Asymptotic behaviour of the null variety for a convex domain in a non-positively curved space form

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Dedicated to Professor Hiroshi Fujita on his sixtieth birthday

By Toshiyuki KOBAYASHI*)

Contents

Chapter 1. Introduction

For a measurable bounded set Ω in $\boldsymbol{R}^n$, the Fourier transform of the characteristic function of Ω is given by,

$$\tilde{\chi}_\Omega(\zeta)=\int_\Omega \exp\Big(\sqrt{-1}\sum_{j=1}^n x_j\zeta_j\Big)dx_1\cdots dx_n,$$

which is an entire function of $\zeta=(\zeta_1,\cdots,\zeta_n)\in \boldsymbol{C}^n$. Then we associate an analytic set (*null variety*) in $\boldsymbol{C}^n$ (or $\boldsymbol{R}^n$) defined by,

$$\mathcal{N}(\Omega):=\{\zeta\in \boldsymbol{C}^n;\ \tilde{\chi}_\Omega(\zeta)=0\},$$

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or

$$\mathcal{N}(\Omega)_R := \mathcal{N}(\Omega) \cap R^n = \{\zeta \in R^n;\ \tilde{\chi}_\Omega(\zeta) = 0\}.$$

In this paper, we concern ourselves to the assignment:

$$\mathcal{N}: \{\text{Bounded sets}\} \ni \Omega \longmapsto \mathcal{N}(\Omega) \in \{\text{Analytic sets}\}.$$

Since the strength of the differentiability assumption will not be the issue, we suppose the boundary $\partial\Omega$ to be of class C^∞.

Our goal is to initiate a new line of investigation by posing the following problems:

Problem A. **How is $\mathcal{N}(\Omega)$?**
Describe $\mathcal{N}(\Omega)$ in terms of the geometrical invariants of Ω.

Problem B. **Does $\mathcal{N}(\Omega)$ determine Ω?**
Is the assignment $\mathcal{N}: \Omega \longmapsto \mathcal{N}(\Omega)$ one to one?

Or more strongly, we ask, "Does a *suitable* subset of $\mathcal{N}(\Omega)$ determine Ω?" Here, a 'suitable subset of $\mathcal{N}(\Omega)$' would mean 'some connected components of $\mathcal{N}(\Omega)$', 'a set of first zero points of $\tilde{\chi}_\Omega(\zeta)$' or 'the intersection of $\mathcal{N}(\Omega)$ and some fixed submanifold in C^n', etc. and such definitions might require certain restrictions on Ω.

Since $\mathcal{N}(\Omega)$ is invariant under parallel displacements of Ω, the injectivity of $\mathcal{N}$ should be considered in the sense of 'up to parallel displacements'. A positive answer to the Problem B would enable us to translate the properties of Ω into those of $\mathcal{N}(\Omega)$ in principle. Thus,

Problem C. **Relate Ω with $\mathcal{N}(\Omega)$.**
What can you tell about Ω when $\mathcal{N}(\Omega)$ has some special properties?

Let us begin with typical examples.

Example (1.1) (Example (2.3.9)). Let Ω be a unit ball in R^n. Then, we have

$$\tilde{\chi}_\Omega(\zeta) = (2\pi)^{n/2} \frac{J_{n/2}(t)}{t^{n/2}},$$

where $t := (\zeta_1^2 + \cdots + \zeta_n^2)^{1/2}$ for $\zeta = (\zeta_1, \cdots, \zeta_n) \in C^n$, and $J_\nu(t)$ denotes the ν-th Bessel function. Let j_m $(m = 1, 2, \cdots)$ be the enumeration of the positive zeros of $J_{n/2}(t)$. Then,

$$\mathcal{N}(\Omega)=\coprod_{m=1}^{\infty}\left\{\zeta\in \boldsymbol{C}^n;\ \sum_{k=1}^{n}\zeta_k^2=j_m^2\right\} \qquad \text{(disjoint union)}.$$

So $\mathcal{N}(\Omega)_R\equiv\mathcal{N}(\Omega)\cap \boldsymbol{R}^n$ consists of countably many concentric hyperspheres.

Example (1.2). Let Ω be a cubic domain $\{x\in \boldsymbol{R}^n;\ |x_j|<1,\ (1\leq j\leq n)\}$. Then, $\tilde{\chi}_\Omega(\zeta)=\prod_{k=1}^{n}\dfrac{2\sin\zeta_k}{\zeta_k}$, for $\zeta=(\zeta_1,\cdots,\zeta_n)\in \boldsymbol{C}^n$. Hence $\mathcal{N}(\Omega)=\bigcup_{k=1}^{n}\bigcup_{m\in \boldsymbol{Z}\setminus\{0\}}\{\zeta\in \boldsymbol{C}^n;\ \zeta_k=\pi m\}$.

A clear distinction in these two example: the null variety $\mathcal{N}(\Omega)_R$ for a ball consists of infinitely many compact components, whereas the one for a cubic domain is connected and noncompact. The former case will be generalized to *strictly* convex domains in Euclidean space and hyperbolic space.

As for the Problem B on the injectivity of $\mathcal{N}$, we must pose a certain condition (connectedness, etc...) on Ω. In fact, we have the following example:

Example (1.3). Fix two positive numbers A and B such that $A\geq 5B$. Set $K:=\left[\dfrac{A-B}{2B}\right]$ (Gaussian integer). For each integer $j\in[1, K]$, put $a_j:=\dfrac{2j+2}{2j+1}A+B$, $b_j:=\dfrac{2j+2}{2j+1}A-B$, $c_j:=\dfrac{2j}{2j+1}A+B$, and $d_j:=\dfrac{2j}{2j+1}A-B$. Then $0<d_j<c_j<b_j<a_j$, $(1\leq j\leq K)$. Set $\Omega_j:=(-a_j, -b_j)\cup(-c_j, -d_j)\cup(d_j, c_j)\cup(b_j, a_j)\subset \boldsymbol{R}$. Then $\tilde{\chi}_{\Omega_j}(\zeta)=\dfrac{8}{\zeta}\sin(B\zeta)\cos(A\zeta)\cos\left(\dfrac{A\zeta}{2j+1}\right)$, and therefore $\mathcal{N}(\Omega_1)=\cdots=\mathcal{N}(\Omega_K)$. A Cartesian product of such Ω's yields the domains in $\boldsymbol{R}^n$ having the same null variety.

Thus we pose

Problem (B.1). Is the assignment

$\mathcal{N}:\{\text{connected bounded sets in } \boldsymbol{R}^n\}\ni\Omega\longmapsto\mathcal{N}(\Omega)\in\{\text{analytic sets in } \boldsymbol{C}^n\}$

one to one up to parallel displacements?

When we restrict our interest on strictly convex domains, this problem contains the following two problems:

Problem (B.2). Is the assignment

$\mathcal{N}$: {strictly convex domains in $\boldsymbol{R}^n$} $\ni \Omega \longmapsto \mathcal{N}(\Omega) \in$ {analytic sets in $\boldsymbol{C}^n$} one to one up to parallel displacements?

Problem (B.3). Suppose the null variety $\mathcal{N}(\Omega)$ for a bounded domain be the (asymptotically) same with that of a strictly convex domain. Is this domain Ω convex?

As for the Problem C, we know a typical example, so called Pompeiu problem (or Schiffer conjecture) (see [20]).

FACT (1.4) (cf. [5]). *The following three conditions on a bounded domain Ω with a connected smooth boundary in $\boldsymbol{R}^n$ are equivalent:*
a) *There exists a nonzero function $u \in C^2(\bar{\Omega})$ and $\lambda \in \boldsymbol{C}$ such that*

$$\Delta u = \lambda u \text{ in } \Omega;\ \frac{\partial u}{\partial n} = 0 \text{ and } u \equiv \text{constant on } \partial\Omega. \tag{1}$$

b) *There exists a nonzero continuous function $f \in C(\boldsymbol{R}^n)$ such that*

$$\int_\Omega f(\sigma \cdot x)\,dx = 0,$$

for any element σ of the Euclidean motion group.
c) *$\mathcal{N}(\Omega)$ contains an $O(n, \boldsymbol{C})$ invariant set.*

Clearly, these conditions are satisfied when Ω is a ball. But whether the converse is true or not is still open even when Ω is assumed to be strictly convex in $\boldsymbol{R}^2$.

If Ω is a centrally symmetric domain, $\tilde{\chi}_\Omega(\zeta)$ is a real valued function on $\boldsymbol{R}^n$. In this case, $\partial\Omega \subset \boldsymbol{R}^n$ and $\mathcal{N}(\Omega)_{\boldsymbol{R}} \subset \boldsymbol{R}^n$ are both codimension one. This fact confirms us that $\mathcal{N}(\Omega)_{\boldsymbol{R}}$ has enough data to determine Ω (ex. Remark (2.3.25)). But unless Ω is centrally symmetric, $\mathcal{N}(\Omega)_{\boldsymbol{R}}$ would have more codimension and less information about Ω. For instance:

Example (1.5). Let $\Omega := \{(x, y) \in \boldsymbol{R}^2;\ 0 < y < \min(1-x, 1+x)\}$ and let $\Omega' := \{(x, y) \in \boldsymbol{R}^2;\ 0 < x < \min(1-y, 1+y)\}$. Then $\tilde{\chi}_\Omega(\xi, \eta) = \dfrac{-2\{\xi(\cos\xi - \cos\eta) + \sqrt{-1}(\eta\sin\xi - \xi\sin\eta)\}}{\xi(\xi^2 - \eta^2)}$, and $\mathcal{N}(\Omega)_{\boldsymbol{R}} = \mathcal{N}(\Omega')_{\boldsymbol{R}} = \mathcal{N}(-\Omega)_{\boldsymbol{R}} = \mathcal{N}(-\Omega')_{\boldsymbol{R}} = \{(m\pi, n\pi) \in \boldsymbol{R}^2;\ m, n \in \boldsymbol{Z},\ m-n \in 2\boldsymbol{Z}, \text{ and } m \neq \pm n\}$.

In fact, some characterization of central symmetry will be given in terms of the codimension of $\mathcal{N}(\Omega)_{\boldsymbol{R}}$ in Corollary (2.3.11).

For a general domain, we will treat $\mathcal{N}(\Omega) \cap S$ instead of $\mathcal{N}(\Omega)_{\boldsymbol{R}} =$

$\mathcal{N}(\Omega)\cap \boldsymbol{R}^n$, where $S:=\{(\zeta\omega_1,\cdots,\zeta\omega_n)\in \boldsymbol{C}^n;\zeta\in \boldsymbol{C},\ (\omega_1,\cdots,\omega_n)\in S^{n-1}\}$. Note that $\boldsymbol{R}^n\subset S\subset \boldsymbol{C}^n$, and $S\setminus\{0\}$ is an $n+1$-dimensional smooth manifold. Then generalizing Example (1.1), we describe the asymptotic behaviour of $\mathcal{N}(\Omega)\cap S$ in Theorem (2.3.6), as a partial answer to Problem A. Let us state it briefly:

THEOREM (1.6) (see Theorem (2.3.6) for details). *Suppose Ω be a strictly convex domain in $\boldsymbol{R}^n$. Then there is a nonnegative integer $m_0\equiv m_0(\Omega)$ such that*

$$\mathcal{N}(\Omega)\cap S=\Big(\coprod_{m\geq m_0}\mathcal{N}_m\Big)\amalg \text{(compact set)}, \qquad \textit{(disjoint union)}, \tag{2}$$

where each $\mathcal{N}_m$ is analytically diffeomorphic to S^{n-1}. More precisely, there are analytic functions $F_m: S^{n-1}\longrightarrow \boldsymbol{C}$ $(m\geq m_0)$ such that

$$\mathcal{N}_m=\{F_m(\omega)\cdot\omega\in \boldsymbol{C}^n;\ \omega\in S^{n-1}\}$$

and

$$F_m(\omega)=\frac{4m+n-1}{2H_\Omega(\omega)}+\sqrt{-1}d_\Omega(\omega)+O(m^{-1}), \qquad \textit{as } m\longrightarrow\infty. \tag{3}$$

Here H_Ω and d_Ω are smooth functions defined on S^{n-1} which is represented by the supporting functions and the curvature of $\partial\Omega$ explicitly (Definition (2.1.14), (2.1.16)).

On the other hand, if Ω is sufficiently near to a ball in a Sobolev norm of the boundary $\partial\Omega$ (Ω needs not convex), the connected components of $\mathcal{N}(\Omega)\cap S$ sufficiently near to the origin are analytically diffeomorphic to S^{n-1}. Now we propose the following conjecture:

CONJECTURE (1.7). *Suppose Ω be a strictly convex domain in Euclidean space. Then* (2) *in Theorem* (1.6) *can be replaced by*

$$\mathcal{N}(\Omega)\cap S=\coprod_{m\geq 1}\mathcal{N}_m, \qquad \textit{(disjoint union)}, \tag{2$'$}$$

where each $\mathcal{N}_m$ is analytically diffeomorphic to S^{n-1}.

As an application of Theorem (1.6), we obtain the following results.

COROLLARY (1.8) (see Corollary (2.3.11)). *We can characterize the geometric properties* (*centrally symmetric, with constant breadth, or globular*) *of a strictly convex domain Ω in $\boldsymbol{R}^n$ in terms of the asymptotics of $\mathcal{N}(\Omega)$.*

In particular, if there are infinitely many eigenvalues for the overdetermined Neumann problem (1), *or more weakly, if* $\mathcal{N}(\Omega)_R$ *contains infinitely many approximating hyperspheres,* Ω *must be a ball.*

A result similar to the last statement was first obtained by Berenstein [1], when Ω is a simply connected domain in $\boldsymbol{R}^2$. Compare also Proposition (1.11).

The preceding Corollary (1.8) is in the line of Problem C. Moreover, Theorem (1.6) itself almost characterize strict convexity. That is, the converse of Theorem (1.6) is almost true:

PROPOSITION (1.9) (cf. Problem (B.3)). *Let* Ω *be a multiply-connected bounded domain with analytic boundaries. If* $\mathcal{N}(\Omega)$ *has the expression* (2) *and* (3) *with some continuous functions* H, d *on* S^{n-1}, *then* Ω *must be convex.*

The proof of this proposition and Proposition (1.11) below will appear in another paper. Let us remark that a null variety for a convex polyhedron does not have even the property (2) in general (for instance, see Example (1.2)). On the other hand, there is a bounded domain Ω which is not convex but whose null variety $\mathcal{N}(\Omega)$ satisfies (2).

Example (1.10). Let $\Omega := \{x \in \boldsymbol{R}^n; r < |x| < 1\}$. Then, $\tilde{\chi}_\Omega(\zeta) = (2\pi)^{n/2} t^{-n/2} \times \{J_{n/2}(t) - r^{n/2} J_{n/2}(rt)\}$, where $t := (\zeta_1^2 + \cdots + \zeta_n^2)^{1/2}$ for $\zeta = (\zeta_1, \cdots, \zeta_n) \in \boldsymbol{C}^n$. Hence, $\mathcal{N}(\Omega)$ satisfies (2) in Theorem (1.6).

Applying Corollary (1.8) and Theorem (1.9) to a special case, we get,

PROPOSITION (1.11). *Let* Ω *be a multiply-connected bounded domain with Lipshitz boundary in* $\boldsymbol{R}^n$. *If the spectrum of* (1) *for* Ω *are asymptotically the same with that of* $B(R)$ *(ball with radius* R*) then* $\Omega = B(R)$.

As another application of Theorem (1.6), we also give a positive answer to Problem (B.2), when the dimension $n=2$. That is:

COROLLARY (1.12). *If two strictly convex domains in* $\boldsymbol{R}^2$ *have the same null variety, then these domains differ from each other by a parallel translation.*

Such injectivity of $\mathcal{N}$ holds in some other cases: First, when restricting to centrally symmetric convex domains in $\boldsymbol{R}^n$, one sees easily that $\mathcal{N}$ is injective (Remark (2.3.25)). Secondly, the injectivity also holds locally when one perturbs a ball. This idea is used in the last Chapter of [17].

When the asymptotic data of $\mathcal{N}(\Omega)\cap S$ is given, Theorem (1.6) enables us to deduce the injectivity problem (Problem (B.2)) from the positive solution of the following problem in differential geometry:

Problem (1.13). Let two strictly convex domain have the same breadth functions and the same ratio of the Gauss-Kronecker curvatures at each point and its antipodal point as a function of normal vectors. Then do these domains differ from each other by a parallel translation?

This problem is equivalent to a uniqueness problem of a certain single differential equation of the second order of the supporting function over S^{n-1} modulo first eigenfunctions of $\Delta_{S^{n-1}}$. This is treated in § 5 of Chapter 2 when $n=2$, but when $n\geq 3$ it is yet unsolved.

Since Corollary (1.12) assures the injectivity of $\mathcal{N}$ in $\boldsymbol{R}^2$ case, we also take an interest in its image. This seems hard, but more weakly we can do this *in an asymptotic sense,* after giving reformulations of Theorem (1.6) and Corollary (1.12) in terms of the coefficients $P_{\mathcal{N}(\Omega)}$, $Q_{\mathcal{N}(\Omega)}$, $R_{\mathcal{N}(\Omega)}, \cdots \in C^\infty(S^{n-1})$ of the asymptotic expansion of the Dirichlet series made from $\mathcal{N}(\Omega)$. Then we obtain,

COROLLARY (1.14) (Proposition (2.3.19), Proposition (2.3.20)). *The assignment:*

$$\left\{\begin{matrix}\textit{Strictly convex domain in } \boldsymbol{R}^2 \\ \textit{up to parallel displacements}\end{matrix}\right\} \ni \Omega \longmapsto (P_{\mathcal{N}(\Omega)}, R_{\mathcal{N}(\Omega)}) \in C^\infty(S^1, \boldsymbol{R}^2)$$

is one to one, and the image can be explicitly characterized.

On a Riemannian symmetric space X, the Fourier transform is also defined with similar properties to those on a Euclidean space ([11]). So the null variety $\mathcal{N}(\Omega)$ is also defined for a bounded domain $\Omega\subset X$. In Chapter 3 we will generalize Theorem (1.6) (Theorem (2.3.6)) to a hyperbolic space $SO_0(n, 1)/SO(n)$ case. To do this, we introduce *H-convex domain,* a notion different from geodesic convexity. Then the analogue of *Gauss-maps* and *supporting functions* are introduced in a noncompact rank one Riemannian symmetric space, both of which are defined in pairs according to the order of the (little) Weyl group. After preparing basic properties of H-convex domains in § 2 and § 3, restricting ourselves to a hyperbolic space, we get the following theorem by using the horospherical method as in Euclidean case.

THEOREM (1.15) (Theorem (3.6.4)). *Suppose Ω be a strictly H-convex domain in a hyperbolic space $X=SO_0(n,1)/SO(n)$. Then*

$$\mathcal{N}(\Omega)=\Big(\coprod_{m\in N}\mathcal{N}_m\Big)\amalg\text{(compact set)}\qquad(\textit{disjoint union}).$$

More precisely, the first approximation of $\mathcal{N}_m$ $(m\to\infty)$ is explicitly expressed in terms of the curvatures and the supporting functions of $\partial\Omega$.

In a hyperbolic space, the null variety $\mathcal{N}(\Omega)$ is not invariant in general under an isometric transformation on X. So a hyperbolic space version of Problem (B.2) is formulated by,

Problem (B.4). Let Ω_1, and Ω_2 be strictly H-convex domains with $\mathcal{N}(\Omega_1)=\mathcal{N}(\Omega_2)$. Then
1) If Ω_1 is not a ball, is $\Omega_1=\Omega_2$?
2) If Ω_1 is a ball, is $\Omega_1=g\cdot\Omega_2$ for some $g\in SO_0(n,1)$?

Theorem (1.15) gives us a method to deal with the injectivity problem of $\mathcal{N}$. In the final section of Chapter 3, we shall prepare more detailed analysis of convex domains when the dimension $n=2$, and illustrate the idea for the injectivity problem which was used in a Euclidean space. In this special case, a uniqueness problem for a periodic solution of a special type of the Duffing equation appears.

In this paper we will use the standard notation N, N_+, $\boldsymbol{Z}$, $\boldsymbol{R}$, $\boldsymbol{R}_+$ and $\boldsymbol{C}$. Here N is the set of non-negative integers and $\boldsymbol{R}_+$ is the set of the positive real numbers and $N_+=N\cap\boldsymbol{R}_+$. For a smooth manifold M, we denote by $C(M)$, $C_0(M)$, $C^{k,\alpha}(M)$, $C^\infty(M)$ and $\mathcal{E}'(M)$ the space of continuous functions, continuous functions with compact support, functions with k-th derivatives satisfying Hölder's condition of order α locally, infinitely differentiable functions and distributions with compact support, defined on M respectively. If M is a complex manifold (resp. a real analytic manifold), we denote by $\mathcal{O}(M)$ (resp. $\mathcal{A}(M)$) the space of holomorphic (resp. real analytic) functions on M.

Parts of the results here were announced in [17]. The author expresses his sincere gratitude to Professor Toshio Oshima for awaking the author's interest in this subject and for his constant encouragement. Thanks are also due to Kaoru Ono and Takashi Kurose for helpful conversations.

Chapter 2. Null variety for a convex domain (Euclidean case)

§ 1. Convex domain in R^n

In this section, we shall review some standard facts about convex domain in Euclidean space.

(2.1.1) Let Ω be a bounded domain whose boundary $\partial\Omega$ is a connected $n-1$ dimensional smooth submanifold of R^n.

We fix an inner product (,) in R^n, and denote the unit sphere by S^{n-1}. Let the Gauss map be

$$\nu\equiv\nu_\Omega:\ \partial\Omega\longrightarrow S^{n-1}, \tag{2.1.2}$$

defined by its outer normal vector field, and the Gauss-Kronecker curvature be,

$$K\equiv K_\Omega:\ \partial\Omega\longrightarrow \boldsymbol{R}, \tag{2.1.3}$$

where we adopt the signature of K so that K is everywhere positive if Ω is a ball. Then the following characterization of (strict) convexity is well known:

FACT (2.1.4) (Hadamard, Chern-Lashof). *Let Ω satisfy* (2.1.1). *Then the following three conditions on Ω are equivalent:*

1) *The Gauss-Kronecker curvature K is positive valued.*

2) *The second fundamental form of the imbedding $\partial\Omega\hookrightarrow \boldsymbol{R}^n$ is non-degenerate.*

3) *The Gauss map ν gives a diffeomorphism from $\partial\Omega$ onto S^{n-1}.*

Moreover one of (therefore all of) the conditions 1)$\sim$3) *implies the following three equivalent conditions:*

4) *Ω is geodesically convex.*
i.e. for any $x, y\in\Omega$ and any $t\in[0, 1]$, $tx+(1-t)y\in\Omega$.

5) *Ω lies in one side with respect to any hyperplane tangent to $\partial\Omega$.*

6) *K is nonnegative valued, and the mapping degree of ν is* 1.

DEFINITION (2.1.5). For a domain $\Omega\subset\boldsymbol{R}^n$ satisfying (2.1.1), Ω is called *strictly convex* if and only if the equivalent conditions 1)$\sim$3) are satisfied. Ω is called *convex* if and only if the equivalent conditions 4)$\sim$6) are satisfied.

REMARK (2.1.6). In Chapter 3, the definition of convexity will be generalized reasonably in a rank one noncompact Riemannian symmetric space, to two different ones, that is, horospherically convex (Definition (3.2.13), Proposition (3.4.2)) and geodesically convex (Remark (3.2.14)).

DEFINITION (2.1.7). For a convex domain Ω, *the supporting function* $h \equiv h_\Omega : S^{n-1} \longrightarrow R$ is given by,

$$h(\omega) := (\omega, \nu^{-1}(\omega)) \tag{2.1.8}$$
$$= \sup_{x \in \Omega}(x, \omega), \qquad \text{for } \omega \in S^{n-1}. \tag{2.1.9}$$

Let $\tilde{h} : \boldsymbol{R}^n \longrightarrow \boldsymbol{R}$ be a linear extension of h, that is,

$$\tilde{h}(x) := |x| h\left(\frac{x}{|x|}\right) \qquad \text{for } x \in \boldsymbol{R}^n \setminus \{0\}, \tag{2.1.10}$$
$$\tilde{h}(0) := 0.$$

Then the following lemma is well known, which states how a strictly convex domain is recovered by its supporting function.

LEMMA (2.1.11) (cf. Corollary (3.4.12), Proposition (3.7.23)). *Let Ω be a strictly convex domain and ν, h and $\tilde{h}$ be as defined in* (2.1.2), (2.1.8) *and* (2.1.10) *respectively. Then for any element ω of S^{n-1},*

$$\nu^{-1}(\omega) = \left(\frac{\partial \tilde{h}}{\partial x^i}(\omega)\right)_{i=1,\cdots,n}. \tag{2.1.12}$$

Or more directly, $\Omega = \bigcap_{\omega \in S^{n-1}} \{x \in \boldsymbol{R}^n ; (x, \omega) < h(\omega)\}$.

REMARK (2.1.13). The definition of $\tilde{h}$ depends on the choice of the origin 0. More precisely, the difference of $\tilde{h}$ by parallel displacements of Ω is just linear functions on $\boldsymbol{R}^n$.

The breadth function of a convex domain Ω is given by,

$$H \equiv H_\Omega : \ S^{n-1} \longrightarrow \boldsymbol{R}_+, \quad H(\omega) := h(\omega) + h(-\omega), \quad \text{for } \omega \in S^{n-1}. \tag{2.1.14}$$

From the definition, H is a positive valued C^∞ function which is invariant under parallel displacements of Ω, and clearly satisfies the following equality:

$$H(\omega) = H(-\omega), \qquad \text{for any } \omega \in S^{n-1}. \tag{2.1.15}$$

For a strictly convex domain Ω, we introduce a new function

$d \equiv d_\Omega : S^{n-1} \longrightarrow \boldsymbol{R}$, as follows:

$$(2.1.16) \qquad d(\omega) := \frac{\log K \circ \nu^{-1}(-\omega) - \log K \circ \nu^{-1}(\omega)}{2H(\omega)}, \qquad (\omega \in S^{n-1}).$$

Since Ω is strictly convex, the Gauss-Kronecker curvature K and the breadth function H are positive valued, so d is a well-defined C^∞ function on S^{n-1}, which is also invariant under parallel displacements of Ω. The following formula is derived from the definition of $d \equiv d_\Omega$ and from (2.1.15):

$$(2.1.17) \qquad d(\omega) + d(-\omega) = 0, \qquad \text{for any } \omega \in S^{n-1}.$$

LEMMA (2.1.18). *Let Ω be a strictly convex domain and d defined in* (2.1.16). *Then the following two conditions on Ω are equivalent*:
1) $d(\omega) \equiv 0$, *for any* $\omega \in S^{n-1}$.
2) *Ω is centrally symmetric with respect to an inner point.*

PROOF. 1)$\rightarrow$2) Put $\Omega^\vee := \{x \in \boldsymbol{R}^n;\ -x \in \Omega\}$. The pullbacks of the Gauss-Kronecker curvatures of Ω and $\Omega^\vee$ by the Gauss maps, namely, $K_\Omega \circ \nu_\Omega^{-1}$ and $K_{\Omega^\vee} \circ \nu_{\Omega^\vee}^{-1}$ coincide because of the assumption 1). Now, by Alexandroff-Fenchel-Jessen's theorem (uniqueness of Minkowski's problem) (cf. [6]), Ω and $\Omega^\vee$ differ each other only by a parallel displacement. Therefore Ω is centrally symmetric with respect to the center of gravity.

