Classification of Two-dimensional Local Conformal Nets with $c < 1$ and 2-cohomology Vanishing for Tensor Categories

YASUYUKI KAWAHIGASHI[*]

Department of Mathematical Sciences
University of Tokyo, Komaba, Tokyo, 153-8914, Japan
e-mail: `yasuyuki@ms.u-tokyo.ac.jp`

ROBERTO LONGO[†]

Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1, I-00133 Roma, Italy
e-mail: `longo@mat.uniroma2.it`

April 14, 2003

Abstract

We classify two-dimensional local conformal nets with parity symmetry and central charge less than 1, up to isomorphism. The maximal ones are in a bijective correspondence with the pairs of *A-D-E* Dynkin diagrams with the difference of their Coxeter numbers equal to 1. In our previous classification of one-dimensional local conformal nets, Dynkin diagrams D_{2n+1} and E_7 do not appear, but now they do appear in this classification of two-dimensional local conformal nets. Such nets are also characterized as two-dimensional local conformal nets with μ-index equal to 1 and central charge less than 1. Our main tool, in addition to our previous classification results for one-dimensional nets, is 2-cohomology vanishing for certain tensor categories related to the Virasoro tensor categories with central charge less than 1.

1 Introduction

The subject of Conformal Quantum Field Theory is particularly interesting in two spacetime dimensions and has indeed been intensively studied in the last two decades with important motivations from Physics (see e.g. [11]) and Mathematics (see e.g. [14]). Basically the richness of structure is due to the fact that conformal group (with respect to the Minkowskian metric) is infinite dimensional in $1+1$ dimensions.

Already at the early stage of investigation, it was realized that such infinite dimensional symmetry group puts rigid constrains on structure and the problem of classification of all models was posed and considered as a major aim. Indeed many important results in this direction were obtained, in particular the central charge $c > 0$, an intrinsic quantum label associated with each model, was shown to split in a discrete range $c < 1$ and a continuos one $c \geq 1$, see [2, 17, 19] refs in [19].

The main purpose of this paper is to achieve a complete classification of the two-dimensional conformal models in the discrete series. In order to formulate such a statement in a precise manner, we need to explain our setting.

The essential, intrinsic structure of a given model is described by a net $\mathcal{A}$ on the two-dimensional Minkowski spacetime $\mathcal{M}$ [22]. With each double cone $\mathcal{O}$ (an open region which is the intersection of the past of one point and the future of a second point) one associates the von Neumann algebra $\mathcal{A}(\mathcal{O})$ generated by the observables localized in $\mathcal{O}$ (say smeared fields integrated with test functions with support in $\mathcal{O}$).

The net $\mathcal{A} : \mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ is then local and covariant with respect to the conformal group. One may restrict $\mathcal{A}$ to the two light rays $x \pm t = 0$ and obtain two local conformal nets $\mathcal{A}_\pm$ on $\mathbb{R}$, hence on its one point compactification S^1. So we have an irreducible two-dimensional subnet

$$\mathcal{B}(\mathcal{O}) \equiv \mathcal{A}_+(I_+) \otimes \mathcal{A}_-(I_-) \subset \mathcal{A}(\mathcal{O}) \ ,$$

where $\mathcal{O} = I_+ \times I_-$ is the double cone associated with the intervals $I_\pm$ of the light rays. The structure of $\mathcal{A}$, thus the classification of local conformal nets, splits in the following two points:

- The classification of local conformal nets on S^1.

- The classification of irreducible local extension of chiral conformal nets.

Here a chiral net is a net that splits in tensor product of two one-dimensional nets on the light rays. Now the conformal group of $\mathcal{M}$ is $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$ [1] thus, restricting the projective unitary covariance representation to the two copies of $\mathrm{Diff}(S^1)$, we get Virasoro nets $\mathrm{Vir}_{c_\pm} \subset \mathcal{A}_\pm$ with central charge $c_\pm$. If there is a parity symmetry, then $c_+ = c_-$, so we may talk of the central charge $c \equiv c_\pm$ of $\mathcal{A}$. If $c < 1$, it turns out by that $\mathcal{A}$ is completely rational [32] and the subnet $\mathrm{Vir}_c \otimes \mathrm{Vir}_c \subset \mathcal{A}$ has finite Jones index, where $\mathrm{Vir}_c \otimes \mathrm{Vir}_c(\mathcal{O}) \equiv \mathrm{Vir}_c(I_+) \otimes \mathrm{Vir}_c(I_-)$.

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[1]More precisely $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$ is the conformal group of the Minkowskian torus $S^1 \times S^1$, the conformal completion of $\mathcal{M} = \mathbb{R} \times \mathbb{R}$ (light ray decomposition), and the covariance group is a central extension of $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$, see Sect. 2.

The classification of two-dimensional local conformal nets with central charge $c < 1$ and parity symmetry thus splits in the following two points:

(a) The classification of Virasoro nets Vir_c on S^1 with $c < 1$.

(b) The classification of irreducible local extensions with finite Jones index of the two-dimensional Virasoro net $\mathrm{Vir}_c \otimes \mathrm{Vir}_c$.

Point (a) has been completely achieved in our recent work [31]. The Virasoro nets on S^1 with central charge less than one are in bijective correspondence with the pairs of A-D_{2n}-$E_{6,8}$ Dynkin diagrams such that the difference of their Coxeter numbers is equal to 1. Among other important aspects of this classification, we mention here the occurrence of nets that are not realized as coset models, in contrast to a long standing expectation. (See Remarks after Theorem 7 of [34] on tihs point. Also, Carpi and Xu recently made a progress on classification for the case $c = 1$ in [10], [54], respectively.)

The aim of this paper is to pursue point (b). We shall obtain a complete classification of the two-dimensional local conformal nets (with parity) with central charge in the discrete series. To this end we first classify the maximal nets in this class. Maximality here means that the net does not admit any irreducible local conformal net extension. Maximality will turn out to be also equivalent to the triviality of the super-selection structure or to μ-index equal to one, that is Haag duality for disconnected union of finitely many double cones.

It is clear at this point that our methods mainly concern Operator Algebras, in particular Subfactor Theory, see [50]. Indeed this was already the case in our previous one-dimensional classification [31]. The use of von Neumann algebras not only provides a clear formulation of the problem, but also suggests the path to follow in the analysis.

Our strategy is the following. The dual canonical endomorphism of $\mathrm{Vir}_c \otimes \mathrm{Vir}_c \subset \mathcal{A}$ decomposes as

$$\theta = \bigoplus_{ij} Z_{ij} \rho_i \otimes \bar{\rho}_j \tag{1}$$

(i.e. the above is the restriction to $\mathrm{Vir}_c \otimes \mathrm{Vir}_c$ of the vacuum representation of $\mathcal{A}$), where $\{\rho_i\}_i$ are representatives of unitary equivalence classes of irreducible DHR endomorphisms of the net Vir_c.

Since $\mu_{\mathcal{A}} = 1$ it turns out, by using the results in [32], that the matrix Z is a modular invariant for the tensor category of representations of the Virasoro net Vir_c [41], and such modular invariants have been classified by Cappelli-Itzykson-Zuber [9].

We shall show that this map $\mathcal{A} \mapsto Z$ sets up a bijective correspondence between the the set of isomorphism classes of two-dimensional maximal local conformal nets with parity and central charge less than one on one hand and the list of Cappelli-Itzykson-Zuber modular invariant on the other hand.

We first prove that the correspondence $\mathcal{A} \mapsto Z$ is surjective. Indeed, by our previous work [31], Z can be realized by α-induction as in [5] for extensions of the

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Virasoro nets. (See [38, 52, 3, 6, 7, 4] for more on α-induction.) Then Rehren's results in [48] imply that θ defined as above (1) is the canonical endomorphism associated with a natural Q-system, and we have a corresponding local extension $\mathcal{A}$ of $\mathrm{Vir}_c \otimes \mathrm{Vir}_c$ and this produces the matrix Z in the above correspondence.

To show the injectivity of the correspondence note that, due to the work of Rehren [47], we have an inclusion

$$\mathrm{Vir}_c(I_+) \otimes \mathrm{Vir}_c(I_-) \subset \mathcal{A}_+(I_+) \otimes \mathcal{A}_-(I_-) \subset \mathcal{A}(\mathcal{O})$$

where $\mathcal{A}_+ \otimes \mathcal{A}_-$ is the maximal chiral subnet. By assumption, $\mathcal{A}_+$ and $\mathcal{A}_-$ are isomorphic with central charge $c < 1$, thus they are in the discrete series classified in [31]. Moreover Z determines uniquely the isomorphism class of $\mathcal{A}_\pm$ and an isomorphism π from a fusion rule of $\mathcal{A}_+$ onto that of $\mathcal{A}_-$ so that the dual canonical endomorphism λ on $\mathcal{A}_+ \otimes \mathcal{A}_-$ decomposes as

$$\lambda = \bigoplus_i \alpha_i \otimes \bar{\alpha}_{\pi(i)}, \tag{2}$$

where $\{\alpha_i\}_i$ is a system of irreducible DHR endomorphisms of $\mathcal{A}_+ = \mathcal{A}_-$.

If Z is a modular invariant of type I, the map π is trivial, so the dual canonical endomorphism has the same form of the Longo-Rehren endomorphism [38]. Thus the classification is reduced to classification of Q-systems in the sense of [36] having the canonical endomorphism of the form given by eq. (1). This type of classification of Q-systems, up to unitary equivalences, was studied by Izumi-Kosaki [27] as a subfactor analogue of 2-cohomology of (finite) groups. In our setting, we now have a 2-cohomology *group* of a tensor category, while the 2-cohomology of Izumi-Kosaki does not have a group structure in general. The group operation comes from a natural composition of 2-cocycles. Then the crucial point in our analysis is the vanishing of this 2-cohomology for certain tensor category as we will explain below, and this vanishing implies that the dual Q-system for the inclusion $\mathcal{A}_+ \otimes \mathcal{A}_- \subset \mathcal{A}$ has a standard dual canonical endomorphism as in the Longo-Rehren Q-system [38], namely $\mathcal{A}_+ \otimes \mathcal{A}_- \subset \mathcal{A}$ is the "quantum double" inclusion constructed in [38]. At this point, as we know the isomorphism class of $\mathcal{A}_\pm$ by our previous classification [31], it follows that the isomorphism class of $\mathcal{A}$ is determined by Z.

If the modular invariant is of type II, then π gives a non-trivial fusion rule automorphism, however π is actually associated with an automorphism of the tensor category acting non-trivially on irreducible objects [4]. We may then extend our arguments of 2-cohomology vanishing and deal also with this case. It turns out that the automorphism π is an automorphism of a *braided* tensor category.

We thus arrive at the following classification: the maximal local two-dimensional conformal nets with $c < 1$ and parity symmetry are in a bijective correspondence with the pairs of the A-D-E Dynkin diagrams such that the difference of their Coxeter numbers is equal to 1, namely Z is a modular invariant listed in Table 1. Note that Dynkin diagrams of type D_{2n+1} and E_7 do appear in the list of present classification of two-dimensional conformal nets, but they were absent in the one-dimensional classification list [31].

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Now, as we shall see, two-dimensional local conformal net $\mathcal{B}$ in the discrete series is a finite-index subnet of a maximal local conformal net $\mathcal{A}$. Moreover $\mathcal{A}$ and $\mathcal{B}$ have the same two-dimensional Virasoro subnet. Using this, we then obtain the classification of all local two-dimensional conformal nets with central charge less $c < 1$. The non-maximal ones correspond bijectively to the pairs $(\mathcal{T}, \alpha)$ where $\mathcal{T}$ is a proper sub-tensor category of the representation tensor category of Vir_c and α is an automorphism of $\mathcal{T}$. There are at most two automorphisms, thus two possible nets for a given $\mathcal{T}$. The complete list is given in Table 2.

As we have mentioned, a crucial point in our analysis is to show the uniqueness up to equivalence of the Q-system associated with the canonical endomorphism of the form (2) in our cases. To this end we consider a cohomology associated with a representation tensor category that we have to show to vanish in our case. Note that our 2-cohomology groups are generalization of the usual 2-cohomology groups of finite groups, so they certainly do not vanish in general.

Before concluding this introduction we make explicit that our classification applies as well to the local conformal nets with central charge less than one on other two-dimensional spacetimes. Indeed if $\mathcal{N}$ is two-dimensional spacetime that is conformally equivalent to $\mathcal{M}$, namely conformally diffeomorphic to a subregion on the Einstein cylinder $S^1 \times \mathbb{R}$, we may then consider the local conformal nets on $\mathcal{N}$ that satisfy the double cone KMS property. These nets are in one-to-one correspondence with the local conformal nets on Minkowski spacetime $\mathcal{M}$, see [21], and so one immediately reads off our classification in these different contexts. An important case where this applies is represented by the two-dimensional de Sitter spacetime.

2 Two-dimensional completely rational nets and central charge

Let $\mathcal{M}$ be the two-dimensional Minkowski spacetime, namely $\mathbb{R}^2$ equipped with the metric $\mathrm{d}t^2 - \mathrm{d}x^2$. We shall also use the light ray coordinates $\xi_\pm \equiv t \pm x$. We have the decomposition $\mathcal{M} = \mathcal{L}_+ \times \mathcal{L}_-$ where $\mathcal{L}_\pm = \{\xi : \xi_\pm = 0\}$ are the two light ray lines. A *double cone* $\mathcal{O}$ is a non-empty open subset of of $\mathcal{M}$ of the form $\mathcal{O} = I_+ \times I_-$ with $I_\pm \subset \mathcal{L}_\pm$ bounded intervals; we denote by $\mathcal{K}$ the set of double cones.

The Möbius group $PSL(2, \mathbb{R})$ acts on $\mathbb{R} \cup \{\infty\}$ by linear fractional transformations, hence this action restricts to a local action on $\mathbb{R}$ (see e.g. [8]), in particular if $F \subset \mathbb{R}$ has compact closure there exists a connected neighborhood $\mathcal{U}$ of the identity in $PSL(2, \mathbb{R})$ such that $gF \subset \mathbb{R}$ for all $g \in \mathcal{U}$. It is convenient to regard this as a local action on $\mathbb{R}$ of the universal covering group $\overline{PSL}(2, \mathbb{R})$ of $PSL(2, \mathbb{R})$. We then have a local (product) action of $\overline{PSL}(2, \mathbb{R}) \times \overline{PSL}(2, \mathbb{R})$ on $\mathcal{M} = \mathcal{L}_+ \times \mathcal{L}_-$. Clearly $\overline{PSL}(2, \mathbb{R}) \times \overline{PSL}(2, \mathbb{R})$ acts by pointwise rescaling the metric $\mathrm{d}\xi_+ \mathrm{d}\xi_-$, i.e. by conformal transformations.

A *local Möbius covariant net* $\mathcal{A}$ on $\mathcal{M}$ is a map

$$\mathcal{A} : \mathcal{O} \in \mathcal{K} \mapsto \mathcal{A}(\mathcal{O})$$

where the $\mathcal{A}(\mathcal{O})$'s are von Neumann algebras on a fixed Hilbert space $\mathcal{H}$, with the following properties:

- *Isotony.* $\mathcal{O}_1 \subset \mathcal{O}_2 \implies \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.

- *Locality.* If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated then $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute elementwise (two points ξ_1 and ξ_2 are spacelike if $(\xi_1 - \xi_2)_+(\xi_1 - \xi_2)_- < 0$).

- *Möbius covariance.* There exists a unitary representation U of $\overline{PSL}(2, \mathbb{R}) \times \overline{PSL}(2, \mathbb{R})$ on $\mathcal{H}$ such that, for every double cone $\mathcal{O} \in \mathcal{K}$,

$$U(g)\mathcal{A}(\mathcal{O})U(g)^{-1} = \mathcal{A}(g\mathcal{O}), \quad g \in \mathcal{U},$$

  with $\mathcal{U} \subset \overline{PSL}(2, \mathbb{R}) \times \overline{PSL}(2, \mathbb{R})$ any connected neighborhood of the identity such that $g\mathcal{O} \subset \mathcal{M}$ for all $g \in \mathcal{U}$.

- *Vacuum vector.* There exists a unit U-invariant vector Ω, cyclic the $\bigcup_{\mathcal{O} \in \mathcal{K}} \mathcal{A}(\mathcal{O})$.

- *Positive energy.* The one-parameter unitary subgroup of U corresponding to time translations has positive generator.

The 2-torus $S^1 \times S^1$ is a conformal completion of $\mathcal{M} = \mathcal{L}_+ \times \mathcal{L}_-$ in the sense that $\mathcal{M}$ is conformally diffeomorphic to a dense open subregion of $S^1 \times S^1$ and the local action of $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ on $\mathcal{M}$ extends to a global conformal action on $S^1 \times S^1$.

But in general the net $\mathcal{A}$ does not extend to a Möbius covariant net on $S^1 \times S^1$; this is related to the failure of timelike commutativity (note that a chiral net, i.e. the tensor product of two local nets on S^1, would extend), indeed we have a covariant unitary representation of $\overline{PSL}(2, \mathbb{R}) \times \overline{PSL}(2, \mathbb{R})$ and not of $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$.

