of A_m diagrams when $m = 2n$ is even. Indeed,
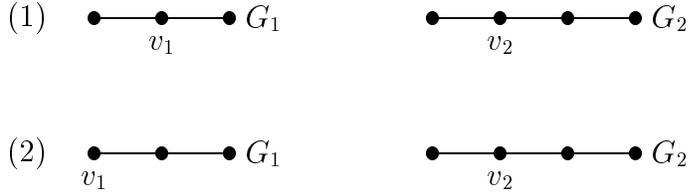
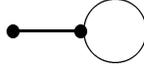Figure 4: Two cases of $(G_1, [v_1], G_2, [v_2])$ for $m = 4$, $c = 7/10$

Figure 5: The tadpole graph T_2

when $\iota : A \hookrightarrow B$ equals $\sigma_{n-1,j} : A \to A$, the even vertices of $G_2 = A_{2n}$ pairwise coincide as B-A sectors with the odd vertices, so that the fusion graph for multiplication by $\sigma_{0,1}$ is T_n. The invariant G_2 in these cases is A_m, nevertheless.

As an example, consider the case $m = 4$, that is, $c = 7/10$. In this case, we have six irreducible DHR sectors for the net $\mathrm{Vir}_{7/10}$. The graphs G_1 and G_2 are automatically A_3 and A_4, respectively, so we have four possibilities for the invariant $(G_1, [v_1], G_2, [v_2])$. If $(G_1, [v_1], G_2, [v_2])$ is as in case (1) of Fig. 4, then the sector ι is given by $\sigma_{1,1}$, thus the four vertices of the graph A_4 give only two mutually inequivalent B-A sectors. That is, the fusion graph of the B-A sectors for multiplication by $\sigma_{0,1}$ is the tadpole graph as in Fig. 5.

In case (2) of Fig. 4 we have four mutually inequivalent B-A sectors for the graph $G_2 = A_4$ for the sector ι given by $\sigma_{0,1}$, and the fusion graph is also A_4.

4   The canonical endomorphism

We want to determine, viewed as a representation of the subtheory $A = \mathrm{Vir}_c$, the vacuum Hilbert space of the local boundary conformal QFT associated with each of the non-local extensions B, classified in the previous section. This representation is given by a DHR endomorphism θ of A whose restriction to a local algebra $A(I)$ (where θ is localized in the interval I) coincides with the canonical endomorphism $\bar{\iota}\iota$ of the subfactors $A(I) \subset B(I)$ classified above. We are therefore interested in the computation of $\bar{\iota}\iota$.
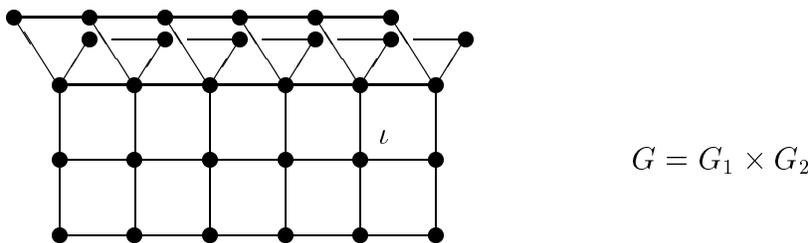
9

$$G = G_1 \times G_2$$

Figure 6: The graph $G_1 \times G_2$ with $G_1 = A_6$ and $G_2 = D_5$. Different vertices may represent the same B-A-sector. ι may be any vertex of G.

By the equality of local and global intertwiners, and by reciprocity, the multiplicity of each irreducible DHR sector σ within θ equals the multiplicity of ι within $\iota\sigma$. We therefore need to control the decomposition of $\iota\sigma$ into irreducibles ("fusion") for all DHR sectors. Because every irreducible sector is a product $\sigma_{j,k} = \sigma_{j,0}\sigma_{0,k}$, and $\sigma_{j,0}$ and $\sigma_{0,k}$ are obtained from the generators $\sigma_{1,0}$ and $\sigma_{0,1}$ by the recursion $\sigma_{0,k+1} = \sigma_{0,k}\sigma_{0,1} \ominus \sigma_{0,k-1}$ and likewise for $\sigma_{j+1,0}$, it suffices to control the fusion with the generators.

We know from the preceding section that the fusion of ι with the generators $\sigma_{1,0}$ and $\sigma_{0,1}$ separately can be described in terms of the two bi-partite graphs G_1 and G_2 such that the vertices of the graphs represent irreducible B-A-sectors and two vertices are linked if the corresponding sectors are connected by the generator. ι corresponds to a distinguished vertex in both graphs. Moreover, we have seen that the fusion of ι with both generators can be described by the "product graph" $G = G_1 \times G_2$ with vertices $\lambda = (v_1 \in G_1, v_2 \in G_2)$ and "horizontal" edges linking (v_1, v_2) with (v_1', v_2) if v_1 and v_1' are linked in G_1, and likewise for "vertical" edges according to the graph G_2. Again, the vertices λ represent irreducible B-A-sectors, and ι is a distinguished vertex of the product graph. See Fig. 6 for an example.