The converse statement is clear, so the lemma is proved. Q.E.D.

§ 2. Asymptotic behavior of $\tilde{\chi}_\Omega(\zeta)$

We shall devote this section to the study of the asymptotic behaviour of $\tilde{\chi}_\Omega$ along some direction in $\boldsymbol{C}^n$. Using the classical Radon transform, we reduce it to the problem of one variable. Next, we show in Proposition (2.2.16) that any zero of a certain Fourier transform has bounded imaginary part, generalizing the well-known fact: *any zero of the ν-th Bessel function $J_\nu(t)$ ($\nu > -1$) is real.* Finally we obtain in Proposition (2.2.32) the asymptotic behaviour of $\tilde{\chi}_\Omega(\omega, \zeta)$ with the imaginary part of ζ bounded. The results in this section will play a basic role in the proof of Theorem (2.3.6).

We denote the characteristic function of Ω by $\chi_\Omega(x)$, where Ω is a bounded measurable set in $\boldsymbol{R}^n$. Namely,

$$\chi_\Omega(x) := \begin{cases} 1 & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

The Fourier-Laplace transform of χ_Ω is given by,

$$\tilde{\chi}_\Omega(z) \equiv \mathcal{F}\chi_\Omega(z) := \int_{R^n} \chi_\Omega(x) e^{i\langle x, z\rangle} dx = \int_\Omega e^{i(x_1 z_1 + x_2 z_2 + \cdots + x_n z_n)} dx_1 dx_2 \cdots dx_n, \tag{2.2.1}$$

for $z = (z_1, \cdots, z_n) \in C^n$.

Since Ω is bounded, $\tilde{\chi}_\Omega(z) \in \mathcal{O}(C^n)$, where $\mathcal{O}(C^n)$ denotes the totality of entire functions on C^n. Set

$$\mathcal{N}(\Omega) := \{z \in C^n;\ \tilde{\chi}_\Omega(z) = 0\}, \tag{2.2.2}$$

and,

$$\mathcal{N}(\Omega)_R := \mathcal{N}(\Omega) \cap R^n = \{z \in R^n;\ \tilde{\chi}_\Omega(z) = 0\}. \tag{2.2.3}$$

Then $\mathcal{N}(\Omega)$ (resp. $\mathcal{N}(\Omega)_R$) is an analytic set in C^n (resp. R^n). We call $\mathcal{N}(\Omega)$ *the null variety* for a given Ω.

REMARK (2.2.4). $\mathcal{N}(\Omega)$ and $\mathcal{N}(\Omega)_R$ are invariant under parallel displacements of Ω, because $\tilde{\chi}_{\Omega + x_0}(z) = e^{i\langle x_0, z\rangle}\tilde{\chi}_\Omega(z)$, where $\Omega + x_0 := \{x + x_0 \in R^n;\ x \in \Omega\}$, for any fixed element x_0 of R^n.

Next, we introduce a collection of special complex lines in C^n with '*real direction*'. That is,

$$l_\omega := C\omega = \{\zeta\omega = (\zeta\omega_1, \zeta\omega_2, \cdots, \zeta\omega_n) \in C^n;\ \zeta \in C\},$$

for each fixed element $\omega \equiv (\omega_1, \omega_2, \cdots, \omega_n) \in S^{n-1} (\subset R^n)$.

We shall study the asymptotic behavior of $\tilde{\chi}_\Omega$ along l_ω. Geometric invariants of Ω will be read from the asymptotics of the intersection of $\mathcal{N}(\Omega)$ and l_ω.

Now, by a little abuse of language, we will use the same letter $\tilde{\chi}_\Omega$ for its restriction to $S^{n-1} \times C$, that is:

$$\tilde{\chi}_\Omega :\ S^{n-1} \times C \ni (\omega, \zeta) \longmapsto \tilde{\chi}_\Omega(\omega, \zeta) \equiv \tilde{\chi}_\Omega(\zeta\omega_1, \zeta\omega_2, \cdots, \zeta\omega_n) \in C^n.$$

From the definition, $\tilde{\chi}_\Omega$ satisfies

$$\tilde{\chi}_\Omega(\omega, z) = \tilde{\chi}_\Omega(-\omega, -z), \qquad \text{for any } \omega \in S^{n-1} \text{ and } z \in C. \tag{2.2.5}$$

Since $\chi_\Omega(x)$ is real valued, $\tilde{\chi}_\Omega$ also satisfies

$$\overline{\tilde{\chi}_\Omega(\omega, \xi+i\eta)}=\tilde{\chi}_\Omega(\omega, -\xi+i\eta), \qquad \text{for any } \omega\in S^{n-1} \text{ and } \xi, \eta\in \boldsymbol{R}. \tag{2.2.6}$$

Here, $\bar{z}$ denotes the complex conjugation of $z\in \boldsymbol{C}$.

The classical method of decomposing a Fourier-Laplace transform into a Radon transform and a Fourier transform of one variable gives the following representation of $\tilde{\chi}_\Omega(\omega, \zeta)$:

$$\tilde{\chi}_\Omega(\omega, \zeta)=\int_{-\infty}^{+\infty} S(\omega, p)e^{ip\zeta}dp=\int_{-h(-\omega)}^{h(\omega)} S(\omega, p)e^{ip\zeta}dp, \tag{2.2.7}$$

where $S(\omega, p)\equiv S_\Omega(\omega, p):=\int_{\boldsymbol{R}^n}\chi_\Omega(x)\delta(p-\langle x, \omega\rangle)dx$ is nothing but the Euclidean area of sectional face of Ω by the hyperplane $\{x\in \boldsymbol{R}^n; \langle x, \omega\rangle=p\}$. Here δ denotes Dirac's delta function of a single valuable.

The next lemma essentially goes back to F. John (see [15]).

LEMMA (2.2.8). *Suppose Ω be strictly convex and retain notation as above. Then, $S\equiv S_\Omega: S^{n-1}\times \boldsymbol{R}\longrightarrow \boldsymbol{R}$ is a continuous function with compact support $\bar{V}$, and $S_\Omega|_V\in C^\infty(V)$. Here, $V:=\{(\omega, p)\in S^{n-1}\times \boldsymbol{R}; \omega\in S^{n-1}, -h(-\omega)<p<h(\omega)\}$, and $\bar{V}$ denotes its closure.*
More precisely, for a sufficiently small constant $\delta>0$, there are two C^∞ functions: $S_j: S^{n-1}\times(-\delta, \delta)\ni(\omega, p)\longmapsto S_j(\omega, p)\in \boldsymbol{R}$, $(j=1, 2)$, such that $S(\omega, p)$ is represented in the neighbourhood of ∂V as follows (see Notation (2.2.12)):

$$\begin{aligned} S(\omega, p) &= \frac{(2\pi)^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}(K\circ\nu^{-1}(-\omega))^{-1/2}(p+h(-\omega))_+^{(n-1)/2} \\ &\quad \times(1+S_1(\omega, p+h(-\omega))(p+h(-\omega))), \qquad \textit{for } |p+h(-\omega)|<\delta. \\ &= \frac{(2\pi)^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}(K\circ\nu^{-1}(\omega))^{-1/2}(p-h(\omega))_-^{(n-1)/2} \\ &\quad \times(1+S_2(\omega, h(\omega)-p)(h(\omega)-p)), \qquad \textit{for } |p-h(\omega)|<\delta. \end{aligned} \tag{2.2.9}$$

PROOF. The first statement is clear. We will first prove the smoothness of $S(\omega, p)$ as a function of ω and $(p-h(\omega))_-^{1/2}$. Since this is a local statement, we fix a trivializing neighbourhood U $(\subset\partial\Omega)$ of the tangent bundle $T(\partial\Omega)\longrightarrow\partial\Omega$. Fix an orthonormal frame on $\partial\Omega$ $(\hookrightarrow \boldsymbol{R}^n)$, which gives the bundle isomorphism

$$U\times \boldsymbol{R}^{n-1} \xrightarrow[\varphi]{\sim} T(\partial\Omega)_{|U} \ (\underset{\mathrm{j}}{\hookrightarrow} U\times \boldsymbol{R}^n \underset{\mathrm{p}}{\longrightarrow} \boldsymbol{R}^n).$$

In the above parenthesis, j is induced from $T(\partial\Omega)\hookrightarrow\partial\Omega\times\boldsymbol{R}^n$ and p is the projection to the second factor.

Define a C^∞ map $\phi: U\times\boldsymbol{R}^{n-1}\times\boldsymbol{R}\longrightarrow\boldsymbol{R}^n$ by,

$$\phi(x; y, s):=x+\mathrm{p}\circ\mathrm{j}\circ\varphi(x, y)+s\ \nu(x), \quad \text{for } (x, y, s)\in U\times\boldsymbol{R}^{n-1}\times\boldsymbol{R}.$$

Here, we look upon an element x of U as in $\boldsymbol{R}^n$ and $\nu:\partial\Omega\longrightarrow S^{n-1}\hookrightarrow\boldsymbol{R}^n$ is the Gauss map (Definition (2.1.2)). From the definition, for each fixed element x of U, $\phi(x;\ ,\ ): \boldsymbol{R}^{n-1}\times\boldsymbol{R}\xrightarrow{\sim}\boldsymbol{R}^n$ gives a moving frame with x its origin. Since $\partial\Omega$ is locally represented by a graph, there is a C^∞ function $f: U\times W\longrightarrow\boldsymbol{R}$ with some open neighbourhood W $(\subset\boldsymbol{R}^{n-1})$ containing the origin 0 such that the following conditions (2.2.10) are satisfied for any $(x, y)\in U\times W$,

$$(2.2.10)\quad \begin{cases} \phi(x; y, f(x, y))\in\partial\Omega, \\ f(x, 0)=\dfrac{\partial f}{\partial y_j}(x, 0)=0 \qquad (1\leq j\leq n-1), \\ \text{The eigenvalues of } \left(\dfrac{\partial^2 f}{\partial y_j\partial y_k}(x, 0)\right)_{j,k} \text{ are } -\kappa_j(x)\ \ (1\leq j\leq n-1). \end{cases}$$

Here $\kappa_j(x)$ $(1\leq j\leq n-1)$ are the principal curvatures of $\partial\Omega$ at x. Set $\tilde{f}$ be the composition map of

$$U\times(-\varepsilon, \varepsilon)\times S^{n-2}\ni(x; r, \theta)\longmapsto(x, r\theta)\in U\times W,$$

and

$$U\times W\ni(x, y)\longmapsto f(x, y)\in\boldsymbol{R},$$

with a small fixed $\varepsilon>0$. Then from (2.2.10), there is a C^∞ function f_1 defined on $U\times(-\varepsilon, \varepsilon)\times S^{n-2}$ such that

$$\tilde{f}(x; r, \theta)=-r^2 f_1(x; r, \theta).$$

Since $f_1(x; 0, \theta)>0$, retaking U and ε if necessary, we may assume $f_1(x; r, \theta)>0$ for any $(x, r, \theta)\in U\times(-\varepsilon, \varepsilon)\times S^{n-2}$.

Then applying the implicit function theorem to the equation

$$r\,(f_1(x; r, \theta))^{1/2}=a,$$

we find a C^∞ function $R: U'\times(-\delta, \delta)\times S^{n-2}\longrightarrow(-\varepsilon, \varepsilon)$ such that for any $(x, a, \theta)\in U'\times(-\delta, \delta)\times S^{n-2}$,

$$R(x; a, \theta)\ (f_1(x; R(x; a, \theta), \theta))^{1/2}=a,$$

and

$$R(x; 0, \theta)=0,$$

with some open neighbourhood $U'\subset U$ $(\subset\partial\Omega)$ and $\delta>0$. The above equalities imply

$$\tilde{f}(x; R(x; a, \theta), \theta)=-a^2,$$

and

$$R(x; a, \theta)>0, \qquad \text{if } a>0.$$

On the other hand, the intersection of Ω and the hyperplane $\{z\in \boldsymbol{R}^n; \langle z, \omega\rangle=p\}$ is represented by the polar coordinate as follows:

$$\{\psi(\nu^{-1}(\omega); r\theta, p-h(\omega));\ 0\leq r<R(\nu^{-1}(\omega); (h(\omega)-p)^{1/2}, \theta),\ \theta\in S^{n-2}\}.$$

The function $S(\omega, p)$ is the Euclidean area of this set, which is obtained by the integration over S^{n-2} of the radial part $R(\nu^{-1}(\omega); (h(\omega)-p)^{1/2}, \theta)$ and its derivatives with respect to θ. Therefore, $S(\omega, p)$ is a C^∞ function of $(\omega, (h(\omega)-p)^{1/2})\in\nu(U')\times[0, \sqrt{\delta})$. This proves the smoothness of $S(\omega, p)$ as a function of ω and $(p-h(\omega))_-^{1/2}$.

The first term of the asymptotic expansion of $S(\omega, p)$ is obtained in [15]. Let us review it. It is the volume of the ellipsoid with the radii of curvature $((h(\omega)-p)/\kappa_i(\nu^{-1}(\omega)))^{1/2}$ $(1\leq i\leq n-1)$, namely, $(2\pi)^{(n-1)/2}\times\Gamma((n+1)/2)^{-1}(K\circ\nu^{-1}(\omega))^{-1/2}(h(\omega)-p)^{(n-1)/2}$.

To complete the proof of the lemma, we only have to show that any coefficient of the power $(p-h(\omega))_-^{n/2+k}$ $(k=0, 1, 2, \cdots)$ vanishes in the expansion of $S(\omega, p)$ at $p=h(\omega)$. Since this claim is $SO(n)$-invariant, we may assume that $\omega=\omega_0\equiv(0, \cdots, 0, 1)\in S^{n-1}$. Set $f_0: W\longrightarrow\boldsymbol{R}$ by $f_0(y):=f(\nu^{-1}(\omega_0), y)$ $(y\in W)$, $\tilde{f}_0: (-\varepsilon, \varepsilon)\times S^{n-2}\longrightarrow\boldsymbol{R}$ by $\tilde{f}_0(r, \theta):=\tilde{f}(\nu^{-1}(\omega_0), r, \theta)=f_0(r\cdot\theta)$, and $R_0: (-\delta, \delta)\times S^{n-2}\longrightarrow(-\varepsilon, \varepsilon)$ by $R_0(t, \theta):=R(\nu^{-1}(\omega_0), t, \theta)$. Let $f_0(y)\sim\sum_{|\alpha|\geq 2} a_\alpha y^\alpha$, $R_0(t, \theta)\sim\sum_{m=1}^{\infty} b_m(\theta)t^m$ be the Taylor expansions with coefficients a_α, $b_m(\theta)\in\boldsymbol{R}$ respectively. Here $\alpha=(\alpha_1, \cdots, \alpha_{n-1})$ $(\alpha_j\in N)$ is a multi-index and the length of α is defined by $|\alpha|:=\sum_{j=1}^{n-1}\alpha_j$. Note that $\sum_{|\alpha|=2} a_\alpha y^\alpha$ are non-degenerate quadratic form because of (2.2.10).

Set $P_k(\theta):=\sum_{|\alpha|=k} a_\alpha\theta^\alpha$. Formal substitution of the above Taylor expansions into the identity $y=\tilde{f}_0(R_0(y^{1/2}, \theta), \theta)$ gives,

$$\begin{aligned} y &\sim \sum_{|\alpha|\geq 2} a_\alpha \theta^\alpha \Big(\sum_{m=1}^{\infty} b_m(\theta) y^{m/2}\Big)^{|\alpha|} \\ &= \sum_{k=2}^{\infty} P_k(\theta) \Big(\sum_{m=1}^{\infty} b_m(\theta) y^{m/2}\Big)^k . \end{aligned}$$

Comparing the coefficients of the power $y^{N/2}$ $(N\geq 2)$ of the both side, we get,

$$\sum_{k=2}^{N} P_k(\theta)(\sum b_{n_1}(\theta) b_{n_2}(\theta)\cdots b_{n_k}(\theta)) = \begin{cases} 1 & (N=2) \\ 0 & (N\geq 3). \end{cases} \tag{2.2.11}$$

Here the sum is taken over $\Big\{(n_1, n_2, \cdots, n_k)\in N_+^k;\ \sum_{j=1}^{k} n_j = N\Big\}$.

Now let us show

$$b_k(-\theta) = (-1)^{k-1} b_k(\theta) \qquad (k\geq 1),$$

by the induction on k.

First note that $P_k(-\theta)=(-1)^k P_k(\theta)$ $(k\geq 1)$, from definition.

From (2.2.11) with $N=2$, we have,

$$P_2(\theta) b_1(\theta)^2 = 1.$$

Therefore, $b_1(\theta)=b_1(-\theta)$.

Suppose $b_k(-\theta)=(-1)^{k-1}b_k(\theta)$ for $1\leq k\leq N-1$. $(N\geq 2)$. From (2.2.11) with N replaced by $N+1$,

$$\begin{aligned} &2P_2(\theta) b_1(\theta) b_N(\theta) \\ &= -P_2(\theta) \sum_{j=2}^{N-1} b_j(\theta) b_{N+1-j}(\theta) - \sum_{k=3}^{N+1} P_k(\theta)(\sum b_{n_1}(\theta) b_{n_2}(\theta)\cdots b_{n_k}(\theta)). \end{aligned}$$

Here the sum is taken over $\Big\{(n_1, n_2, \cdots, n_k)\in N_+^k;\ \sum_{j=1}^{k} n_j = N+1\Big\}$. In particular, b_j appears in the right hand side only when $j\leq N-1$.

Replacing θ by $-\theta$ in the above identity and using the assumption of the induction, we get,

$$\begin{aligned} &2P_2(\theta) b_1(\theta) b_N(-\theta) \\ &= -P_2(\theta)(-1)^{N+1} \sum_{j=2}^{N-1} b_j(\theta) b_{N+1-j}(\theta) - \sum_{k=3}^{N+1} (-1)^k P_k(\theta) \\ &\quad \times (\sum (-1)^{\Sigma(n_j-1)} b_{n_1}(\theta)\cdots b_{n_k}(\theta)) \\ &= (-1)^{N+1}\Big(-P_2(\theta) \sum_{j=2}^{N-1} b_j(\theta) b_{N+1-j}(\theta) - \sum_{k=3}^{N+1} P_k(\theta)(\sum b_{n_1}(\theta)\cdots b_{n_k}(\theta))\Big) \\ &= (-1)^{N+1} 2P_2(\theta) b_1(\theta) b_N(\theta). \end{aligned}$$

Hence $b_N(-\theta)=(-1)^{N-1}b_N(\theta)$ and the induction is now completed.

Since the volume $\{r\theta\in \boldsymbol{R}^{n-1}; 0\leq r<R_0(t, \theta), \theta\in S^{n-2}\}$ is given by the pairing of $R_0(t, \cdot)$ and an $SO(n-2)$-invariant distribution F on S^{n-2}, we have

$$\begin{aligned} S(\omega_0, -t^2) &= \int_{S^{n-1}} F(\theta)R_0(t, \theta)d\theta \\ &= \frac{1}{2}\int_{S^{n-1}} F(\theta)(R_0(t, \theta)+R_0(t, -\theta))d\theta. \end{aligned}$$

Here, we write the distribution $F(\theta)$ as if it were a usual function for convenience. Since the Taylor expansion of $\frac{1}{2}(R_0(t, \theta)+R_0(t, -\theta))$ is given by $\sim\sum_{n=0}^{\infty} b_{2n+1}(\theta)t^{2n+1}$, any coefficient of the power $(p-h(\omega_0))_-^{n/2+k}$ ($k=0, 1, 2, \cdots$) vanishes in the expansion of $S(\omega_0, p)$ at $p=h(\omega_0)$.

Hence the proof of the lemma is completed. Q.E.D.

Paley-Wiener's theorem asserts that the singularity of a function is reflected by the growth of its Fourier transform. We shall go into details of the asymptotic behaviour of the functions having special singularities. To do this, it is convenient to list up some notations here which will be used repeatedly throughout this section.

Notation (2.2.12). For $\mu>0$, we put

$$x_+^\mu := \begin{cases} x^\mu & \text{if } x>0, \\ 0 & \text{if } x\leq 0. \end{cases} \qquad x_-^\mu := \begin{cases} 0 & \text{if } x\geq 0, \\ |x|^\mu & \text{if } x<0. \end{cases}$$

$$(\xi+i0)^{-\mu-1} := \begin{cases} \xi^{-\mu-1} & \text{if } \xi>0, \\ e^{\pi i\mu}|\xi|^{-\mu-1} & \text{if } \xi<0. \end{cases}$$

(We will need this not in the case when ξ is near 0 but only when $|\xi|$ is very large.)

Let φ be a C^∞ function on $\boldsymbol{R}$ having the following three properties:

$$\text{(2.2.13)} \qquad \begin{cases} \varphi(x)=\varphi(-x) & \text{for any } x\in \boldsymbol{R}, \\ \varphi(x)=1 & \text{if } x\in[-\delta, \delta], \\ \operatorname{supp}\varphi\subset[-2\delta, 2\delta]. \end{cases}$$

Here δ is a positive constant.

Let λ be a real positive number. Let $f(x)$ be a continuous function on $\boldsymbol{R}$ with compact support contained in $[0, A]$ and have the following expression for some $N\in N$ and for some φ satisfying (2.2.13) with $4\delta<A$:

$$f(x)=\sum_{j=0}^{N} a_j x_+^{\lambda+j}\varphi(x)+\sum_{j=0}^{N} b_j(x-A)_-^{\lambda+j}\varphi(x-A)+g(x), \tag{2.2.14)-(N}$$

where $a_j,\ b_j\in \boldsymbol{C}\ (a_0\neq 0)$, $g(x)\in C^{[\lambda]+N,1}(\boldsymbol{R})$, and $\operatorname{supp} g\subset[0, A]$. Note that A, a_j, and b_j are unique for a given function $f(x)$ and not dependent on the choice of φ. Clearly, $f(x)|_{\boldsymbol{R}\setminus[0,A]}\in C^{\infty}(\boldsymbol{R}\setminus[0, A])$.

Finally, set

$$p(\lambda):=\Gamma(\lambda+1)\exp\left(\frac{\pi i(\lambda+1)}{2}\right). \tag{2.2.15}$$

It is well known that *all the zeros of the ν-th Bessel function $J_\nu(t)$ $(\nu>-1)$ are real.* The following proposition is a generalization of this fact (see Example (2.2.28), Example (2.2.29)). Recall that $\mathcal{F}f$ stands for the Fourier-Laplace transform of f (see (2.2.1)).

PROPOSITION (2.2.16). *Let f be a continuous function on $\boldsymbol{R}$ having an expression as in* (2.2.14) *with $N=1$. Then,*

$$\sup\{|\operatorname{Im}(\zeta)|;\ \mathcal{F}f(\zeta)=0\}<\infty. \tag{2.2.17}$$

PROOF. From the assumption, $f(x)$ has the expression as in (2.2.14). That is,

$$f(x)=\sum_{j=0}^{1} a_j x_+^{\lambda+j}\varphi(x)+\sum_{j=0}^{1} b_j(x-A)_-^{\lambda+j}\varphi(x-A)+g(x).$$

Here $g\in C^{[\lambda]+1,1}(\boldsymbol{R})$. Set $h(x):=f(A-x)$. Then h has also the expression (2.2.14), and $\mathcal{F}h(\zeta)=\exp(i\zeta A)\mathcal{F}f(-\zeta)$. Therefore it is enough to prove

$$\sup\{\operatorname{Im}(\zeta);\ \mathcal{F}f(\zeta)=0\}<\infty, \tag{2.2.18}$$

instead of (2.2.17). Before proving (2.2.18), we shall prepare the following three Lemmas (2.2.19), (2.2.22) and (2.2.24).

LEMMA (2.2.19). *Let φ be a function satisfying* (2.2.13). *For any $\lambda>0$, and any integer $k\geq\lambda$, there is a constant $C>0$ such that*

$$|\mathcal{F}(x_+^{\lambda}\varphi(x))(\zeta)-p(\lambda)\zeta^{-\lambda-1}|\leq C|\zeta|^{-k}\eta^{-1}\exp(-\delta\eta), \tag{2.2.20}$$

for any $\zeta=\xi+i\eta$ with $\eta>0$.

Note that if $\eta>1$, the right hand side of (2.2.20) *can be replaced by $C|\zeta|^{-k}$.*

PROOF. Set $\phi(x):=x_+^\lambda\varphi(x)-x_+^\lambda$. Then $\phi(x)$ is a smooth function on $\boldsymbol{R}$ with its support contained in $[\delta,\infty)$. Since $k>\lambda$,

$$C:=\sup_{x\in\boldsymbol{R}}\left|\left(\frac{d}{dx}\right)^k\phi(x)\right|<\infty.$$

If $\eta=\mathrm{Im}(\zeta)>0$, using integral by parts, we get,

$$\begin{aligned}|\zeta^k\mathcal{F}\phi(\zeta)|&\le\int_\delta^\infty\left|\left(\frac{d}{dx}\right)^k\phi(x)\right||\exp(i\zeta x)|dx\\&\le C\int_\delta^\infty\exp(-\eta x)dx\\&=C\eta^{-1}\exp(-\delta\eta).\end{aligned}$$

On the other hand, the next formula holds ([9] vol. 1 p. 171) when $\lambda>0$ and $\eta>0$.

$$\int_{-\infty}^{+\infty}x_+^\lambda e^{ix(\xi+i\eta)}dx=p(\lambda)(\xi+i\eta)^{-\lambda-1}. \qquad \text{(Notation (2.2.15))} \tag{2.2.21}$$

Therefore

$$|\mathcal{F}(x_+^\lambda\varphi(x))(\zeta)-p(\lambda)\zeta^{-\lambda-1}|\le|\mathcal{F}\phi(\zeta)|\le C|\zeta|^{-k}\eta^{-1}\exp(-\delta\eta).$$

Hence the lemma. Q.E.D.