Let however G be the quotient of $\overline{PSL}(2, \mathbb{R}) \times \overline{PSL}(2, \mathbb{R})$ modulo the relation $(r_{2\pi}, r_{-2\pi}) = (\mathrm{id}, \mathrm{id})$ (spatial 2π-rotation is the identity).

Proposition 2.1. *The representation U of $\overline{PSL}(2, \mathbb{R}) \times \overline{PSL}(2, \mathbb{R})$ factors through a representation of G.*

The above proposition holds as a consequence of spacelike locality, it is a particular case of the conformal spin-statistics theorem and can be proved as in [20].

Because of the above Prop. 2.1, $\mathcal{A}$ does extend to a local G-covariant net on the Einstein cylinder $\mathcal{E} = \mathbb{R} \times S^1$, the cover of the 2-torus obtained by lifting the time coordinate from S^1 to $\mathbb{R}$.

Explicitly, $\mathcal{M}$ is conformally equivalent to a double cone $\mathcal{O}_\mathcal{M}$ of $\mathcal{E}$. By parametrizing $\mathcal{E}$ with coordinates (t', θ), $-\infty < t' < \infty$, $-\pi \leq \theta < \pi$, the transformation

$$\xi_\pm = \tan(\tfrac{1}{2}(t' \pm \theta)) \tag{3}$$

is a diffeomorphism of the subregion $\mathcal{O}_\mathcal{M} = \{(t', \theta) : -\pi < t' \pm \theta < \pi\} \subset \mathcal{E}$ with $\mathcal{M}$, which is a conformal map when $\mathcal{E}$ is equipped with the metric $\mathrm{d}s^2 \equiv \mathrm{d}t'^2 - \mathrm{d}\theta^2$.

G acts globally on $\mathcal{E}$ and the net $\mathcal{A}$ extends uniquely to a G-covariant net of $\mathcal{E}$ with U the unitary covariant action (see [8]). We shall denote by the same symbol $\mathcal{A}$ both the net on $\mathcal{M}$ and the extended net on $\mathcal{E}$.

If $\mathcal{O}_1 \subset \mathcal{M}$ (or $\mathcal{O}_1 \subset \mathcal{E}$) we shall denote by $\mathcal{A}(\mathcal{O}_1)$ the von Neumann algebra generated by the $\mathcal{A}(\mathcal{O})$'s as $\mathcal{O}$ varies in the double cones contained in $\mathcal{O}_1$. If $\mathcal{O} \in \mathcal{K}$ we shall denote by $\Lambda_{\mathcal{O}}$ the one-parameter subgroup of G defined as follows: $\Lambda_{\mathcal{O}} = g\Lambda_W g^{-1}$ if W is a wedge, Λ_W is the boost one-parameter group associated with W, and $g\mathcal{O} = W$ with $g \in PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$, see [23].

We collect in the next proposition a few basic properties of a local Möbius covariant net. The proof is either in the references or can be immediately obtained from those. All the statements also hold true (with obvious modifications) in any spacetime dimension. We shall use the lattice symbol $\vee$ to denote the von Neumann algebra generated.

Proposition 2.2. *Let $\mathcal{A}$ be a local Möbius covariant net on $\mathcal{M}$ as above. The following hold:*

(i) Double cone KMS property. *If $\mathcal{O} \subset \mathcal{E}$ is a double cone, then the unitary modular group associated with $(\mathcal{A}(\mathcal{O}), \Omega)$ has the geometrical meaning $\Delta_{\mathcal{O}}^{it} = U(\Lambda_{\mathcal{O}}(-2\pi t))$ [8].*

(ii) Haag duality on $\mathcal{E}$; wedge duality on $\mathcal{M}$. *If $\mathcal{O} \subset \mathcal{E}$ is a double cone then $\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'$. Here $\mathcal{O}'$ is the causal complement of $\mathcal{O}$ in $\mathcal{E}$ (note that $\mathcal{O}'$ is still a double cone.) In particular $\mathcal{A}(W') = A(W)'$, where W is a wedge in $\mathcal{M}$, say $W = (-\infty, a) \times (-\infty, b)$ and W' its causal complement in $\mathcal{M}$, thus $W' = (a, \infty) \times (b, \infty)$ [8, 21].*

(iii) Modular PCT symmetry. *There is a anti-unitary involution Θ on $\mathcal{H}$ such that $\Theta\mathcal{A}(\mathcal{O})\Theta = \mathcal{A}(-\mathcal{O})$, $\Theta U(g)\Theta = U(\theta(g))$ and $\theta\Omega = \Omega$. Here $\mathcal{O}$ is any double one in $\mathcal{E}$ and θ is the automorphism of G associated with space and time reflection [8].*

(iv) Additivity. *Let $\mathcal{O}$ be a double cone and $\{\mathcal{O}_i\}$ a family of open sets such that $\bigcup_i \mathcal{O}_i$ contains the axis of $\mathcal{O}$. Then $\mathcal{A}(\mathcal{O}) \subset \bigvee_i \mathcal{A}(\mathcal{O}_i)$ [15].*

(v) Equivalence between rreducibility and uniqueness of the vacuum. *$\mathcal{A}$ is irreducible on $\mathcal{M}$ (that is $\left(\bigcup_{\mathcal{O} \in \mathcal{K}} \mathcal{A}(\mathcal{O})\right)'' = B(\mathcal{H})$), iff $\mathcal{A}$ is irreducible on $\mathcal{E}$, iff Ω is the unique U-invariant vector (up to a phase) [20].*

(vi) Decomposition into irreducibles. *$\mathcal{A}$ has a unique direct integral decomposition in terms of local irreducible Möbius covariant nets. If $\mathcal{A}$ is conformal (see below) then the fibers in the decomposition are also conformal [20].*

By the above point (vi) we shall always assume our nets to be *irreducible*.

Let $\mathrm{Diff}(\mathbb{R})$ denote the group of positively oriented diffeomorphisms of $\mathbb{R}$ that are smooth at infinity (with the identification $\mathbb{R} = S^1 \smallsetminus \{\infty\}$, $\mathrm{Diff}(\mathbb{R})$ is the subgroup of

Diff(S^1) of orientation preserving diffeomorphisms of S^1 that fix the point ∞). By identifying $\mathcal{M}$ with the double cone $\mathcal{O}_{\mathcal{M}} \subset \mathcal{E}$ as above, we may identify elements of Diff$(\mathbb{R}) \times$ Diff$(\mathbb{R})$ with conformal diffeomorphisms of $\mathcal{O}_{\mathcal{M}}$. Such diffeomorphisms uniquely extend (by periodicity) to global conformal diffeomorphisms of $\mathcal{E}$. Namely the element $(r_{2\pi}, \mathrm{id})$ of G generates a subgroup of G (isomorphic to $\mathbb{Z}$) for which $\mathcal{O}_{\mathcal{M}}$ is a fundamental domain in $\mathcal{E}$. We may then extend an element of Diff$(\mathbb{R}) \times$ Diff$(\mathbb{R})$ from $\mathcal{O}_{\mathcal{M}}$ to all $\mathcal{E}$ by requiring commutativity with this $\mathbb{Z}$-action; this is the unique conformal extension to $\mathcal{E}$.

Let Conf$(\mathcal{E})$ denote the group of global, orientation preserving conformal diffeomorphisms of $\mathcal{E}$. Conf$(\mathcal{E})$ is generated by Diff$(\mathbb{R}) \times$ Diff$(\mathbb{R})$ and G (note that Diff$(\mathbb{R}) \times$ Diff$(\mathbb{R})$ intersects G in the "Poincaré-dilation" subgroup). Indeed if $\varphi \in$ Conf$(\mathcal{E})$, then $\varphi\mathcal{O}_{\mathcal{M}}$ is a maximal double cone of $\mathcal{E}$, namely the causal complement of a point. Thus there exists an element $g \in \mathsf{G}$ such that $g\mathcal{O}_{\mathcal{M}} = \varphi\mathcal{O}_{\mathcal{M}}$. Then $\psi \equiv g^{-1}\varphi$ maps $\mathcal{O}_{\mathcal{M}}$ onto $\mathcal{O}_{\mathcal{M}}$ and so $\psi \in$ Diff$(\mathbb{R}) \times$ Diff$(\mathbb{R})$ and $\varphi = g \cdot \psi$. Note that, by the same argument, any element of Conf$(\mathcal{E})$ is uniquely the product of an element of Diff$(\mathbb{R}) \times$ Diff$(\mathbb{R})$, a space rotation and time translation on $\mathcal{E}$.

A *local conformal net* $\mathcal{A}$ on $\mathcal{M}$ is a Möbius covariant net such that the unitary representation U of G extends to a projective unitary representation of Conf$(\mathcal{E})$ (still denoted by U) such that so that the extended net on $\mathcal{E}$ is covariant. In particular

$$U(g)\mathcal{A}(\mathcal{O})U(g)^{-1} = \mathcal{A}(g\mathcal{O}), \quad g \in \mathcal{U} \ ,$$

if $\mathcal{U}$ is a connected neighborhood of the identity of Conf$(\mathcal{E})$, $\mathcal{O} \in \mathcal{K}$, and $g\mathcal{O} \subset \mathcal{M}$ for all $g \in \mathcal{U}$. We further assume that

$$U(g)XU(g)^{-1} = X, \quad g \in \mathrm{Diff}(\mathbb{R}) \times \mathrm{Diff}(\mathbb{R}) \ ,$$

if $X \in \mathcal{A}(\mathcal{O}_1)$, $g \in$ Diff$(\mathbb{R}) \times$ Diff$(\mathbb{R})$ and g acts identically on $\mathcal{O}_1$. We may check the conformal covariance on $\mathcal{M}$ by the local action of Diff$(\mathbb{R}) \times$ Diff$(\mathbb{R})$.

Given a Möbius covariant net $\mathcal{A}$ on $\mathcal{M}$ and a bounded interval $I \subset \mathcal{L}_+$ we set

$$\mathcal{A}_+(I) \equiv \bigcap_{\mathcal{O}=I \times J} \mathcal{A}(\mathcal{O}) \tag{4}$$

(intersection over all intervals $J \subset \mathcal{L}_-$), and analogously define $\mathcal{A}_-$. By identifying $\mathcal{L}_\pm$ with $\mathbb{R}$ we then get two Möbius covariant local nets $\mathcal{A}_\pm$ on $\mathbb{R}$, *the chiral components of $\mathcal{A}$*, but for the cyclicity of Ω; we shall also denote $\mathcal{A}_\pm$ by $\mathcal{A}_R$ and $\mathcal{A}_L$. By the Reeh-Schlieder theorem the cyclic subspace $\mathcal{H}_\pm \equiv \overline{\mathcal{A}_\pm(I)\Omega}$ is independent of the interval $I \subset \mathcal{L}_\pm$ and $\mathcal{A}_\pm$ restricts to a (cyclic) Möbius covariant local net on $\mathbb{R}$ on the Hilbert space $\mathcal{H}_\pm$. Since Ω is separating for every $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$, the map $X \in \mathcal{A}_\pm(I) \mapsto X \upharpoonright \mathcal{H}_\pm$ is an isomorphism for any interval I, so we will often identify $\mathcal{A}_\pm$ with its restriction to $\mathcal{H}_\pm$.

Proposition 2.3. *Let $\mathcal{A}$ be a Möbius covariant (resp. conformal) net on $\mathcal{M}$. Setting $\mathcal{A}_0(\mathcal{O}) \equiv \mathcal{A}_+(I_+) \vee \mathcal{A}_-(I_-)$, $\mathcal{O} = I_+ \times I_-$, then $\mathcal{A}_0$ is a Möbius covariant (resp.*

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conformal) subnet of $\mathcal{A}$, there exists a consistent family of vacuum preserving conditional expectations $\varepsilon_{\mathcal{O}} : \mathcal{A}(O) \to \mathcal{A}_0(\mathcal{O})$ and the natural isomorphism from the product $\mathcal{A}_+(I_+) \cdot \mathcal{A}_-(I_-)$ to the algebraic tensor product $\mathcal{A}_+(I_+) \odot \mathcal{A}_-(I_-)$ extends to a normal isomorphism between $\mathcal{A}_+(I_+) \vee \mathcal{A}_-(I_-)$ and $\mathcal{A}_+(I_+) \otimes \mathcal{A}_-(I_-)$.

Thus we may identify $\mathcal{H}_+ \otimes \mathcal{H}_-$ with $\mathcal{H}_0 \equiv \overline{\mathcal{A}_0(\mathcal{O})\Omega}$ and $\mathcal{A}_+(I_+) \otimes \mathcal{A}_-(I_-)$ with $\mathcal{A}_0(\mathcal{O})$.

It is easy to see that $\mathcal{A}_0$ is the unique maximal chiral subnet of $\mathcal{A}$, namely it coincides with the subnet $\mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max}$ in Rehren's work [47, 48]. That is to say $\mathcal{A}_L^{\max}(\mathcal{O}) \otimes 1 = \mathcal{A}(\mathcal{O}) \cap U(\{\mathrm{id}\} \times PSL(2, \mathbb{R}))'$ and similarly for $\mathcal{A}_R^{\max}$. Indeed $\mathcal{A}_L^{\max} \otimes \mathcal{A}_L^{\max}$, being chiral, is clearly contained in $\mathcal{A}_0$; on the other hand $\mathcal{A}_+$ commutes with $U \upharpoonright \mathrm{id} \times PSL(2, \mathbb{R})$ so $\mathcal{A}_+ \subset \mathcal{A}_L^{\max}$ and analogously $\mathcal{A}_- \subset \mathcal{A}_R^{\max}$.

Now suppose that $\mathcal{A}$ is conformal. We have

Proposition 2.4. *If $\mathcal{A}$ is conformal then $\mathcal{A}_0 \equiv \mathcal{A}_L^{\max} \otimes \mathcal{A}_L^{\max}$ is also conformal, moreover $\mathcal{A}_0$ extends to a local $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$-covariant net on the 2-torus, namely $\mathcal{A}_\pm$ are local conformal nets on S^1.*

Assuming $\mathcal{A}$ to be conformal we set

$$\mathrm{Vir}_+(I) \equiv \{U(g) : g \in \mathrm{Diff}(I) \times \{\mathrm{id}\}\} \ I \subset \mathcal{L}_+ \tag{5}$$

$$\mathrm{Vir}_-(I) \equiv \{U(g) : g \in \{\mathrm{id}\} \times \mathrm{Diff}(I)\}, \ I \subset \mathcal{L}_- \tag{6}$$

$$\mathrm{Vir}(\mathcal{O}) \equiv \mathrm{Vir}_+(I_+) \otimes \mathrm{Vir}_-(I_-), \ I_\pm \subset \mathcal{L}_\pm \tag{7}$$

Proposition 2.5. $\mathrm{Vir}_\pm(I) \subset \mathcal{A}_\pm(I)$, $I \subset \mathcal{L}_\pm$, *and* $\mathrm{Vir}_+(I_+) \vee \mathrm{Vir}_-(I_-)$ *is naturally isomorphic to* $\mathrm{Vir}_+(I_+) \otimes \mathrm{Vir}_-(I_-)$, $I_\pm \subset \mathcal{L}_\pm$.

$\mathrm{Vir}_\pm$ *is the restriction to $\mathcal{L}_\pm$ of the Virasoro subnet of $\mathcal{A}_\pm$.*

In this paper we shall use only the a priori weaker form of conformal covariance given by the above proposition. Indeed we shall just need that $\mathcal{A}_\pm$ are conformal nets on S^1, with central charge less than one.

2.1   Complete rationality

Let $\mathcal{A}$ be a local conformal net on the two-dimensional Minkowski spacetime $\mathcal{M}$. We shall say that $\mathcal{A}$ is *completely rational* if the following three conditions hold:

a) *Haag duality on $\mathcal{M}$.* For any double cone $\mathcal{O}$ we have $\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}')'$. Here $\mathcal{O}'$ is the causal complement of $\mathcal{O}$ in $\mathcal{M}$

b) *Split property.* If $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ and the closure of $\bar{\mathcal{O}}_1$ of $\mathcal{O}_1$ is contained in $\mathcal{O}_2$, the natural map $\mathcal{A}(\mathcal{O}_1) \cdot \mathcal{A}(\mathcal{O}_2)' \to \mathcal{A}(\mathcal{O}_1) \odot \mathcal{A}(\mathcal{O}_2)'$ extends to a normal isomorphism $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)' \to \mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)'$.

c) *Finite μ-index.* Let $E = \mathcal{O}_1 \cup \mathcal{O}_2 \subset \mathcal{M}$ be the union of two double cones $\mathcal{O}_1, \mathcal{O}_2$ such that $\bar{\mathcal{O}}_1$ and $\bar{\mathcal{O}}_2$ are spacelike separated. Then the Jones index $[\mathcal{A}(E')' : \mathcal{A}(E)]$ is finite. This index is denoted by $\mu_{\mathcal{A}}$, *the μ-index of $\mathcal{A}$.*

The notion of complete rationality has been introduced and studied in [32] for a local net $\mathcal{C}$ on $\mathbb{R}$. If $\mathcal{C}$ is conformal, the definition of complete rationality strictly parallels the above one in the two-dimensional case. In general, the above (one-dimensional version) of the above three conditions must be supplemented by the following two conditions

d) *Strong additivity.* If $I_1, I_2 \subset \mathbb{R}$ are open intervals and I is the interior of $\overline{I_1 \cup I_2}$, then $\mathcal{C}(I) = \mathcal{C}(I_1) \vee \mathcal{C}(I_2)$.

e) *Modular PCT symmetry.* There is a vector Ω, cyclic and separating for all the $\mathcal{C}(I)$'s, such that if $a \in \mathbb{R}$ the modular conjugation J of $(\mathcal{C}(a, \infty), \Omega)$ satisfies $J\mathcal{C}(I)J = \mathcal{C}(I + 2a)$, for all intervals I.