From the product graph G, the fusion of each of its vertices with any DHR sector can be computed in terms of vertices of G, i.e., $\lambda\sigma$ can be decomposed into irreducibles represented by the vertices of G. But different vertices may represent identical B-A-sectors; we only know that within each horizontal or vertical subgraph, the even vertices represent pairwise inequivalent sectors, and so do the odd vertices, cf. Remark 3.5. In order to compute the canonical endomorphism, we have to determine all identifications between vertices of G as B-A-sectors.

We continue to assume that m is odd and hence $G_1 = A_{m-1}$. The Coxeter number $m' = 2n$ of G_2 is either $m + 1$ or $m - 1$. We exploit the fact that $\sigma_{m-2,0}$ and $\sigma_{0,m'-2}$ represent the same DHR sector τ, and that τ is simple (it has dimension 1).

10

Hence fusion with τ, as a horizontal sector $\sigma_{m-2,0}$, yields an automorphism α_1 of the graph G_1 such that $\alpha_1(v_1) = v_1\sigma_{m-2,0}$, and, as a vertical sector, similarly yields an automorphism α_2 of G_2. It follows that the vertices $\lambda = (v_1, v_2)$ and $\alpha(\lambda) = (\alpha_1(v_1), \alpha_2(v_2))$ of the product graph represent the same sectors.

Because τ connects even vertices of G_1 with odd ones, α_1 must be the unique non-trivial automorphism of G_1. To determine α_2, one may use the above-mentioned recursion to compute the fusion of the vertices of G_2 with $\tau = \sigma_{0,m'-2}$. We find that α_2 is the unique non-trivial automorphism if G_2 is either an A graph or E_6 or D_{2n+1}, and it is trivial if G_2 is E_7, E_8, or D_{2n}.

We now claim that the identifications due to α give *all* pairs of vertices of G which represent the same B-A-sector.

Proposition 4.1. *The graph*

$$(G_1 \times G_2)/(\alpha_1 \times \alpha_2)$$

is the fusion graph of ι with respect to $\sigma_{1,0}$ and $\sigma_{0,1}$, i.e., its vertices represent inequivalent irreducible B-A-sectors, and its horizontal and vertical edges correspond to fusion with the two generators.

Proof By the Perron-Frobenius theory, the dimensions of the B-A-sectors represented by the vertices $\lambda = (v_1, v_2)$ of G are common multiples of $\nu(v_1)\mu(v_2)$ where $\nu(v_1)$ and $\mu(v_2)$ are the components of the Perron-Frobenius eigenvectors ν of G_1 and μ of G_2. Let now $\lambda = (v_1, v_2)$ and $\lambda' = (v_1', v_2')$ be two vertices of G which represent the same B-A-sector. Then clearly

$$\nu(v_1)\mu(v_2) = \nu(v_1')\mu(v_2').$$

If v_1 it at distance j from an extremal vertex $\tilde{v}_1$ of G_1, then $\tilde{\lambda} = (\tilde{v}_1, v_2)$ is a subsector of $\lambda\sigma_{j,0}$ and consequently of $\lambda'\sigma_{j,0}$. Hence $\tilde{\lambda}$ is equivalent to some subsector $(\tilde{v}_1', v_2')$ of $\lambda'\sigma_{j,0}$, implying $\mu(v_2)/\mu(v_2') = \nu(\tilde{v}_1')/\nu(\tilde{v}_1) = d_k$ for some k. Lemma 3.2 tells us that this is only possible if $d_k = 1$. It follows that $\mu(v_2') = \mu(v_2)$ and $\nu(v_1') = \nu(v_1)$.

This means in particular that $v_1' = v_1$ or $v_1' = \alpha_1(v_1)$, and that v_2 and v_2' and $\alpha_2(v_2)$ are all even or all odd. If $v_1' = v_1$, then λ and λ' are two even or two odd vertices within the same vertical subgraph representing the same sector. This is only possible if $\lambda = \lambda'$. If on the other hand $v_1' = \alpha_1(v_1)$, then the same argument applies to $\alpha(\lambda)$ and λ', giving $\lambda' = \alpha(\lambda)$. $\qquad\square$

Having determined the fusion graph, it is now straightforward to compute (as described above) the canonical endomorphism for every possible position of ι as a distinguished vertex of the fusion graph, and hence to determine the vacuum Hilbert space for each local boundary conformal QFT with $c < 1$.

We display below the canonical endomorphism $\theta_{\tilde{v}_1, \tilde{v}_2}$ whenever $\tilde{v}_1$ and $\tilde{v}_2$ are extremal vertices of G_1 and G_2. All other cases are then easily obtained by the following argument: If v_1 is at distance j from an extremal vertex $\tilde{v}_1$ of G_1, then $v_1 = \tilde{v}_1\sigma_{j,0}$.