LEMMA (2.2.22). *Let $g\in C^{k-1,1}(\boldsymbol{R})$ $(k\in N_+)$ whose support is contained in $[0,A]$ $(0<A<\infty)$. Then,*

$$|\mathcal{F}g(\zeta)|\le\left\|\left(\frac{d}{dx}\right)^k g\right\|_\infty|\zeta|^{-k}\eta^{-1}, \tag{2.2.23}$$

for any $\zeta=\xi+i\eta$ with $\eta>0$.

Note that if $\eta>1$, the right hand side of (2.2.23) can be replaced by $\left\|\left(\frac{d}{dx}\right)^k g\right\|_\infty|\zeta|^{-k}$.

This lemma is proved in the same way as in the last lemma, so we omit the proof.

LEMMA (2.2.24). *Let φ be a function as in (2.2.13). Fix any $\lambda>0$, and any nonnegative integer N. Put $L:=[\lambda]+N+1\in N$. Then there is a constant $C>0$ such that*

$$\left|\mathcal{F}(x_-^\lambda\varphi(x))(\zeta)-\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}p(\lambda+n)(-\bar{\zeta})^{-\lambda-n-1}\right| \tag{2.2.25}$$
$$\leq C|\zeta|^{-L}\eta^{-1}\{(1+\eta^{[\lambda]})\eta^{N+1}\exp(4\delta\eta)+(1+\eta^N)\exp(-\delta\eta)\},$$

for any $\zeta=\xi+i\eta$ *with* $\eta>0$.

Note that if $\eta>1$, *the right hand side of* (2.2.20) *can be written (possibly after changing the constant* $C>0$) *as* $C|\zeta|^{-L}\exp(A\eta)$, *where* $A>0$ *is a constant larger than* 4δ.

PROOF. Set $h(x):=x_+^\lambda\varphi(x)\exp(2\eta)-\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}x_+^{\lambda+n}\varphi(x)$. Then $h(x)=\sum_{n=N+1}^{\infty}\frac{(2\eta)^n}{n!}x_+^{\lambda+n}\varphi(x)\in C^{L-1,1}(\boldsymbol{R})$, and $\operatorname{supp}h\subset[0,2\delta]$.

Let $x\in[0,2\delta]$,

$$\left|\left(\frac{d}{dx}\right)^L h(x)\right|\leq\sum_{n=0}^{\infty}\frac{(2\eta)^{n+N+1}}{(n+N+1)!}\left|\left(\frac{d}{dx}\right)^L\{x_+^{\lambda+n+N+1}\varphi(x)\}\right|$$
$$\leq 2^L\sup\left\{\left|\left(\frac{d}{dx}\right)^k\varphi(x)\right|;\ x\in\boldsymbol{R},\ 0\leq k\leq L\right\}$$
$$\times\sum_{n=0}^{\infty}\frac{(2\eta)^{n+N+1}}{(n+N+1)!}\max_{0\leq k\leq L}\left(\prod_{j=0}^{k}(\lambda+n+N+1-j)\right)x^{\lambda+n+N+1-k}$$
$$\leq C_1\eta^{N+1}\left(1+\sum_{n=1}^{\infty}\frac{n^{[\lambda]}}{n!}(2\eta x)^n\right)$$
$$\leq C_1\eta^{N+1}\left(1+\left(x\frac{d}{dx}\right)^{[\lambda]}\exp(2\eta x)\right)$$
$$\leq C_2\eta^{N+1}(1+\eta^{[\lambda]})\exp(2\eta x). \tag{2.2.26}$$

Therefore, if $\eta=\mathrm{Im}(\zeta)>0$,

$$\left|\mathcal{F}(x_-^\lambda\varphi(x))(\xi+i\eta)-\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}p(\lambda+n)(-\xi+i\eta)^{-\lambda-n-1}\right|$$
$$=\left|\mathcal{F}(x_+^\lambda\varphi(x))(-\xi-i\eta)-\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}p(\lambda+n)(-\xi+i\eta)^{-\lambda-n-1}\right|$$
$$=\left|\mathcal{F}(x_+^\lambda\varphi(x)\exp(2x\eta))(-\xi+i\eta)-\mathcal{F}\left(\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}x_+^{\lambda+n}\right)(-\xi+i\eta)\right|$$
$$\leq|\mathcal{F}h(-\xi+i\eta)|+\sum_{n=0}^{N}\frac{(2\eta)^n}{n!}|\mathcal{F}(x_+^{\lambda+n}-x_+^{\lambda+n}\varphi(x))(-\xi+i\eta)|.$$

From Lemma (2.2.19) and Lemma (2.2.22),

$$\leq \left\| \left(\frac{d}{dx}\right)^L h(x) \right\|_\infty |\zeta|^{-L}\eta^{-1} + C'\left(\sum_{n=0}^{N} \frac{(2\eta)^n}{n!}\right)|\zeta|^{-L}\eta^{-1}\exp(-\delta\eta)$$
$$\leq C_2|\zeta|^{-L}\eta^{-1}(\eta^{N+1}(1+\eta^{[\lambda]})\exp(4\delta\eta) + (1+\eta^N)\exp(-\delta\eta)),$$

whence the lemma. Q.E.D.

Now, let us complete the proof of Proposition (2.2.16). It is sufficient for (2.2.18) to show

(2.2.18)′ $\quad \sup\{\mathrm{Im}(\zeta);\ \mathcal{F}f(\zeta)=0,\ \mathrm{Im}(\zeta)>0, \text{ and } |\zeta|\geq K\}<\infty,$

with some large constant $K>0$. (If the set in (2.2.18)′ is empty, there is nothing to prove!.) Now we will prove (2.2.18)′. From Lemma (2.2.19), Lemma (2.2.24), and Lemma (2.2.22), there is a constant $C>0$ such that the following three inequalities hold for any $\zeta=\xi+i\eta$ with $\eta>0$, and $0\leq j\leq 1$,

$$|a_j|\left|\mathcal{F}(x_+^{\lambda+j}\varphi(x))(\zeta) - p\left(\lambda+\frac{j}{2}\right)\zeta^{-\lambda-j-1}\right| \leq C|\zeta|^{-[\lambda]-2},$$

$$|b_j|\left|\mathcal{F}((x-A)_-^{\lambda+j}\varphi(x-A))(\zeta) - \exp(iA\zeta)\left(p\left(\lambda+\frac{j}{2}\right)(-\bar{\zeta})^{-\lambda-j-1}\right.\right.$$
$$\left.\left. +2\eta p\left(\lambda+1+\frac{j}{2}\right)(-\bar{\zeta})^{-\lambda-j-2}\right)\right| \leq C|\zeta|^{-[\lambda]-2},$$

and

$$|\mathcal{F}g(\zeta)|\leq C|\zeta|^{-[\lambda]-2}.$$

Therefore,

$$|\mathcal{F}f(\zeta) - a_0 p(\lambda)\zeta^{-\lambda-1} - \exp(iA\zeta)b_0 p(\lambda)(-\bar{\zeta})^{-\lambda-1}|$$
$$\leq |b_0||\mathcal{F}(2\eta(x-A)_-^{\lambda+1})(\zeta)| + |b_1||\mathcal{F}((x-A)_-^{\lambda+1}(1+2\eta(x-A)))(\zeta)|$$
$$+|a_1||\mathcal{F}(x_+^{\lambda+1})(\zeta)| + \sum_{j=0}^{1}|a_j||\mathcal{F}(x_+^{\lambda+j}(\varphi(x)-1))(\zeta)|$$
$$+\sum_{j=0}^{1}|b_j||\mathcal{F}((x-A)_-^{\lambda+j}(\varphi(x-A)-1-2\eta(x-A)))(\zeta)| + |\mathcal{F}g(\zeta)|$$
$$\leq |b_0 2\eta\exp(-A\eta)p(\lambda+1)(-\bar{\zeta})^{-\lambda-2}| + |a_1||p(\lambda+1)\zeta^{-\lambda-2}|$$
$$+|b_1||\exp(-A\eta)(p(\lambda+1)(-\bar{\zeta})^{-\lambda-2} + 2\eta p(\lambda+2)(-\bar{\zeta})^{-\lambda-3})| + 5C|\zeta|^{-[\lambda]-2}.$$

Since the coefficients such as $\eta\exp(-A\eta)$ are bounded if $\eta>0$, there is a constant $M>0$ independent of η such that

(2.2.27) $\quad |\mathcal{F}f(\zeta) - a_0 p(\lambda)\zeta^{-\lambda-1} - \exp(iA\zeta)b_0 p(\lambda)(-\bar{\zeta})^{-\lambda-1}| \leq M|\zeta|^{-[\lambda]-2}.$

Set

$$K:=\left(\frac{2M}{|a_0p(\lambda)|}\right),$$

and

$$\varepsilon:=[\lambda]-\lambda+1\ \ (>0).$$

Then multiplying the last inequality (2.2.27) by $|\zeta|^{\lambda+1}$, we obtain

$$\left|\mathcal{F}f(\zeta)\zeta^{\lambda+1}-a_0p(\lambda)-\exp(iA\zeta)b_0p(\lambda)\left(-\frac{\bar{\zeta}}{\zeta}\right)^{-\lambda-1}\right|\leq M|\zeta|^{-\varepsilon},$$

for any $\zeta=\xi+i\eta$ with $\eta>1$.

Suppose $\mathcal{F}f(\zeta)=0$, $\eta=\mathrm{Im}(\zeta)>1$, and $|\zeta|\geq K^{1/2}$. Then we have,

$$|p(\lambda)|\left|a_0+b_0\exp(iA\zeta)\left(-\frac{\bar{\zeta}}{\zeta}\right)^{-\lambda-1}\right|\leq M|\zeta|^{-\varepsilon}.$$

Transposing the first term to the other side, and then using the assumption $|\zeta|\geq K^{1/2}$, we get,

$$|b_0|\exp(-A\eta)\geq|a_0|-\frac{M|\zeta|^{-\varepsilon}}{p(\lambda)}\geq\frac{1}{2}|a_0|,$$

which implies (2.2.18)′. Proposition (2.2.16) is thus proved. Q.E.D.

Now we give the following typical examples of Proposition (2.2.6).

Example (2.2.28). Let $\lambda>-1$ and

$$f(x):=\begin{cases}(1-x^2)^{\lambda} & \text{if } |x|<1,\\ 0 & \text{if } |x|\geq 1.\end{cases}$$

Then $\mathcal{F}f(\zeta)=\sqrt{\pi}\,\Gamma(\lambda+1)\left(\frac{\zeta}{2}\right)^{-\lambda-1}J_{\lambda+1/2}(\zeta)$, and all the zeros of $\mathcal{F}f(\zeta)$ are real.

This is derived from the following well-known fact: *All the zeros of the ν-th Bessel function $J_\nu(\zeta)$ $(\nu>-1)$ are real.*

Example (2.2.29). Let $\lambda>-1$ and

$$f(x):=\begin{cases}\cos^{\lambda}x & \text{if } |x|<\pi/2.\\ 0 & \text{if } |x|\geq\pi/2.\end{cases}$$

Then $\mathcal{F}f(\zeta)=\pi\Gamma(\lambda+1)2^{-\lambda-1}\Gamma\left(\frac{\lambda+\zeta+2}{2}\right)^{-1}\Gamma\left(\frac{\lambda-\zeta+2}{2}\right)^{-1}$ ([G-R] 3.631) and the set of the zeros of $\mathcal{F}f(\zeta)$ is given by $\{\pm\zeta\in \boldsymbol{C};\zeta-\lambda-2\in 2\boldsymbol{N}\}$.

In order to study the zeros of $\tilde{\chi}_{\Omega}(\omega,\zeta)$, we may assume $|\mathrm{Im}\,\zeta|$ is bounded owing to Proposition (2.2.16).

LEMMA (2.2.30). *Let f be a function satisfying* (2.2.14) *with* $N=1$. *For any* $E>0$, *there is a constant* $M>0$ *such that*

$$(2.2.31)\qquad \begin{aligned}&|\mathcal{F}f(\zeta)-a_0p(\lambda)(\xi+i0)^{-\lambda-1}-\exp(iA\zeta)b_0p(\lambda)(-\xi+i0)^{-\lambda-1}|\\ &\leq M|\zeta|^{-[\lambda]-2},\end{aligned}$$

for any $\zeta=\xi+i\eta$ *with* $|\xi|\geq 1$, *and* $E>|\eta|$.

PROOF. Put $f_1(x):=f(x)\exp(-2xE)$. Then $f_1(x)$ has also the expression (2.2.14) with the same N and a_0, and with b_0 replaced by $b_0\exp(-2AE)$. By (2.2.27), there is a constant $M_1>0$ such that for any $\zeta=\xi+i\eta$ with $\eta>E$,

$$|\mathcal{F}f_1(\zeta)-a_0p(\lambda)\zeta^{-\lambda-1}-\exp(iA\zeta)b_0\exp(-2AE)p(\lambda)(-\bar{\zeta})^{-\lambda-1}|\leq M_1|\zeta|^{-[\lambda]-2}.$$

Suppose $|\xi|\geq 1$ and $3E>\eta>E$. Then we have,

$$|\zeta|\geq|\xi|\geq\frac{1}{3(E+1)}|\zeta|,$$

$$|\zeta^{-\lambda-1}-(\xi+i0)^{-\lambda-1}|\leq M_2|\xi|^{-\lambda-2}\leq M_2|\xi|^{-[\lambda]-2},$$

and

$$|(-\bar{\zeta})^{-\lambda-1}-(-\xi+i0)^{-\lambda-1}|\leq M_2|\xi|^{-\lambda-2}\leq M_2|\xi|^{-[\lambda]-2},$$

with some positive constant $M_2>0$.
Put $M:=M_2|p(\lambda)|(|a_0|+|b_0|\exp(-3AE))+M_1$.

$$\begin{aligned}&|\mathcal{F}f_1(\zeta)-a_0p(\lambda)(\xi+i0)^{-\lambda-1}-\exp(iA(\zeta+2iE))b_0p(\lambda)(-\xi+i0)^{-\lambda-1}|\\ &\leq|a_0p(\lambda)||\zeta^{-\lambda-1}-(\xi+i0)^{-\lambda-1}|\\ &\quad+|b_0\exp(-2AE)p(\lambda)\exp(-A\eta)||(-\bar{\zeta})^{-\lambda-1}-(-\xi+i0)^{-\lambda-1}|\\ &\quad+|\mathcal{F}f_1(\zeta)-a_0p(\lambda)\zeta^{-\lambda-1}-\exp(iA\zeta)b_0\exp(-2AE)p(\lambda)(-\bar{\zeta})^{-\lambda-1}|\\ &\leq((|a_0|+|b_0|\exp(-2AE)\exp(-A\eta))|p(\lambda)|M_2+M_1)|\xi|^{-[\lambda]-2}\\ &\leq M|\xi|^{-[\lambda]-2},\qquad \text{for any } \zeta=\xi+i\eta \text{ with } |\xi|\geq 1, \text{ and } 3E>\eta>E.\end{aligned}$$

Since $\mathcal{F}f(\zeta)=\mathcal{F}f_1(\zeta+2iE)$, the above inequality implies (2.2.30). Thus we have proved this lemma. Q.E.D.

In §1, we defined a function $d\equiv d_{\Omega}: S^{n-1}\longrightarrow \boldsymbol{R}$ by,

$$d(\omega):=\frac{\log K\circ\nu^{-1}(-\omega)-\log K\circ\nu^{-1}(\omega)}{2H(\omega)}, \qquad (\omega\in S^{n-1}). \tag{2.1.16}$$

Using this function we state the asymptotic behaviour of $\tilde{\chi}_{\Omega}(\zeta)$ in the following proposition.

PROPOSITION (2.2.32). *Suppose Ω be strictly convex and with notation as above. Then,*
1) $\tilde{\chi}_{\Omega}(\omega,\zeta)\in\mathcal{A}(S^{n-1}\times \boldsymbol{C})$, *and $\tilde{\chi}_{\Omega}(\omega,\zeta)$ is an entire function of $\zeta\in \boldsymbol{C}$ for each fixed $\omega\in S^{n-1}$.*
2) $\tilde{\chi}_{\Omega}(\omega,\xi+i\eta)=\overline{\tilde{\chi}_{\Omega}(\omega,-\xi+i\eta)}=\tilde{\chi}_{\Omega}(-\omega,-\xi-i\eta)=\overline{\tilde{\chi}_{\Omega}(-\omega,\xi-i\eta)}$, *for any $\omega\in S^{n-1}$ and any $\xi,\eta\in \boldsymbol{R}$.*
3) $\sup\{|\mathrm{Im}\,\zeta|;\ \tilde{\chi}_{\Omega}(\omega,\zeta)=0,\ \omega\in S^{n-1}\}<\infty$.
4) *When $|\eta|<C$ (C is any constant), $\tilde{\chi}_{\Omega}$ has the following asymptotics:*

$$\begin{aligned}\tilde{\chi}_{\Omega}(\omega,\xi+i\eta) &\sim(2\pi)^{(n-1)/2}e^{\pi i(n+1)/4}(K\circ\nu^{-1}(\omega))^{-1/2}e^{ih(\omega)(\xi+i\eta)}\\ &\quad\times(e^{H(\omega)(\eta-d(\omega)-i\xi)}+e^{-\pi i(n+1)/2})|\xi|^{-(n+1)/2}+O(|\xi|^{-(n+3)/2}) \qquad as\ \xi\to+\infty,\\ &\sim(2\pi)^{(n-1)/2}e^{-\pi i(n+1)/4}(K\circ\nu^{-1}(\omega))^{-1/2}e^{ih(\omega)(\xi+i\eta)}\\ &\quad\times(e^{H(\omega)(\eta-d(\omega)-i\xi)}+e^{-\pi i(n+1)/2})|\xi|^{-(n+1)/2}+O(|\xi|^{-(n+3)/2}) \qquad as\ \xi\to-\infty.\end{aligned} \tag{2.2.33}$$

PROOF. 1) follows from the compactness of Ω and 2) is a restatement of (2.2.5) and (2.2.6). 3) is obtained by applying Proposition (2.2.16) to the formula (2.2.7) on the basis of Lemma (2.2.8).

For 4), we need only to prove the case when $\xi\longrightarrow+\infty$, because 2) implies that

$$\tilde{\chi}_{\Omega}(\omega,\xi+i\eta)\ (\text{as }\xi\to-\infty)\equiv\overline{\tilde{\chi}_{\Omega}(\omega,\xi'+i\eta)}\ (\text{as }\xi'\to+\infty),\ \text{where }\xi':=-\xi.$$

By a partition of unity and from Lemma (2.2.8), $S(\omega,p+h(-\omega))$ has the expression (2.2.14) for any $N\in N$ with $\lambda=\frac{n-1}{2}$, $a_0=(2\pi)^{(n-1)/2}\times \Gamma\left(\frac{n+1}{2}\right)^{-1}(K\circ\nu^{-1}(-\omega))^{-1/2}$, $b_0=(2\pi)^{(n-1)/2}\Gamma\left(\frac{n+1}{2}\right)^{-1}(K\circ\nu^{-1}(-\omega))^{-1/2}$, and $A=h(\omega)+h(-\omega)=H(\omega)$ (Notation (2.1.14)). Applying Lemma (2.2.30) to (2.2.7), we get the asymptotics as $\xi\longrightarrow+\infty$ with $|\eta|<C$:

$$\begin{aligned}
&\tilde{\chi}_\Omega(\omega, \xi+i\eta)=\int_{-\infty}^{+\infty} S(\omega, p)e^{ip(\xi+i\eta)}dp \\
&=\frac{(2\pi)^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}\times\Gamma\left(\frac{n+1}{2}\right)\Big((K\circ\nu^{-1}(-\omega))^{-1/2}e^{\pi i(n+1)/4}e^{h(-\omega)(\eta-i\xi)} \\
&\quad+(K\circ\nu^{-1}(\omega))^{-1/2}e^{-\pi i(n+1)/4}e^{h(\omega)(-\eta+i\xi)}\Big)\times|\xi|^{-(n+1)/2}+O(|\xi|^{-(n+3)/2}) \\
&=(2\pi)^{(n-1)/2}(K\circ\nu^{-1}(\omega))^{-1/2}e^{\pi i(n+1)/4}e^{h(\omega)(-\eta+i\xi)} \\
&\quad\times\Big((K\circ\nu^{-1}(\omega))^{1/2}(K\circ\nu^{-1}(-\omega))^{-1/2}e^{(h(\omega)+h(-\omega))(\eta-i\xi)}+e^{-\pi i(n+1)/2}\Big) \\
&\quad\times|\xi|^{-(n+1)/2}+O(|\xi|^{-(n+3)/2}).
\end{aligned}$$

In terms of the definition of $H(\omega)$ and $d(\omega)$ ((2.1.14) and (2.1.16)), this is equivalent to,

$$\begin{aligned}
&=(2\pi)^{(n-1)/2}(K\circ\nu^{-1}(\omega))^{-1/2}e^{\pi i(n+1)/4}e^{h(\omega)(-\eta+i\xi)} \\
&\quad\times(e^{H(\omega)(\eta-d(\omega)-i\xi)}+e^{-\pi i(n+1)/2})\times|\xi|^{-(n+1)/2}+O(|\xi|^{-(n+3)/2}).
\end{aligned}$$

Hence the proof of the proposition is completed. Q.E.D.

REMARK (2.2.34). It is not new thing to consider the Fourier transform of a characteristic function of a domain or to obtain its asymptotic behaviour. F. John [1937] got the asymptotic behaviour along a direction of $\boldsymbol{R}^n$ when Ω is centrally symmetric strictly convex body in the study of a certain homogeneous integral equation. Vinberg [1967] utilize $\tilde{\chi}_\Omega(\zeta)$ when Ω is a convex cone in the study of complex homogeneous domains. (In this case, the domain of convergence of $\tilde{\chi}_\Omega(z)$ in $i\boldsymbol{R}^n$ is the dual cone of Ω). Berenstein [1980] used the asymptotic behaviour of $\tilde{\chi}_\Omega(\zeta)$ when Ω is a domain in $\boldsymbol{R}^2$, in the study of a certain free boundary problem related to Pompeiu problem. His method of obtaining the asymptotics is to reduce it to the line integral on $\partial\Omega$ by using Green's formula and the direction of variables of tending to ∞ is rather different from our l_ω, although there is no essential difference from ours. Our method is originated in [15].

REMARK (2.2.35). The assumption of strict convexity is not important in getting asymptotics of $\tilde{\chi}_\Omega(\omega, \zeta)$. This assumption is essentially used when we state how $\mathcal{N}(\Omega)$ is. Even if Ω is not necessarily convex, we can easily obtain the asymptotics along the direction of l_ω ($\omega\in S^{n-1}$), where ω satisfies,

$$K(q)\neq 0 \text{ for } \forall q\in\partial\Omega \text{ such that } \nu(q)=\pm\omega. \tag{2.2.36}$$

§ 3. Statement of results

In this section, we shall state our main theorem and corollaries in $\boldsymbol{R}^n$ case. Suppose Ω be strictly convex. Then, $\mathcal{N}(\Omega)$ are distributed along $S^{n-1}\underset{Z_2}{\times}\boldsymbol{C}^\times$ (notation as below) in an asymptotically regular fashion with bounded imaginary part. This description is stated in Theorem (2.3.6), from which we can read some geometric properties of Ω from those of the null variety $\mathcal{N}(\Omega)$ (Corollary (2.3.10)). Corollary (2.3.11) asserts that the assignment $\Omega\longmapsto\mathcal{N}(\Omega)$ is one to one up to parallel displacements when we assume that Ω is strictly convex and that the dimension $n=2$. This gives a partial answer of our Problem B. These results are also reformulated in terms of Dirichlet series made from $\mathcal{N}(\Omega)$. These reformulations and some characterization of the injective image of the above assignment $\Omega\longmapsto\mathcal{N}(\Omega)$ are stated in Proposition (2.3.15), Proposition (2.3.19), and Proposition (2.3.20).

Theorem (2.3.6) will be proved in § 4; Corollary (2.3.10) and Corollary (2.3.11) will be in § 5; Proposition (2.3.15) together with Proposition (2.3.19) and Proposition (2.3.20) will be in § 6.

Let Ω be a strictly convex domain in $\boldsymbol{R}^n$ (Definition (2.1.5)). Let us recall notations in § 1.

$\nu\equiv\nu_\Omega : \partial\Omega\longrightarrow S^{n-1}$, the Gauss map ((2.1.2)).
$K\equiv K_\Omega: \partial\Omega\longrightarrow\boldsymbol{R}$, the Gauss-Kronecker curvature ((2.1.3)).
$H\equiv H_\Omega: S^{n-1}\longrightarrow\boldsymbol{R}_+$, the breadth function ((2.1.14)).

$$d\equiv d_\Omega : S^{n-1}\ni\omega\longmapsto d(\omega)=\frac{\log K\circ\nu^{-1}(-\omega)-\log K\circ\nu^{-1}(\omega)}{2H(\omega)}\in\boldsymbol{R},\quad ((2.1.16)).$$

Note that all these maps are invariant under parallel displacements of Ω.

We defined in § 2 the null variety ((2.2.2) and (2.2.3)):

$$\mathcal{N}(\Omega)=\{\zeta\in\boldsymbol{C}^n;\ \tilde{\chi}_\Omega(\zeta)=0\},\ \text{and}\ \mathcal{N}(\Omega)_{\boldsymbol{R}}=\mathcal{N}(\Omega)\cap\boldsymbol{R}^n. \tag{2.3.1}$$

Now we introduce a new function which will be useful in the description of the asymptotics of $\mathcal{N}(\Omega)$.

$$f:\ S^{n-1}\times\boldsymbol{R}\ni(\omega,m)\overset{C^\infty}{\longmapsto}f(\omega,m)\equiv f_m(\omega):=\frac{\pi(4m+n-1)}{2H(\omega)}+i\ d(\omega)\in\boldsymbol{C}. \tag{2.3.2}$$

Let $n+1$ dimensional real analytic manifold $S:=S^{n-1}\underset{Z_2}{\times}\boldsymbol{C}^\times$ be the image of the following map Π.

$$\Pi = \iota \circ \pi : S^{n-1} \times C^{\times} \xrightarrow{\pi} S^{n-1} \underset{Z_2}{\times} C^{\times} \equiv S^{n-1} \times C^{\times} / \sim \overset{\iota}{\hookrightarrow} C^n. \tag{2.3.3}$$

Here two elements (ω, ζ) and (ω', ζ') of $S^{n-1} \times C^{\times}$ are *equivalent* if and only if

$$(\omega', \zeta') = (\omega, \zeta) \quad \text{or} \quad (-\omega, -\zeta). \tag{2.3.4}$$

We denote by $S := S^{n-1} \underset{Z_2}{\times} C^{\times} \equiv S^{n-1} \times C^{\times} / \sim$ the corresponding quotient space of $S^{n-1} \times C^{\times}$, and by π the corresponding quotient map. ι is defined by,

$$\iota((\omega, \zeta) \bmod \sim) := \zeta\omega = (\zeta\omega_1, \zeta\omega_2, \cdots, \zeta\omega_n) \in C^n, \tag{2.3.5}$$

for $\zeta \in C$ and $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$. Via the injection ι, we will regard $S = S^{n-1} \underset{Z_2}{\times} C^{\times}$ as a subset of C^n.