If $\mathcal{C}$ is conformal, then $d)$ and $e)$ follows from $a), b), c)$. In any case all conditions $a)$ to $e)$ have the strong consequences on the structure of $\mathcal{A}$ [32]. In particular

$$\mu_{\mathcal{C}} = \sum_i d(\rho_i)^2$$

where the ρ_i form a system of irreducible sectors of $\mathcal{C}$.

Returning to the two-dimensional local conformal net $\mathcal{A}$, consider the time-zero net

$$\mathcal{C}(I) \equiv \mathcal{A}(\mathcal{O}),$$

where I is an interval of the $t = 0$ line in $\mathcal{M}$ and $\mathcal{O} = I''$ is the double cone with basis I. Note that $\mathcal{C}$ is local but *not* conformal (positivity of energy does not hold). However $\mathcal{C}$ inherits all properties from $a)$ to $e)$ from $\mathcal{A}$. Thus we may define $\mathcal{A}$ to be completely rational by requiring $\mathcal{C}$ to be completely rational. In this way all results in [32] immediately applies to the two-dimensional context.

3   Modular invariance and μ-index of a net

Rehren raised a question in [49, page 351, lines 8–13] about modular invariant arising from a decomposition of a two-dimensional net and its μ-index. Müger has then solved the problem affirmatively in [41]. We recall some notions and results necessary for our work here.

In [47, 48, 49], Rehren studied 2-dimensional local conformal quantum field theory $\mathcal{B}(O)$ which irreducible extends a given pair of chiral theories $\mathcal{A} = \mathcal{A}_L \otimes \mathcal{A}_R$. That is, the mathematical structure studied there is an irreducible inclusion of nets, $\mathcal{A}_L(I) \otimes \mathcal{A}_R(J) \subset \mathcal{B}(O)$, where I, J light ray intervals and O a double cone $I \times J$. Note that here $\mathcal{A}_L$ and $\mathcal{A}_R$ can be distinct. For such an extension, we decompose the dual canonical endomorphism θ on $\mathcal{A}_L \otimes \mathcal{A}_R$ as

$$\theta = \bigoplus_{ij} Z_{ij} \alpha_i^L \otimes \alpha_j^R,$$

where $\{\alpha_i^L\}_i$ and $\{\alpha_j^R\}_j$ are systems of irreducible DHR endomorphisms of $\mathcal{A}_L$ and $\mathcal{A}_R$, respectively. The matrix $Z = (Z_{ij})$ is called a coupling matrix. The two nets $\mathcal{A}_L$ and $\mathcal{A}_R$ define S- and T-matrices, S_L, T_L, S_R, T_R, respectively, as in [46]. We are interested in the case where the S-matrices are invertible. (By the results in [32], this invertibility, which is called non-degeneracy of the braiding, holds if the nets are completely rational in the sense of [32].) Then Rehren considered when the following two intertwining relations hold.

$$T_L Z = Z T_R, \qquad S_L Z = Z S_R. \tag{8}$$

Note that if $\mathcal{A}_L = \mathcal{A}_R$ and the non-degeneracy of the braiding holds, this condition implies the usual modular invariance of Z. (We always have $Z_{00} = 1$ and $Z_{ij} \in \{0, 1, 2, \dots\}$.) He considered natural situations where the above equalities (8) hold, but also pointed out that it is not necessarily valid in general by showing a very easy counter-example to the intertwining property (8). He then continues as follows. "A possible criterium to exclude models like the counter examples, and hopefully to enforce the intertwining property, could be that the local 2D theory $\mathcal{B}$ does not possess nontrivial superselection sectors, but I have no proof that this condition indeed has the desired consequences." Müger [41] has proved that this triviality of the superselection structures is indeed sufficient (and necessary) for the intertwining property (8), when the nets $\mathcal{A}_L$ and $\mathcal{A}_R$ are completely rational.

Theorem 3.1 (Müger [41]). *Under the above conditions, the following are equivalent.*

1. *The net $\mathcal{B}$ has only the trivial superselection sector.*

2. *The μ-index μ_B is 1.*

3. *The matrix Z has the intertwining property (8),*

$$T_L Z = Z T_R, \qquad S_L Z = Z S_R.$$

In the case where we can naturally identify $\mathcal{A}_L$ and $\mathcal{A}_R$, the above theorem gives a relation between the classification problem of the modular invariants and the classification problem of the local extension of $\mathcal{A}_L \otimes \mathcal{A}_R$ with μ-index equal to 1.

4 Longo-Rehren subfactors and 2-cohomology of a tensor category

Let M be a type III factor. We say that a finite subset $\Delta \subset \text{End}(M)$ is a *system of endomorphisms* of M if the following conditions hold, as in [5, Definition 2.1].

1. Each $\lambda \in \Delta$ is irreducible and has finite statistical dimension.

2. The endomorphisms in Δ are mutually inequivalent.

3. We have $\mathrm{id}_M \in \Delta$.

4. For any $\lambda \in \Delta$, we have an endomorphism $\bar{\lambda} \in \Delta$ such that $[\bar{\lambda}]$ is the conjugate sector of $[\lambda]$.

5. The set Δ is closed under composition and subsequent irreducible decomposition, i.e., for any $\lambda, \mu \in \Delta$, we have non-negative integers $N_{\lambda,\mu}^{\nu}$ with $[\lambda][\mu] = \sum_{\nu \in \Delta} N_{\lambda,\mu}^{\nu}[\nu]$ as sectors.

Two typical examples of systems of endomorphism are as follows. First, if we have a subfactor $N \subset M$ with finite index, then consider representatives of unitary equivalence classes of irreducible endomorphisms appearing in irreducible decompositions of powers γ^n of the canonical endomorphism γ for the subfactor. If the set of representatives is finite, that is, if the subfactor is of finite depth, then we obtain a finite system of endomorphisms. Second, if we have a local conformal net $\mathcal{A}$ on the circle, we consider representatives of unitary equivalence classes of irreducible DHR endomorphisms of this net. If the set of representatives is finite, that is, if the net is rational, then we obtain a finite system of endomorphisms of $M = \mathcal{A}(I)$, where I is some fixed interval of the circle.

Recall the definition of a Q-system in [36]. Let θ be an endomorphism of a type III factor. A triple (θ, V, W) is called a Q-system if we have the following properties.

$$
\begin{aligned}
V &\in \mathrm{Hom}(\mathrm{id}, \theta), & (9)\\
W &\in \mathrm{Hom}(\theta, \theta^2), & (10)\\
V^*V &= 1, & (11)\\
W^*W &= 1, & (12)\\
V^*W &= \theta(V^*)W \in \mathbb{R}_+, & (13)\\
W^2 &= \theta(W)W, & (14)\\
\theta(W^*)W &= WW^*. & (15)
\end{aligned}
$$

Actually, it has been proved in [39] that Condition (15) is redundant. (It has been also proved in [27] that Condition (14) is redundant if (15) is assumed.) In this case, θ is a canonical endomorphism of a certain subfactor of the original factor.

For a finite system Δ as above, Longo and Rehren constructed a subfactor $M \otimes M^{\mathrm{opp}} \subset R$ in [38, Proposition 4.10] such that the dual canonical endomorphism has a decomposition $\theta = \bigoplus_{\lambda \in \Delta} \lambda \otimes \lambda^{\mathrm{opp}}$, by explicitly writing down a Q-system (θ, V, W). We, however, could have an inequivalent Q-system for the same dual canonical endomorphism θ. (We say that two Q-systems (θ, V_1, W_1) and (θ, V_2, W_2) are *equivalent* if we have a unitary $u \in \mathrm{Hom}(\theta, \theta)$ satisfying

$$
V_2 = uV_1, \qquad W_2 = u\theta(u)W_1u^*.
$$

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This equivalence of Q-systems is equivalent to inner conjugacy of the corresponding subfactors [27].) We study this problem of uniqueness of the Q-systems below. Classification of Q-systems for a given dual canonical endomorphism was studied as a subfactor analogue of 2-cohomology of a group in [27]. We show that for a Longo-Rehren Q-system, we naturally have a 2-cohomology *group* of a tensor category, while 2-cohomology in [27] is not a group in general.

Suppose we have a family $(C_{\lambda\mu})_{\lambda,\mu\in\Delta}$ with $C_{\lambda\mu} \in \mathrm{Hom}(\lambda\mu, \lambda\mu)$. An intertwiner $C_{\lambda\mu}$ naturally defines an operator $C_{\lambda\mu}^{\nu} \in \mathrm{End}(\mathrm{Hom}(\nu, \lambda\mu))$ for $\nu \in \Delta$ by composition from the left. For $\lambda, \mu, \nu, \pi \in \Delta$, we have a decomposition

$$\mathrm{Hom}(\pi, \lambda\mu\nu) = \bigoplus_{\sigma\in\Delta} \mathrm{Hom}(\sigma, \lambda\mu) \otimes \mathrm{Hom}(\pi, \sigma\nu).$$

We have

$$\bigoplus_{\sigma\in\Delta} C_{\lambda\mu}^{\sigma} \otimes C_{\sigma\nu}^{\pi} \in \mathrm{End}(\mathrm{Hom}(\pi, \lambda\mu\nu))$$

according to this decomposition. We similarly have

$$\bigoplus_{\tau\in\Delta} C_{\lambda\tau}^{\pi} \otimes C_{\mu\nu}^{\tau} \in \mathrm{End}(\mathrm{Hom}(\pi, \lambda\mu\nu))$$

based on the last expression of the decompositions

$$\begin{aligned}
\mathrm{Hom}(\pi, \lambda\mu\nu) &\cong \bigoplus_{\tau\in\Delta} \mathrm{Hom}(\pi, \lambda\tau) \otimes \lambda(\mathrm{Hom}(\tau, \mu\nu)) \\
&\cong \bigoplus_{\tau\in\Delta} \mathrm{Hom}(\pi, \lambda\tau) \otimes \mathrm{Hom}(\tau, \mu\nu).
\end{aligned}$$

We now consider the following conditions.

Definition 4.1. We say that a family $(C_{\lambda\mu})_{\lambda,\mu\in\Delta}$ is a *unitary 2-cocycle* of Δ, if the following conditions hold.

1. For $\lambda, \mu \in \Delta$, each $C_{\lambda\mu}$ is a unitary operator in $\mathrm{Hom}(\lambda\mu, \lambda\mu)$.

2. For $\lambda \in \Delta$, we have $C_{\lambda\mathrm{id}} = 1$ and $C_{\mathrm{id}\lambda} = 1$.

3. For $\lambda, \mu, \nu, \pi \in \Delta$, we have

$$\bigoplus_{\sigma\in\Delta} C_{\lambda\mu}^{\sigma} \otimes C_{\sigma\nu}^{\pi} = \bigoplus_{\tau\in\Delta} C_{\lambda\tau}^{\pi} \otimes C_{\mu\nu}^{\tau}$$

as an identity in $\mathrm{End}(\mathrm{Hom}(\pi, \lambda\mu\nu))$ with respect to the above decompositions of $\mathrm{Hom}(\pi, \lambda\mu\nu)$.

We always assume unitarity for $C_{\lambda\mu}$ in this paper, so we simply say a 2-cocycle for a unitary 2-cocycle. For a 2-cocycle $(C_{\lambda\mu})_{\lambda,\mu\in\Delta}$, we define $C^{\pi}_{\lambda\mu\nu} \in \mathrm{End}(\mathrm{Hom}(\pi, \lambda\mu\nu))$ by

$$\bigoplus_{\sigma\in\Delta} C^{\sigma}_{\lambda\mu} \otimes C^{\pi}_{\sigma\nu}.$$

Similarly, we can define

$$C^{\mu_1\mu_2\cdots\mu_m}_{\lambda_1\lambda_2\cdots\lambda_n} \in \mathrm{End}(\mathrm{Hom}(\mu_1\mu_2\cdots\mu_m, \lambda_1\lambda_2\cdots\lambda_n)).$$

(Note that well-definedness follows from the Condition 3 in Definition 4.1.) In this notation, we have $C^{\lambda\mu}_{\lambda\mu} \in \mathrm{End}(\mathrm{Hom}(\lambda\mu, \lambda\mu))$ and this endomorphism is given as the left multiplication of $C_{\lambda\mu} \in \mathrm{Hom}(\lambda\mu, \lambda\mu)$ on $\mathrm{Hom}(\lambda\mu, \lambda\mu)$, where the product structure on $\mathrm{Hom}(\lambda\mu, \lambda\mu)$ is given by composition. In this way, we can identify $C^{\lambda\mu}_{\lambda\mu} \in \mathrm{End}(\mathrm{Hom}(\lambda\mu, \lambda\mu))$ and $C_{\lambda\mu} \in \mathrm{Hom}(\lambda\mu, \lambda\mu)$.

We next consider a strict C^*-tensor category $\mathcal{T}$, with conjugates, subobjects, and direct sums, whose objects are given as finite direct sums of endomorphisms in Δ. We then study an automorphism Φ of $\mathcal{T}$ such that $\Phi(\lambda)$ and λ are unitarily equivalent for all objects λ in $\mathcal{T}$. For all $\lambda \in \Delta$, we choose a unitary u_λ with $\Phi(\lambda) = \mathrm{Ad}(u_\lambda) \cdot \lambda$. By adjusting Φ with $(\mathrm{Ad}(u_\lambda))_{\lambda\in\Delta}$, we may and do assume that $\Phi(\lambda) = \lambda$. Then such an automorphism Φ gives a family of automorphisms

$$\Phi^{\mu_1\mu_2\cdots\mu_m}_{\lambda_1\lambda_2\cdots\lambda_n} \in \mathrm{Aut}(\mathrm{Hom}(\mu_1\mu_2\cdots\mu_m, \lambda_1\lambda_2\cdots\lambda_n)),$$

for $\lambda_1, \lambda_2, \cdots, \lambda_n, \mu_1, \mu_2, \cdots, \mu_m \in \Delta$, with the compatibility condition

$$\Phi^{\nu_1\nu_2\cdots\nu_k}_{\lambda_1\lambda_2\cdots\lambda_n} = \bigoplus_{\mu_1,\mu_2,\cdots,\mu_m\in\Delta} \Phi^{\mu_1\mu_2\cdots\mu_m}_{\lambda_1\lambda_2\cdots\lambda_n} \otimes \Phi^{\nu_1\nu_2\cdots\nu_k}_{\mu_1\mu_2\cdots\mu_m}$$

on the decomposition

$$\begin{aligned}
&\mathrm{Hom}(\nu_1\nu_2\cdots\nu_k, \lambda_1\lambda_2\cdots\lambda_n) \\
&= \bigoplus_{\mu_1,\mu_2,\cdots,\mu_m\in\Delta} \mathrm{Hom}(\mu_1\mu_2\cdots\mu_m, \lambda_1\lambda_2\cdots\lambda_n) \otimes \mathrm{Hom}(\nu_1\nu_2\cdots\nu_k, \mu_1\mu_2\cdots\mu_m).
\end{aligned}$$

It is clear that a family $(C^{\mu_1\mu_2\cdots\mu_m}_{\lambda_1\lambda_2\cdots\lambda_n})$ arising from a 2-cocycle $(C_{\lambda\mu})$ is an automorphism of a tensor category in this sense.

Conversely, suppose that we have an automorphism Φ of a tensor category acting on objects trivially as above. Then using the isomorphism

$$\mathrm{Hom}(\lambda\mu, \lambda\mu) \cong \bigoplus_{\nu\in\Delta} \mathrm{Hom}(\nu, \lambda\mu) \otimes \mathrm{Hom}(\lambda\mu, \nu),$$

the family $(\Phi^{\nu}_{\lambda\mu})$ gives a unitary intertwiner in $\mathrm{Hom}(\lambda\mu, \lambda\mu)$. We denote this intertwiner by $C_{\lambda\mu}$ and then it is clear that the family $(C_{\lambda\mu})$ gives a 2-cocycle in the above sense. Thus in this correspondence, we can identify a 2-cocycle on Δ and an automorphism of the tensor category arising from Δ that fixes each object in the category.