If v_2 is at distance k from the extremal vertex $\tilde{v}_2$ on the same "leg" of G_2, then $v_2 = \tilde{v}_2 \sigma_{0,k}$. (If v_2 is the trivalent vertex of the D or E graphs, then this is true for each of the three legs.) It then follows that $(v_1, v_2) = (\tilde{v}_1, \tilde{v}_2)\sigma_{j,k}$, and hence

$$\theta_{v_1,v_2} = \theta_{\tilde{v}_1,\tilde{v}_2}\sigma_{j,k}^2.$$

The canonical endomorphisms $\theta_{\tilde{v}_1,\tilde{v}_2}$ for all pairs of extremal vertices of G_1 and G_2 are listed in the following table.

G_2	m'	dist.	$\theta_{\tilde{v}_1,\tilde{v}_2}$
A_n	$n+1$	—	$\sigma_{0,0}$
D_n	$2n-2$	1	$\sigma_{0,0} \oplus \sigma_{0,4} \oplus \sigma_{0,8} \oplus \ldots \oplus \sigma_{0,4[n/2]-4}$
D_n	$2n-2$	$n-3$	$\sigma_{0,0} \oplus \sigma_{0,2n-4}$
E_6	12	1	$\sigma_{0,0} \oplus \sigma_{0,4} \oplus \sigma_{0,6} \oplus \sigma_{0,10}$
E_6	12	2	$\sigma_{0,0} \oplus \sigma_{0,6}$
E_7	18	1	$\sigma_{0,0} \oplus \sigma_{0,4} \oplus \sigma_{0,6} \oplus \sigma_{0,8} \oplus \sigma_{0,10} \oplus \sigma_{0,12} \oplus \sigma_{0,16}$
E_7	18	2	$\sigma_{0,0} \oplus \sigma_{0,6} \oplus \sigma_{0,10} \oplus \sigma_{0,16}$
E_7	18	3	$\sigma_{0,0} \oplus \sigma_{0,8} \oplus \sigma_{0,16}$
E_8	30	1	$\sigma_{0,0} \oplus \sigma_{0,4} \oplus \sigma_{0,6} \oplus \sigma_{0,8} \oplus 2\sigma_{0,10} \oplus \sigma_{0,12} \oplus 2\sigma_{0,14} \oplus$ $\oplus\sigma_{0,16} \oplus 2\sigma_{0,18} \oplus \sigma_{0,20} \oplus \sigma_{0,22} \oplus \sigma_{0,24} \oplus \sigma_{0,28}$
E_8	30	2	$\sigma_{0,0} \oplus \sigma_{0,6} \oplus \sigma_{0,10} \oplus \sigma_{0,12} \oplus \sigma_{0,16} \oplus \sigma_{0,18} \oplus \sigma_{0,22} \oplus \sigma_{0,28}$
E_8	30	4	$\sigma_{0,0} \oplus \sigma_{0,10} \oplus \sigma_{0,18} \oplus \sigma_{0,28}$

Table 4.1. The canonical endomorphisms $\theta_{\tilde{v}_1,\tilde{v}_2}$ for all pairs of extremal vertices of G_1 and G_2. The entry in the third column indicates the distance of $\tilde{v}_2$ from the trivalent vertex, i.e., the length of the "leg" of G_2 on which $\tilde{v}_2$ is the extremal vertex.

The local chiral extensions classified earlier [5] are precisely those cases where G_2 is A, D_{2n}, E_6, or E_8, and both v_1 and v_2 are extremal vertices (on the respective longest leg in the D and E cases).

In the non-local cases, the local algebras of the associated BCFT on the half-space are the relative commutants as described in the introduction. Note that, in order to determine the resulting factorizing chiral charge structure [9] of the local fields, more detailed information about the DHR category and the Q-system is needed, than the simple combinatorial data exploited in this work.

References

[1] J. Böckenhauer, D. E. Evans & Y. Kawahigashi, *Chiral structure of modular invariants for subfactors*, Commun. Math. Phys. **210** (2000) 733–784.

[2] S. Doplicher, R. Haag & J. E. Roberts, *Local observables and particle statistics*, I. Commun. Math. Phys. **23** (1971) 199–230; II. **35** (1974) 49–85.

[3] D. E. Evans & Y. Kawahigashi, "Quantum symmetries on operator algebras", Oxford University Press, 1998.

[4] F. Goodman, P. de la Harpe & V. F. R. Jones, *Coxeter graphs and towers of algebras*, MSRI publications 14, Berlin, Springer, 1989

[5] Y. Kawahigashi & R. Longo, *Classification of local conformal nets. Case $c < 1$*, Ann. of Math. **160** (2004), 493–522. math-ph/0201015.

[6] Y. Kawahigashi & R. Longo, *Classification of two-dimensional local conformal nets with $c < 1$ and 2-cohomology vanishing for tensor categories*, Commun. Math. Phys. **244** (2004) 63–97. math-ph/0304022.

[7] Y. Kawahigashi, R. Longo & M. Müger, *Multi-interval subfactors and modularity of representations in conformal field theory*, Commun. Math. Phys. **219** (2001) 631–669.

[8] R. Longo & K.-H. Rehren, *Nets of subfactors*, Rev. Math. Phys. **7** (1995) 567–597.

[9] R. Longo & K.-H. Rehren, *Local fields in boundary CFT*, Rev. Math. Phys. **16** (2004) 909–960.