Then the following theorem is our main theorem in this chapter which describes the asymptotic behaviour of $\mathcal{N}(\Omega) \cap S$.

THEOREM (2.3.6). *Let $S = S^{n-1} \underset{Z_2}{\times} C^{\times}$ be an $n+1$-dimensional manifold defined as above. Let Ω be a strictly convex domain in R^n. Then there is an integer $m_0 \equiv m_0(\Omega)$ dependent only on Ω such that*

$$\mathcal{N}(\Omega) \cap S = \Big(\coprod_{m=m_0}^{\infty} \mathcal{N}_m \Big) \amalg (\text{compact set}), \qquad (\textit{disjoint union}),$$

where for each integer $m \geq m_0$, $\mathcal{N}_m$ is a regular submanifold in S $(\subset C^n)$, and is analytically diffeomorphic to S^{n-1}.

More precisely, for each integer $m \geq m_0$, there exists an analytic map $F_m : S^{n-1} \longrightarrow C$ such that the correspondence

$$\Pi \circ (\mathrm{id} \times F_m) : S^{n-1} \ni \omega \longmapsto F_m(\omega) \cdot \omega \in C^n,$$

gives an analytic isomorphism from S^{n-1} onto $\mathcal{N}_m$ $(\subset C^n)$. Moreover, $\{F_m; m \geq m_0\}$ satisfies the following two conditions (Notation (2.3.2)): *For any element ω of S^{n-1},*

$$F_m(\omega) = f_m(\omega) + O(m^{-1}) \qquad \text{as } N \ni m \longrightarrow \infty, \tag{2.3.7}$$

and,

$$F_m(\omega) = \overline{F_m(-\omega)}, \qquad (\omega \in S^{n-1}). \tag{2.3.8}$$

In (2.3.7), *the estimate is uniform with respect to* $\omega \in S^{n-1}$.

Example (2.3.9). When Ω is a unit ball, $\tilde{\chi}_\Omega(z)=(2\pi)^{n/2}\frac{J_{n/2}(\zeta)}{\zeta^{n/2}}$, where $\zeta:=(z_1^2+z_2^2+\cdots+z_n^2)^{1/2}$ for $z=(z_1, z_2, \cdots, z_n)\in C^n$, and $J_\nu(\zeta)$ denotes the ν-th Bessel function. When $\nu>-1$, $\frac{J_\nu(\zeta)}{\zeta^\nu}$ is an entire function of $\zeta \in C$, whose zeros are all real. Let $j(\nu, m)$ $(m\in N_+)$ be the enumeration of positive zeros (increasing order). Set

$$\mathcal{M}_m:=\left\{z=(z_1, z_2, \cdots, z_n)\in C^n;\ z_1^2+z_2^2+\cdots+z_n^2=j\left(\frac{n}{2}, m\right)^2\right\},$$

for $m\in N_+$. Then we have,

$$\mathcal{N}(\Omega)=\coprod_{m=1}^{\infty}\mathcal{M}_m.$$

On the other hand, the following asymptotic behavior of $j(\nu, m)$ is well known:

$$j(\nu, m)=\frac{\pi(4m+2\nu-1)}{4}+O(m^{-1}), \quad \text{as } m\longrightarrow\infty.$$

In this case, $f(\omega, m)=\frac{\pi(4m+n-1)}{4}$, and $d(\omega)=0$, so $\mathcal{N}_m$ (in Theorem (2.3.6)) is precisely the m-th connected component of $\mathcal{N}(\Omega)\cap S$.

By using the description of the asymptotics of $\mathcal{N}(\Omega)$ in Theorem (2.3.6), the question "Does $\mathcal{N}(\Omega)$ determine Ω?" turns out to be "yes", when we require that Ω be strictly convex in $\boldsymbol{R}^2$.

COROLLARY (2.3.10). *If two strictly convex domains in* $\boldsymbol{R}^2$ *have the same null variety, then these domains differ from each other by a parallel translation.*

Next corollary relates the asymptotics of $\mathcal{N}(\Omega)\cap S$ with some geometric properties of Ω.

COROLLARY (2.3.11). *The following three conditions* a)$\sim$c) (*in* 1), 2), *and* 3), *respectively*) *on a strictly convex domain* Ω *in* $\boldsymbol{R}^n$ *are equivalent.*

1) a) *Ω is centrally symmetric with respect to an inner point.*

b) $\mathcal{N}(\Omega)_R$ (*Definition* (2.2.3)) *contains countably many hypersurfaces in* $\boldsymbol{R}^n$ *as connected components.*

c) $\mathcal{N}(\Omega)_R$ *is a disjoint union of countably many closed hypersurfaces without boundaries in* $\boldsymbol{R}^n$ *and a compact set.*

2) a) Ω *is a ball.*

b) $\mathcal{N}(\Omega)_R$ *contains countably many hyperspheres in* $\boldsymbol{R}^n$ *as connected components.*

c) $\mathcal{N}(\Omega)_R$ *consists of countably many concentric hyperspheres whose common center is the origin.*

3) a) $\partial\Omega$ *is a hypersurface with constant breadth.*

b) $\mathrm{pr}(\mathcal{N}(\Omega)\cap S)$ $(\subset \boldsymbol{R}^n)$ *contains a sequence of hypersurfaces which approximates hyperspheres asymptotically.*

c) $\mathrm{pr}(\mathcal{N}(\Omega)\cap S)$ $(\subset \boldsymbol{R}^n)$ *consists of hypersurfaces which approximates hyperspheres asymptotically.*

Here the projection $\mathrm{pr}: S = S^{n-1}\underset{\boldsymbol{Z}_2}{\times}\boldsymbol{C}^{\times} \longrightarrow \boldsymbol{R}$ *is defined by,*

$$\mathrm{pr}:\ S \ni (\omega, \xi+i\eta) \bmod \boldsymbol{Z}_2 \longmapsto (\xi\omega_1, \xi\omega_2, \cdots, \xi\omega_n) \in \boldsymbol{R}^n,$$

and precisely speaking, 'asymptotically' in 3) b) (*it is the same with* c)) *is in the following sense:*
There exist hypersurfaces $X_j \subset \mathrm{pr}(\mathcal{N}(\Omega)\cap S)$ $(j\in N)$, *increasing sequence* $\boldsymbol{R}\ni R_j\uparrow\infty$, *and a constant* $C>0$ *such that*

$$\mathrm{dist}(X_j, S(R_j)) < CR_j^{-\varepsilon}, \qquad \textit{for any } j\in N.$$

Here, a positive constant $0<\varepsilon\leq 1$ *is chosen arbitrary, giving the same condition in* b) *or* c) *of* 3). *And we define*

$$\text{(2.3.12)} \qquad \mathrm{dist}(X_j, S(R_j)) := \max_{x\in X_j}\min_{y\in S(R_j)}|x-y| + \max_{y\in S(R_j)}\min_{x\in X_j}|x-y|,$$

and, $S(R_j) := \{x\in \boldsymbol{R}^n;\ |x|=R_j\}$.

Now, Corollary (2.3.10) is a kind of a uniqueness theorem. How about 'an existence theorem'? To our regret, it seems hard to determine the necessary and sufficient conditions for an analytic set in $\boldsymbol{C}^n$ to coincide with a null set of $\tilde{\chi}_\Omega$ for some convex domain Ω, except when $n=1$. But in much weaker sense, namely, about the conditions of the asymptotics of $\mathcal{N}(\Omega)$, we can give a kind of existence theorem at least when $n=2$. To do this, it is convenient to give a reformulation of Theorem

(2.3.6) in terms of Dirichlet series made from $\mathcal{N}(\Omega)$.

Let us recall $l_\omega := \boldsymbol{C}\cdot\omega = \{\zeta\omega = (\zeta\omega_1, \zeta\omega_2, \cdots, \zeta\omega_n) \in \boldsymbol{C}^n; \zeta \in \boldsymbol{C}\}$, for each element $\omega = (\omega_1, \cdots, \omega_n) \in S^{n-1}$ (see § 2).

DEFINITION (2.3.13). A subset $\mathcal{M}$ in $\boldsymbol{C}^n$ is called *annual ring-like* when there are three functions $P, Q, R : S^{n-1} \longrightarrow \boldsymbol{R}$, such that the following conditions hold: For each $\omega \in S^{n-1}$, the intersection l_ω and $\mathcal{M}$ consists of countably many points. Let $p(\omega, k)$ $(k \in \boldsymbol{N})$ be the enumeration of them. (Our definition does not depend on this enumeration.) Then for each $\omega \in S^{n-1}$, Dirichlet series $\sum_{k=0}^{\infty} e^{-p(\omega,k)t}$ is absolutely convergent for any $t > 0$ and,

$$\lim_{t\downarrow 0}\left|\sum_{k=0}^{\infty} e^{-p(\omega,k)t} - \frac{1}{4\pi t}(P(\omega) + (Q(\omega) + iR(\omega))t)\right| = 0.$$

Our concern is the asymptotics of $\mathcal{M}$ and we are sometimes obliged to neglect finite subset of $\mathcal{M} \cap l_\omega$. But if $\mathcal{M}$ satisfies the above conditions, the following formula holds for any $N \in \boldsymbol{N}$.

$$\lim_{t\downarrow 0}\left|\sum_{k=N}^{\infty} e^{-p(\omega,k)t} - \frac{1}{4\pi t}(P(\omega) + (Q(\omega) - 4N\pi + iR(\omega))t)\right| = 0.$$

So it is natural to consider $\tilde{Q} : S^{n-1} \longrightarrow S^1 \simeq \boldsymbol{R}/4\pi\boldsymbol{Z}$, instead of Q.

Notation (2.3.14). We use the notation $P_{\mathcal{M}}$, $\tilde{Q}_{\mathcal{M}}$, and $R_{\mathcal{M}}$ instead of P, $\tilde{Q}$, and R respectively, when we want to emphasize these functions are invariants of an annual ring-like set $\mathcal{M}$.

Then the following proposition is a reformulation of Theorem (2.3.6).

PROPOSITION (2.3.15). *Suppose Ω be strictly convex domain in $\boldsymbol{R}^n$. Then the null variety $\mathcal{N}(\Omega)$ $(\subset \boldsymbol{C}^n)$ is annual ring-like* (Definition (2.3.13)). *Moreover, with notation as above and as in* (2.1.2), (2.1.3) *and* (2.1.14), *the following three formulae hold*:

$$P_{\mathcal{N}(\Omega)}(\omega) = 2H_\Omega(\omega). \tag{2.3.16}$$

$$\tilde{Q}_{\mathcal{N}(\Omega)}(\omega) \equiv (3-n)\pi \mod 4\pi\boldsymbol{Z}. \tag{2.3.17}$$

$$R_{\mathcal{N}(\Omega)}(\omega) = -\log\frac{K_\Omega \circ \nu_\Omega^{-1}(-\omega)}{K_\Omega \circ \nu_\Omega^{-1}(\omega)}. \tag{2.3.18}$$

And a uniqueness theorem (Corollary (2.3.10)) is reformulated as follows:

PROPOSITION (2.3.19). *The assignment*

$$\left\{\begin{matrix}\text{Strictly convex domain in } \boldsymbol{R}^2 \\ \text{up to parallel displacements}\end{matrix}\right\} \ni \Omega \longmapsto (P_{\mathfrak{N}(\Omega)}, R_{\mathfrak{N}(\Omega)}) \in C^\infty(S^1, \boldsymbol{R}^2)$$

is one to one.

Conversely, we are ready to state a kind of existence theorem, which describes the precise image of the assignment in the above Proposition (2.3.19).

PROPOSITION (2.3.20). $(P, R) \in C^\infty(S^1, \boldsymbol{R}^2)$ *belongs to the injective image of the assignment in Proposition* (2.3.19) *if and only if the following four conditions are satisfied:*
For any $\theta \in S^1 \simeq \boldsymbol{R}/2\pi\boldsymbol{Z}$,

$$P(\theta) = P(\theta+\pi), \tag{2.3.21}$$

$$P(\theta) + P''(\theta) > 0, \tag{2.3.22}$$

$$R(\theta) + R(\theta+\pi) = 0. \tag{2.3.23}$$

$$\int_0^{2\pi} \frac{P(\theta)+P''(\theta)}{1+\exp(R(\theta))} e^{\pm i\theta} d\theta = 0. \tag{2.3.24}$$

Here $P''(\theta) := \dfrac{d^2}{d\theta^2} P(\theta)$.

REMARK (2.3.25). Things would go easy, if we assume in addition Ω is centrally symmetric. Say, a uniqueness theorem (cf. Corollary (2.3.10) or Proposition (2.3.19)) holds for any dimension $n \in N_+$. That is,

$$\left\{\begin{matrix}\text{Strictly convex domains in } \boldsymbol{R}^n \\ \text{which is centrally symmetric} \\ \text{with respect to the origin}\end{matrix}\right\} \ni \Omega \longmapsto (P_{\mathfrak{N}(\Omega)}, R_{\mathfrak{N}(\Omega)}) \in C^\infty(S^{n-1}, \boldsymbol{R}^2)$$

is one to one.

And our existence theorem is formulated as follows:
$(P, R) \in C^\infty(S^{n-1}, \boldsymbol{R}^2)$ *belongs to the above injective image if and only if the following four conditions are satisfied:*

For any $\omega\in S^{n-1}$, $P(\omega)=P(-\omega)$, $\det(D^2P+P)(\omega)>0$, $R(\omega)\equiv 0$, *and for any linear function* $v(\omega)$ *on* $\boldsymbol{R}^n$, $\int_{S^{n-1}} v(\omega)\det(D^2P+P)(\omega)d\omega=0$.
Here, the Hessian D^2, and det$\equiv$determinant of a tensor field of $(0,2)$-type are defined by the standard Riemannian metric on S^{n-1}.

REMARK (2.3.26). When Ω is centrally symmetric convex domain (not necessarily strictly convex), $\mathcal{N}(\Omega)$ is also annual ring-like and $\tilde{Q}_{\mathcal{N}(\Omega)}$ (Notation (2.3.14); cf. Proposition (2.3.15)) turns out to be a (crude) measure of the degeneracy of the Hessian.

§ 4. Proof of the main theorem

In this section, Ω is strictly convex and retain notations in § 3.

Put a nonnegative constant D by,

$$D:=\max\{|d(\omega)|;\ \omega\in S^{n-1}\}.$$

Lemma (2.1.18) says that $D=0$ if and only if Ω is centrally symmetric with respect to some inner point. By Proposition (2.2.32), Theorem (2.3.6) is deduced from the following

THEOREM (2.3.6)′. *Let* Ω *be a strictly convex domain in* $\boldsymbol{R}^n$. *There is an integer* $m_0\equiv m_0(\Omega)$ *dependent only on* Ω. *Let* W *be an open subset in* $S:=S^{n-1}\underset{Z_2}{\times}\boldsymbol{C}^\times$ $(\subset\boldsymbol{C}^n)$ *given by,*

$$W:=\Pi\left(\left\{(\omega,\xi+i\eta);\ \xi,\eta\in\boldsymbol{R},\ \omega\in S^{n-1},\ |\eta|<D+1,\ |\xi|>f\left(\omega,m_0-\frac{1}{2}\right)\right\}\right).$$

Then we have

$$\mathcal{N}(\Omega)\cap W=\coprod_{m=m_0}^{\infty}\mathcal{N}_m \qquad (\textit{disjoint union of connected components}),$$

where for each integer $m\geq m_0$, $\mathcal{N}_m$ *is a regular submanifold in* $\boldsymbol{C}^n$, *satisfying the conditions in Theorem* (2.3.6).

Define an open set in $\boldsymbol{C}$ by,

$$U:=\{\zeta=\xi+i\eta\in\boldsymbol{C};\ \xi>0,\ |\eta|<D+1\} \tag{2.4.1}$$

and set $\phi_j: S^{n-1}\times U\longrightarrow\boldsymbol{C}$ $(j=1,2)$ as follows.

$$\begin{aligned}(2.4.2)\quad \phi_1(\omega,\zeta):=&(2\pi)^{(n-1)/2}e^{\pi i(n+1)/4}(K\circ\nu^{-1}(\omega))^{-1/2}e^{ih(\omega)\zeta}\\ &\times(e^{H(\omega)(-d(\omega)-i\zeta)}+e^{-\pi i(n+1)/2})\zeta^{-(n+1)/2},\qquad(\omega\in S^{n-1},\zeta\in U),\end{aligned}$$

$$(2.4.3)\qquad \phi_2(\omega,\zeta):=\tilde{\chi}_\Omega(\omega,\zeta)-\phi_1(\omega,\zeta),\qquad(\omega\in S^{n-1},\zeta\in U).$$

Here, we choose a branch of $\zeta^{-(n+1)/2}$ so that it takes positive value on the real positive axis.

It is clear from the definition that ϕ_j $(j=1,2)$ are C^∞ function on $S^{n-1}\times U$ and holomorphic as a function of $\zeta\in U$ for each fixed $\omega\in S^{n-1}$.

The following lemma is an immediate corollary of Proposition (2.2.32).

LEMMA (2.4.4). *There exists a constant $C\equiv C(\Omega)>0$ such that for any $\zeta=\xi+i\eta\in U$ satisfying* $\mathrm{Re}\,\zeta\equiv\xi>1$ *and for any $\omega\in S^{n-1}$,*

$$(2.4.5)\qquad |\phi_2(\omega,\zeta)|\le C|\xi|^{-(n+3)/2}.$$

Now we will prepare the following three lemmas, namely Lemma (2.4.6), (2.4.7), and (2.4.12). Lemma (2.4.4) and Lemma (2.4.7) imply that, roughly speaking, the null set of $\tilde{\chi}_\Omega(\omega,\zeta)$ is approximated by that of $\phi_1(\omega,\zeta)$. Therefore the description of the null set of $\phi_1(\omega,\zeta)$ in Lemma (2.4.6) gives the asymptotic behaviour of the null set of $\tilde{\chi}_\Omega(\omega,\zeta)$, which is the contents of Lemma (2.4.12).

LEMMA (2.4.6). *The null set of $\phi_1(\omega,\zeta)$ is given by,*

$$\left\{(\omega,f_m(\omega));\ \omega\in S^{n-1},\ m\in \boldsymbol{Z},\ m>\frac{1-n}{4}\right\}\subset S^{n-1}\times U.$$

Furthermore, $\zeta=f_m(\omega)$ $\left(\text{for any } m\in\boldsymbol{Z},\ m>\frac{1-n}{4}\right)$ is a simple root of $\phi_1(\omega,\zeta)=0$ for each fixed element ω of S^{n-1} (Notation (2.3.2)).

LEMMA (2.4.7). *There are constants $m_1>0$, $r_1>0$, and $C_1>0$ such that the following statements* 1)~3) *hold.*

1) *Let $m\in N$ and $z\in\boldsymbol{C}$. If $m\ge m_1$ and $|z|<r_1$, then*

$$(2.4.8)\qquad |\phi_1(\omega,f(\omega,m)+z)|\ge C_1m^{-(n+1)/2}|z|.$$

2) *Let $m\in N$ and $\eta\in\boldsymbol{R}$. If $m\ge m_1$ and $|\eta|\le D+1$, then*

$$(2.4.9)\qquad \left|\phi_1\!\left(\omega,f\!\left(\omega,m+\frac{1}{2}\right)+i\eta\right)\right|\ge C_1\!\left(\mathrm{Re}\,f\!\left(\omega,m+\frac{1}{2}\right)\right)^{-(n+1)/2}.$$

3) *Let $\xi\in\boldsymbol{R}$. If $\xi>\mathrm{Re}\,f(\omega,m_1)$, then*

$$(2.4.10) \qquad |\phi_1(\omega, \xi+i(D+1))| \geq C_1 \xi^{-(n+1)/2},$$

and

$$(2.4.11) \qquad |\phi_1(\omega, \xi-i(D+1))| \geq C_1 \xi^{-(n+1)/2}.$$

LEMMA (2.4.12). *There is an integer $m_2 \equiv m_2(\Omega)$ and there exists a map F_m: $S^{n-1} \longrightarrow C$ for each integer $m \geq m_2$ such that the following conditions are satisfied.*
Define an $n+1$-dimensional submanifold W_2 in C^n by,

$$W_2 := \Pi\left(\left\{(\omega, \xi+i\eta);\ \xi, \eta \in \boldsymbol{R}, \omega \in S^{n-1}, |\eta| < D+1, |\xi| > f\left(\omega, m_2 - \frac{1}{2}\right)\right\}\right).$$

(Recall Π: $S^{n-1} \times C^{\times} \longrightarrow S \subset C^n$ is the natural quotient map defined in (2.3.3)). *Then*

$$(2.4.13) \qquad \mathcal{N}(\Omega) \cap W_2 = \coprod_{m=m_2}^{\infty} \mathcal{N}_m \quad \textit{(disjoint union of connected components)}.$$

Moreover for any $m \geq m_2$,

$$(2.4.14) \qquad \mathcal{N}_m = \Pi \circ (\mathrm{id} \times F_m)(S^{n-1}) = \Pi(\{(\omega, F_m(\omega));\ \omega \in S^{n-1}\}),$$

and

$$(2.4.15) \qquad |F_m(\omega) - f_m(\omega)| \leq C_2 m^{-1},$$

for some constant C_2 depending only on Ω.

Now, let us prove these three lemmas. Then the proof of Theorem (2.3.6)$'$ will be completed except showing the analyticity of F_m.

PROOF OF LEMMA (2.4.6). For $\omega \in S^{n-1}$ and $\zeta = \xi + i\eta \in U$,

$$\phi_1(\omega, \zeta) = 0 \longleftrightarrow e^{H(\omega)(-d(\omega)-i\zeta)} = -e^{-\pi i(n+1)/2}, \qquad \text{and } \zeta \in U.$$

On the other hand, for $\omega \in S^{n-1}$ and $\zeta = \xi + i\eta \in U$,

$$e^{H(\omega)(-d(\omega)-i\zeta)} = -e^{-\pi i(n+1)/2}.$$

$$\begin{aligned} \longleftrightarrow \quad H(\omega)(-d(\omega)-i\zeta) &= -\frac{\pi i(n+1)}{2} - (2m-1)\pi i \\ &= -\pi i\left(2m + \frac{n-1}{2}\right), \qquad \text{for some } m \in \boldsymbol{Z}. \end{aligned}$$

$$\longleftrightarrow \quad \zeta = \frac{\pi\left(2m+\frac{n-1}{2}\right)}{H(\omega)} + i\ d(\omega), \qquad \text{for some } m \in \boldsymbol{Z}.$$

Finally, $\zeta = \dfrac{\pi\left(2m+\frac{n-1}{2}\right)}{H(\omega)} + i\ d(\omega) \in U.$

$(U \equiv \{\zeta = \xi + i\eta \in \boldsymbol{C};\ \xi > 0,\ |\eta| < D+1\}$ was defined in (2.4.1))

$$\longleftrightarrow \quad m > \frac{1-n}{4}.$$

Thus the lemma is proved. Q.E.D.

PROOF OF LEMMA (2.4.7). Each assumption in 1)$\sim$3) makes the imaginary part of each second variable in $\phi_1(\omega, *)$, namely, $\mathrm{Im}(f(\omega, m)+z) = d(\omega) + \mathrm{Im}\, z$, $\mathrm{Im}\left(f\left(\omega, m+\frac{1}{2}\right)+i\eta\right) = d(\omega)+\eta$, and $\mathrm{Im}(\xi + i(D\pm 1)) = D \pm 1$, bounded.

When $\omega \in S^{n-1}$ and $|\mathrm{Im}\,\zeta|$ is bounded and $\mathrm{Re}\,\zeta$ is sufficiently large, $|(2\pi)^{(n-1)/2} e^{\pi i(n+1)/4} (K \circ \nu^{-1}(\omega))^{-1/2} e^{ih(\omega)\zeta}|$ is bounded from 0 and ∞, and $|\zeta^{-(n+1)/2}| \sim |\xi^{-(n+1)/2}|$. Therefore to prove this lemma, it is enough to estimate

$$A(\omega, \zeta) := |e^{H(\omega)(-d(\omega)-i\zeta)} + e^{-\pi i(n+1)/2}|, \tag{2.4.16}$$

for $\omega \in S^{n-1}$, $\zeta \in \boldsymbol{C}$, when ζ takes special values according to each case 1)$\sim$3).

Recall that the breadth function of Ω (Definition (2.1.14)) $H \equiv H_\Omega : S^{n-1} \longrightarrow \boldsymbol{R}$ is a positive valued C^∞ function on S^{n-1}, so there are positive constants $H_j \equiv H_j(\Omega) > 0$ $(j=1, 2)$ such that for any $\omega \in S^{n-1}$,

$$H_1 < H(\omega) < H_2. \tag{2.4.17}$$

Now, let us estimate $A(\omega, *)$ under each assumption 1)$\sim$3) of this lemma.