We now have the following definition.

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Definition 4.2. (1) We say that 2-cocycles $(C_{\lambda\mu})_{\lambda\mu}$ and $(C'_{\lambda\mu})_{\lambda\mu}$ are *equivalent* if we have a family $(\omega_\lambda)_\lambda$ of scalars of modulus 1 such that

$$C_{\lambda\mu}^\nu = \omega_\nu/(\omega_\lambda\omega_\mu)C''^\nu_{\lambda\mu} \in \mathrm{End}(\mathrm{Hom}(\nu, \lambda\mu)).$$

If a 2-cocycle $(C_{\lambda\mu})_{\lambda\mu}$ is equivalent to $(1)_{\lambda\mu}$, then we say that it is *trivial*.

(2) We say that a 2-cocycle $(C_{\lambda\mu})_{\lambda\mu}$ is *scalar-valued* if all $C^\nu_{\lambda\mu}$'s are scalar operators on $\mathrm{Hom}(\lambda\mu, \nu)$.

(3) We say that an automorphism Φ of the tensor category as above is *trivial* if we have a family $(\omega_\lambda)_\lambda$ of scalars of modulus 1 satisfying

$$\Phi^{\mu_1\mu_2\cdots\mu_m}_{\lambda_1\lambda_2\cdots\lambda_n} = \omega_{\mu_1}\cdots\omega_{\mu_m}/(\omega_{\lambda_1}\cdots\omega_{\lambda_n}).$$

Note that if a 2-cocycle is trivial, then it is scalar-valued, in particular.

We now recall the definition of the Longo-Rehren subfactor [38, Proposition 4.10] as follows. (See [40], [43], [44] for related or more general definitions.) Let $\Delta = \{\lambda_k \mid k = 0, 1, \ldots, n\}$ be a finite system of endomorphisms of a type III factor M where $\lambda_0 = \mathrm{id}$. We choose a system $\{V_k \mid k = 0, 1, \ldots, n\}$ of isometries with $\sum_{k=0}^n V_k V_k^* = 1$ in the factor $M \otimes M^{\mathrm{opp}}$, where M^{opp} is the opposite algebra of M and we denote the anti-linear isomorphism from M onto M^{opp} by j. Then we set

$$\rho(x) = \sum_{k=0}^n V_k((\lambda_k \otimes \lambda_k^{\mathrm{opp}})(x))V_k^*,$$

for $x \in M \otimes M^{\mathrm{opp}}$, where $\lambda^{\mathrm{opp}} = j \cdot \lambda \cdot j^{-1}$. We set $V = V_0 \in \mathrm{Hom}(\mathrm{id}, \rho)$ and define $W \in \mathrm{Hom}(\rho, \rho^2)$ as follows.

$$W = \sum_{k,l,m=0}^n \sqrt{\frac{d_k d_l}{w d_m}} V_k(\lambda_k \otimes \lambda_k^{\mathrm{opp}})(V_l)T_{kl}^m V_m^*,$$

where d_k is the statistical dimension of λ_k, w is the global index of the system, $w = \sum_{k=0}^n d_k^2$, and

$$T_{kl}^m = \sum_{i=1}^{N_{kl}^m} (T_{kl}^m)_i \otimes j((T_{kl}^m)_i).$$

Here N_{kl}^m is the structure constant, $\dim \mathrm{Hom}(\lambda_m, \lambda_k\lambda_l)$, and $\{(T_{kl}^m)_i \mid i = 1, 2, \ldots, N_{kl}^m\}$ is a fixed orthonormal basis of $\mathrm{Hom}(\lambda_m, \lambda_k\lambda_l)$. Note that the operator T_{kl}^m does not depend on the choice of the orthonormal basis. Proposition 4.10 in [38] says that the triple (ρ, V, W) is a Q-system. Thus we have a subfactor $M \otimes M^{\mathrm{opp}} \subset R$ with index w corresponding to the dual canonical endomorphism ρ. We call this a *Longo-Rehren subfactor* arising from the system Δ.

Furthermore, if Δ is a subsystem of all the irreducible DHR endomorphisms of a local conformal net $\mathcal{A}$, then any Q-system having this dual canonical endomorphism gives an extension $\mathcal{B} \supset \mathcal{A} \otimes \mathcal{A}^{\mathrm{opp}}$. This 2-dimensional net $\mathcal{B}$ is local if and only if $\varepsilon(\rho, \rho)W = W$ by [38, Proposition 4.10], where ε is the braiding. In general, if the

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system Δ has a braiding ε, and this condition $\varepsilon(\rho, \rho)W = W$ holds, we say that the Q-system (ρ, V, W) satisfies *locality*.

We now would like to characterize a general Q-system having the same dual canonical endomorphism ρ. First, we have the following simple lemma.

Lemma 4.3. *Let F, F' be finite dimensional complex Hilbert spaces and j an anti-linear isomorphism from F onto F'. For any vector $\xi \in F \otimes F'$, we define a linear map $A : F \to F$ by $\xi = \sum_k A\xi_k \otimes j(\xi_k)$ where $\{\xi_k\}$ is an orthonormal basis of F. Then this linear map A is independent of the choice of the orthonormal basis $\{\xi_k\}$.*

Proof This is straightforward by the anti-isomorphism property of j. $\square$

The next Theorem gives our characterization of Q-systems.

Theorem 4.4. *Let Δ, ρ, V, W be as above. If another triple (ρ, V, W') with $W' \in \mathrm{Hom}(\rho, \rho^2)$ is a Q-system, we have a 2-cocycle $(C_{\lambda\mu})_{\lambda,\mu\in\Delta}$ such that*

$$W' = \sum_{k,l,m=0}^{n} \sqrt{\frac{d_k d_l}{w d_m}} V_k (\lambda_k \otimes \lambda_k^{\mathrm{opp}})(V_l)(C_{\lambda_k \lambda_l} \otimes 1)T_{kl}^m V_m^*. \tag{16}$$

Conversely, if we have a 2-cocycle $(C_{\lambda\mu})_{\lambda,\mu\in\Delta}$, then the triple (ρ, V, W') with W' defined as in (16) is a Q-system.

The Q-system (ρ, V, W') is equivalent to the above canonical Q-system (ρ, V, W) if and only if the corresponding 2-cocycle $(C_{\lambda\mu})_{\lambda,\mu\in\Delta}$ is trivial, if and only if the corresponding automorphism of the tensor category arising from Δ is trivial.

Moreover, suppose that the system Δ has a braiding $\varepsilon^\pm$. Then the Q-system (ρ, V, W') satisfies locality if and only if the corresponding 2-cocycle $(C_{\lambda\mu})_{\lambda,\mu\in\Delta}$ satisfies the following symmetric condition.

$$C_{\lambda\mu} = \varepsilon_{\mu\lambda}^{-} C_{\mu\lambda} \varepsilon_{\lambda\mu}^{+}, \tag{17}$$

for all $\lambda, \mu \in \Delta$. If this symmetric condition holds, the corresponding automorphism of the tensor category arising from Δ is an automorphism of a braided tensor category.

Proof If (ρ, V, W') with $W' \in \mathrm{Hom}(\rho, \rho^2)$ is a Q-system, then we have a system of intertwiners $(C_{\lambda\mu})_{\lambda,\mu\in\Delta}$ such that identity (16) holds and the intertwiners $(C_{\lambda\mu})$ are uniquely determined by Lemma 4.3. Expanding the both hand sides of identity (14), we obtain the following identity.

$$\sum_{k,l,m,p,q=0}^{n} \sqrt{\frac{d_k d_l d_m}{w^2 d_p}} V_k (\lambda_k \otimes \lambda_k^{\mathrm{opp}})(V_l)(\lambda_k \lambda_l \otimes \lambda_k^{\mathrm{opp}} \lambda_l^{\mathrm{opp}})(V_m)$$
$$(\lambda_k \otimes \lambda_k^{\mathrm{opp}})((C_{\lambda_l \lambda_m} \otimes 1)T_{lm}^q)(C_{\lambda_k \lambda_q} \otimes 1)T_{kq}^p V_p^*$$
$$= \sum_{k,l,m,p,r=0}^{n} \sqrt{\frac{d_k d_l d_m}{w^2 d_p}} V_k (\lambda_k \otimes \lambda_k^{\mathrm{opp}})(V_l)(\lambda_k \lambda_l \otimes \lambda_k^{\mathrm{opp}} \lambda_l^{\mathrm{opp}})(V_m)$$
$$(C_{\lambda_k \lambda_l} \otimes 1)T_{kl}^r (C_{\lambda_r \lambda_m} \otimes 1)T_{rm}^p V_p^*. \tag{18}$$

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We decompose

$$
\begin{aligned}
\mathrm{Hom}(\lambda_p, \lambda_k \lambda_l \lambda_m) &\cong \bigoplus_{q=0}^{n} \mathrm{Hom}(\lambda_p, \lambda_k \lambda_q) \otimes \mathrm{Hom}(\lambda_q, \lambda_l \lambda_m) \\
&\cong \bigoplus_{r=0}^{n} \mathrm{Hom}(\lambda_r, \lambda_k \lambda_l) \otimes \mathrm{Hom}(\lambda_p, \lambda_r \lambda_m),
\end{aligned}
$$

as above, and apply Lemma 4.3 to the above identity (18) to obtain Condition 3 in Definition 4.1. Similarly, Condition 2 in Definition 4.1 follows from identity (13).

We next prove unitarity of $C_{\lambda\mu} \in \mathrm{Hom}(\lambda\mu, \lambda\mu)$. First note that the operator $C_{\lambda\bar{\lambda}}^{\mathrm{id}}, C_{\bar{\lambda}\lambda}^{\mathrm{id}}$ are scalar multiples of the identity because $\mathrm{Hom}(\mathrm{id}, \lambda\bar{\lambda})$, $\mathrm{Hom}(\mathrm{id}, \bar{\lambda}\lambda)$ are both 1-dimensional.

Since the triple (ρ, V, W') also satisfies identity (15), we expand the both side hands of identity (15) and use Lemma 4.3 as in the above arguments. Then we obtain the following. The intertwiner space $\mathrm{Hom}(\lambda\mu, \nu\sigma)$ for $\lambda, \mu, \nu, \sigma \in \Delta$ can be decomposed in two ways as follows.

$$
\mathrm{Hom}(\lambda\mu, \nu\sigma) \cong \bigoplus_{\tau \in \Delta} \mathrm{Hom}(\lambda, \nu\tau) \otimes \mathrm{Hom}(\tau\mu, \sigma) \tag{19}
$$

$$
\cong \bigoplus_{\pi \in \Delta} \mathrm{Hom}(\lambda\mu, \pi) \otimes \mathrm{Hom}(\pi, \nu\sigma). \tag{20}
$$

On one hand, Lemma 4.3 applied to the left hand side of identity (15) produces a map in $\mathrm{End}(\mathrm{Hom}(\lambda\mu, \nu\sigma))$ which maps $T_i \otimes S_j^* \in \mathrm{Hom}(\lambda, \nu\tau) \otimes \mathrm{Hom}(\tau\mu, \sigma)$, identified with $\nu(S_j^*)T_i \in \mathrm{Hom}(\lambda\mu, \nu\sigma)$, to $\nu(S_j^* C_{\tau\mu}^*)C_{\nu\tau}T_i \in \mathrm{Hom}(\lambda\mu, \nu\sigma)$, where T_i and S_j are isometries in $\mathrm{Hom}(\lambda, \nu\tau)$ and $\mathrm{Hom}(\sigma, \tau\mu)$, respectively. On the other hand, Lemma 4.3 applied to the right hand side of identity (15) produces a map in $\mathrm{End}(\mathrm{Hom}(\lambda\mu, \nu\sigma))$ which maps $T_i'^* \otimes S_j' \in \mathrm{Hom}(\lambda\mu, \pi) \otimes \mathrm{Hom}(\pi, \nu\sigma)$, identified with $S_j'T_i'^* \in \mathrm{Hom}(\lambda\mu, \nu\sigma)$, to $C_{\nu\sigma}S_j'T_i'^* C_{\lambda\mu}^* \in \mathrm{Hom}(\lambda\mu, \nu\sigma)$, where T_i' and S_j' are isometries in $\mathrm{Hom}(\pi, \lambda\mu)$ and $\mathrm{Hom}(\pi, \nu\sigma)$, respectively. These two maps are equal in $\mathrm{End}(\mathrm{Hom}(\lambda\mu, \nu\sigma))$. In the above decomposition (20), we set $\lambda = \sigma = \mathrm{id}$ and $\mu = \nu$, then we have $\tau = \bar{\mu}$ and $\pi = \mu$ in the summations. With Frobenius reciprocity as in [25] and the above identity of two maps in $\mathrm{End}(\mathrm{Hom}(\lambda\mu, \nu\sigma))$, we obtain the identity

$$
C_{\mu\bar{\mu}}^{\mathrm{id}} \overline{C_{\bar{\mu}\mu}^{\mathrm{id}}} = 1. \tag{21}
$$

We next apply identity (12) to (16) and obtain the following equality

$$
\sum_{\lambda, \mu \in \Delta} d_\lambda d_\mu K_{\lambda\mu}^\nu = w d_\nu, \tag{22}
$$

where we have set $K_{\lambda\mu}^\nu = \mathrm{Tr}((C_{\lambda\mu}^\nu)^* C_{\lambda\mu}^\nu)$ and Tr is the non-normalized trace on $\mathrm{Hom}(\lambda\mu, \lambda\mu)$. Setting $\nu = \mathrm{id}$ in (22), we obtain

$$
\sum_{\lambda \in \Delta} d_\lambda^2 |C_{\lambda\bar{\lambda}}^{\mathrm{id}}|^2 = w,
$$

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which, together with (21), implies $|C_{\lambda\lambda}^{\text{id}}| = 1$ for all $\lambda \in \Delta$.

In the above decomposition (20), we now set $\lambda = \text{id}$, then we have $\tau = \bar{\nu}$ and $\pi = \mu$ in the summations. With Frobenius reciprocity as in [25] and the above identity of two maps in $\text{End}(\text{Hom}(\lambda\mu, \nu\sigma))$, we obtain the identity

$$C_{\nu\bar{\nu}}^{\text{id}} \nu((C_{\bar{\nu}\mu}^{\sigma})^* \tilde{T}) R_{\nu\bar{\nu}} = \sqrt{\frac{d_\mu}{d_\nu d_\sigma}} C_{\nu\sigma}^\mu T, \tag{23}$$

for all $T \in \text{Hom}(\mu, \nu\sigma)$, where $\tilde{T} \in \text{Hom}(\bar{\nu}\mu, \sigma)$ is the Frobenius dual of T and $R_{\nu\bar{\nu}} \in \text{Hom}(\text{id}, \nu\bar{\nu})$ is the canonical isometry. This identity (23), Condition 3 in Definition 4.1, already proved, and identity (21) imply the following identity,

$$\begin{aligned}
\langle C_{\bar{\nu}\mu}^{\sigma} T, C_{\bar{\nu}\mu}^{\sigma} S \rangle &= (C_{\nu\bar{\nu}}^{\text{id}})^* R_{\bar{\nu}\nu}^* \bar{\nu}(C_{\nu\sigma}^{\mu} \tilde{S}^*) C_{\bar{\nu}\mu}^{\sigma} T \\
&= (C_{\nu\bar{\nu}}^{\text{id}})^* C_{\bar{\nu}\nu}^{\text{id}} S^* T \\
&= \langle T, S \rangle,
\end{aligned}$$

where we have $T, S \in \text{Hom}(\sigma, \bar{\nu}\mu)$ and the inner product is given by $\langle T, S \rangle = S^* T \in \mathbb{C}$. This is the desired unitarity of $C_{\bar{\nu}\mu}$.

The converse also holds in the same way and the remaining parts are straightforward. □

It is easy to see that we can multiply 2-cocycles and the multiplication on the equivalences classes of 2-cocycles is well-defined. In this way, we obtain a group and this is called the 2-*cohomology group* of Δ (or of the corresponding tensor category). It is also easy to see that the multiplication gives the composition of the corresponding automorphisms of the tensor category.

The part of the above theorem on a bijective correspondence between Q-systems (ρ, V, W') with locality and automorphisms of the braided tensor category has been also announced by Müger in [41].

Example 4.5. If all the endomorphisms in Δ are automorphisms, then the fusion rules determine a finite group G. It is easy to see that the Longo-Rehren Q-system gives a crossed product by an outer action of G and the above 2-cohomology group for Δ is isomorphic to the usual 2-cohomology group of G.

Furthermore, if the system Δ has a braiding, then the group G is abelian. In this case, the symmetric condition of a cocycle means $c_{g,h} = c_{h,g}$ for the corresponding usual 2-cocycle c of the finite abelian group G. It is well-known that such a 2-cocycle is trivial. (See [1, Lemma 3.4.2], for example.)

When all the 2-cocycles for Δ are trivial, we say that we have a 2-cohomology vanishing for Δ. Thus, 2-cohomology vanishing implies uniqueness of the Longo-Rehren subfactor in the following sense.

Corollary 4.6. *Let Δ be as above. If we have a 2-cohomology vanishing for Δ and $\rho = \bigoplus_{\lambda \in \Delta} \lambda \otimes \lambda^{\text{opp}}$ is a dual canonical endomorphism for a subfactor $M \otimes M^{\text{opp}} \subset P$, then this subfactor is inner conjugate to the Longo-Rehren subfactor $M \otimes M^{\text{opp}} \subset R$.*

5 2-cohomology vanishing and classification

In this section, we first study a general theory of 2-cohomology for a C^*-tensor category and then apply it to the tensor categories related to the Virasoro algebra. We consider a strict C^*-tensor category $\mathcal{T}$ (with conjugates, subobjects, and direct sums) in the sense of [13, 39] and we assume that we have only finitely many equivalence classes of irreducible objects in $\mathcal{T}$ and that each object has a decomposition into a finite direct sum of irreducible objects. Such a tensor category is often called *rational*. We may and do assume that our tensor categories are realized as those of endomorphisms of a type III factor. Choose a system Δ of endomorphisms of a type III factor M corresponding to the C^*-tensor category $\mathcal{T}$. Suppose we have a 2-cocycle $(C_{\lambda\mu})_{\lambda,\mu\in\Delta}$.

We introduce some basic notions. Suppose that we have $\sigma \in \Delta$ such that for any $\lambda \in \Delta$, there exists $k \geq 0$ such that $\lambda \prec \sigma^k$. Then we say that σ is a *generator* of Δ. In the following, we consider only the case $\sigma = \bar{\sigma}$. In this case, we say that Δ *has a self-conjugate generator σ.*

Suppose $\sigma \in \Delta$ is a self-conjugate generator of Δ. We further assume that for all $\lambda, \mu \in \Delta$, we have $\dim \mathrm{Hom}(\lambda\sigma, \mu) \in \{0, 1\}$. In this case, we say that *multiplications by σ have no multiplicities*.

Take $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \Delta$ and assume

$$\dim \mathrm{Hom}(\lambda_1\sigma, \lambda_2) = \dim \mathrm{Hom}(\sigma\lambda_1, \lambda_3)$$
$$= \dim \mathrm{Hom}(\lambda_3\sigma, \lambda_4) = \dim \mathrm{Hom}(\sigma\lambda_2, \lambda_4) = 1.$$

Choose isometric intertwiners

$$T_1 \in \mathrm{Hom}(\lambda_2, \lambda_1\sigma), \quad T_2 \in \mathrm{Hom}(\lambda_4, \sigma\lambda_2),$$
$$T_3 \in \mathrm{Hom}(\lambda_3, \sigma\lambda_1), \quad T_4 \in \mathrm{Hom}(\lambda_4, \lambda_3\sigma).$$

Then the composition $T_4^* T_3^* \sigma(T_1) T_2$ is in $\mathrm{Hom}(\lambda_4, \lambda_4) = \mathbb{C}$. This values is the *connection* as in [42], [14, Chapter 9]. We denote this complex number by $W(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. (Note that this value depends on T_1, T_2, T_3, T_4 though they do not appear in the notation.) If all these complex numbers are non-zero, then we say that *the connections of Δ with respect to the generator σ are non-zero*. This condition is independent of the choices of isometric intertwiners T_j's, because we now assume that multiplications by σ have no multiplicities.

Suppose we have a map $g : \Delta \to \mathbb{Z}/2\mathbb{Z}$. For an endomorphism σ that is a direct sum of elements λ_j's with $g(\lambda_j) = k \in \mathbb{Z}/2\mathbb{Z}$, we also set $g(\sigma) = k$. If we have $g(\lambda\mu) = g(\lambda) + g(\mu)$, then we say that Δ has a $\mathbb{Z}/2\mathbb{Z}$-grading. An endomorphism $\lambda \in \Delta$ is called *even* [resp. *odd*] when $g(\lambda) = 0$ [resp. $g(\lambda) = 1$].

Theorem 5.1. *Suppose we have a finite system Δ of endomorphisms with a self-conjugate generator $\sigma \in \Delta$ satisfying all the following conditions.*

1. *Multiplications by σ have no multiplicities.*

2. *One of the following holds.*

(a) *We have $\sigma \prec \sigma^2$.*

(b) *The system Δ has a $\mathbb{Z}/2\mathbb{Z}$-grading and the generator σ is odd.*

3. *The connections of Δ with respect to the generator σ are non-zero.*

4. *For any $\lambda, \nu_1, \nu_2 \in \Delta$ with $\nu_1 \prec \sigma^n$, $\nu_2 \prec \sigma^n$, $\lambda \prec \sigma\nu_1$, and $\lambda \prec \sigma\nu_2$, we have $\mu \in \Delta$ with $\mu \prec \sigma^{n-1}$, $\nu_1 \prec \sigma\mu$, and $\nu_2 \prec \sigma\mu$.*

Then any 2-cocycle $(C_{\lambda\mu})_{\lambda\mu}$ of Δ is trivial.

Before presenting a proof, we make a comment on Condition 4. Consider the Bratteli diagram for the higher relative commutants of a subfactor $\sigma(M) \subset M$. We number the steps of the Bratteli diagrams as $0, 1, 2, \ldots$. Then Condition 4 says the following. (Recall that σ is self-conjugate.) Suppose we have vertices corresponding to ν_1 and ν_2 at the n-th step of the Bratteli diagrams, and they are connected to the vertex λ in the $n + 1$-st step. Then there exists a vertex μ in the $n - 1$-st step that is connected to ν_1 and ν_2. Note that if ν_1 and ν_2 already appear in the $n - 2$-nd step, then this condition trivially holds by taking $\mu = \lambda$. Thus, if the subfactor $\sigma(M) \subset M$ is of finite depth, then checking finitely many cases is sufficient for verifying Condition 4, and this can be done by drawing the principal graph of the subfactor $\sigma(M) \subset M$.

Proof Using Conditions 1, 3 and 4, we first prove that the unitary operator

$$C^\lambda_{\sigma\sigma\cdots\sigma} \in \mathrm{End}(\mathrm{Hom}(\lambda, \sigma\sigma\cdots\sigma))$$

is scalar for any $\lambda \in \Delta$. Let the number of σ's in $C^\lambda_{\sigma\sigma\cdots\sigma}$ be k and we prove the above property $C^\lambda_{\sigma\sigma\cdots\sigma} \in \mathbb{C}$ by induction on k. Note that the intertwiner space $\mathrm{Hom}(\lambda, \sigma\sigma\cdots\sigma)$ is decomposed as

$$\bigoplus \mathrm{Hom}(\lambda_1, \sigma\sigma) \otimes \mathrm{Hom}(\lambda_2, \lambda_1\sigma) \otimes \cdots \otimes \mathrm{Hom}(\lambda, \lambda_{k-2}\sigma),$$

and each of the space

$$\mathrm{Hom}(\lambda_1, \sigma\sigma) \otimes \mathrm{Hom}(\lambda_2, \lambda_1\sigma) \otimes \cdots \otimes \mathrm{Hom}(\lambda, \lambda_{k-2}\sigma)$$

is one-dimensional by Condition 1. Each such one-dimensional subspace gives a non-zero eigenvector of the unitary operator $C^\lambda_{\sigma\sigma\cdots\sigma}$ with eigenvalue

$$C^{\lambda_1}_{\sigma\sigma} C^{\lambda_2}_{\lambda_1\sigma} \cdots C^{\lambda}_{\lambda_{k-2}\sigma}$$

and what we have to prove is these eigenvalues are all identical. Note that the decomposition of $\mathrm{Hom}(\lambda, \sigma\sigma\cdots\sigma)$ as above is depicted graphically in Figure 1. Another picture Figure 2 gives another decomposition into a direct sum of one-dimensional eigenspaces. Roughly speaking, what we prove is that if a unitary matrix has several "different" decompositions into direct sums of one-dimensional eigenspaces, then the unitary matrix need to be a scalar multiple of the identity matrix.
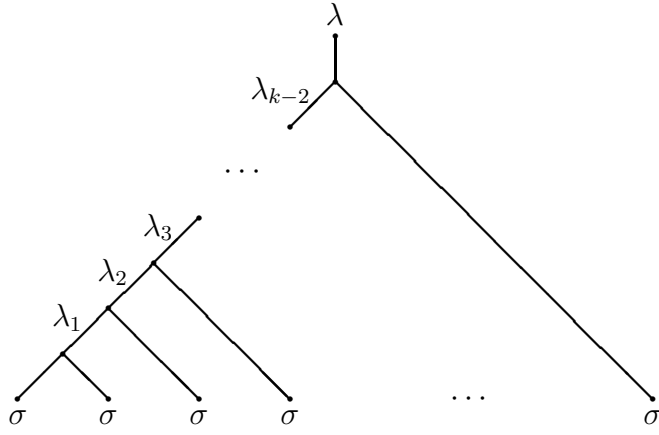
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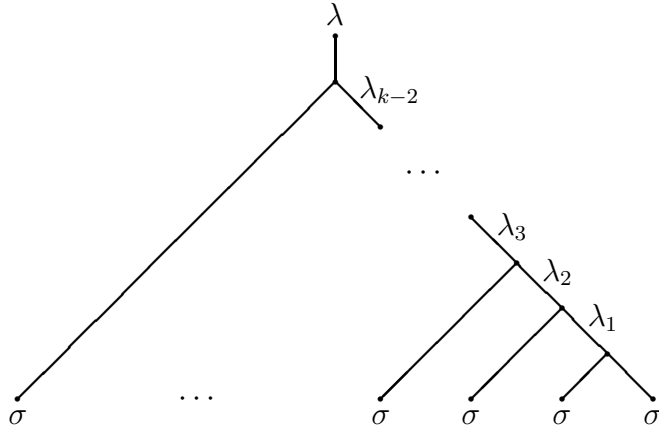Figure 1: Decomposition into a direct sum of one-dimensional eigenspaces

Figure 2: Decomposition into a direct sum of one-dimensional eigenspaces

First, let $k = 2$. By Condition 1, the space $\mathrm{Hom}(\lambda, \sigma\sigma)$ is one-dimensional for any $\lambda \in \Delta$, so we obviously have $C^\lambda_{\sigma\sigma} \in \mathbb{C}$.

Suppose now we have $C^\lambda_{\sigma\sigma\cdots\sigma} \in \mathbb{C}$ for any $\lambda \in \Delta$ if the number of σ's is less than or equal to k. We will prove $C^\lambda_{\sigma\sigma\cdots\sigma} \in \mathbb{C}$ for any $\lambda \in \Delta$ when the number of σ's is $k + 1$. First note that we have $C^\lambda_{\sigma\sigma\cdots\sigma} C^\mu_{\lambda\sigma} \in \mathbb{C}$ by the induction hypothesis and Condition 1. What we have to prove is that this scalar is independent of λ when μ is fixed. That is, suppose we have $\lambda, \lambda', \mu \in \Delta$, $\lambda \prec \sigma^k$, $\lambda' \prec \sigma^k$, $\mu \prec \lambda\sigma$, $\mu \prec \lambda'\sigma$. We will prove

$$C^\lambda_{\sigma\sigma\cdots\sigma} C^\mu_{\lambda\sigma} = C^{\lambda'}_{\sigma\sigma\cdots\sigma} C^\mu_{\lambda'\sigma} \in \mathbb{C}.$$

By Condition 4, there exists $\nu \in \Delta$ such that $\nu \prec \sigma^{k-1}$, $\lambda \prec \sigma\nu$, and $\lambda' \prec \sigma\nu$. Then there exists $\tau \in \Delta$ such that $\tau \prec \nu\sigma$ and $\mu \prec \sigma\tau$. Note that we have

$$C^\lambda_{\sigma\sigma\cdots\sigma} C^\mu_{\lambda\sigma} = C^\lambda_{\sigma\nu} C^\nu_{\sigma\sigma\cdots\sigma} C^\mu_{\lambda\sigma} \in \mathbb{C},$$

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where the number of σ's in $C^\nu_{\sigma\sigma\cdots\sigma}$ is $k-1$. The scalar $C^\lambda_{\sigma\nu}C^\mu_{\lambda\sigma}$ is the eigenvalue of the operator $C^\mu_{\sigma\nu\sigma}$ corresponding to the eigenvector given by the one-dimensional intertwiner space $\mathrm{Hom}(\lambda, \sigma\nu) \otimes \mathrm{Hom}(\mu, \lambda\sigma)$. Similarly, the scalar $C^\tau_{\nu\sigma}C^\mu_{\sigma\tau}$ is the eigenvalue of the same operator $C^\mu_{\sigma\nu\sigma}$ corresponding to the eigenvector given by the one-dimensional intertwiner space $\mathrm{Hom}(\tau, \nu\sigma) \otimes \mathrm{Hom}(\mu, \sigma\tau)$. Condition 3 implies that these two eigenvectors are not orthogonal, thus the two eigenvalues are equal, because the operator $C^\mu_{\sigma\nu\sigma}$ has an orthonormal basis of eigenvectors and thus it is normal. In this way, we obtain the identities

$$C^\lambda_{\sigma\nu}C^\mu_{\lambda\sigma} = C^\tau_{\nu\sigma}C^\mu_{\sigma\tau} = C^{\lambda'}_{\sigma\nu}C^\mu_{\lambda'\sigma},$$

which implies

$$C^\lambda_{\sigma\sigma\cdots\sigma}C^\mu_{\lambda\sigma} = C^\lambda_{\sigma\nu}C^\nu_{\sigma\sigma\cdots\sigma}C^\mu_{\lambda\sigma} = C^{\lambda'}_{\sigma\nu}C^\nu_{\sigma\sigma\cdots\sigma}C^\mu_{\lambda'\sigma} = C^{\lambda'}_{\sigma\sigma\cdots\sigma}C^\mu_{\lambda'\sigma} \in \mathbb{C},$$

as desired, where the numbers of σ's in $C^\lambda_{\sigma\sigma\cdots\sigma}$, $C^\nu_{\sigma\sigma\cdots\sigma}$, and $C^{\lambda'}_{\sigma\sigma\cdots\sigma}$ are k, $k-1$, and k, respectively.

We next prove the triviality of the cocycle C by using Condition 2.

First we assume we have 2 (a) of the assumptions in the Theorem, that is, $\sigma \prec \sigma^2$. Set $\omega_{\mathrm{id}} = 1$. Since $\mathrm{id} \prec \sigma^2$, the condition $C^\sigma_{\sigma\sigma\sigma} \in \mathbb{C}$ implies that $C^\sigma_{\sigma\sigma}C^\sigma_{\sigma\sigma} = C^{\mathrm{id}}_{\sigma\sigma}C^\sigma_{\mathrm{id}\sigma}$. By unitariry of C in Theorem 4.4, we have $|C^\sigma_{\sigma\sigma}| = 1$, we thus set $\omega_\sigma = (C^\sigma_{\sigma\sigma})^{-1} \in \mathbb{C}$. (Recall that we have already proved $C^\sigma_{\sigma\sigma}$ is a scalar.) Then this implies $\mathbb{C}^{\mathrm{id}}_{\sigma\sigma} = \omega_{\mathrm{id}}/\omega^2_\sigma$. For $\lambda \in \Delta$ not equivalent to id, σ, we choose a minimum positive integer k with $\lambda \prec \sigma^k$. We set $\omega_\lambda = \omega^k_\sigma C^\lambda_{\sigma\cdots\sigma} \in \mathbb{C}$, where the number of σ's in $C^\lambda_{\sigma\cdots\sigma}$ is k. For any $m > k$, we can represent the scalar $C^\lambda_{\sigma\cdots\sigma}$, where σ appears for m times, as $C^\sigma_{\sigma\sigma} \cdots C^\sigma_{\sigma\sigma}C^\lambda_{\sigma\cdots\sigma}$, where the number of $C^\sigma_{\sigma\sigma}$'s is $m-k$ and the number of σ's in $C^\lambda_{\sigma\cdots\sigma}$ is k. This implies $C^\lambda_{\sigma\cdots\sigma} = \omega_\lambda/\omega^m_\sigma$, where the number of σ's in $C^\lambda_{\sigma\cdots\sigma}$ is m. Now choose arbitrary $\lambda, \mu, \nu \in \Delta$ with $\lambda \prec \sigma^l$, $\mu \prec \sigma^m$. We can represent $C^\nu_{\sigma\cdots\sigma} \in \mathbb{C}$ with σ appearing for $l+m$ times, as the product $C^\nu_{\lambda\mu}C^\lambda_{\sigma\cdots\sigma}C^\mu_{\sigma\cdots\sigma}$ where σ's appear for l and m times, respectively, and then we obtain

$$C^\nu_{\lambda\mu}\frac{\omega_\lambda}{\omega^l_\sigma}\frac{\omega_\mu}{\omega^m_\sigma} = \frac{\omega_\nu}{\omega^{l+m}_\sigma},$$

which gives $C^\nu_{\lambda\mu}\omega_\lambda\omega_\mu = \omega_\nu$. Unitarity in Theorem 4.4 gives $\omega_\lambda\omega_\mu \neq 0$, we thus have $C^\nu_{\lambda\mu} = \omega_\nu/(\omega_\lambda\omega_\mu)$.