1) In this case, $\zeta = f(\omega, m) + z$. So $d(\omega) + i\zeta = \dfrac{\pi i\left(2m+\frac{n-1}{2}\right)}{H(\omega)} + iz$. Therefore,

$$\begin{aligned} A(\omega, f(\omega, m)+z) &= |e^{-\pi i(2m+(n-1)/2) - iH(\omega)z} + e^{-\pi i(n+1)/2}| \\ &= |-e^{-iH(\omega)z} + 1|. \end{aligned}$$

Choosing a positive constant $r_0\,(>0)$ so small that $r_0 e^{r_0}<\frac{1}{2}$, and put $r_1:=\frac{r_0}{H_2}$, then we get, $|-e^{-iH(\omega)z}+1|\geq\frac{1}{2}|z|H_1$, for any $z\in \boldsymbol{C}$ with $|z|<r_0$, because of the following inequality (2.4.18).

$$|1-e^x|\geq\frac{1}{2}|x|, \tag{2.4.18}$$

for any $x\in \boldsymbol{C}$ with $|x|<r_0$, which follows from

$$|1-e^x|\geq|x|-|1-e^x-x|=|x|-\left|x^2\sum_{n=2}^{\infty}\frac{x^{n-2}}{n!}\right|\geq|x|-|x|^2e^{|x|}\geq\frac{1}{2}|x|.$$

2) In this case, $\zeta=f\left(\omega, m+\frac{1}{2}\right)+i\eta$. So $d(\omega)+i\zeta=\dfrac{\pi i\left(2m+\frac{n+1}{2}\right)}{H(\omega)}-\eta$. Therefore,

$$\begin{aligned}A\left(\omega, f\left(\omega, m+\frac{1}{2}\right)+i\eta\right)&=|e^{-\pi i(2m+(n+1)/2)+H(\omega)\eta}+e^{-\pi i(n+1)/2}|\\&=|e^{H(\omega)\eta}+1|\\&\geq 1, \qquad \text{for any } \eta\in \boldsymbol{R}.\end{aligned}$$

3) In this case, $\zeta=\xi\pm i(D+1)$. So $d(\omega)+i\zeta=d(\omega)+i\xi\mp(D+1)$. Therefore, for any $\omega\in S^{n-1}$ and any $\xi\in \boldsymbol{R}$,

$$\begin{aligned}A(\omega, \xi\pm(D+1))&=|e^{H(\omega)(-i\xi\pm(D+1)-d(\omega))}+e^{-\pi i(n+1)/2}|\\&=|1-e^{H(\omega)(\pm(D+1)-d(\omega))}e^{-i\xi H(\omega)+\pi i(n+1)/2}|\\&\geq|1-e^{H(\omega)(\pm(D+1)-d(\omega))}|\\&\geq|1-e^{\pm H(\omega)}|\\&\geq|1-e^{\pm H_1}|\ (>0). \qquad \text{(Notation (2.4.17))}\end{aligned}$$

From the above inequalities of $A(\omega, *)$, each estimate (2.4.8)~(2.4.11) of $\phi_1(\omega, *)$ is proved. Q.E.D.

PROOF OF LEMMA (2.4.12). Set

$$C_2:=\frac{2C}{C_1}\left(\frac{\pi}{H_2}\right)^{-(n+3)/2}, \tag{2.4.19}$$

where constants C, C_1, and H_2 are as in Lemma (2.4.4), Lemma (2.4.7), and (2.4.17) respectively.

For each fixed $\omega \in S^{n-1}$ and $m \in N$, we define two open sets in C as follows:

$$V_1(\omega, m) := \{\zeta \in C;\ |\zeta - f(\omega, m)| < C_2 m^{-1}\} \subset C,$$

$$V_2(\omega, m) := \Big\{\zeta = \xi + i\eta \in C;\ |\eta| < D+1,$$
$$\operatorname{Re} f\Big(\omega, m - \frac{1}{2}\Big) < \xi < \operatorname{Re} f\Big(\omega, m + \frac{1}{2}\Big)\Big\} \subset C.$$

Choose a positive integer m_2 so large that the following two conditions are satisfied for any $\omega \in S^{n-1}$ and any integer $m \geq m_2$:

$$V_1(\omega, m) \subset V_2(\omega, m), \tag{2.4.20}$$

$$|\phi_1(\omega, \zeta)| > |\phi_2(\omega, \zeta)|;\ \text{for any } \zeta \in \partial V_j(\omega, m)\ \ (j=1, 2). \tag{2.4.21}$$

This is possible because (2.4.20) and (2.4.21) are satisfied if

$$m_2 \geq \max\Big\{C_2(D+1)^{-1}, C_2 H_2 \pi^{-1}, m_1+1, \frac{1}{2}(H_2\pi^{-1}+1), \frac{1}{2}(2H_2 C C_1^{-1}\pi^{-1}+1)\Big\},$$

where m_1 is the constant in Lemma (2.4.7).

In fact, suppose we choose m_2 as above. We will show (2.4.20) and (2.4.21).

First, note that $V_1(\omega, m) \subset V_2(\omega, m)$ if and only if

$$C_2 m^{-1} \leq \min\Big\{D+1, \frac{\pi}{H(\omega)}\Big\}.$$

On the other hand, the assumption $m \geq m_2 \geq \max\{C_2(D+1)^{-1}, C_2 H_2 \pi^{-1}\}$ implies

$$C_2 m^{-1} \leq \min\Big\{D+1, \frac{\pi}{H_2}\Big\} \leq \min\Big\{D+1, \frac{\pi}{H(\omega)}\Big\}. \tag{2.4.22}$$

So it follows $V_1(\omega, m) \subset V_2(\omega, m)$, for any $\omega \in S^{n-1}$.

Next, the assumption $m \geq m_2 \geq \max\Big\{m_1+1, \frac{1}{2}(H_2\pi^{-1}+1)\Big\}$ implies

$$\operatorname{Re} f\Big(\omega, m \pm \frac{1}{2}\Big) = \operatorname{Re} f(\omega, m_1) + \frac{\pi}{H(\omega)}(2(m-m_1)\pm 1) > \operatorname{Re} f(\omega, m_1),$$

and

$$\operatorname{Re} f\left(\omega, m-\frac{1}{2}\right)=\frac{\pi\left(2m+\frac{n-3}{2}\right)}{H(\omega)}\geq\frac{\pi(2m-1)}{H_2}\geq 1.$$

So we can apply Lemma (2.4.4) and Lemma (2.4.7). Thus if $\zeta=\xi+i\eta\in\partial V_j(\omega, m)$ $(j=1, 2)$,

$$|\phi_2(\omega, \zeta)|\leq C\xi^{-(n+3)/2}\leq C\left(\operatorname{Re} f\left(\omega, m-\frac{1}{2}\right)\right)^{-(n+3)/2}.$$

As for $\phi_1(\omega, \zeta)$, we treat it separately according to $\zeta\in\partial V_1(\omega, m)$ or $\zeta\in\partial V_2(\omega, m)$.
When $\zeta=\xi+i\eta\in\partial V_1(\omega, m)$,

$$|\phi_1(\omega, \zeta)|\geq C_1 m^{-(n+1)/2}\times C_2 m^{-1}=C_1C_2 m^{-(n+3)/2}.$$

From Definition (2.4.19) of C_2,

$$=2C\left(\frac{\pi m}{H_2}\right)^{-(n+3)/2}.$$

By (2.4.17) and $m+\frac{n-3}{2}\geq 1-1\geq 0$,

$$\geq 2C\left(\frac{\pi\left(2m+\frac{n-3}{2}\right)}{H(\omega)}\right)^{-(n+3)/2}$$
$$=2C\left(\operatorname{Re} f\left(\omega, m-\frac{1}{2}\right)\right)^{-(n+3)/2}.$$

Thus $|\phi_1(\omega, \zeta)|>|\phi_2(\omega, \zeta)|$ for any $\omega\in S^{n-1}$ and any $\zeta\in\partial V_1(\omega, m)$ where $m\geq m_2$.
When $\zeta=\xi+i\eta\in\partial V_2(\omega, m)$, from 2) and 3) in Lemma (2.4.7),

$$|\phi_1(\omega, \zeta)|\geq C_1\xi^{-(n+1)/2}.$$

The assumption $m\geq m_2\geq\frac{1}{2}(2H_2CC_1^{-1}\pi^{-1}+1)$ implies,

$$\xi\geq\operatorname{Re} f\left(\omega, m-\frac{1}{2}\right)=\frac{\pi\left(2m+\frac{n-3}{2}\right)}{H(\omega)}\geq\frac{\pi(2m-1)}{H_2}\geq 2\left(\frac{C}{C_1}\right),$$

so $C_1\xi^{-(n+1)/2} > C\xi^{-(n+3)/2}$ follows.
Hence $|\phi_1(\omega,\zeta)| > |\phi_2(\omega,\zeta)|$ for any $\omega \in S^{n-1}$ and any $\zeta \in \partial V_2(\omega, m)$ with $m \geq m_2$.
So we have proved our m_2 is large enough for (2.4.20) and (2.4.21).

Now, for each fixed $\omega \in S^{n-1}$ and each fixed integer $m \geq m_2$, owing to (2.4.20) and (2.4.21), we can apply Rouché's theorem to $\phi_1(\omega,\zeta)$ and $\phi_2(\omega,\zeta)$ in the domain $V_1(\omega, m)$ or $V_2(\omega, m)\setminus\overline{V_1(\omega, m)}$. Then $\tilde{\chi}_\Omega(\omega,\zeta) = \phi_1(\omega,\zeta) + \phi_2(\omega,\zeta)$ and $\phi_1(\omega,\zeta)$ have the same number of null points in $\overline{V_1(\omega, m)}$ (resp. in $\overline{V_2(\omega, m)}\setminus V_1(\omega, m)$). So, it follows from Lemma (2.4.6) that $\tilde{\chi}_\Omega(\omega,\zeta)$ has just one zero point in $\overline{V_2(\omega, m)}$ and it lies in fact in $V_1(\omega, m)$. Therefore $F_m: S^{n-1} \longrightarrow C$ is well defined for each integer $m \geq m_2$ as a map characterized by the following equivalent property (2.4.23) or (2.4.24): That is,

(2.4.23) $\quad F_m(\omega) \in V_1(\omega, m)$, and $\tilde{\chi}_\Omega(\omega, F_m(\omega)) = 0$, for $\omega \in S^{n-1}$.

(2.4.24) $\quad F_m(\omega) \in \overline{V_2(\omega, m)}$, and $\tilde{\chi}_\Omega(\omega, F_m(\omega)) = 0$, for $\omega \in S^{n-1}$.

By (2.4.24), for each $\omega \in S^{n-1}$,

$$\left\{\zeta = \xi + i\eta \in C;\ \tilde{\chi}_\Omega(\omega,\zeta) = 0,\ \xi \geq \operatorname{Re} f\left(\omega, m_2 - \frac{1}{2}\right),\ |\eta| < D+1\right\}$$
$$= \bigcup_{m=m_2}^{\infty} \{\zeta = \xi + i\eta \in \overline{V_2(\omega, m)};\ \tilde{\chi}_\Omega(\omega,\zeta) = 0\}$$
$$= \bigcup_{m=m_2}^{\infty} \{F_m(\omega)\}.$$

Thus, $\{F_m; m \geq m_2\}$ satisfies the conditions (2.4.13) and (2.4.14) in our lemma. While (2.4.15) is clear from (2.4.23), so Lemma (2.4.12) is proved. Q.E.D.

Now, in order to complete our proof of Theorem (2.3.6)′, we only have to prove $F_m: S^{n-1} \longrightarrow C$ (defined in Lemma (2.4.12)) is analytic if $m \in N$ is large enough. By using the implicit function theorem, it is enough to prove the following Lemma (2.4.25), since $\tilde{\chi}_\Omega(\omega,\zeta)$ is an analytic function on $S^{n-1} \times C$ and holomorphic with respect to $\zeta \in C$ for each fixed $\omega \in S^{n-1}$.

LEMMA (2.4.25). *There is a constant $m_0 = m_0(\Omega) \in N$ such that: if $\zeta = \xi + i\eta \in C$ and $\omega \in S^{n-1}$ satisfies*

(2.4.26) $\quad \tilde{\chi}_\Omega(\omega,\zeta) = 0 \quad$ *and* $\quad \xi > \operatorname{Re} f\left(\omega, m_0 - \frac{1}{2}\right) \quad$ *and* $\quad |\eta| < D+1,$

then $\dfrac{d}{d\zeta}\tilde{\chi}_\Omega(\omega,\zeta)\neq 0$, *where* $\dfrac{d}{d\zeta}$ *is a holomorphic derivative.*

PROOF. We can prove the following asymptotic behaviour in the same way as in Proposition (2.2.32):

$$
\begin{aligned}
(2.4.27)\quad \frac{d}{d\zeta}\tilde{\chi}_\Omega(\omega,\zeta) &= \int_{-\infty}^{+\infty} S(\omega,p)ipe^{ip\zeta}dp \\
&= \frac{(2\pi)^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}\times\Gamma\left(\frac{n+1}{2}\right)(-ih(-\omega)(K\circ\nu^{-1}(-\omega))^{-1/2}e^{\pi i(n+1)/4}e^{h(-\omega)(\eta-i\xi)} \\
&\quad +ih(\omega)(K\circ\nu^{-1}(\omega))^{-1/2}e^{-\pi i(n+1)/4}e^{h(\omega)(-\eta+i\xi)})\times|\xi|^{-(n+1)/2}+O(|\xi|^{-(n+3)/2}) \\
&= (2\pi)^{(n-1)/2}(K\circ\nu^{-1}(\omega))^{-1/2}e^{\pi i(n-1)/4}e^{h(\omega)(-\eta+i\xi)} \\
&\quad \times(h(-\omega)(K\circ\nu^{-1}(\omega))^{1/2}(K\circ\nu^{-1}(-\omega))^{-1/2}e^{(h(\omega)+h(-\omega))(\eta-i\xi)}-h(\omega)e^{-\pi i(n+1)/2}) \\
&\quad \times|\xi|^{-(n+1)/2}+O(|\xi|^{-(n+3)/2}) \\
&= (2\pi)^{(n-1)/2}(K\circ\nu^{-1}(\omega))^{-1/2}e^{\pi i(n-1)/4}e^{h(\omega)(-\eta+i\xi)} \\
&\quad \times(h(-\omega)e^{H(\omega)(\eta-d(\omega)-i\xi)}-h(\omega)e^{-\pi i(n+1)/2})\times|\xi|^{-(n+1)/2}+O(|\xi|^{-(n+3)/2})
\end{aligned}
$$

as $\xi\longrightarrow+\infty$ and $|\eta|<D+1$.

We will soon choose a suitable constant $m_0\,(\geq m_2)$ in the statement of this lemma. Then, if $(\omega,\zeta)\in S^{n-1}\times C$ satisfies (2.4.26), Lemma (2.4.12) says that there exist an integer m and a complex number z, both uniquely determined with the following property:

$$(2.4.28)\qquad \zeta=f(\omega,m)+z,\quad |z|\leq C_2m^{-1},\quad\text{and } m\geq m_2,$$

where $m_2\in N$ and $C_2>0$ are the constants appeared in Lemma (2.4.12). Set $B\colon S^{n-1}\times C\longrightarrow C$ by

$$
\begin{aligned}
B(\omega,\zeta) := {} & (2\pi)^{(n-1)/2}(K\circ\nu^{-1}(\omega))^{-1/2}e^{\pi i(n-1)/4}e^{ih(\omega)\zeta} \\
& \times(h(-\omega)e^{H(\omega)(-d(\omega)-i\zeta)}-h(\omega)e^{-\pi i(n+1)/2}).
\end{aligned}
$$

Then from the asymptotic behaviour (2.4.27) of $\dfrac{d}{d\zeta}\tilde{\chi}_\Omega(\omega,\zeta)$, there is a constant $\tilde{C}>0$ depending only on Ω, such that for any $\xi,\eta\in\boldsymbol{R}$ with $\xi\geq 1$ and $|\eta|<D+1$,

$$(2.4.29)\qquad \left|\frac{d}{d\zeta}\tilde{\chi}_\Omega(\omega,\zeta)\right|\geq|B(\omega,\zeta)|\,|\xi|^{-(n+1)/2}-\tilde{C}|\xi|^{-(n+3)/2}.$$

We want to show $\frac{d}{d\zeta}\tilde{\chi}_{\Omega}(\omega,\zeta)\neq 0$ when (ω,ζ) satisfies (2.2.26), or more weakly it is represented as (2.4.28). By the above inequality (2.4.29), it is reduced to show $|B(\omega,\zeta)|$ is 'sufficiently large'. Let us show it.

Now, under the condition (2.4.28),

$$\begin{aligned}
(2.4.30)\qquad & |B(\omega,f(\omega,m)+z)| \\
&=(2\pi)^{(n-1)/2}(K\circ\nu^{-1}(\omega))^{-1/2}e^{-h(\omega)d(\omega)}|e^{ih(\omega)z}| \\
&\quad\times|h(-\omega)e^{H(\omega)(-d(\omega)-if(\omega,m)-iz)}-h(\omega)e^{-\pi i(n+1)/2}| \\
&=(2\pi)^{(n-1)/2}(K\circ\nu^{-1}(\omega))^{-1/2}e^{-h(\omega)d(\omega)}|e^{ih(\omega)z}| \\
&\quad\times|h(-\omega)e^{H(\omega)(-iz-\pi i(2m+(n-1)/2))}-h(\omega)e^{-\pi i(n+1)/2}| \\
&=(2\pi)^{(n-1)/2}(K\circ\nu^{-1}(\omega))^{-1/2}e^{-h(\omega)d(\omega)}|e^{ih(\omega)z}||h(-\omega)e^{-iH(\omega)z}+h(\omega)|.
\end{aligned}$$

Choose an integer $m_3=m_3(\Omega)$ such that the following two inequalities hold:
For any $z\in C$ which satisfies $|z|<C_2m_3^{-1}$,

$$(2.4.31)\qquad H_1-\Big(\max_{\omega\in S^{n-1}}|h(-\omega)|\Big)H_2|z|e^{H_2|z|}\geq\frac{1}{2}H_1,$$

and

$$(2.4.32)\qquad \min_{\omega\in S^{n-1}}|e^{ih(\omega)z}|\geq\frac{1}{2}.$$

Put a positive constant by

$$(2.4.33)\qquad C_3\equiv C_3(\Omega):=\frac{H_1}{4}(2\pi)^{(n-1)/2}\min_{\omega\in S^{n-1}}\{(K\circ\nu^{-1}(\omega))^{-1/2}e^{-h(\omega)d(\omega)}\}.$$

If $|z|<C_2m_3^{-1}$, we have

$$\begin{aligned}
|h(-\omega)e^{-iH(\omega)z}+h(\omega)|&\geq(h(\omega)+h(-\omega))-|h(-\omega)(e^{-iH(\omega)z}-1)| \\
&\geq H(\omega)-|h(-\omega)||H(\omega)||z|e^{|H(\omega)||z|} \\
&\geq H_1-\Big(\max_{\omega\in S^{n-1}}|h(-\omega)|\Big)H_2|z|e^{H_2|z|}.
\end{aligned}$$

From this inequality and the definition of C_3, we obtain

$$(2.4.34)\qquad |B(\omega,f(\omega,m)+z)|\geq C_3,$$

for any $m\in N$, $z\in C$, and $\omega\in S^{n-1}$ such that $m\geq m_3$, $|z|<C_2m^{-1}$.

Substitution of this inequality (2.4.34) into (2.4.29) gives

$$(2.4.35)\qquad \left|\frac{d}{d\zeta}\tilde{\chi}_{\Omega}(\omega, f(\omega, m)+z)\right| \geq C_3\xi^{-(n+1)/2}-\tilde{C}\xi^{-(n+3)/2}=\xi^{-(n+3)/2}(C_3\xi-\tilde{C}),$$

where $\xi := \mathrm{Re}(f(\omega, m)+z)=\dfrac{\pi\left(2m+\dfrac{n-1}{2}\right)}{H(\omega)}+\mathrm{Re}(z)$.

Now, put $m_0 := \max\left\{m_2, m_3, \dfrac{1}{2}(1+H_2\pi^{-1}(2\tilde{C}C_3^{-1}+D+1))\right\}$.
Then if $m\geq m_0$, $|z|\leq C_2m^{-1}$ and $\omega\in S^{n-1}$,

$$\begin{aligned}|\mathrm{Re}(f(\omega, m)+z)| &= \left|\frac{\pi\left(2m+\frac{n-1}{2}\right)}{H(\omega)}-\mathrm{Re}\, z\right| \\ &\geq \frac{\pi(2m-1)}{H_2}-Cm_2^{-1} \\ &\geq 2\left(\frac{C}{C_3}\right)+D+1-C_2m_2^{-1}.\end{aligned}$$

By (2.4.22),

$$\geq \frac{2\tilde{C}}{C_3}.$$

Substitution of this inequality into the right side of (2.4.35) gives

$$\left|\frac{d}{d\zeta}\tilde{\chi}_{\Omega}(\omega, \zeta)\right| \geq \xi^{-(n+3)/2}(C_3\xi-\tilde{C})>0,$$

where $\xi := \mathrm{Re}\,\zeta\equiv\mathrm{Re}(f(\omega, m)+z)$, under the condition (2.4.28). This inequality is what we wanted. Q.E.D.

The proof of Theorem (2.3.6) is thus completed.

§ 5. Proof of corollaries

Let Ω be a strictly convex domain and with notations as in § 3. When the null variety $\mathcal{N}(\Omega)$ is given, we can tell the data $d\equiv d_{\Omega}$ and $H\equiv H_{\Omega}$ of Ω (Definition (2.1.16) and (2.1.14)) from the asymptotics of $\mathcal{N}(\Omega)\cap(S^{n-1}\underset{Z_2}{\times}C^{\times})$ by Theorem (2.3.6). Since $d_{\Omega}(\omega)\equiv\dfrac{\log K\circ\nu^{-1}(-\omega)-\log K\circ\nu^{-1}(\omega)}{2H(\omega)}$, we can also tell $\dfrac{K\circ\nu^{-1}(\omega)}{K\circ\nu^{-1}(-\omega)}$ from $\mathcal{N}(\Omega)$.

PROOF OF COROLLARY (2.3.10). We parametrize S^1 by $\theta \in [0, 2\pi)$ via $S^1 \simeq \boldsymbol{R}/2\pi\boldsymbol{Z}$.
The radius of curvature is given by

$$(2.5.1) \qquad \rho \equiv \rho_\Omega : S^1 \ni \theta \longmapsto (K \circ \nu^{-1}(\theta))^{-1} \in \boldsymbol{R}_+.$$

Then ρ is represented by the supporting function as follows (cf. Proposition (3.7.21)).

$$(2.5.2) \qquad \rho(\theta) = h(\theta) + h''(\theta).$$

Note that the right-hand side is invariant under parallel translation of Ω, although $h(\theta)$ is not so.

As we remarked at the starting of this section, when $\mathcal{N}(\Omega)$ is given, two function A, B: $S^1 \xrightarrow{C^\infty} \boldsymbol{R}_+$ are read from the asymptotic data of $\mathcal{N}(\Omega) \cap (S^{n-1} \underset{Z_2}{\times} \boldsymbol{C}^\times)$ such that

$$(2.5.3) \qquad \frac{K \circ \nu^{-1}(\theta+\pi)}{K \circ \nu^{-1}(\theta)} \equiv \frac{\rho(\theta)}{\rho(\theta+\pi)} = A(\theta),$$

and,

$$(2.5.4) \qquad H(\theta) \equiv h(\theta) + h(\theta+\pi) = B(\theta).$$

It is clear that A and B also satisfy

$$(2.5.5) \qquad A(\theta) > 0, \ B(\theta) > 0, \qquad \text{for any } \theta \in S^1,$$

as well as obvious parity conditions.
Substitution of (2.5.2) into (2.5.3) gives

$$h(\theta) + h''(\theta) = A(\theta)(h(\theta+\pi) + h''(\theta+\pi)).$$

By (2.5.4),

$$= A(\theta)(B(\theta) - h(\theta) + B''(\theta) - h''(\theta)).$$

Then we get the following linear ordinary differential equation of $h(\theta)$ on S^1:

$$(2.5.6) \qquad h''(\theta) + h(\theta) = \frac{A(\theta)}{1 + A(\theta)}(B(\theta) + B''(\theta)).$$

The right-hand side is well-defined C^∞ function on S^1 because of (2.5.5). Therefore $h(\theta)$ is uniquely solved up to

$$\{h(\theta) \in C^\infty(S^1);\ h''(\theta)+h(\theta)=0\}=\boldsymbol{R}\langle\cos\theta\rangle+\boldsymbol{R}\langle\sin\theta\rangle.$$

The right-hand side is just a restriction of linear functions of $\boldsymbol{R}^2$ to S^1. By Lemma (2.1.11) and Remark (2.1.13), the injectivity up to parallel displacements of the correspondence: $\Omega \longmapsto \mathfrak{N}(\Omega)$ is now proved. Q.E.D.

PROOF OF COROLLARY (2.3.11).

1) a)→c) Let Ω be centrally symmetric. Then Ω can be moved to $\Omega^\vee := \{x \in \boldsymbol{R}^n;\ -x \in \Omega\}$ by parallel displacement. By Remark (2.2.4),

$$\mathfrak{N}(\Omega)=\mathfrak{N}(\Omega^\vee). \tag{2.5.7}$$

Let F_m^Ω and $F_m^{\Omega^\vee}$ $(m \in N)$ be the functions in Theorem (2.3.6) for Ω and respectively. Then, for any $\omega \in S^{n-1}$,

$$\begin{aligned} F_m^\Omega(\omega) &= F_m^{\Omega^\vee}(\omega) && ((2.5.7)) \\ &= F_m^\Omega(-\omega) && (\text{from definition of } F_m^\Omega) \\ &= \overline{F_m^\Omega(\omega)} && ((2.3.9)). \end{aligned}$$

Therefore, $F_m^\Omega\colon S^{n-1} \longrightarrow \boldsymbol{C}$ is real valued. With notations in Theorem (2.3.6), $\mathfrak{N}(\Omega) \cap W = \coprod_{m=m_0}^{\infty} \Pi\{(\omega, F_m(\omega));\ \omega \in S^{n-1}\} \subset \boldsymbol{R}^n$. So $\mathfrak{N}(\Omega)_{\boldsymbol{R}} \cap W = \mathfrak{N}(\Omega) \cap W = \coprod_{m=m_0}^{\infty} \Pi\{(\omega, F_m(\omega));\ \omega \in S^{n-1}\}$. This implies c).

c)→b) is trivial.

b)→a) By Theorem (2.3.6), there exists a countably many subset $N \subset N$ such that $F_m = F_m^\Omega\colon S^{n-1} \longrightarrow \boldsymbol{C}$ is real valued on an open set (possibly dependent on m) for any $m \in N$. Since F_m is analytic, such F_m is real valued on *a whole set* of S^{n-1}. Recall that (2.3.8) and (2.3.2) say that

$$F_m(\omega)=\frac{\pi(4m+n-1)}{2H(\omega)}+i\,d(\omega)+O(m^{-1}) \qquad \text{as } m \longrightarrow \infty.$$

Therefore it follows $d_\Omega(\omega)=0$ for any $\omega \in S^{n-1}$.
Now Lemma (2.1.18) implies a), namely, Ω is centrally symmetric.

3) The equivalence is an immediate consequence of Theorem (2.3.6) and of the following formula.

$$\mathrm{pr}\circ\Pi(\omega, F_m(\omega))=\frac{\pi(4m+n-1)}{2H(\omega)}\omega+O(m^{-1}), \qquad \text{as } N \ni m \longrightarrow \infty.$$

Here, the estimate is uniform with respect to $\omega \in S^{n-1}$.