We next deal with the the case 2 (b), that is, we now assume that the system Δ has a $\mathbb{Z}/2\mathbb{Z}$-grading and the generator σ is odd. We first set $\omega_{\mathrm{id}} = 1$. Since $\mathrm{id} \prec \sigma^2$, we next set ω_σ to be a square root of $(C^{\mathrm{id}}_{\sigma\sigma})^{-1}$. (Note that $|C^{\mathrm{id}}_{\sigma\sigma}| = 1$ by unitarity in Theorem 4.4.) It does not matter which square root we choose. For $\lambda \in \Delta$ not equivalent to id, σ, we choose a minimum positive integer k with $\lambda \prec \sigma^k$ in the same way as above in the case of 2 (a). We again set $\omega_\lambda = \omega^k_\sigma C^\lambda_{\sigma\cdots\sigma} \in \mathbb{C}$, where the number of σ's in $C^\lambda_{\sigma\cdots\sigma}$ is k. For any $m > k$, we can represent the scalar $C^\lambda_{\sigma\cdots\sigma}$, where σ appears for m times, as $C^{\mathrm{id}}_{\sigma\sigma} \cdots C^{\mathrm{id}}_{\sigma\sigma}C^\lambda_{\sigma\cdots\sigma}$, where the number of $C^{\mathrm{id}}_{\sigma\sigma}$'s is $(m-k)/2$ and the number of σ's in $C^\lambda_{\sigma\cdots\sigma}$ is k, because $m-k$ is now even, due to the $\mathbb{Z}/2\mathbb{Z}$-grading. Then we obtain

$$C^\lambda_{\sigma\cdots\sigma} = \frac{1}{\omega^{m-k}_\sigma}\frac{\omega_\lambda}{\omega^k_\sigma} = \frac{\omega_\lambda}{\omega^m_\sigma},$$

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where the number of σ's in $C^\lambda_{\sigma\cdots\sigma}$ is m. Then the same argument as in the above case of 2 (a) proves the triviality of the cocycle $C_{\lambda\mu}$. $\qquad\square$

Remark 5.2. The 2-cohomology does not vanish in general, as well-known in the finite group case. For example, if the system Δ arises from an outer action of a finite group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, it is known that we have a non-trivial unitary 2-cocycle for this group G. So as in Example 4.5, the 2-cohomology for the corresponding tensor category does not vanish.

In [31, Theorem 2.4, Theorem 4.1], we have classified local extensions of the conformal nets $SU(2)_k$ and Vir_c with $k = 1, 2, 3, \ldots$ and $c = 1 - 6/m(m+1)$, $m = 2, 3, 4, \ldots$. (Here the symbol Vir_c denotes the Virasoro net with central charge c.) We use the symbols $SU(2)_k$ and Vir_c also for the corresponding C^*-tensor categories. We also say that the corresponding C^*-tensor categories of these local extensions of the nets $SU(2)_k$ and Vir_c are *extensions* of the tensor categories $SU(2)_k$ and Vir_c. Furthermore, the tensor category $SU(2)_k$ has a natural $\mathbb{Z}/2\mathbb{Z}$-grading and the even objects make a sub-tensor category. We call it *the even part of $SU(2)_k$*. We then have the following theorem.

Theorem 5.3. *Any finite system Δ of endomorphisms corresponding to one of the following tensor categories has a self-conjugate generator σ satisfying all the Conditions in Theorem 5.1, and thus, we have 2-cohomology vanishing for these tensor categories.*

1. *The $SU(2)_k$-tensor categories and their extensions.*

2. *The sub-tensor categories of those in Case 1.*

3. *The Virasoro tensor categories Vir_c with $c < 1$ and their extensions.*

4. *The sub-tensor categories of those in Case 3.*

Proof We deal with the following cases separately. Here for the extensions of $SU(2)_k$-tensor categories and the Virasoro tensor categories Vir_c, we use the labels by (pairs of) Dynkin diagrams as in [31, Theorem 2.4, Theorem 4.1], which arise from the labels of modular invariants by Cappelli-Itzykson-Zuber [9]. (These also correspond to the type I modular invariants listed in Table 1 in this paper.) Note that the braiding does not matter now, so we ignore the braiding structure here.

1. The $SU(2)_k$-tensor categories and their extensions.

   (a) Tensor categories A_n.
   (b) Tensor categories D_{2n}.
   (c) Tensor category E_6.
   (d) Tensor category E_8.

2. The (non-trivial) sub-tensor categories of those in Case 1.

    (a) The group $\mathbb{Z}/2\mathbb{Z}$.

    (b) The even parts of the $SU(2)_k$-tensor categories.

3. The Virasoro tensor categories Vir_c with $c < 1$ and their extensions.

    (a) Tensor categories (A_{n-1}, A_n).

    (b) Tensor categories (A_{4n}, D_{2n+2}).

    (c) Tensor categories (D_{2n+2}, A_{4n+2}).

    (d) Tensor category (A_{10}, E_6).

    (e) Tensor category (E_6, A_{12}).

    (f) Tensor category (A_{28}, E_8).

    (g) Tensor ategory (E_8, A_{30}).

4. The (non-trivial) sub-tensor categories of those in Case 3.

    (a) The sub-tensor categories of those in Case 3 (a).

    (b) The sub-tensor categories of those in Case 3 (b).

    (c) The sub-tensor categories of those in Case 3 (c).

    (d) The sub-tensor categories of those in Case 3 (d).

    (e) The sub-tensor categories of those in Case 3 (e).

    (f) The sub-tensor categories of those in Case 3 (f).

    (g) The sub-tensor categories of those in Case 3 (g).

Case 1 (a). We label the irreducible objects of the tensor category A_{k+1} with $0, 1, 2, \ldots, k$, as usual. Let σ be the standard generator 1. Condition 1 of Theorem 5.1 clearly holds. Since the fusion rule of the tensor category $SU(2)_k$ has a $\mathbb{Z}/2\mathbb{Z}$-grading and this generator 1 is odd, Condition 2 (b) also holds. Now the connection values with respect to this σ are the usual connection values of the paragroup A_{k+1} as in [42], [30], [14, Section 11.5], and they are non-zero and Condition 3 holds. The multiplication rule by the generator σ is described with the usual Bratteli diagram for the principal graph A_{k+1} as in [28], [14, Chapter 9], so we see that Condition 4 holds.

Case 1 (b). The irreducible objects of the tensor category are labeled with the even vertices of the Dynkin diagram D_{2n}. (So we also use the name D_{2n}^{even} for this tensor category.) If $2n = 4$, then this tensor category is given by the group $\mathbb{Z}/3\mathbb{Z}$, and we can verify the conclusion directly, so we assume that $n > 2$. We label σ as in Figure 3.

Then we can easily verify Conditions 1, 2 (a) and 4. We next verify Condition 3. We label four irreducible objects as in Figure 4. (If $n = 3$, we set $\lambda_1 = \mathrm{id}$.) Note that the connection with respect to the generator σ has a principal graph as in Figure 5. (See [24], for example, for the fusion rules of a subfactor with principal graph D_{2n}.)
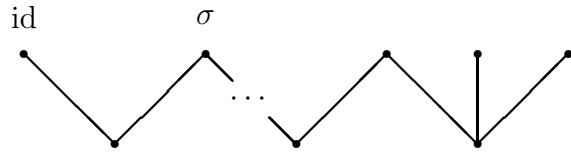
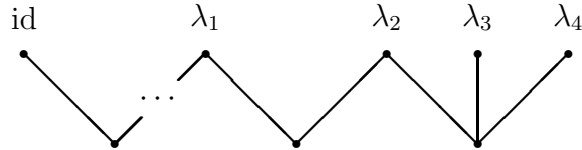Figure 3: The principal graph for the subfactor D_{2n}

Figure 4: The principal graph for the subfactor D_{2n}

We first claim that if the vertices λ_3 and λ_4 are not involved, then the connection values with respect to the the generator σ are non-zero. As in [3, II, Section 3], we may assume that the irreducible objects of the tensor category are realized as $\{\alpha_0, \alpha_2, \ldots, \alpha_{2n-4}, \alpha_{2n-2}^{(1)}, \alpha_{2n-2}^{(2)}\}$, arising from α-induction applied to the system $SU(2)_{4n-4}$ having the irreducible objects $\{0, 1, 2, \ldots, 4n-4\}$. (Note that it does not matter whether we use α^+ or α^-, so we have dropped the $\pm$ symbol.) We denote, by $W(i, j, k, l)$, the connection value with respect to the generator $\sigma = \alpha_2$ given by the square in Figure 6. (Note that the value $W(i, j, k, l)$ depends on the choices of intertwiners, but the absolute value $|W(i, j, k, l)|$ is independent of such choices, since the intertwiner spaces are now all one-dimensional.) For example, assume $n > 4$ and
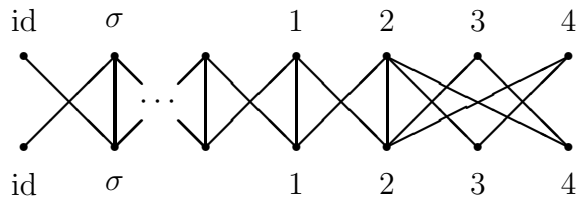

Figure 5: The principal graph for the subfactor $\sigma(M) \subset M$

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consider the connection value $W(\alpha_4, \alpha_6, \alpha_4, \alpha_6)$. By [3, II, Section 3], all the four intertwiners involved in this connection come from the intertwiners for $SU(2)_{4n-4}$, and thus the connection value is given by the connection $W(4, 6, 4, 6)$ for $SU(2)_{4n-4}^{\mathrm{even}}$ with respect to the generator 2. This value is given as a single term of $6j$-symbols of $SU(2)_{4n-4}$ and it is non-zero by [29]. The general case is dealt with in the same method.
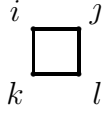
$$
\begin{array}{cc}
i & j \\
\square & \\
k & l
\end{array}
$$

Figure 6: A connection value for D_{2n}^{even}

Thus, we consider the remaining case where all the four vertices of the connection value are one of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. In the below, we denote the vertices $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ simply by $1, 2, 3, 4$. Denote this statistical dimensions of $1, 2, 3, 4$ by d_1, d_2, d_3, d_4 respectively. Their explicit values are as follows.

$$
\begin{aligned}
d_1 &= \frac{\sin \frac{2n-5}{4n-2}\pi}{\sin \frac{\pi}{4n-2}}, \\
d_2 &= \frac{\sin \frac{2n-3}{4n-2}\pi}{\sin \frac{\pi}{4n-2}}, \\
d_3 &= d_4 = \frac{1}{2 \sin \frac{\pi}{4n-2}}.
\end{aligned}
\tag{24}
$$

For a fixed pair (i, l), we denote the unitary matrix $(W(i, j, k, l))_{j,k}$ by W_{il}. Using the bi-unitarity Axioms 1 and 4 in [14, Chapter 10], originally due to [42], we compute several matrices W_{il} below. Recall that the renormalization Axiom 4 in [14, Chapter 10] now implies

$$
|W(i, j, k, l)| = \sqrt{\frac{d_j d_k}{d_i d_l}} |W(j, i, l, k)|.
$$

If $i = 1$ and $l \neq 3, 4$, then the entries in W_{il} are again given as single terms of the $6j$-symbols of $SU(2)_{4n-4}$ and thus, they are non-zero. The unitary matrices W_{13} and W_{14} have size 1×1, so the entries are obviously non-zero.

The unitary matrix W_{21} has a size 2×2, and all the entries in W_{il} are again given as single terms of the $6j$-symbols of $SU(2)_{4n-4}$ and thus, they are non-zero.

The unitary matrix W_{22} has a size 4×4. The entry $W(2, 1, 1, 2)$ is non-zero because we have already seen that W_{11} has no zero entries and we have the renormalization axiom. Similarly, the entries $W(2, 2, 1, 2)$, $W(2, 1, 2, 2)$, $W(2, 3, 1, 2)$, $W(2, 1, 3, 2)$, $W(2, 4, 1, 2)$, and $W(2, 1, 4, 2)$ are non-zero.

The entry $W(2, 2, 2, 2)$ is also given as a single term of the $6j$-symbols of $SU(2)_{4n-4}$ and thus, it is non-zero.

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We assume $W(2,3,2,2) = 0$ and will derive a contradiction. Using the renormalization axiom twice, we obtain $W(2,2,3,2) = 0$. Another use of the renormalization axiom gives $W(3,2,2,2) = 0$. Since the 2×2 matrix W_{32} is unitary, this implies $|W(3,4,2,2)| = 1$. The renormalization axiom then gives $|W(2,2,3,4)| = 1$. Since the 2×2 matrix W_{24} is unitary, this gives $W(2,3,3,4) = W(2,2,2,4) = 0$. These two equalities then give $W(3,4,2,3) = 0$ and $W(2,2,4,2) = 0$ with the renormalization axiom, respectively. Thus we have verified the $(2,4)$-entry of the 4×4 unitary matrix W_{22} is zero. Similarly, its $(4,2)$-entry is also zero. The identity $W(3,4,2,3) = 0$ and unitarity of the 2×2 matrix W_{33} give $|W(3,2,2,3)| = 1$. The renormalization axiom then produces $|W(2,3,3,2)| = d_3/d_2$. The 1×1 matrix W_{43} is unitary, thus the renormalization axiom gives $|W(2,4,3,2)| = d_3/d_2$. Similarly, we obtain $|W(2,3,4,2)| = d_3/d_2$. The 1×1 matrix W_{13} is unitary, thus the renormalization axiom gives $|W(2,1,3,2)| = \sqrt{d_1 d_3}/d_2$. Now we use the orthogonality of the second and third row vectors of the 4×4 unitary matrix W_{22}. We have so far obtained that the $(2,3)$, $(2,4)$, $(3,2)$-entries are zero and the $(3,1)$-entry is non-zero. We thus know that the $(2,1)$-entry is zero, but this is a contradiction because we have already seen above that the $(2,1)$-entry $W(2,1,2,2)$ is non-zero. We have thus proved $W(2,3,2,2) \neq 0$. By a similar method, we can prove that $W(2,4,2,2)$, $W(2,2,3,2)$ and $W(2,2,4,2)$ are all non-zero.

We next assume $W(2,3,3,2) = 0$. For the same reason as above, we obtain

$$|W(2,3,1,2)| = |W(2,4,1,2)| = |W(2,1,3,2)| = |W(2,1,4,2)| = \frac{\sqrt{d_1 d_3}}{d_2}, \quad (25)$$

$$|W(2,3,4,2)| = |W(2,4,3,2)| = \frac{d_3}{d_2}. \quad (26)$$

Since $W(2,3,3,2) = 0$, the renormalization axiom implies $W(3,2,2,3) = 0$. Since the 2×2-matrix W_{33} is unitary, we obtain $|W(3,2,4,3)| = 1$. The renormalization axiom gives $|W(2,3,3,4)| = \sqrt{d_3/d_2}$. Unitarity of the 2×2-matrix W_{24} then gives $|W(2,2,2,4)| = \sqrt{d_3/d_2}$, which then gives $|W(2,2,4,2)| = |W(2,4,2,2)| = d_3/d_2$ with the renormalization axiom. The identities (24), together with a simple computation of trigonometric functions, give

$$d_1 d_3 + 2d_3^2 = d_2^2. \quad (27)$$

Since the third row vector, the fourth row vector, and the third column vector of the unitary matrix W_{22} have a norm 1, this identity (27), together with (25), (26) gives $|W(2,2,3,2)| = |W(2,3,2,2)| = d_3/d_2$ and $W(2,4,4,2) = 0$. Thus the matrix $A = (A_{jk})_{jk} = (|W(2,k,j,2)|)_{jk}$ is given as follows, where α, β, γ are non-negative

real numbers.

$$
\begin{pmatrix}
\alpha & \beta & \dfrac{\sqrt{d_1 d_3}}{d_2} & \dfrac{\sqrt{d_1 d_3}}{d_2} \\[2ex]
\beta & \gamma & \dfrac{d_3}{d_2} & \dfrac{d_3}{d_2} \\[2ex]
\dfrac{\sqrt{d_1 d_3}}{d_2} & \dfrac{d_3}{d_2} & 0 & \dfrac{d_3}{d_2} \\[2ex]
\dfrac{\sqrt{d_1 d_3}}{d_2} & \dfrac{d_3}{d_2} & \dfrac{d_3}{d_2} & 0
\end{pmatrix}
\tag{28}
$$

Orthogonality of the first and third row vectors of W_{22} implies

$$
\frac{\sqrt{d_1 d_3}\, d_3}{d_2^2} \leqq \alpha \frac{\sqrt{d_1 d_3}}{d_2} + \beta \frac{d_3}{d_2}.
\tag{29}
$$

Since the first row vector of W_{22} has a norm 1, we also have

$$
\alpha^2 + \beta^2 = 1 - \frac{2 d_1 d_3}{d_2^2}.
\tag{30}
$$

The Cauchy-Schwarz inequality with (29), (30), we obtain

$$
\frac{\sqrt{d_1}\, d_3}{d_2} \leqq \sqrt{d_1 + d_3} \sqrt{1 - \frac{2 d_1 d_3}{d_2^2}},
$$

which, together with (27), implies

$$
\sqrt{d_1}\, d_3 \leqq \sqrt{d_1 + d_3} \sqrt{2 d_3^2 - d_1 d_3}.
$$

This implies $d_1^2 \leqq 2 d_3$, which gives

$$
\sin^2 \frac{2n - 5}{4n - 2} \pi \leqq \frac{1}{2}
\tag{31}
$$

by (24). This inequality (31) fails, if we have $(2n-5)/(4n-2) > 1/4$, that is, $n > 9/2$. Since we now assume $n \geq 3$, this has produced a contradiction and we have shown $W(2,3,3,2) \neq 0$, unless $n = 3, 4$. We deal with the remaining two cases $n = 3, 4$ by direct computations of the connection as follows.