2) a)$\to$c) is clear from Example (2.3.9).

c)$\to$b) is trivial.

b)$\to$a) 2)-b) implies 1)-b) and 3)-b) which are equivalent to 1)-a) and 3)-a), then in turn, these two conditions imply 2)-a). Hence the proof of Corollary (2.3.11) is completed. Q.E.D.

§ 6. Dirichlet series for an annual ring-like set

In this section we shall give the proof of Proposition (2.3.15), (2.3.19) and (2.3.20) in § 3, which are reformulations of our main results and a kind of the characterization on $\mathcal{N}(\Omega)$ in Dirichlet series.

Let Ω be a strictly convex domain in $\boldsymbol{R}^n$. The first statement of Proposition (2.3.15) asserts that the null variety $\mathcal{N}(\Omega)$ in an annual ring-like set (Definition (2.3.16)). This is deduced from the following simple

LEMMA (2.6.1). *Let $f: N_+ \longrightarrow \boldsymbol{C}$ have the asymptotic behaviour*

$$f(n) = an + b + O(n^{-1}) \qquad (n \to \infty)$$

with $a>0$ and $b \in \boldsymbol{C}$. Then $\sum_{n=1}^{\infty} \exp(-f(n)t)$ converges absolutely if $t>0$ and has the following asymptotic behaviour

$$\sum_{n=1}^{\infty} \exp(-f(n)t) = \frac{1}{at} - \left(\frac{1}{2} + \frac{b}{a}\right) + O(t \log t), \qquad (t \downarrow 0).$$

PROOF. From the assumption, there is a positive constant $C>0$ such that

$$|1 - \exp((an+b-f(n))t)| < \frac{Ct}{n},$$

for any $n \in N_+$ and $t \in \boldsymbol{R}$ with $|t|<1$.
Therefore for $0<t<1$,

$$\begin{aligned} &\left| \sum_{n=1}^{\infty} \exp(-f(n)t) - \sum_{n=1}^{\infty} \exp(-(an+b)t) \right| \\ &\leq \sum_{n=1}^{\infty} \exp(-(an+b)t) |\exp((an+b-f(n))t) - 1| \\ &\leq Ct \exp(-bt) \sum_{n=1}^{\infty} \frac{\exp(-ant)}{n} \\ &= Ct \exp(-bt)(at - \log(\exp(at) - 1)) \end{aligned}$$

$$=O(t\log t).$$

On the other hand, we have

$$\sum_{n=1}^{\infty}\exp(-(an+b)t)=\frac{\exp(-bt)}{\exp(at)-1}$$
$$=\frac{1}{at}-\left(\frac{1}{2}+\frac{b}{a}\right)+O(t).$$

This proves the lemma. Q.E.D.

Now let us complete the proof of Proposition (2.3.15). Let $p(\omega, m)$ $(m\in N_+, \omega\in S^{n-1})$ be the enumeration of $\mathcal{N}(\Omega)\cap l_\omega$ (Notation (2.3.13)). Theorem (2.3.6) asserts that

$$p(\omega, m+m_1)=\frac{1}{2H_\Omega(\omega)}\left(4m\pi+(n-1)\pi+i\log\frac{K_\Omega\circ\nu_\Omega^{-1}(-\omega)}{K_\Omega\circ\nu_\Omega^{-1}(\omega)}\right)+O(m^{-1})$$

as $m\to\infty$, where $m_1=m_1(\omega)\in Z$ is some constant.
Therefore we have

$$P_{\mathcal{N}(\Omega)}(\omega)=4\pi\times\frac{2H_\Omega(\omega)}{4\pi}=2H_\Omega(\omega),$$

$$\tilde{Q}_{\mathcal{N}(\Omega)}(\omega)+iR_{\mathcal{N}(\Omega)}(\omega)=4\pi\left(-\frac{1}{2}-\frac{1}{4\pi}\left((n-1)\pi+i\log\frac{K_\Omega\circ\nu_\Omega^{-1}(-\omega)}{K_\Omega\circ\nu_\Omega^{-1}(\omega)}\right)\right)$$
$$=(-n-1)\pi-i\log\frac{K_\Omega\circ\nu_\Omega^{-1}(-\omega)}{K_\Omega\circ\nu_\Omega^{-1}(\omega)},\qquad \mathrm{mod}\ 4\pi Z.$$

Thus the proof of Proposition (2.3.15) is completed.

Proposition (2.3.19) is an immediate consequence of Corollary (2.3.10) and its proof (see § 5). Now let us prove Proposition (2.3.20).

PROOF OF PROPOSITION (2.3.20). We use the notation in the proof of Corollary (2.3.10).

First let P, R be the coefficients of the asymptotic behaviour of the null variety $\mathcal{N}(\Omega)$. Then the parity condition (2.3.21) and (2.3.23) is clear from (2.3.16) and (2.3.18). As we saw in the proof of Corollary (2.3.10), the radius of curvature is given by

$$\rho(\theta)=h(\theta)+h''(\theta)$$
$$=\frac{1}{2}\frac{P(\theta)+P''(\theta)}{1+\exp(R(\theta))},$$

from (2.5.2) and (2.5.6). (In the notation there $A(\theta)=\exp(-R(\theta))$ and $B(\theta)=\frac{1}{2}H(\theta)$.) Since $\rho(\theta)>0$ for any $\theta\in S^1$, and $\int_0^{2\pi}(h(\theta)+h''(\theta))d\theta=0$, the condition (2.3.22) and (2.3.24) follows.

Conversely, let $(P, R)\in C^\infty(S^1, \boldsymbol{R}^2)$ satisfy the four conditions (2.3.21) $\sim$(2.3.24). Let $h(\theta)\in C^\infty(S^1)$ be a solution of the following differential equation.

$$h''(\theta)+h(\theta)=\frac{1}{2}\frac{1}{1+\exp R(\theta)}(P(\theta)+P''(\theta)). \tag{2.6.2}$$

The condition (2.3.24) assures the existence of h and the condition (2.3.22) assures the strict convexity of the domain Ω defined from $h(\theta)$ by using (2.1.12).

We have from (2.3.21) and (2.3.23)

$$\begin{aligned} h''(\theta+\pi)+h(\theta+\pi)&=\frac{1}{2}\frac{1}{1+\exp R(\theta+\pi)}(P(\theta+\pi)+P''(\theta+\pi)) \\ &=\frac{1}{2}\frac{\exp R(\theta)}{1+\exp R(\theta)}(P(\theta)+P''(\theta)). \end{aligned} \tag{2.6.3}$$

Therefore $H''_\Omega(\theta)+H_\Omega(\theta)=\frac{1}{2}(P(\theta)+P''(\theta))$, where $H_\Omega(\theta)=h(\theta)+h(\theta+\pi)$ as usual. Thus $H_\Omega(\theta)=\frac{1}{2}P(\theta)+a\cos\theta+b\sin\theta$ with some a, $b\in\boldsymbol{R}$. Then $H_\Omega(\theta+\pi)=\frac{1}{2}P(\theta+\pi)-a\cos\theta-b\sin\theta$. Using the parity condition (2.1.15) and (2.3.21) again, we have $a=b=0$ and $H_\Omega(\theta)=\frac{1}{2}P(\theta)$.

From (2.6.2) and (2.6.3) we have

$$\exp(R(\theta))=\frac{h''(\theta+\pi)+h(\theta+\pi)}{h''(\theta)+h(\theta)}=\frac{\rho(\theta+\pi)}{\rho(\theta)}=\frac{\kappa(\theta)}{\kappa(\theta+\pi)}.$$

This formula and Proposition (2.3.15) imply that (P, R) coincides with $(P_{\mathfrak{N}(\Omega)}, R_{\mathfrak{N}(\Omega)})$ (notation in Proposition (2.3.19)). Thus the proof of the proposition is completed.

Chapter 3. Null variety for a convex domain (hyperbolic case)

§ 1. Notation

In this section, we shall prepare some standard notations concerning Riemannian globally symmetric space and Radon-Fourier transform on it which will be utilized throughout this chapter.

Let G be a noncompact connected semisimple Lie group, $\mathfrak{g}$ the Lie algebra of G, and B the Killing form of $\mathfrak{g}$. Fix a Cartan involution θ of G. We also use the same letter θ for its derivative. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces for θ (Cartan decomposition), then B is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$. Let $\mathfrak{a}_C:=\mathfrak{a}\otimes_{\boldsymbol{R}} \boldsymbol{C}$, $\mathfrak{a}^*:=\mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{a}, \boldsymbol{R})$, and $\mathfrak{a}_C^*:=\mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{a}, \boldsymbol{C})\simeq\mathrm{Hom}_{C}(\mathfrak{a}_C, \boldsymbol{C})$. For each $\alpha\in\mathfrak{a}^*$, the root space of α is given by,

$$\mathfrak{g}(\mathfrak{a};\alpha):=\{X\in\mathfrak{g};\ [H,X]=\alpha(H)X,\quad \text{for all } H\in\mathfrak{a}\}.$$

A finite set $\Sigma\equiv\Sigma(\mathfrak{g},\mathfrak{a}):=\{\alpha\in\mathfrak{a}^*;\ \mathfrak{g}(\mathfrak{a};\alpha)\neq\{0\}\}\setminus\{0\}$ is called *the restricted root system* of $\mathfrak{g}$. Fix a positive system $\Sigma^+\equiv\Sigma^+(\mathfrak{g},\mathfrak{a})$ of the root system Σ. Then the positive Weyl chamber is given by,

$$\mathfrak{a}^+:=\{H\in\mathfrak{a}^*;\ \alpha(H)>0 \text{ for any } \alpha\in\Sigma^+\}.$$

Set a maximal nilpotent subalgebra $\mathfrak{n}:=\sum_{\alpha\in\Sigma^+}\mathfrak{g}(\mathfrak{a};\alpha)$ $(\subset\mathfrak{g})$. We denote by N, A, and K the analytic subgroups having $\mathfrak{n}$, $\mathfrak{a}$, and $\mathfrak{k}$ as its Lie algebras. Set $A: G\longrightarrow\mathfrak{a}$ be a projection corresponding to the Iwasawa decomposition $G=NAK$. That is, for each element g of G, the element $A(g)$ of $\mathfrak{a}$ is a uniquely determined by the following property:

$$g\in N\exp A(g)K.$$

Here, $\exp:\mathfrak{a}\longrightarrow A$ denote an exponential mapping.

Let $M:=Z_K(\mathfrak{a})$ (resp. $M':=N_K(\mathfrak{a})$), the centralizer (resp. normalizer) of $\mathfrak{a}$ in K. Then the quotient group $W:=M'/M$ is identified with the Weyl group associated with the root system Σ, and $P:=MAN$ is the Langlands decomposition of a minimal parabolic subgroup of G. It follows from the Iwasawa decomposition that the inclusion $K\hookrightarrow G$ induces a isomorphism $K/M\simeq G/P$, which is diffeomorphic to a sphere if $\dim_{\boldsymbol{R}}\mathfrak{a}=1$.

Let ρ be the element of $\mathfrak{a}^*$ defined by,

$$\rho(H):=\frac{1}{2}\ \mathrm{trace}(\mathrm{ad}(H)_{|\mathfrak{n}}), \qquad \text{for } H\in\mathfrak{a}^*.$$

Let $m_\alpha:=\dim_{\boldsymbol{R}}\mathfrak{g}(\mathfrak{a},\alpha)$ be the multiplicity of α, then $\rho=\frac{1}{2}\sum_{\alpha\in\Sigma^+}m_\alpha\alpha\in\mathfrak{a}^*$.

Fix a G-invariant Riemannian metric g on $X:=G/K$ (which always exists since K is compact) and denote the volume element by dx, Levi-Civita connection by ∇. Then (X,∇) is called a *Riemannian symmetric space*. The Iwasawa decomposition assures that the mapping:

$$A\times N\ni(a,n)\longmapsto anK\in G/K=X$$

is a surjective analytic diffeomorphism. Let da, dn be the measures on A, N induced from the Riemannian metric on X via the above isomorphism. Then da (resp. dn) gives a Haar measure on a unimodular group A (resp. N) (ref. [14] Ch. 1. Cor. 5.3). Let $o:=eK\in G/K$, and we denote by T_oX the tangent space of X at o. Then T_oX is identified with $\mathfrak{p}$ by the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. When restricted to $T_oX\simeq\mathfrak{p}$, the Riemannian metric tensor g is a scalar multiple of the Killing form B on each irreducible component of X, while ∇ is the unique G-invariant connection (ref. [13] Ch. 4 Cor. 4.3). The dimension of $\mathfrak{a}$ does not depend on the choice of a maximally abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and called *the split rank of* G or *the rank of Riemannian symmetric space* $X=G/K$.

Throughout this chapter, $\mathrm{Exp}:T_pX\longrightarrow X$ $(p\in X)$ denotes an exponential mapping at p in a manifold X with affine connection ∇, and $\exp:\mathfrak{g}\longrightarrow G$ denotes an exponential mapping for a Lie algebra (including the case $\mathfrak{g}=\boldsymbol{R}$).

When the rank of X is one, we define an element a_0 of A by,

$$\|\log(a_0)\|=1, \quad \text{and} \quad \log(a_0)\in\mathfrak{a}^+. \tag{3.1.1}$$

Here, $\|\ \|$ denotes the norm on $\mathfrak{p}\simeq T_oX$ induced from the Riemannian metric g at o, and $\log:A\longrightarrow\mathfrak{a}$ is the inverse of the bijective map $\exp:\mathfrak{a}\longrightarrow A$. Set $a_0^t:=\exp(t\log(a_0))\in A$, for $t\in\boldsymbol{R}$. Then $A\cdot o=\{a_0^t\cdot o;t\in\boldsymbol{R}\}$ is a geodesic in (X,g) parametrized by the arc-length t. Note that $da=dt$ in terms of our notation in this rank one case.

Furthermore, in the rank one case, the Weyl group $W=M'/M\simeq\boldsymbol{Z}_2$.

Notation (3.1.2). Fix an element m_{-1} of M' so that it represents the non-trivial element in $W\simeq\boldsymbol{Z}_2$, and let $m_1:=e\in M'$ $(\subset G)$.

Let $\mathcal{E}(X)$ denote the space of infinitely many differentiable functions on X endowed with the Fréchet topology of uniform convergence of functions and their derivatives on compact sets. The topological dual of $\mathcal{E}(X)$ is $\mathcal{E}'(X)$, the space of Schwartz distributions on X with compact support.

The following definition of Radon-Fourier transform was introduced by S. Helgason ([11]).

DEFINITION (3.1.3). The (Radon-)Fourier transform on X $\mathcal{F}:\mathcal{E}'(X)\longrightarrow \mathcal{A}(\mathfrak{a}_C^*\times K/M)$ is given by,

$$(3.1.4)\qquad \mathcal{F}F(\zeta, kM)\equiv\tilde{F}(\zeta, kM):=\langle F(x), \exp\langle i\zeta+\rho, A(k^{-1}x)\rangle\rangle = \int_X F(x)\exp\langle i\zeta+\rho, A(k^{-1}x)\rangle dx,$$

for $F\in\mathcal{E}'(X)$, $\zeta\in\mathfrak{a}_C^*$, and $k\in K$. Note that $K/M\times X\ni(kM, x)\longmapsto A(k^{-1}x)\in\mathfrak{a}$ is well-defined, because $A((km)^{-1}gb)=A(k^{-1}g)$ for any $m\in M$ and any $b\in K$.

§ 2. Submanifolds in X

In this section, we treat some submanifolds in a rank one noncompact Riemannian symmetric space, which will play an important role in getting the asymptotics of $\mathcal{N}(\Omega)$. Finally we give a definition of *the Gauss maps* and *the supporting functions* of a *strictly H-convex* domain in X.

Let $X=G/K$ be a Riemannian symmetric space of rank one and retain notations in § 1.

DEFINITION (3.2.1). A *horosphere* in X is an image of the map $N\ni n\longmapsto g_1ng_2^{-1}\cdot o\in X$, for two fixed elements g_1, g_2 in G. It is known that this map is diffeomorphic into X with a closed image. Denote by Ξ the totality of horospheres in X. Let $S(TX)$ be the unit tangent sphere bundle of the Riemannian manifold (X, g), and $L(X)$ be the totality of oriented complete geodesics in X.

LEMMA (3.2.2). *Let $X=G/K$ be a noncompact Riemannian symmetric space of rank one and let a_0 be the element of A defined in* (3.1.1). *Then the mappings,*

$$K/M\times A\ni(kM, a)\longmapsto kaMN\in G/MN,$$
$$G/MN\ni gMN\longmapsto gN\cdot o\in\Xi,$$

$$G/M \ni gM \longmapsto \frac{d}{dt}\bigg|_{t=0} ga_0^t \cdot o \in S(TX),$$

and

$$G/MA \ni gMA \longmapsto \{ga_0^t \cdot o \in X;\ -\infty < t < \infty\} \in L(X),$$

give the following bijections respectively.

$$K/M \times A \simeq G/MN \simeq \Xi, \tag{3.2.3}$$

$$G/M \simeq S(TX), \tag{3.2.4}$$

and,

$$G/MA \simeq L(X). \tag{3.2.5}$$

REMARK (3.2.6). Since the Riemannian manifold (X, g) is complete, Exp is everywhere defined. Then the surjective mapping:

$$S(TX) \ni (p, V) \longmapsto \{\mathrm{Exp}_p(tV) \in X;\ -\infty < t < \infty\} \in L(X),$$

induces the following commutative diagrams:

$$\begin{array}{ccc} G/M & \xrightarrow{\sim} & S(TX) \\ \downarrow & & \downarrow \\ G/MA & \xrightarrow{\sim} & L(X), \end{array} \tag{3.2.7}$$

where $G/M \longrightarrow G/MA$ is the natural quotient map.

PROOF OF LEMMA (3.2.2) AND REMARK (3.2.6). It is well known that (3.2.3) holds without the assumption $\mathrm{rank}(G/K)=1$ ([12] Proposition 1.1).

For (3.2.4), first we show the transitivity of G-action on $S(TX)$. G acts on X transitively and the isotropy subgroup K at $o \in X$ also acts transitively on the projective space $\boldsymbol{P}^1(\mathfrak{p}) \simeq \boldsymbol{P}^1(T_oX)$ because of the rank one condition ([13] Ch. 5 Lemma 6.3). Since the element m_{-1} of M' reverses the one dimensional subspace $\mathfrak{a}$ $(\subset \mathfrak{p})$, G acts transitively on $S(\mathfrak{p}) \simeq S(TX)$. Next, the stabilizer of G at $(o, \log(a_0)) \in S(TX)$ is $Z_K(\{\log(a_0)\}) = Z_K(\mathfrak{a}) = M$. Thus (3.2.4) is proved.

Now, let us prove (3.2.5). For $Y \in \mathfrak{p} \simeq T_oX$, $\exp Y \cdot o = \mathrm{Exp}_o(Y)$ (ref. [13] Ch. 4 Thm. 3.3), so $a_0^t \cdot o = \mathrm{Exp}_o(t \log(a_0))$ $(t \in \boldsymbol{R})$ is a geodesic parametrized by the arc-length t, which shows the well-definedness of the mapping

$$G/MA \ni gMA \longmapsto \{ga_0^t \cdot o \in X;\ -\infty < t < \infty\} \in L(X).$$

For $g\in G$,

$$\begin{aligned} ga_0^t\cdot o &= g\exp(t\log(a_0))\cdot o \\ &= g\,\mathrm{Exp}_o(t\log(a_0)) \qquad (\text{via } \mathfrak{p}\simeq T_oX) \\ &= g\cdot \mathrm{Exp}_o\Big(t\frac{d}{ds}\Big|_{s=0}(a_0^s\cdot o)\Big) \\ &= \mathrm{Exp}_{g\cdot o}\Big(t\frac{d}{ds}\Big|_{s=0}(ga_0^s\cdot o)\Big), \end{aligned}$$

showing the commutativity of the diagram (3.2.7) and hence the surjectivity of $G/MA\longrightarrow L(X)$. Injectivity is followed by the same consideration as the proof of in (3.2.4). Thus we have proved Lemma (3.2.2). Q.E.D.

LEMMA (3.2.8). *Retain notations as in Lemma* (3.2.2). *For each element g of G, there are just two horospheres orthogonal to $gM\in G/M\simeq S(TX)$ at $g\cdot o\in X$, that is, $gm_\varepsilon N\cdot o$ $(\varepsilon=\pm 1)$. (Recall $m_\varepsilon\in M'$ is a representative of the Weyl group $W=M'/M\simeq \boldsymbol{Z}_2$.* See (3.1.2).)

PROOF. Since G acts on Ξ and $S(TX)$ both transitively and since the Riemannian metric g on X is G-invariant, we may assume $g=e$. We may also assume that the Riemannian metric g at o is a scalar multiple of the Killing form B when restricted to $\mathfrak{p}\simeq T_oX$.

Suppose a horosphere $kaN\cdot o$ $((kM, a)\in K/M\times A\simeq\Xi)$ go through the origin $o\in X$. Then $kaNK\supset K$, which implies $a=e$ because of the uniqueness of the Iwasawa decomposition. Suppose $a=e$, then $kaN\cdot o=kN\cdot o$ goes through o and the tangent space at o is $\{Y-\theta Y;\ Y\in \mathrm{Ad}(k)\mathfrak{n}\}\subset\mathfrak{p}\simeq T_oX$. The necessary and sufficient condition that this subspace is orthogonal to $M\in G/M\simeq S(TX)$ is,

(3.2.9) $\quad B(H, Y-\theta Y)=0$, for any $Y\in\mathrm{Ad}(k)\mathfrak{n}$ and any $H\in\mathfrak{a}$.

Since $B(H, Y+\theta Y)=B(H+\theta H, Y)=0$, the condition (3.2.9) is equivalent to,

$$B(H, Y)=B(H,\theta Y)=0, \text{ for any } Y\in\mathrm{Ad}(k)\mathfrak{n} \text{ and any } H\in\mathfrak{a}.$$

Since the Killing form B is G-invariant and θ commutes with an element of K, this is also equivalent to,

$$B(H, Y)=B(H,\theta Y)=0, \text{ for any } Y\in\mathfrak{n} \text{ and any } H\in\mathrm{Ad}(k)^{-1}\mathfrak{a}.$$

That is, $\mathrm{Ad}(k)^{-1}\mathfrak{a}\subset\{H\in\mathfrak{p};\, B(H, \mathfrak{n}+\theta\mathfrak{n})=0\}$. The right hand side is nothing but $\mathfrak{a}$, because $\mathfrak{g}=\mathfrak{m}+\mathfrak{a}+\mathfrak{n}+\theta\mathfrak{n}$ gives an orthogonal decomposition for B and because the Lie algebra $\mathfrak{m}$ of M is contained in $\mathfrak{k}$. So k normalizes $\mathfrak{a}$, namely $k\in M'$. Thus we have shown that when a horosphere $kaN\cdot o$ go through the origin o and is orthogonal to $\left.\frac{d}{dt}\right|_{t=0} a_0^t\cdot o\in T_oX$ at this point, then $kM\in M'/M$ and $a=e$. The converse is similar and easier. Thus we have proved our lemma in the case when $g=e$. Q.E.D.

In the final of this section, we give a definition of the analogue of the Gauss map ψ_ε ($\varepsilon=\pm1$), a horospherical convex domain and its supporting functions h_ε ($\varepsilon=\pm1$) in a rank one Riemannian symmetric space. In the case of $X=SL(2, \boldsymbol{R})/SO(2)$, this is the same with the one introduced by [2] ($\psi_\varepsilon=G$, G^* in the notation there).

ASSUMPTION (3.2.10). Let Ω be a bounded domain in a rank one Riemannian symmetric space X whose boundary $\partial\Omega$ is a connected $n-1$ dimensional regular submanifold of X.

DEFINITION (3.2.11). Suppose Ω satisfies the assumption (3.2.10). Let $\nu\in\Gamma(\partial\Omega, TX_{|\partial\Omega})$ be the outer unit normal vector field over $\partial\Omega$. The map $\psi_\varepsilon:\partial\Omega\longrightarrow G/P\simeq K/M$ ($\varepsilon=\pm1$) is defined by the composition of the following three maps:

$$\partial\Omega\ni p\longmapsto(p, \nu(p))\in S(TX),$$
$$S(TX)\xrightarrow{\sim}G/M, \qquad ((3.2.4))$$

and

$$\pi_\varepsilon:\ G/M\ni gM\longmapsto gm_\varepsilon MAN\in G/MAN\simeq K/M. \tag{3.2.12}$$

We call this map $\psi_\varepsilon:\partial\Omega\longrightarrow G/P\simeq K/M$ ($\varepsilon=\pm1$) the *Gauss map* for the embedding $\partial\Omega\hookrightarrow X$.

Since $\pi_\varepsilon: G/M\longrightarrow G/MAN\simeq K/M$ does not depend on the choice of the representative $m_\varepsilon\in M'$ of the Weyl group $W\simeq M'/M$ (Notation (3.1.2)), ψ_ε is well-defined.

In $\boldsymbol{R}^n$, one of the characterizations of a strictly convex domain (Fact (2.1.4)) is that the Gauss map is diffeomorphic. Generalizing the notion of convexity by this property, we give the following definition.

DEFINITION (3.2.13). A domain Ω in a noncompact Riemannian symmetric space of rank one satisfying (3.2.10) is called *strictly horospherically convex* (*strictly H-convex*) when both $\psi_\varepsilon : \partial\Omega \longrightarrow G/P \simeq K/M$ ($\varepsilon = \pm 1$) are surjective diffeomorphisms, and called *horospherically convex* (*H-convex*) when $\partial\Omega$ lies in one side with respect to any two horospheres tangent to $\partial\Omega$ (ref. Lemma (3.2.8)).

Some other characterizations of a strict H-convexity analogous to Fact (2.1.4) in a Euclidean space will be given in § 4.

REMARK (3.2.14). In a simply connected complete Riemannian manifold with nonpositive sectional curvature (e.g. a noncompact Riemannian symmetric space), there exists a unique geodesic which goes through two given points. Therefore *geodesical convexity* is defined just as in $\boldsymbol{R}^n$ (Fact (2.1.6) property (4)). But in a rank one noncompact Riemannian symmetric space, geodesical convexity does not imply horospherical convexity. This is because a horosphere is not totally geodesic.

Let $\{f_1, \cdots, f_{n-1}\}$ be an orthonormal frame on $\partial\Omega$ near x ($\in \partial\Omega$). The following definition on the principal curvatures is traditional.