If $n = 3$, we have the Dynkin diagram D_6. A subfactor with principal with D_6 is realized as the asymptotic inclusion [42, page 137], [14, Definition 12.23], [26, Section 2], of a subfactor with principal graph A_4 as in [43, Section III.1], [14, page 663], [26, Theorem 4.1]. Thus the tensor category D_6^{even} is realized as a self-tensor product of the tensor category of A_4^{even} and that our current generator σ is realized as a tensor product of the standard generators in two copies of A_4^{even}. As in Case 2 (b) below, the connection values are non-zero for A_4^{even}, thus our current connection values are also non-zero as products of two non-zero values.
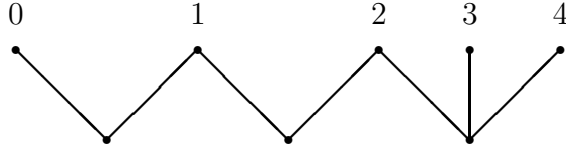
Figure 7: The principal graph for the subfactor D_8

We finally deal with the case $n = 4$. We label the even vertices of the principal graph D_8 as in Figure 7.

We continue the computations of $|W(i,j,k,l)|$'s using the matrix (28), where the non-negative real numbers α, β, γ have been defined. The renormalization axiom gives $|W(1,1,2,1)| = \sqrt{d_2/d_1}|W(1,1,1,2)|$ and unitarity of the 2×2-matrix W_{12} gives $|W(1,2,2,2)| = |W(1,1,1,2)|$. So we have

$$|W(1,1,2,1)| = |W(1,2,1,1)| = \sqrt{\frac{d_2}{d_1}}|W(1,2,2,2)| = |W(2,1,2,2)| = \beta, \qquad (32)$$

again by the renormalization. We also have

$$|W(1,2,1,1)| = \beta. \qquad (33)$$

Unitarity of the 1×1-matrix W_{02} gives $|W(0,1,1,2)| = 1$ and thus, the renormalization axiom gives

$$|W(1,0,2,1)| = |W(1,2,0,1)| = \frac{\sqrt{d_2}}{d_1}, \qquad (34)$$

since $d_0 = 1$. Similarly, unitarity of the 1×1-matrix W_{01} gives

$$|W(1,0,1,1)| = |W(1,1,0,1)| = \frac{1}{\sqrt{d_1}}, \qquad (35)$$

and unitarity of the 1×1-matrix W_{00} gives

$$|W(1,0,0,1)| = \frac{1}{d_1}. \qquad (36)$$

We also have

$$|W(1,2,2,1)| = \frac{d_2}{d_1}|W(2,1,1,2)| = \frac{d_2}{d_1}\alpha, \qquad (37)$$

Thus the 3×3-matrix $B = (B_{jk})_{jk} = (|W(1, k, j, 1)|)_{jk}$ is given as follows, where δ is a non-negative real number, by (32), (33), (34), (35), (36), (37).

$$
\begin{pmatrix}
\dfrac{1}{d_1} & \dfrac{1}{\sqrt{d_1}} & \dfrac{\sqrt{d_2}}{d_1} \\
\dfrac{1}{\sqrt{d_1}} & \delta & \beta \\
\dfrac{\sqrt{d_2}}{d_1} & \beta & \dfrac{d_2}{d_1}\alpha
\end{pmatrix}
\tag{38}
$$

The first row vector of the matrix (28) has a norm 1, thus we have

$$
\alpha^2 + \beta^2 = 1 - \frac{2d_1 d_3}{d_2^2}.
\tag{39}
$$

The third row vector of the matrix (38) has a norm 1, thus we have

$$
\frac{d_2}{d_1^2} + \beta^2 + \frac{d_2^2}{d_1^2}\alpha^2 = 1.
\tag{40}
$$

Equations (39) and (40) give the following value for β^2.

$$
\beta^2 = \frac{d_2^2 - 2d_1 d_3 - d_1^2 + d_2}{d_2^2 - d_1^2}.
\tag{41}
$$

Note that the denominator is not zero. Let t be the index of the subfactor with principal graph D_8. (That is, $t = 4\cos^2 \pi/14$.) Then the Perron-Frobenius theory gives the following identities.

$$
\begin{aligned}
d_1 &= t - 1, \\
d_2 &= t^2 - 3t + 1, \\
d_3 &= \frac{t^3 - 5t^2 + 6t - 1}{2}.
\end{aligned}
$$

Then these imply $d_2^2 - 2d_1 d_3 - d_1^2 + d_2 = 0$ in (41), we thus obtain $\beta = 0$, which has been already excluded above. We have thus reached a contradiction and shown $W(2, 3, 3, 2) \neq 0$.

Similarly, we can prove $W(2, 4, 4, 2) \neq 0$.

The unitary matrix W_{34} has a size 1×1, so the renormalization axiom implies $W(2, 4, 3, 2) \neq 0$. Similarly, we have $W(2, 3, 4, 2) \neq 0$. We have thus proved that all the entries of W_{22} are non-zero.

The unitary matrix W_{23} has a size 2×2. If this matrix has a zero entry, we have either $W(2, 2, 2, 3) = W(2, 4, 4, 3) = 0$ or $W(2, 2, 4, 3) = W(2, 4, 2, 3) = 0$. The former case, together with the renormalization axiom, implies $W(2, 2, 3, 2) = 0$, which is already excluded in the above study of W_{22}. The latter case gives $|W(2, 4, 4, 3)| = 1$, which, together with the renormalization axiom, implies $|W(4, 2, 3, 4)| = \sqrt{d_2/d_4} > 1$ by (24). This is against the unitarity axiom and thus cannot happen.

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The 2×2 unitary matrix W_{24} is dealt with in a similar way to the case W_{23}.

The unitary matrices W_{31} and W_{34} also have size 1×1, so the entries are again non-zero. The matrices W_{32} and W_{33} have size 2×2. The entries of W_{32} have the same absolute values as the entries of W_{23}, so the above arguments for W_{23} show that they are non-zero. We next consider W_{33}. If this 2×2 unitary matrix contains a zero entry, then we have either $W(3,2,2,3) = W(3,4,4,3) = 0$ or $W(3,2,4,3) = W(3,4,2,3) = 0$. The former case, together with the renormalization axiom, implies $W(2,3,3,2) = 0$, which is already excluded in the above study of W_{22}. The latter case, together with the renormalization axiom, implies $W(2,3,3,4) = 0$, which is already excluded in the above study of W_{24}.

The four matrices W_{4l} can be dealt with in the same way as above for W_{3l}.

Thus we are done for Case 1 (b).

Case 1 (c). Only fusion rules and $6j$-symbols matter, and the braiding does not matter, for the Conditions in Theorem 5.1, so our tensor category can be identified with $SU(2)_2$ and this is a special case of Case 1 above.

Case 1 (d). In a similar way to the above case, this tensor category can be identified with the even part of the tensor category $SU(2)_3$, so this is a special case of Case 2 (b) below.

Case 2 (a). This is trivial.

Case 2 (b). We label the irreducible objects of the tensor category $SU(2)_k$ with index $0, 1, 2, \ldots, k$, as above. (We also use the name A_{k+1}^{even} for this tensor category.) Let σ be the generator 2 this time. Conditions 1 and 2 (a) of Theorem 5.1 clearly hold. Since all $6j$-symbols for $SU(2)_k$ have non-zero values as in [29], Condition 3 holds, in particular. The multiplication rule by the generator σ is described with the even steps of the usual Bratteli diagram for the principal graph A_{k+1} as in [28], [14, Chapter 9], so we see that Condition 4 holds.

Case 3 (a). This is the Virasoro tensor category with central charge $c = 1 - 6/n(n+1)$. We recall the description of the irreducible objects in the tensor category given by [53, Theorem 4.6] applied to $SU(2)_{n-1} \subset SU(2)_{n-2} \otimes SU(2)_1$, as follows. (Also see [31, Section 3] for our notations.) We now have a net of subfactors $\mathrm{Vir}_c \otimes SU(2)_{n-1} \subset SU(2)_{n-2} \otimes SU(2)_1$ with finite index and apply the α-induction to this inclusion. The irreducible representations of the net Vir_c are labeled as

$$\{\sigma_{j,k} \mid j = 0, 1, \ldots, n-2, \quad k = 0, 1, \ldots, n-1, \quad j+k \in 2\mathbb{Z}\}.$$

Xu's result [53, Theorem 4.6] then shows the following. First, the systems $\{\sigma_{j,k}\}$ and $\{\alpha_{\sigma_{j,k} \otimes \mathrm{id}}\}$ have the isomorphic fusion rules and $6j$-symbols. Furthermore, the latter system is isomorphic to the system

$$\{(\lambda'_j \otimes \mathrm{id})(\alpha_{\mathrm{id} \times \lambda_k}) \mid j = 0, 1, \ldots, n-2, \quad k = 0, 1, \ldots, n-1, \quad j+k \in 2\mathbb{Z}\},$$

where $\{\lambda_k \mid k = 0, 1, \ldots, n-1\}$ and $\{\lambda'_j \mid j = 0, 1, \ldots, n-2\}$ are the system of irreducible DHR endomorphisms of the nets $SU(2)_{n-1}$ and $SU(2)_{n-2}$, respectively. This system have further isomorphic fusion rules and $6j$-symbols to the system

$$\{\lambda'_j \otimes \lambda_k \mid j = 0, 1, \ldots, n-2, \quad k = 0, 1, \ldots, n-1, \quad j+k \in 2\mathbb{Z}\}, \qquad (42)$$

of irreducible DHR endomorphisms of the net $SU(2)_{n-2} \otimes SU(2)_{n-1}$. (Note that we have a restriction $j + k \in 2\mathbb{Z}$, so this system is a subsystem of that of the all the irreducible DHR endomorphisms of the net $SU(2)_{n-2} \otimes SU(2)_{n-1}$.) As in [31, Section 3], we can identify the system of these $\sigma_{j,k}$'s with the system of characters of the minimal models [11, Subsection 7.3.4] whose fusion rules are given in [11, Subsection 7.3.3]. We take the DHR endomorphism $\sigma_{1,1}$ as σ in Theorem 5.1 and then, from these fusion rules, we easily see that Condition 1 holds. It is also easy to see that we have a natural $\mathbb{Z}/2\mathbb{Z}$-grading such that σ is an odd generator, so Condition 2 (b) holds. By considering the connection of the system (42), we know that the connection value with respect to the generator σ is a product of the two connection values of the systems $SU(2)_{n-2}$ and $SU(2)_{n-1}$ with respect to the standard generators. Since these two connection values for $SU(2)_{n-2}$ and $SU(2)_{n-1}$ are the usual connection values for the paragroups labeled with the Dynkin diagrams A_{n-1} and A_n, and they are non-zero by [42], [30], [14, Section 11.5], we conclude that Condition 3 holds. From the fusion rule described as above, we verify that Condition 4 also holds. (Recall the comment on Condition 4 after the statement of Theorem 5.1 and draw the principal graph for a subfactor given by $\sigma_{1,1}$.)

Case 3 (b). The tensor category is produced with α-induction and a simple current extension of index 2 as in [3, II, Section 3]. The fusion rules and $6j$-symbols are given by a direct product of the two systems A_{4n}^{even} and D_{2n+2}^{even}. We can use the direct product of the σ in Figure 3 and the σ in Case 2 (b) as the current σ for Theorem 5.1. Then Conditions 1, 2 (a), and 4 easily follow and the connection values are non-zero as products of non-zero values in Cases 1 (b) and 2.

Case 3 (c). This case is proved in a similar way to the above proof of case 3 (b).

Case 3 (d). The tensor category is produced with α-induction as in [31, Section 4.2]. The irreducible objects of the tensor category are labeled with pairs (j, k) with $j = 0, 1, \ldots, 9$ and $k = 0, 1, 2$ with $j + k \in 2\mathbb{Z}$. The fusion rules of the objects $\{(j, 0) \mid j = 0, 1, \ldots, 9\}$ obey the A_{10} fusion rule and those of $\{(0,0), (0,1), (0,2)\}$ obey the A_3 fusion rule. Let σ be the object $(1, 1)$. Then as in Case 1, we can verify Conditions 1, 2 (b), 3 and 4.

Case 3 (e). This case is proved in a similar way to the above proof of case 3 (d).

Case 3 (f). The tensor category is again produced with α-induction as in [31, Section 4.2]. The fusion rules and $6j$-symbols are given as the direct product of the two systems A_{28}^{even} and A_4^{even}. The irreducible objects of the former system are labeled with $0, 2, \ldots, 26$ as usual, and the latter system is given as $\{\text{id}, \tau\}$ with $\tau^2 = \text{id} \oplus \tau$. Then we can choose $(14, \tau)$ as σ and verify Conditions 1, 2 (a), 3 and 4, using the same arguments as in Cases 2 and 3 (b).

Case 3 (g). This case is proved in a similar way to the above proof of case 3 (f).

Case 4 (a). Now, the only non-trivial sub-tensor categories are $\mathbb{Z}/2\mathbb{Z}$, $SU(2)_{n-2}$, $SU(2)_{n-2}^{\text{even}}$, $SU(2)_{n-1}$, $SU(2)_{n-1}^{\text{even}}$ and the even parts with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading described in the above proof of Case 3 (a). The conclusion trivially holds for the first case. The next four cases have been already dealt with in Cases 1(a) and 2 (b). In the last case, we can identify the tensor category with the direct product of two tensor

categories $SU(2)_{n-2}^{\mathrm{even}}$ and $SU(2)_{n-1}^{\mathrm{even}}$. We use the same labeling of the irreducible DHR sectors as in the proof of the Case 3 (a) and then we can use the generator $\sigma_{2,2}$ as σ in Theorem 5.1.

Case 4 (b). The only non-trivial sub-tensor categories we have are now A_{4n}^{even} and D_{2n+2}^{even}. Thus, we have the conclusion by Cases 2 (b) and 1 (b), respectively.

Case 4 (c). This case is proved in a similar way to the above proof of case 4 (b).

Case 4 (d). The only non-trivial sub-tensor categories we have are now $\mathbb{Z}/2\mathbb{Z}$, A_{10}^{even}, their direct product, and A_3. We can deal with the group $\mathbb{Z}/2\mathbb{Z}$ trivially. The cases A_{10}^{even} and A_3 are particular cases of Cases 2 (b) and 1 (a), respectively. For the case of the direct product of A_{10}^{even} and $\mathbb{Z}/2\mathbb{Z}$, we can choose $\sigma = (2,2)$ in the notation of the proof for Case 3 (d).

Case 4 (e). This case is proved in a similar way to the above proof of case 4 (d).

Case 4 (f). The only non-trivial sub-tensor categories we have are now A_4^{even} and A_{28}^{even}. Both are special cases of Case 2 (b).

Case 4 (g). This case is proved in a similar way to the above proof of case 4 (f). $\qquad\square$

Remark 5.4. We have the following application of the above theorem. Consider the tensor category corresponding to the WZW-model $SU(2)_{28}$. Regard the irreducible objects as irreducible endomorphisms of a type III factor M and label them as $\mathrm{id} = \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{28}$ as usual. Then the endomorphism $\gamma = \lambda_0 \oplus \lambda_{10} \oplus \lambda_{18} \oplus \lambda_{28}$ is a dual canonical endomorphism and uniqueness of Q-system (γ, V, W) up to unitary equivalence was shown in [33, Section 6] based on a result in vertex operator algebras. (This uniqueness was used in our previous work [31].) Izumi has also given another proof of this uniqueness with a more direct method. We remark that our above theorem also gives a different proof of this uniqueness as follows.