DEFINITION (3.2.15). Let Ω satisfy (3.2.10) (Here the assumption of boundedness of Ω is used only for the orientation of $\partial\Omega$). For $x \in \partial\Omega$, *the principal curvatures* at x for the imbedding $\partial\Omega \hookrightarrow X$ are the eigenvalues $\lambda_j(x) \equiv \lambda_{\partial\Omega, j}(x)$ $(1 \leq j \leq n-1)$ of the second fundamental form $(-g_x(\nabla_{f_j(x)} f_k, \nu(x)))_{1 \leq j, k \leq n-1}$.

This definition does not depend on the choice of the orthonormal frame $\{f_1, \cdots, f_{n-1}\}$. The analogue of the Gauss-Kronecker curvature in a space form will be defined in § 4 by some symmetric polynomials of the principal curvature (Definition (3.4.1)).

In a Euclidean space, the supporting function (Definition (2.1.7)) plays an important role in analyzing a convex domain. Let us introduce the analogue of the supporting function of a strictly H-convex domain in a rank one noncompact Riemannian symmetric space. Its geometrical interpretation will be given in Lemma (3.5.2) in a hyperbolic space $X = SO_0(n, 1)/SO(n)$ and the analogue of the Gauss-Kronecker curvature (some symmetric polynomials of the principal curvatures) will be represented by the differential equation of the supporting function in § 7 when $X = SO_0(2, 1)/SO(2)$.

DEFINITION (3.2.16). For a strictly H-convex domain Ω, *the support* *ing function* $h_\varepsilon \equiv h_{\Omega,\varepsilon} : G/P \longrightarrow \boldsymbol{R}$, $(\varepsilon = \pm 1)$ is the composition of the following five maps:

$$\phi_\varepsilon^{-1} : G/P \xrightarrow{\sim} \partial\Omega, \qquad \text{(Definition (3.2.13))},$$
$$\partial\Omega \ni p \longmapsto (p, \nu(p)) \in S(TX),$$
$$S(TX) \xrightarrow{\sim} G/M, \qquad \text{(Lemma (3.2.2))},$$

$$G/M \ni gM \longmapsto -A(m_\varepsilon^{-1} g^{-1}) \in \mathfrak{a}, \qquad \text{(Notation (3.1.2))}, \tag{3.2.17}$$

and

$$\mathfrak{a} \ni c \log a_0 \longmapsto c \in \boldsymbol{R}. \qquad \text{(Notation (3.1.1))}.$$

The breadth function of a strictly H-convex domain Ω is given by,

$$H \equiv H_\Omega := h_1 - h_{-1} : G/P \simeq K/M \longrightarrow \boldsymbol{R}. \tag{3.2.18}$$

Since the Iwasawa projection $A : G = NAK \longrightarrow \mathfrak{a}$ satisfies

$$A(mg) = A(g), \qquad \text{for any } m \in M \text{ and } g \in G,$$

the mapping (3.2.17) is well-defined and does not depend on the choice of $m_\varepsilon \in M'$ (Notation (3.1.2)).

The following lemma asserts that the Gauss map (resp. the supporting function) is G-equivariant (resp. K-equivariant).

LEMMA (3.2.19). *Let Ω satisfy* (3.2.10). *Given any $g \in G$, let $g \cdot \Omega := \{g \cdot x \in X; x \in \Omega\}$. Then,*

$$g\phi_{\Omega,\varepsilon}(x) = \phi_{g\cdot\Omega,\varepsilon}(g \cdot x), \qquad (x \in \Omega). \tag{3.2.20}$$

In particular, if Ω is strictly H-convex, then $g \cdot \Omega$ is also strictly H-convex.

Suppose Ω be strictly H-convex. Then for any element b of K,

$$h_{b^{-1}\cdot\Omega,\varepsilon}(gP) = h_{\Omega,\varepsilon}(bgP), \qquad (gP \in G/P). \tag{3.2.21}$$

PROOF. The first statement is derived from the commutativity of the diagram below:

$$\begin{array}{ccccccc}
\partial\Omega & \longrightarrow & S(TX) & \xrightarrow{\sim} & G/M & \xrightarrow{\pi_\varepsilon} & G/P \\
g\cdot\downarrow & & g\cdot\downarrow & & g\cdot\downarrow & & g\cdot\downarrow \\
\partial\Omega & \longrightarrow & S(TX) & \xrightarrow{\sim} & G/M & \xrightarrow[\pi_\varepsilon]{} & G/P.
\end{array}$$

On the other hand, the mapping (3.2.17) is factored by $G/M \longrightarrow$

$K\backslash G/M \longrightarrow \mathfrak{a}$ because the Iwasawa projection $A: G=NAK \longrightarrow \mathfrak{a}$ is right K-invariant as well as left M-invariant. Therefore the last statement is shown in a similar way. Q.E.D.

REMARK (3.2.22). In Euclidean space, the Gauss map (Definition (2.1.2)) is the outer unit normal vector field via the identification $T_p(\boldsymbol{R}^n) \simeq \boldsymbol{R}^n$ for each element $p \in \boldsymbol{R}^n$, by using parallel translations. The existence of this identification is essential in defining the Gauss map.

The Gauss map is also defined by the tangent hyperplane: First note that the equivalence class by parallel translations of oriented hyperplanes in $\boldsymbol{R}^n$ is isomorphic to S^{n-1}. Then the Gauss map is the correspondence to the equivalence class to which the oriented tangent space $T_p(\partial\Omega)$ belongs (the orientation is given by which side of $T_p(\partial\Omega)$ Ω lies in).

In the case of a noncompact rank one Riemannian symmetric space G/K, our ψ_ε has also analogous geometric interpretations. First recall that a Hadamard manifold (i.e. simply connected nonpositively curved manifold) X has a compactification $\bar{X}=X \cup \partial X$ whose boundary ∂X is called *geometric boundary*, defined by certain equivalence class of oriented geodesics ([8]). In the case of $X=G/K$, let $L(X) \longrightarrow \partial X$ be the above quotient map. Here $L(X)$ stands for the totality of oriented complete geodesics (Definition (3.2.1)). Let $G/MA \longrightarrow G/MAN$ be the natural quotient map, and $G/MAN \longrightarrow \partial X$ be defined by,

(3.2.23) $$gMAN \longmapsto \lim_{t \to +\infty} g a_0^t \cdot o,$$

where the limit is taken in $\bar{X}$. Then together with (3.2.7), we have the following commutative diagram:

(3.2.24) $$\begin{array}{ccc} G/M & \xrightarrow{\sim} & S(TX) \\ \downarrow & & \downarrow \\ G/MA & \xrightarrow{\sim} & L(X) \\ \downarrow & & \downarrow \\ G/MAN & \xrightarrow{\sim} & \partial X. \end{array}$$

The mapping $S(TX) \longrightarrow \partial X$ factored by $L(X)$ induces the following surjective isomorphism:

(3.2.25) $$S(T_pX) \xrightarrow{\sim} \partial X,$$

for each point $p \in X$. As we saw it, this Euclidean version is essential in defining the Gauss map. Thus our Gauss map $\psi_1: \partial\Omega \longrightarrow G/MAN$ is nothing but the correspondence to the outer unit normal vector field over $\partial\Omega$ by using the identification (3.2.25) at each point. ψ_{-1} is similar because $\psi_{\varepsilon}:\partial\Omega\longrightarrow G/MAN$ is $\lim_{t\to+\infty}\mathrm{Exp}(\varepsilon t\nu(p))\in\partial X\ (\subset\bar{X})$ via $\partial X\simeq G/MAN$ ((3.2.23)). This is one interpretation of ψ_{ε}. Another is by using a horosphere. Now let us explain it.

We call two horospheres are *parallel* when they have the same image of the correspondence $\Xi\longrightarrow\partial X$ which is defined so that the following diagram commutes:

$$\begin{tikzcd} G/MN \arrow[r, "\sim"] \arrow[d] & \Xi \arrow[d] \\ G/MAN \arrow[r, "\sim"] & \partial X. \end{tikzcd}$$

Here, $G/MN\longrightarrow G/MAN$ is the natural quotient map, $G/MN\longrightarrow\Xi$ is in (3.2.3) and $G/MAN\longrightarrow\partial X$ is in (3.2.24). We call the corresponding point in ∂X *the infinite point of the horosphere.* Then $\psi_{\varepsilon}(p)$ $(\varepsilon=\pm1)$ are also regarded as the limit points of the horospheres which are tangent to $T_p(\partial\Omega)$ (From Lemma (3.2.8), there are two such horospheres.).

In short, the Gauss map $\psi_1:\partial\Omega\longrightarrow\partial X$ is the composition of the following map (3.2.26) and either (3.2.24) or (3.2.27), and ψ_{-1} is similar.

(3.2.26) $$\partial\Omega\ni p\longmapsto(p,\nu(p))\in S(TX),$$

(3.2.24) $$\begin{tikzcd} G/M \arrow[r, "\sim"] \arrow[d] & S(TX) \arrow[d] & \text{geodesics} \\ G/MA \arrow[r, "\sim"] \arrow[d] & L(X) \arrow[d] & \text{geometric boundary} \\ G/MAN \arrow[r, "\sim"] & \partial X & \end{tikzcd}$$

(3.2.27) $$\begin{tikzcd} G/M \arrow[r, "\sim"] \arrow[d] & S(TX) \arrow[d] & \text{tangent horosphere} \\ G/MN \arrow[r, "\sim"] \arrow[d] & \Xi \arrow[d] & \text{infinite point} \\ G/MAN \arrow[r, "\sim"] & \partial X & \end{tikzcd}$$

REMARK (3.2.28). Let us define an involutive diffeomorphism of $\partial X\simeq K/M$ by σ: $K/M\ni kM\longmapsto km_{-1}M\in K/M$. On the other hand, the Cartan involution θ of G induces an involutive diffeomorphism of G/K, and we also denote it by θ. Then the following diagrams commute for $\varepsilon=\pm1$.

$$\begin{tikzcd} \partial\Omega \arrow[r, "\psi_{\Omega,\varepsilon}"] \arrow[d, "\theta"'] & K/M \arrow[d, "\sigma"] \\ \partial(\theta\Omega) \arrow[r, "\psi_{\theta\Omega,\varepsilon}"'] & K/M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K/M \arrow[r, "h_{\Omega,\varepsilon}"] \arrow[d, "\sigma"'] & \boldsymbol{R} \arrow[d, equal, "\mathrm{id.}"] \\ K/M \arrow[r, "h_{\theta\Omega,\varepsilon}"'] & \boldsymbol{R} \end{tikzcd}$$

The proof is derived from the definitions, by making use of the formula $A(\theta(m_{-1}g))=A(g)$ $(g\in G)$.

§ 3. A space with constant negative curvature

Now we shall deal with the case $G=SO_0(n, 1)$, the identity component of the matrix group: $SO(n, 1):=\{g\in GL(n+1, \boldsymbol{R}); gI_{n,1}{}^tg=I_{n,1}\}$, where $I_{n,1}=\mathrm{diag}(1, \cdots, 1, -1)\in GL(n, \boldsymbol{R})$.

The correspondence $\theta: G\ni g\longmapsto {}^tg^{-1}\in G$ gives a Cartan involution of G and the corresponding maximal compact subgroup is $K=SO(n)$ naturally imbedded in $G=SO_0(n, 1)$. It is well known that $X=G/K$ is a rank one Riemannian symmetric space with constant sectional curvature (a simply connected negatively curved space form, hyperbolic space).

The Lie algebra of G is given by,

$$\mathfrak{g}=\mathfrak{so}(n, 1)=\{Y\in M(n, \boldsymbol{R}); YI_{n,1}+I_{n,1}{}^tY=0\}.$$

Set $\mathfrak{a}:=\boldsymbol{R}H\subset\mathfrak{p}$, $H:=E_{n,n+1}+E_{n+1,n}\in\mathfrak{g}$, where $E_{i,j}$ $(1\leq i, j\leq n+1)$ is the matrix unit. Define an element α of $\mathfrak{a}^*$ by the equation $\alpha(H)=1$. Then the set of restricted root of $\mathfrak{g}$ is given by $\Sigma=\{\pm\alpha\}$ with each multiplicity $n-1$. Choose a positive root system of $\mathfrak{g}$ so that $\Sigma^+:=\{\alpha\}$, then the corresponding maximal nilpotent Lie algebra is given by,

$$\mathfrak{n}=\sum_{j=1}^{n-1}\boldsymbol{R}N_j,$$

where $N_j:=E_{n,j}+E_{n+1,j}-E_{j,n}+E_{j,n+1}$.
Then we define a minimal parabolic subgroup $P=MAN$ of G and so on as we did in § 1.

We identify $\mathfrak{a}_C^*$ with $\boldsymbol{C}$ by,

$$\mathfrak{a}_C^*\ni\zeta\alpha\longleftrightarrow\zeta\in\boldsymbol{C}. \tag{3.3.1}$$

Via this identification, we look upon the Fourier transform $\mathcal{F}$ (Definition (3.1.3)) as a map from $\mathcal{E}'(X)$ to $\mathcal{A}(\boldsymbol{C}\times K/M)$.

Fix a positive number k and fix the Riemannian metric g on X so that X has constant sectional curvature $\equiv -k^2$. Then $a_0=\exp(kH)\in A$. (notation (3.1.1))

Let us recall here briefly some standard models of the negatively curved space form $X=G/K$.

For $x={}^t(x_1, \cdots, x_n, x_{n+1})\in\boldsymbol{R}^{n+1}$, define a bilinear form of $\boldsymbol{R}^n$ by,

$$(3.3.2) \qquad [x, x] := -\sum_{j=1}^{n} (x_j)^2 + (x_{n+1})^2.$$

1) (Quadric model)

$$\begin{aligned} X_1 &:= \{x = {}^t(x_1, \cdots, x_n, x_{n+1}) \in \boldsymbol{R}^{n+1};\ [x, x] > 0,\ x_{n+1} > 0\} / \boldsymbol{R}_+^\times \\ &\simeq \{x = {}^t(x_1, \cdots, x_n, x_{n+1}) \in \boldsymbol{R}^{n+1};\ [x, x] = 1,\ x_{n+1} > 0\} \\ \partial X_1 &:= \{x = {}^t(x_1, \cdots, x_n, x_{n+1}) \in \boldsymbol{R}^{n+1};\ [x, x] = 0,\ x_{n+1} > 0\} / \boldsymbol{R}_+^\times. \end{aligned}$$

Here, $\boldsymbol{R}_+^\times$ acts on $\boldsymbol{R}^{n+1}$ diagonally and ∂X is the geometric boundary of X (Remark (3.2.22)).

2) (Unit ball model)

$$X_2 := \{u = {}^t(u_1, \cdots, u_n) \in \boldsymbol{R}^n;\ u_1^2 + \cdots + u_n^2 < 1\},$$

with the Riemannian structure

$$ds^2 = \frac{4}{k^2} \frac{du_1^2 + \cdots + du_n^2}{(1 - (u_1^2 + \cdots + u_n^2))^2}.$$

$$\partial X_2 := \{u = {}^t(u_1, \cdots, u_n) \in \boldsymbol{R}^n;\ u_1^2 + \cdots + u_n^2 = 1\},$$

with the Riemannian structure

$$ds^2 = \frac{1}{k^2} (du_1^2 + \cdots + du_n^2).$$

Note that on ∂X_2, $\sum_{j=1}^{n} u_j du_j = 0$.

3) (Upper half-space model)

$$X_3 := \{y = {}^t(y_1, \cdots, y_n) \in \boldsymbol{R}^n;\ y_n > 0\},$$

with the Riemannian structure

$$ds^2 = \frac{1}{k^2} \frac{dy_1^2 + \cdots + dy_n^2}{y_n^2}.$$

$$\partial X_3 := \{y = {}^t(y_1, \cdots, y_n) \in \boldsymbol{R}^n;\ y_n = 0\} \cup \{\infty\},$$

with the Riemannian structure

$$ds^2 = \frac{4}{k^2} \frac{dy_1^2 + \cdots + dy_{n-1}^2}{(1 + y_1^2 + \cdots + y_{n-1}^2)^2}.$$

The natural action of $G = SO_0(n, 1)$ on $\boldsymbol{R}^{n+1}$ preserves the bilinear form $[\ ,\ ]$, so X_1 and ∂X_1 are stable under this action. The G-action on

X_j and ∂X_j $(j=2,3)$ is defined through the following isomorphisms $\Phi_{ji}:\bar{X}_i \longrightarrow \bar{X}_j$, $(1\leq i,j\leq 3)$ which are defined by,

$$\Phi_{jj}=\mathrm{id}_{|\bar{X}_j}, \qquad \Phi_{ij}\circ\Phi_{jk}=\Phi_{ik}$$

and,

$$(3.3.3) \quad (u_1,\cdots,u_n)=\Phi_{21}(x_1,\cdots,x_n)=\frac{1}{[x,x]^{1/2}+x_{n+1}}(x_1,\cdots,x_n),$$

$$(3.3.4) \quad (y_1,\cdots,y_n)=\Phi_{32}(u_1,\cdots,u_n) = \frac{1}{u_1^2+\cdots+u_{n-1}^2+(u_n-1)^2}\left(2u_1,\cdots,2u_{n-1},1-\sum_{j=1}^n u_j^2\right).$$

Then,

$$(3.3.5)(a) \quad (u_1,\cdots,u_n)=\Phi_{23}(y_1,\cdots,y_n) = \frac{1}{y_1^2+\cdots+y_{n-1}^2+(y_n+1)^2}\left(2y_1,\cdots,2y_{n-1},\sum_{j=1}^n y_j^2-1\right),$$

$$(3.3.5)(b) \quad (y_1,\cdots,y_n)=\Phi_{31}(x_1,\cdots,x_n)=\frac{1}{x_{n+1}-x_n}(x_1,\cdots,x_{n-1},1),$$

$$(3.3.5)(c) \quad (x_1,\cdots,x_{n+1})=\Phi_{13}(y_1,\cdots,y_n) = \frac{1}{2y_n}\left(2y_1,\cdots,2y_{n-1},\sum_{j=1}^n y_j^2-1,\sum_{j=1}^n y_j^2+1\right).$$

Here $x=(x_1,\cdots,x_{n+1})$ is normalized such that $[x,x]=1$.

Let $o_1:={}^t(0,\cdots,0,1)\in X_1\subset \boldsymbol{R}^{n+1}$, $o_2:={}^t(0,\cdots,0)\in X_2\subset \boldsymbol{R}^n$, $o_3:={}^t(0,\cdots,0,1)\in X_3\subset \boldsymbol{R}^n$, and $v_1:={}^t(0,\cdots,0,1,1)\in\partial X_1\subset \boldsymbol{R}^{n+1}$, $v_2:={}^t(0,\cdots,0,1)\in\partial X_2\subset \boldsymbol{R}^n$, $v_3:=\{\infty\}\in\partial X_3$. Then $\Phi_{ij}(o_j)=o_i$, $\Phi_{ij}(v_j)=v_i$, and the isotropy subgroup of G at $o_j\in X_j$ $(j=1,2,3)$ is K, and at $v_j\in\partial X_j$ $(j=1,2,3)$ is $P=MAN$.

In models 2) and 3), we have defined a G-invariant (resp. K-invariant) Riemannian metric on X (resp. on ∂X), which is conformal to the standard metric in $\boldsymbol{R}^n$ in each model. Moreover it is easy to see that the above isomorphism $\Phi_{32}:X_2 \xrightarrow{\sim} X_3$ and $\partial X_2 \xrightarrow{\sim} \partial X_3$ give isometries. The following fact on geodesics and horospheres is classical.

FACT (3.3.6). *In a model 2), any geodesic is the intersection of X_2 and either a circle or a line (diameter) orthogonal to ∂X_2 in $\boldsymbol{R}^n$, while any horosphere is the intersection of X_2 and a hypersphere tangent to*

∂X_2 in $\boldsymbol{R}^n$. In a model 3), any geodesic is either a half-circle or a half-line orthogonal to ∂X_3 in $\boldsymbol{R}^n$, while any horosphere is either a hypersphere tangent to ∂X_3 in $\boldsymbol{R}^n$ removed by the point of tangency or a hyperplane parallel to ∂X_3 in $\boldsymbol{R}^n$.

In a model 3), let us calculate the principal curvatures of $\partial\Omega$ locally represented as a graph. Recall that in a model 3), $(y_1, \cdots, y_n)$ is a (global) coordinate on $X_3 \equiv X$. Let $\frac{\partial}{\partial y_j} \in \mathcal{X}(X)$ $(1 \leq j \leq n)$, and $g_{ij} := g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) \in C^\infty(X)$ $(1 \leq i, j \leq n)$. Let $(g^{ij})_{1\leq i,j\leq n} \in C^\infty(X, GL(n, \boldsymbol{R}))$ be the inverse matrix of $(g_{ij})_{1\leq i,j\leq n} \in C^\infty(X, GL(n, \boldsymbol{R}))$. The Christoffel symbols are given by,

$$\Gamma^i_{jl} := \frac{1}{2} \sum_{m=1}^{n} g^{im}\left(\frac{\partial g_{jm}}{\partial y_l} + \frac{\partial g_{lm}}{\partial y_j} - \frac{\partial g_{jl}}{\partial y_m}\right), \qquad (1 \leq i, j, l \leq n).$$

Then simple calculation yields,

$$g_{ij} = k^{-2}(y_n)^{-2}\delta_{ij}, \qquad (1 \leq i, j \leq n) \tag{3.3.7}$$

$$g^{ij} = k^2(y_n)^2\delta_{ij}, \qquad (1 \leq i, j \leq n) \tag{3.3.8}$$

and

$$\Gamma^i_{jl} = \begin{array}{llll} \text{when} & i \neq n, & = -y_n^{-1} & \text{if } (j, l) = (n, i) \text{ or } (i, n), \\ & & = 0 & \text{otherwise}, \\ \text{when} & i = n, & = -y_n^{-1} & \text{if } j = l = n, \\ & & = y_n^{-1} & \text{if } j = l \neq n, \\ & & = 0 & \text{if } j \neq l. \end{array} \tag{3.3.9}$$

LEMMA (3.3.10). *Retain notations as above. Let $\Omega \subset X_3$ satisfy* (3.2.10). *Given $x \in \partial\Omega$ $(\subset X \equiv X_3 \subset \boldsymbol{R}^n)$, let $x' := (y_1(x), \cdots, y_{n-1}(x)) \in \boldsymbol{R}^{n-1}$ $(\subset \boldsymbol{R}^{n-1} \cup \{\infty\} \simeq \partial X_3)$ be its orthogonal projection from $\boldsymbol{R}^n$ to $\boldsymbol{R}^{n-1}$. Let V be an open neighbourhood in ∂X_3 containing x'. Suppose $x \in \partial\Omega$ be a critical point of $y_{n|\partial\Omega}$. Then $\partial\Omega$ is represented locally as a graph:*

$$\{(z, f(z)) \in \boldsymbol{R}^n;\ z = (z_1, \cdots, z_{n-1}) \in V\},$$

where $f: V \longrightarrow \boldsymbol{R}$ is a C^∞ function satisfying

$$\frac{\partial f}{\partial z_j}(x') = 0, \qquad (1 \leq j \leq n-1). \tag{3.3.11}$$

Define a sign $\varepsilon=-1$ *or* $+1$ *according as* Ω *lies where* $f(z)<z_n$ *or* $f(z)>z_n$ *in the neighbourhood of* x. *Denote by*

$$B_l \equiv B_l(x) \qquad (1\leq l\leq n-1), \tag{3.3.12}$$

the eigenvalues of the Hessian $\left(\frac{\partial^2 f}{\partial z_i \partial z_j}(x')\right)_{1\leq i,j\leq n-1}$. *Then the principal curvature of* $\partial\Omega$ *at* x *is given by,*

$$\lambda_l(x)=-\varepsilon k(1+y_n(x)B_l(x)), \qquad (1\leq l\leq n-1). \tag{3.3.13}$$

PROOF. By a little abuse of language, denote by $\nu:=\sum_{j=1}^{n}\nu_j\frac{\partial}{\partial y_j}$ a local extension of the outer unit normal vector field over $\partial\Omega$ ($\hookrightarrow \boldsymbol{R}^n$), where ν_j is a smooth function defined near x. Then from the assumption (3.3.11) and our definition of ε, we have,

$$\nu_j(x)=\varepsilon k\delta_{jn}y_n(x), \qquad (1\leq j\leq n). \tag{3.3.14}$$

Define vector fields over X by,

$$e_j:=\frac{\partial}{\partial y_j}+\frac{\partial f}{\partial y_j}\frac{\partial}{\partial y_n}, \qquad (1\leq j\leq n-1).$$

Then each e_j is a local extension of a tangent vector field of $\partial\Omega$ ($\hookrightarrow \boldsymbol{R}^n$), and at x,

$$\|e_j\|_x\equiv(g(e_j(x),e_j(x)))^{1/2}=k^{-1}(y_n(x))^{-1}, \qquad (1\leq j\leq n-1). \tag{3.3.15}$$

Now, from our definition of e_j,

$$\nabla_{e_i}e_j=\nabla_{\frac{\partial}{\partial y_i}}\left(\frac{\partial}{\partial y_j}+\frac{\partial f}{\partial y_j}\frac{\partial}{\partial y_n}\right)+\frac{\partial f}{\partial y_i}\nabla_{\frac{\partial}{\partial y_n}}\left(\frac{\partial}{\partial y_j}+\frac{\partial f}{\partial y_j}\frac{\partial}{\partial y_n}\right).$$

Using (3.3.11) and (3.3.9), we see,

$$\begin{aligned}\nabla_{e_i(x)}e_j&=\nabla_{\frac{\partial}{\partial y_i}}\Big|_x\frac{\partial}{\partial y_j}+\frac{\partial^2 f}{\partial y_i\partial y_j}(x)\frac{\partial}{\partial y_n}\Big|_x\\&=\left(\frac{\delta_{ij}}{y_n(x)}+\frac{\partial^2 f}{\partial y_i\partial y_j}(x)\right)\frac{\partial}{\partial y_n}\Big|_x.\end{aligned} \tag{3.3.16}$$

Normalize e_j by putting $f_j:=\frac{e_j}{\|e_j\|}$. Then from (3.3.14), (3.3.15) and (3.3.16), we get,

$$(3.3.17)\qquad -g_x(\nabla_{f_i}f_j, \nu) = -\varepsilon k y_n(x)\Big(\frac{\delta_{ij}}{y_n(x)} + \frac{\partial^2 f}{\partial y_i \partial y_j}(x)\Big)$$
$$= -\varepsilon k\Big(\delta_{ij} + y_n(x)\frac{\partial^2 f}{\partial y_i \partial y_j}(x)\Big).$$

The principal curvatures are the eigenvalues of the matrix of the left side (Definition (3.2.15)), while the eigenvalues of the right side one are $-\varepsilon k(1+y_n(x)B_l(x))$, $(1\leq l\leq n-1)$. This completes the proof of the equation (3.3.13). Q.E.D.