We may assume that M is injective. Suppose that we have two endomorphisms ρ_1, ρ_2 of M such that $\rho_1 \bar{\rho}_1 = \rho_2 \bar{\rho}_2 = \gamma$. As in [6, Proposition A.3], we can prove that the two subfactors $\rho_1(M) \subset M$ and $\rho_2(M) \subset M$ have the isomorphic higher relative commutants, and then we conclude by [45, Corollary 6.4] that the two subfactors are isomorphic via $\theta \in \mathrm{Aut}(M)$. We then may and do assume $\rho_2 = \theta \cdot \rho_1$ and now we have $\theta \cdot \gamma \cdot \theta^{-1} = \gamma$. Since $\gamma = \lambda_0 \oplus \lambda_{10} \oplus \lambda_{18} \oplus \lambda_{28}$ and powers of γ produce all of $\lambda_0, \lambda_2, \lambda_4, \ldots, \lambda_{28}$, we know that $[\theta \cdot \lambda_{2j} \cdot \theta^{-1}] = [\lambda_{2j}]$ for $j = 0, 1, 2, \ldots, 14$, where the sqare brackets denote the unitary equivalence classes. Then we have a map

$$\theta : \mathrm{Hom}(\lambda, \mu) \ni t \mapsto \theta(t) \in \mathrm{Hom}(\theta \cdot \lambda \cdot \theta^{-1}, \theta \cdot \mu \cdot \theta^{-1})$$

giving an automorphism of the tensor category generated by powers of γ. By Case 2 of Theorem 5.3, this automorphism θ is trivial in the sense of Definition 4.2. The automorphism θ sends the Q-system (γ, V_1, W_1) for ρ_1 to the one (γ, V_2, W_2) for ρ_2, and now the triviality of θ implies that these two systems are unitarily equivalent.

Using the above Theorem 5.3, we obtain the following classification result of 2-dimensional completely rational nets. The meaning of the condition that the μ-index is 1 will be further studied in the next section.

Consider a 2-dimensional local completely rational conformal net $\mathcal{B}$ with central charge $c = 1 - 6/m(m+1) < 1$ and μ-index $\mu_\mathcal{B} = 1$. By [47], we have inclusions

$$\mathcal{A}_L \otimes \mathcal{A}_R \subset \mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max} \subset \mathcal{B},$$

where $\mathcal{A}_L, A_R, \mathcal{A}_L^{\max}, \mathcal{A}_R^{\max}$ are one-dimensional local conformal nets. By assumption, $\mathcal{A}_L^{\max}$ and $\mathcal{A}_R^{\max}$ have the same central charge c. Rehren's result [47, Corollary 3.5] and our results [32, Proposition 24] together imply that the fusion rules of the systems of entire irreducible DHR endomorphisms of the two nets $\mathcal{A}_L^{\max}, \mathcal{A}_R^{\max}$ are isomorphic, and our previous result [31, Theorem 5.1] implies that the two nets $\mathcal{A}_L^{\max}, \mathcal{A}_R^{\max}$ are isomorphic as nets. Since both $\mathcal{A}_L^{\max}, \mathcal{A}_R^{\max}$ contain Vir_c as subnets, we obtain an irreducible inclusion $\mathrm{Vir}_c \otimes \mathrm{Vir}_c \subset \mathcal{B}$. A decomposition of a vacuum sector of $\mathcal{B}$ restricted on $\mathrm{Vir}_c \otimes \mathrm{Vir}_c$ produces a decomposition matrix $(Z_{\lambda\mu})_{\lambda\mu}$, where λ, μ are representatives of uintary equivalence classes of irreducible DHR endomorphisms of the net Vir_c. Since $\mu_\mathcal{B} = 1$, by Theorem 3.1, due to Müger [41], we know that this matrix Z is a modular invariant of the Virasoro tensor category Vir_c and such modular invariants have been classified by Cappelli-Itzykson-Zuber [9] as in Table 1.

m	Labels for modular invariants in [9]	Type
n	(A_{n-1}, A_n)	I
$4n$	(D_{2n+1}, A_{4n})	II
$4n+1$	(A_{4n}, D_{2n+2})	I
$4n+2$	(D_{2n+2}, A_{4n+2})	I
$4n+3$	(A_{4n+2}, D_{2n+3})	II
11	(A_{10}, E_6)	I
12	(E_6, A_{12})	I
17	(A_{16}, E_7)	II
18	(E_7, A_{18})	II
29	(A_{28}, E_8)	I
30	(E_8, A_{30})	I

Table 1: Modular invariants for the Virasoro tensor category Vir_c

We claim that this correspondence from $\mathcal{B}$ to Z is bijective.

Theorem 5.5. *The above correspondence from $\mathcal{B}$ to Z gives a bijection from the set of isomorphism classes of such two-dimensional nets to the set of modular invariants Z in Table 1.*

Proof We first prove that this correspondence is surjective. Take a modular invariant Z in Table 1. By [31, Subsections 4.1, 4.2, 4.3], we conclude that this modular invariant can be realized with α-induction as in [5, Corollary 5.8] for extensions of the Virasoro nets. Then Rehren's results in [48, Theorem 1.4, Proposition 1.5] imply that we have a corresponding Q-system and a local extension $\mathcal{B} \supset \mathrm{Vir}_c \otimes \mathrm{Vir}_c$ and that this $\mathcal{B}$ produces the matrix Z in the above correspondence.

We next show injectivity of the map. Suppose that we have inclusion

$$\mathcal{A}_L \otimes \mathcal{A}_R \subset \mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max} \subset \mathcal{B},$$

where $\mathcal{A}_L, \mathcal{A}_R$ are isomorphic to Vir_c and that this decomposition gives a matrix Z. We have to prove that the net $\mathcal{B}$ is uniquely determined up to isomorphism. Recall that the nets $\mathcal{A}_L^{\max}$ and $\mathcal{A}_R^{\max}$ are among those classified by [31, Theorem 5.1]. As we have seen above, $\mathcal{A}_L^{\max}$ and $\mathcal{A}_R^{\max}$ are isomorphic as nets and we can naturally identify them. This isomorphism class and an isomorphism π from a fusion rule of $\mathcal{A}_L^{\max}$ onto that of $\mathcal{A}_R^{\max}$ are uniquely determined by Z by [31, Theorem 5.1] (Also see [4].)

If the modular invariant is of type I, then we can naturally identify $\mathcal{A}_L^{\max}$ and $\mathcal{A}_R^{\max}$ and the map π is trivial. Then the Q-system for the inclusion $\mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max} \subset \mathcal{B}$ has a standard dual canonical endomorphisms as in the Longo-Rehren Q-system and the above results Corollary 4.6, Theorems 5.1, 5.3 imply that this Q-system is equivalent to the Longo-Rehren Q-system.

If the modular invariant is of type II, then we have a non-trivial fusion rule automorphism π. We then know by [4, Lemma 5.3] that this fusion rule automorphism π actually gives an automorphism of the tensor category acting non-trivially on irreducible objects. The same arguments as in the proof of Theorem 4.4 show that 2-cohomology vanishing implies uniqueness of the Q-system. Again, the above results Corollary 4.6, Theorems 5.1, 5.3 give the 2-cohomology vanishing, thus we have the desired uniqueness of the Q-system for the inclusion $\mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max} \subset \mathcal{B}$. □

Remark 5.6. In the case the modular invariant Z above is of type II, the automorphism π of the tensor category above is actually an autmorphism of a braided tensor category, as seen from the above proof.

In the above classification, we have shown 2-cohomology vanishing without assuming locality. In the context of classification of two-dimensional nets, this means that any (relatively local irreducible) extension $\mathcal{B}$ of $\mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max}$ with μ-index being 1 is automatically local.

6 The μ-index, maximality of extensions, and classification of non-maximal nets

In Theorem 5.5, we have classified 2-dimensional completely rational local conformal nets and central charge less than 1 under the assumption that the μ-index is 1. In this section, we clarify the meaning of this condition on the μ-index. As we have seen above, this condition is equivalent to triviality of the superselection structure of the net. We further show that this condition is equivalent to maximality of extensions of the 2-dimensional net, when we have a parity symmetry for the net $\mathcal{B}$. Here the net $\mathcal{B}$ is said to have a parity symmetry if we have a vacuum-fixing unitary involution P such that $P\mathcal{B}(O)P = \mathcal{B}(pO)$, where p maps $x + t \mapsto x - t$ in the two-dimensional

Minkowski space. In this case, P clearly implements an isomorphism of $\mathcal{A}_L$ and $\mathcal{A}_R$ and thus, an isomorphism of $\mathcal{A}_L^{\max}$ and $\mathcal{A}_R^{\max}$.

Suppose we have a local extension $\mathcal{C}$ of the two-dimensional completely rational local conformal net $\mathcal{B}$ and the inclusion $\mathcal{B} \subset \mathcal{C}$ is strict. Then we have $\mu_\mathcal{B} > \mu_C \geq 1$ by [32, Proposition 24]. That is, if the net $\mathcal{B}$ is not maximal with respect to local extensions, then we have $\mu_\mathcal{B} > 1$. This argument does not require a parity symmetry condition.

Conversely, suppose we have $\mu_\mathcal{B} > 1$. Then the results in [41] show that the dual canonical endomorphism for the inclusion $\mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max} \subset \mathcal{B}$ is of the form $\bigoplus \lambda \otimes \pi(\lambda)$, where both $\mathcal{A}_L^{\max}$ and $\mathcal{A}_R^{\max}$ are local extensions of Vir_c and λ runs through a proper subsystem of the system of the irreducible DHR endomorphisms of the net $\mathcal{A}_L^{\max}$ and π is an isomorphism from such system onto another subsystem of irreducible DHR endomorphisms of the net $\mathcal{A}_R^{\max}$. Both $\mathcal{A}_L^{\max}$ and $\mathcal{A}_R^{\max}$ are in the classification list of [31, Theorem 5.1], and now they are isomorphic. Recall that at least one of the two subsystems is a proper subsystem, since $\mu_\mathcal{B} > 1$, and the parity symmetry condition now implies that both subsystems are proper.

First suppose that the map π is trivial. Then the Q-system for the inclusion $\mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max} \subset \mathcal{B}$ is the usual Longo-Rehren Q-system arising from the subsystem by Corollary 4.6, Theorem 5.1 and Case 4 of Theorem 5.3. Then, Izumi's Galois correspondence [26, Theorem 2.5] shows that we have a further extension $\mathcal{C} \supset \mathcal{B}$ such that the Q-system for $\mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max} \subset \mathcal{C}$ is the Longo-Rehren Q-system using the entire system of the irreducible DHR endomorphisms of $\mathcal{A}_L^{\max}$ and the index $[\mathcal{C} : \mathcal{B}]$ is strictly larger than 1. We know that the extension $\mathcal{C}$ arising from the Longo-Rehren Q-system is local. That is, the net $\mathcal{B}$ is not maximal with respect to local extensions.

Next suppose that the map π is non-trivial. By checking the representation categories of the local extensions of the Virasoro nets classified in [31, Theorem 5.1], we know that only such non-trivial isomorphisms arise from interchanging of $2j$ and $4n - 2 - 2j$ of the system $SU(2)_{4n-2}$, where $j = 0, 1, \ldots, 2n - 1$, or the well-known non-trivial automorphism of the system D_{10}^{even}. In both cases, the map π can be extended to an automorphism of the entire system of irreducible DHR endomorphism of $\mathcal{A}_L^{\max}$ and we can obtain a proper extension $\mathcal{C} \supset \mathcal{B}$ in a similar way to the above case. Thus, again, the net $\mathcal{B}$ is not maximal with respect to local extensions. We summarize these proper sub-tensor categories of the extensions of the Virasoro tensor categories Vir_c ($c < 1$) with trivial or non-trivial automorphisms as in Table 2. Each entry "nontrivial" means that we have a unique nontrivial automorphism for the sub-tensor category. For example, the sub-tensor category $SU(2)_6^{\mathrm{even}}$ of (A_7, A_8) appears in the case $n = 8$ of the 4th entry having a trivial automorphism and the case $n = 2$ of the 5th entry having a nontrivial automorphism. We thus have exactly two non-maximal local conformal nets for this sub-tensor category.

Thus we have proved that the net $\mathcal{B}$ with parity symmetry has $\mu_\mathcal{B} = 1$ if and only if it is maximal with respect to local extensions. In such a case, we say that $\mathcal{B}$ is a maximal net. These results, together with Theorem 5.5, imply the following main theorem of this paper immediately.

m	Tensor category	Sub-tensor category	Automorphism
n	(A_{n-1}, A_n)	$\{\mathrm{id}\}$	trivial
n	(A_{n-1}, A_n)	$\mathbb{Z}/2\mathbb{Z}$	trivial
n	(A_{n-1}, A_n)	$SU(2)_{n-2}$	trivial
n	(A_{n-1}, A_n)	$SU(2)_{n-2}^{\mathrm{even}}$	trivial
$4n$	(A_{4n-1}, A_{4n})	$SU(2)_{4n-2}^{\mathrm{even}}$	nontrivial
n	(A_{n-1}, A_n)	$SU(2)_{n-1}$	trivial
$4n-1$	(A_{4n-2}, A_{4n-1})	$SU(2)_{4n-2}^{\mathrm{even}}$	nontrivial
n	(A_{n-1}, A_n)	$SU(2)_{n-2}^{\mathrm{even}} \times SU(2)_{n-1}^{\mathrm{even}}$	trivial
$4n$	(A_{4n-1}, A_{4n})	$SU(2)_{4n-2}^{\mathrm{even}} \times SU(2)_{4n-1}^{\mathrm{even}}$	nontrivial
$4n-1$	(A_{4n-2}, A_{4n-1})	$SU(2)_{4n-3}^{\mathrm{even}} \times SU(2)_{4n-2}^{\mathrm{even}}$	nontrivial
$4n+1$	(A_{4n}, D_{2n+2})	$\{\mathrm{id}\}$	trivial
$4n+1$	(A_{4n}, D_{2n+2})	$SU(2)_{4n-1}^{\mathrm{even}}$	trivial
$4n+1$	(A_{4n}, D_{2n+2})	D_{2n+2}^{even}	trivial
$4n+2$	(D_{2n+2}, A_{4n+2})	$\{\mathrm{id}\}$	trivial
$4n+2$	(D_{2n+2}, A_{4n+2})	$SU(2)_{4n+1}^{\mathrm{even}}$	trivial
$4n+2$	(D_{2n+2}, A_{4n+2})	D_{2n+2}^{even}	trivial
11	(A_{10}, E_6)	$\{\mathrm{id}\}$	trivial
11	(A_{10}, E_6)	$\mathbb{Z}/2\mathbb{Z}$	trivial
11	(A_{10}, E_6)	$SU(2)_2$	trivial
11	(A_{10}, E_6)	$SU(2)_9^{\mathrm{even}}$	trivial
11	(A_{10}, E_6)	$\mathbb{Z}/2\mathbb{Z} \times SU(2)_9^{\mathrm{even}}$	trivial
12	(E_6, A_{12})	$\{\mathrm{id}\}$	trivial
12	(E_6, A_{12})	$\mathbb{Z}/2\mathbb{Z}$	trivial
12	(E_6, A_{12})	$SU(2)_2$	trivial
12	(E_6, A_{12})	$SU(2)_{11}^{\mathrm{even}}$	trivial
12	(E_6, A_{12})	$\mathbb{Z}/2\mathbb{Z} \times SU(2)_{11}^{\mathrm{even}}$	trivial
17	(A_{16}, D_{10})	D_{10}^{even}	nontrivial
18	(D_{10}, A_{18})	D_{10}^{even}	nontrivial
29	(A_{28}, E_8)	$\{\mathrm{id}\}$	trivial
29	(A_{28}, E_8)	$SU(2)_3^{\mathrm{even}}$	trivial
29	(A_{28}, E_8)	$SU(2)_{27}^{\mathrm{even}}$	trivial
30	(E_8, A_{30})	$\{\mathrm{id}\}$	trivial
30	(E_8, A_{30})	$SU(2)_3^{\mathrm{even}}$	trivial
30	(E_8, A_{30})	$SU(2)_{29}^{\mathrm{even}}$	trivial

Table 2: Proper sub-tensor categories of extensions of the Virasoro tensor categories Vir_c with automorphisms

Theorem 6.1. *The above correspondence from $\mathcal{B}$ to Z in Theorem 5.5 gives a bijection from the set of isomorphism classes of such maximal two-dimensional nets with parity symmetry and central charge less than 1 to the set of modular invariants Z in Table 1.*

Furthermore, the above discussions on the possible proper sub-tensor categories of the extensions of the Virasoro tensor categories Vir_c ($c < 1$) with trivial or non-trivial automorphisms imply that non-maximal two-dimensional local conformal nets with parity symmetry and central charge less than 1 are classified according to Table 2, since we have 2-cohomology vanishing for all these tensor categories by Theorem 5.3.

Theorem 6.2. *The non-maximal two-dimensional local conformal nets with parity symmetry and central charge less than 1 are classified bijectively, up to isomorphism, according to the entries in Table 2.*

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