LEMMA (3.3.18). *With notations as above,*

$$(3.3.19)\qquad \frac{\partial \nu_j}{\partial y_i}(x) = -\varepsilon k y_n(x)\frac{\partial^2 f}{\partial y_i \partial y_j}(x), \qquad (1\leq i, j\leq n-1).$$

PROOF. Differentiating $\nu := \sum_{j=1}^{n} \nu_j \frac{\partial}{\partial y_j}$ with respect to y_i, we have,

$$\nabla_{\frac{\partial}{\partial y_i}}\nu = \sum_{m=1}^{n}\Big(\frac{\partial \nu_m}{\partial y_i}\frac{\partial}{\partial y_m} + \sum_{l=1}^{m}\nu_m \Gamma^l_{im}\frac{\partial}{\partial y_l}\Big).$$

By substituting (3.3.11), (3.3.9), and (3.3.14), this equation evaluated at x is

$$(3.3.20)\qquad \nabla_{e_i(x)}\nu = \nabla_{\frac{\partial}{\partial y_i}\big|_x}\nu$$
$$= \sum_{m=1}^{n}\Big(\frac{\partial \nu_m}{\partial y_i}(x)\frac{\partial}{\partial y_m}\Big|_x + \varepsilon k y_n(x)\Big(-\frac{1}{y_n(x)}\Big)\frac{\partial}{\partial y_i}\Big|_x\Big).$$
$$= \sum_{m=1}^{n}\Big(\frac{\partial \nu_m}{\partial y_i}(x)\frac{\partial}{\partial y_m}\Big|_x - \varepsilon k\frac{\partial}{\partial y_i}\Big|_x\Big).$$

Substituting (3.3.14), (3.3.16), and (3.3.20) into

$$g_x(\nabla_{e_i}\nu, e_j) + g_x(\nu, \nabla_{e_i}e_j) = e_i(x)\cdot g(\nu, e_j)$$
$$=0, \qquad (1\leq i, j\leq n-1),$$

we obtain

$$\Big(\frac{\partial \nu_j}{\partial y_i}(x) - \varepsilon k\delta_{ij}\Big)\|e_j\|_x^2 + \varepsilon k y_n(x)\Big(\frac{\delta_{ij}}{y_n(x)} + \frac{\partial^2 f}{\partial y_i \partial y_j}(x)\Big)\Big\|\frac{\partial}{\partial y_n}\Big\|_x^2 = 0,$$

for $1\leq i$, $j\leq n-1$. Since $\|e_j\|_x = \Big\|\frac{\partial}{\partial y_n}\Big\|_x = k^{-1}(y_n(x))^{-1}$ by (3.3.15), this is equivalent to

$$\frac{\partial \nu_j}{\partial y_i}(x) = -\varepsilon k y_n(x) \frac{\partial^2 f}{\partial y_i \partial y_j}(x), \qquad (1 \leq i, j \leq n-1),$$

proving the lemma. Q.E.D.

Making use of Lemma (3.3.10), we can easily calculate the principal curvatures of a hypersphere and a horosphere.

Example (3.3.21). Let $\Omega \equiv B(R) := \left\{ u \in \boldsymbol{R}^n;\ \sum_{j=1}^n u_j^2 < r^2 \right\} \subset X_2$ (unit ball model) for $0<r<1$. Then $B(R)$ is a ball with radius $R := \frac{1}{k} \log \frac{1+r}{1-r}$ in the Riemannian manifold (X_2, ds^2). In the neighbourhood of $x := {}^t(0, \cdots, 0, r) \in \partial\Omega$, $\partial\Omega$ is represented as a graph:

$$u_n = r - \frac{1}{2r} \sum_{j=1}^{n-1} u_j^2 + O\left(\left(\sum_{j=1}^{n-1} u_j^2\right)^{3/2}\right).$$

Then by transforming from X_2 to X_3 by Φ_{32}, $\partial\Omega$ is represented near $x_3 = \Phi_{32}(x_2) = {}^t\left(0, \cdots, 0, \frac{1+r}{1-r}\right) \in X_3$ as a graph:

$$y_n = \frac{1+r}{1-r} - \frac{1-r^2}{4r} \sum_{j=1}^{n-1} y_j^2 + O\left(\left(\sum_{j=1}^{n-1} y_j^2\right)^{3/2}\right).$$

With notation as in Lemma (3.3.10), $B_l(x_3) = -\frac{1-r^2}{2r}$, $y_n(x_3) = \frac{1+r}{1-r}$, and hence the principal curvatures $\lambda_l(x) = -k\left(1 - \frac{1+r}{1-r} \frac{1-r^2}{2r}\right) = \frac{k}{\tanh(kR)}$, $(1 \leq l \leq n-1)$. So all the principal curvatures of a sphere with radius R are $\frac{k}{\tanh(kR)}$. We also find that the supporting function $h_{B(R),\varepsilon} \equiv \varepsilon R$ after Lemma (3.5.2), $K_\varepsilon \equiv \left(\frac{k(1+\varepsilon r)^2}{2r}\right)^{n-1} = \left(\frac{2\varepsilon k \exp(2\varepsilon kR)}{\exp(2\varepsilon kR)-1}\right)^{n-1}$ after Definition (3.4.1), and that $d_\Omega \equiv (4R)^{-1}\left(\log\left(\frac{k^{n-1}(1-r)^{2(r-1)}}{(2r)^{n-1}}\right) - \log\left(\frac{k^{n-1}(1+r)^{2(r-1)}}{(2r)^{n-1}}\right)\right) = -\frac{k(n-1)}{2}$ after Definition (3.5.12).

As for a horosphere, by taking a limit $R \to \infty$ in the above formula or by applying Lemma (3.3.10) directly with $B_l(x)=0$ $(1\leq l\leq n-1)$, we see that all the principal curvatures of a horosphere are k everywhere. (If we choose the opposite orientation of a horosphere, all the principal curvatures would be $-k$.)

§ 4. Convex domain in X

We shall devote this section to some characterizations of (strict) horospherical convexity, generalizing Fact (2.1.4) in Euclidean space. In a hyperbolic space, geodesical convexity is weaker than horospherical convexity, and either or sometimes both of them share the property analogous to each one in Fact (2.1.4) with a suitable formulation. But here we treat only the properties which enable us to use the same method as in Euclidean space when obtaining the asymptotic behaviour of $\tilde{\chi}_{\Omega}(\zeta, bM)$ in § 5.

Throughout this section, $X=SO_0(n,1)/SO(n)$ has constant sectional curvature $-k^2$.

DEFINITION (3.4.1). Let Ω satisfy (3.2.10). Define $K_\varepsilon\equiv K_{\partial\Omega,\varepsilon}:\partial\Omega\longrightarrow \boldsymbol{R}$, $(\varepsilon=\pm 1)$, by $K_\varepsilon(x)=\prod_{j=1}^{n-1}(\lambda_j(x)+\varepsilon k)$, for $x\in\partial\Omega$, where $\lambda_j(x)\equiv\lambda_{j,\partial\Omega}(x)$ are the principal curvatures of $\partial\Omega$. (Definition (3.2.15)). We sometimes call K_ε $(\varepsilon=\pm 1)$ *the Gauss curvature.*

PROPOSITION (3.4.2). *Let Ω satisfy* (3.2.10).

1) *The following three conditions on Ω are equivalent*:

a) *Ω is strictly horospherical convex* (*strictly H-convex*) (*Definition* (3.2.13)). (*i.e. ϕ_ε $(\varepsilon=\pm 1)$ give a diffeomorphism from $\partial\Omega$ onto K/M*).

a)′ *$\phi_{-1}:\partial\Omega\longrightarrow K/M$ is locally diffeomorphic.*

b) *$K_\varepsilon\equiv K_{\partial\Omega,\varepsilon}:\partial\Omega\longrightarrow \boldsymbol{R}$, $(\varepsilon=\pm 1)$ are positive valued.*

2) *Strict H-convexity implies H-convexity.* (*Definition* (3.2.13)).

3) *H-convexity implies*

$$\lambda_{\partial\Omega,j}(x)\geq k,\quad (1\leq j\leq n-1),\qquad \textit{for any } x\in\partial\Omega.$$

The proof of this result rests on the following Lemma (3.4.4). Before stating it, let us prepare some notations.

Put a positive valued smooth functions $P:X\longrightarrow \boldsymbol{R}$ by,

$$(3.4.3)\qquad P(x):=\left(k\left(1+\sum_{l=1}^{n-1}y_l(x)^2\right)\right)^{-n+1}y_n(x)^{n-1},\qquad \text{for } x\in X\equiv X_3,$$

(a upper half space model X_3; notation in § 3).

Induce a Riemannian metric on $\partial\Omega$ from X and recall that we have defined a K-invariant metric on $\partial X \simeq G/P \simeq K/M$ in § 3. So we can define the Jacobian $\det(J\psi_\varepsilon): \partial\Omega \longrightarrow \boldsymbol{R}$, of the Gauss map $\psi_\varepsilon: \partial\Omega \longrightarrow G/P \simeq \partial X$, ($\varepsilon = \pm 1$). Then the following Lemma (3.4.4) is due to [2] (when the dimension $n=2$), which asserts that the Jacobian of the 'Gauss map' is a multiple of the Gaussian curvature.

LEMMA (3.4.4). *Suppose Ω satisfy Assumption* (3.2.10), *and retain notations as above. Then,*

$$\det(J\psi_\varepsilon)(x) = \varepsilon^{n-1} P(x) K_\varepsilon(x), \qquad \textit{for } \varepsilon = \pm 1 \textit{ and } x \in \partial\Omega. \tag{3.4.5}$$

PROOF. $\psi_\varepsilon: \partial\Omega \longrightarrow G/P$ is a G-equivariant map and both X and G/P have K-invariant Riemannian structures. Hence, shifting by an element of K, we may assume that $\partial\Omega$ satisfies the conditions in Lemma (3.3.10) at x and that $\psi_{-\varepsilon}(x) = \lim\limits_{t \to +\infty} \operatorname{Exp}(-t\varepsilon\nu(q)) = \infty$ in the upper half space model $\bar{X}_3 = X_3 \cup \partial X_3 \simeq G/K \cup G/P$. We will use the notations in Lemma (3.3.10) and in its proof. Let $\nu = \sum\limits_{j=1}^{n} \nu_j \dfrac{\partial}{\partial y_j}$ be a local extension of the outer unit normal vector field of $\partial\Omega \subset X_3$. We want to find $\psi_\varepsilon(q) = \lim\limits_{t \to \infty} \operatorname{Exp}(t\varepsilon\nu(q))$ explicitly for an element q in a small neighbourhood of $\partial\Omega$ containing x. First recall that in a half-space model, a geodesic is a half-circle in $\boldsymbol{R}^n$ with its center on $y_n = 0$. (Fact (3.3.6)).

Let the equation of the hypersphere containing the half-circle $\{\operatorname{Exp}(t\varepsilon\nu(q)); t \in \boldsymbol{R}\}$ $(\subset X_3) \subset \boldsymbol{R}^n$ and orthogonal to $\boldsymbol{R}^{n-1} \times \{0\}$ be

$$\sum_{j=1}^{n-1} (y_j - a_j)^2 + y_n^2 = r^2,$$

where $a_j \in \boldsymbol{R}$, and $r > 0$ are constants determined by $q \in \partial\Omega$. Then we have

$$\sum_{j=1}^{n-1} (y_j(q) - a_j)^2 + y_n^2(q) = r^2, \tag{3.4.6}$$

and

$$\sum_{j=1}^{n-1} (y_j(q) - a_j)\nu_j(q) + y_n(q)\nu_n(q) = 0. \tag{3.4.7}$$

The limit points $\lim\limits_{t \to \pm\infty} \operatorname{Exp}(t\varepsilon\nu(q)) \in \boldsymbol{R}^n$ can be written ${}^t(y_1(q) + c\nu_1(q), \cdots,$

$y_{n-1}(q)+c\nu_{n-1}(q), 0)$ with some constant $c\equiv c(q)\in \boldsymbol{R}$. Let us find the constant $c(q)$. Put $|\nu|(q):=\left(\sum_{j=1}^{n}\nu_j^2(q)\right)^{1/2}=ky_n(q)$.

$$\begin{aligned}
&\left(c(q)\sum_{j=1}^{n-1}\nu_j^2(q)-y_n(q)\nu_n(q)+\varepsilon y_n(q)|\nu|(q)\right)\\
&\quad\times\left(c(q)\sum_{j=1}^{n-1}\nu_j^2(q)-y_n(q)\nu_n(q)-\varepsilon y_n(q)|\nu|(q)\right)\\
&=\left(c(q)\sum_{j=1}^{n-1}\nu_j^2(q)-y_n(q)\nu_n(q)\right)^2-y_n^2(q)\sum_{j=1}^{n}\nu_j^2(q)\\
&=\left(c(q)^2\sum_{j=1}^{n-1}\nu_j^2(q)-2c(q)y_n(q)\nu_n(q)\right)\sum_{j=1}^{n-1}\nu_j^2(q)+y_n^2(q)\nu_n^2(q)-y_n^2(q)\sum_{j=1}^{n}\nu_j^2(q)\\
&=\left(c(q)^2\sum_{j=1}^{n-1}\nu_j^2(q)-2c(q)y_n(q)\nu_n(q)-y_n^2(q)\right)\sum_{j=1}^{n-1}\nu_j^2(q).
\end{aligned}$$

From (3.4.7),

$$\begin{aligned}
&=\left(c(q)^2\sum_{j=1}^{n-1}\nu_j^2(q)+2c(q)\sum_{j=1}^{n-1}(y_j(q)-a_j)\nu_j(q)-y_n^2(q)\right)\sum_{j=1}^{n-1}\nu_j^2(q)\\
&=\left(\sum_{j=1}^{n-1}(y_j(q)+c(q)\nu_j(q)-a_j)^2-\left(\sum_{j=1}^{n-1}(y_j(q)-a_j)^2+y_n^2(q)\right)\right)\sum_{j=1}^{n-1}\nu_j^2(q).
\end{aligned}$$

From (3.4.6),

$$\begin{aligned}
&=(r^2-r^2)\sum_{j=1}^{n-1}\nu_j^2(q)\\
&=0.
\end{aligned}$$

Therefore

$$\begin{aligned}
c(q)&=\left(\sum_{j=1}^{n-1}\nu_j^2(q)\right)^{-1}y_n(q)(\nu_n(q)\pm\varepsilon|\nu|(q))\\
&=(|\nu|^2(q)-\nu_n^2(q))^{-1}y_n(q)(\nu_n(q)\pm\varepsilon|\nu|(q))\\
&=(-\nu_n(q)\pm\varepsilon|\nu|(q))^{-1}y_n(q)\\
&=(-\nu_n(q)\pm\varepsilon ky_n(q))^{-1}y_n(q).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\lim_{t\to\pm\infty}\operatorname{Exp}(t\varepsilon\nu(q))\\
&={}^t\!\left(y_1(q)+\frac{y_n(q)\nu_1(q)}{-\nu_n(q)\pm\varepsilon ky_n(q)},\cdots,y_{n-1}(q)+\frac{y_n(q)\nu_{n-1}(q)}{-\nu_n(q)\pm\varepsilon ky_n(q)},0\right)\\
&\in\boldsymbol{R}^{n-1}\times\{0\}\subset\partial X_3.
\end{aligned}$$

Taking care of the signature, we get

$$\text{(3.4.8)}\qquad \phi_\varepsilon(q)={}^t\Big(y_1(q)+\frac{y_n(q)\nu_1(q)}{-\nu_n(q)+\varepsilon k y_n(q)},\ \cdots,\ y_{n-1}(q)+\frac{y_n(q)\nu_{n-1}(q)}{-\nu_n(q)+\varepsilon k y_n(q)},\ 0\Big)$$
$$\in \partial X_3\ \ (\simeq \boldsymbol{R}^{n-1}\cup\{\infty\}).$$

Put $f_j:=\frac{e_j}{\|e_j\|}$, $e_j:=\frac{\partial}{\partial y_j}+\frac{\partial f}{\partial y_j}\frac{\partial}{\partial y_n}$ $(1\le j\le n-1)$, where $\partial\Omega$ is represented as a graph of $f(y_1,\cdots,y_{n-1})$. (Notations here are the same as in the proof of Lemma (3.3.10)). Then for $1\le j$, $l\le n-1$, $\varepsilon\in\{\pm 1\}$, and $x\in\partial\Omega$,

$$\begin{aligned}
&\Big\|\frac{\partial}{\partial y_l}\Big\|_{\phi_\varepsilon(x)} f_j(x)\ \text{(the l-th component of ϕ_ε)}\\
&=\Big\|\frac{\partial}{\partial y_l}\Big\|_{\phi_\varepsilon(x)} f_j(x)\Big(y_l+\frac{y_n\nu_l}{-\nu_n+\varepsilon k y_n}\Big)\\
&=\frac{2}{k\Big(1+\sum_{m=1}^{n-1}y_m^2(x)\Big)}k y_n(x)\frac{\partial}{\partial y_j}\Big|_x\Big(y_l+\frac{y_n\nu_l}{-\nu_n+\varepsilon k y_n}\Big).
\end{aligned}$$

From (3.3.14) and Lemma (3.3.18),

$$\begin{aligned}
&=2y_n(x)\Big(1+\sum_{m=1}^{n-1}y_m^2(x)\Big)^{-1}\Big(\delta_{jl}+\frac{y_n(x)}{2\varepsilon k y_n(x)}\varepsilon k y_n(x)\frac{\partial^2 f}{\partial y_j\partial y_l}(x)\Big)\\
&=2y_n(x)\Big(1+\sum_{m=1}^{n-1}y_m^2(x)\Big)^{-1}\Big(\delta_{jl}+\frac{1}{2}y_n(x)\frac{\partial^2 f}{\partial y_j\partial y_l}(x)\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\det(J\phi_\varepsilon)(x)\\
&=\Big(2y_n(x)\Big(1+\sum_{m=1}^{n-1}y_m^2(x)\Big)^{-1}\Big)^{n-1}\det\Big(\delta_{jl}+\frac{1}{2}y_n(x)\frac{\partial^2 f}{\partial y_j\partial y_l}(x)\Big)_{1\le j,l\le n-1}\\
&=\Big(y_n(x)\Big(1+\sum_{m=1}^{n-1}y_m^2(x)\Big)^{-1}\Big)^{n-1}\prod_{l=1}^{n-1}(2+y_n(x)B_l(x))\qquad\text{(notation (3.3.12))}\\
&=\Big(y_n(x)\Big(1+\sum_{m=1}^{n-1}y_m^2(x)\Big)^{-1}\Big)^{n-1}\prod_{l=1}^{n-1}(2+\varepsilon k^{-1}\lambda_l(x)-1)\quad\text{(by (3.3.13))}\\
&=\varepsilon^{n-1}\Big(y_n(x)k^{-1}\Big(1+\sum_{m=1}^{n-1}y_m^2(x)\Big)^{-1}\Big)^{n-1}\prod_{l=1}^{n-1}(\lambda_l(x)+\varepsilon k).
\end{aligned}$$

From our Definition (3.4.1) and (3.4.3),

$$=\varepsilon^{n-1}P(x)K_\varepsilon(x).$$

Hence the lemma is proved. Q.E.D.

Now, we are ready to finish the proof of Proposition (3.4.2).

PROOF OF PROPOSITION (3.4.2). 1) a)$\to$a)$'$ is clear.
a)$'\to$b) In a upper half space model, let x be the point of ∂X realizing the minimum of $\{y_n(q); q\in\partial\Omega\}$. Then applying Lemma (3.3.10) ($\varepsilon=-1$, $B_j(x)\geq 0$), we obtain that any principal curvature of $\partial\Omega$ satisfies $\lambda_j(x)\geq k$, $(1\leq j\leq n-1)$.

On the other hand, since ψ_{-1} is locally diffeomorphic from the assumption a)$'$, the Jacobian $\det(J\psi_{-1})(x)\neq 0$. From Lemma (3.4.4), this implies $K_{-1}(x)=\prod_{j=1}^{n-1}(\lambda_j(x)-k)\neq 0$. Therefore $\lambda_j(x)>k$, $(1\leq j\leq n-1)$, which also implies $\lambda_j(q)>k$, $(1\leq j\leq n-1)$ for any $q\in\partial\Omega$, because $\partial\Omega$ is connected and because eigenvalues are continuous with respect to matrix elements. Therefore $K_\varepsilon(q)\equiv\prod_{j=1}^{n-1}(\lambda_j(q)+\varepsilon k)>0$, $(\varepsilon=\pm 1)$ for any $q\in\partial\Omega$.
b)$\to$a)$'$ Let Ω satisfy b). Then from Lemma (3.4.4), $\det(J\psi_\varepsilon)(x)=\varepsilon^{n-1}P(x)K_\varepsilon(x)\neq 0$. Therefore $\psi_\varepsilon:\partial\Omega\longrightarrow K/M$ gives a local diffeomorphism. For b)$\to$a), it is enough to prove ψ_ε is injective, because a local diffeomorphism ψ_ε from a compact set $\partial\Omega$ is a covering map. The injectivity will be shown in the next proof 2). (When $n\geq 3$, the injectivity is derived also from $\pi_1(S^{n-1})=1$.)
2) Let Ω satisfy a)$'$ and b). Then

$$\lambda_j(q)>k,\quad (1\leq j\leq n-1)\qquad \text{for any } q\in\partial\Omega. \tag{3.4.9}$$

So $\partial\Omega$ lies *locally* in one side with respect to any pair of horospheres tangent to $\partial\Omega$. For a global statement, we use Morse theory. In a half space model, the n-th coordinate function

$$y_n:\ \partial\Omega\longrightarrow \boldsymbol{R},$$

is a Morse function (i.e. has no degenerate critical points). Indeed, suppose $x\in\partial\Omega$ be a degenerate critical points. Then from Lemma (3.3.10) there must be at least one principal curvature λ_j whose absolute value is k, which contradicts to $\lambda_j>k$ ((3.4.9)).

The index at the critical point x is defined to be the number of negative eigenvalues (with multiplicity) of the matrix $\left(\dfrac{\partial^2 y_n}{\partial z_i\partial z_j}\right)_{1\leq i,j\leq n-1}$, where $\{z_i\}_{1\leq i\leq n-1}$ is any local coordinate near x. We denote by C_j the number of critical points of index j on $\partial\Omega$. Then from Lemma (3.3.10) and (3.4.9),

(3.4.10)
$$\begin{aligned} C_j&=0 \quad \text{if } j\neq 0,\ n-1.\\ C_0&=\text{the number of points on } \partial\Omega \text{ where } y_{n|\partial\Omega} \text{ is maximal}\\ &=\#\{\psi_1^{-1}(\{\infty\})\subset\partial\Omega\}\\ &\geq 1.\\ C_{n-1}&=\text{the number of points on } \partial\Omega \text{ where } y_{n|\partial\Omega} \text{ is minimal}\\ &=\#\{\psi_{-1}^{-1}(\{\infty\})\subset\partial\Omega\}\\ &\geq 1. \end{aligned}$$

We denote by R_j the j-th Betti number of $\partial\Omega$. Since $\partial\Omega$ is orientable closed $n-1$ manifold,

(3.4.11)
$$R_0=R_{n-1}=1.$$

From the Morse inequalities (see [18] p. 30), we have

$$\begin{aligned} R_0&\leq C_0 &&=C_0,\\ R_1-R_0&\leq C_1-C_0 &&=-C_0,\\ R_2-R_1+R_0&\leq C_2-C_1+C_0 &&=C_0,\\ &\cdots\cdots \end{aligned}$$
$$R_{n-2}-R_{n-3}+\cdots+(-1)^nR_0\leq C_{n-2}-C_{n-3}+\cdots+(-1)^nC_0=(-1)^nC_0,$$
$$R_{n-1}-R_{n-2}+\cdots-(-1)^nR_0=C_{n-1}-C_{n-2}+\cdots-(-1)^nC_0=C_{n-1}-(-1)^nC_0.$$

Then by (3.4.10) and (3.4.11), we obtain

$$R_1=R_2=\cdots=R_{n-2}=0,$$
$$C_0=R_0=1,$$

and

$$C_{n-1}=R_{n-1}=1.$$

In particular, the last formula implies that the minimum of $y_{n|\partial\Omega}$ is attained at only one point (say, x_0) on $\partial\Omega$. Since the horosphere tangent to $\partial\Omega$ at x_0 is either the hyperplane (in a half space model):

$$y_n=y_n(x_0),$$

or the hypersphere (removed by the point on ∂X):

$$\sum_{j=1}^{n-1}(y_j-y_j(x_0))^2+\Big(y_n-\frac{1}{2}y_n(x_0)\Big)^2=\Big(\frac{1}{2}y_n(x_0)\Big)^2,$$

these two horospheres lie in one side with respect to $\partial\Omega$.

For any point q of $\partial\Omega$, we can find an element b of K so that

$y_{n|b\cdot\partial\Omega}$ is minimal at $b\cdot q$. Applying the above argument to $b\cdot\partial\Omega$, it follows H-convexity of $\partial\Omega$. From the second equality in (3.4.10), this also implies the injectivity of ψ_ε ($\varepsilon=\pm 1$). Thus the proof of 1) is completed and 2) is proved.

3) From Lemma (3.3.10) and Example (3.3.21), if Ω is H-convex, then the principal curvatures of $\partial\Omega$ satisfies either

$$\lambda_{\partial\Omega,j}(q)\geq k,\quad (1\leq j\leq n-1),\qquad \text{for any } q\in\partial\Omega.$$

or

$$\lambda_{\partial\Omega,j}(q)\leq -k,\quad (1\leq j\leq n-1),\qquad \text{for any } q\in\partial\Omega.$$

Using the same argument as in the proof of a)$'\rightarrow$b) in 1), only the possible case is the first one. Q.E.D.

COROLLARY (3.4.12) (cf. Lemma (2.1.1) and Proposition (3.7.23)). *Given a strictly convex domain Ω in $X=SO_0(n,1)/SO(n)$, Ω is recovered by one of the supporting functions $h_{\Omega,\varepsilon}$ as follows:*

$$\Omega=\bigcap_{b\in\Omega}\{ba_0^t n\cdot o\in X;\ -\infty<t<h_{\Omega,1}(b),\ n\in N\},$$

or

$$\Omega=\bigcap_{b\in\Omega}\{ba_0^t n\cdot o\in X;\ h_{\Omega,-1}(b)<t<+\infty,\ n\in N\}.$$

PROOF. From the next Lemma (3.5.2), one sees that the horosphere $a_0^{h_{b^{-1}\Omega,\varepsilon}(e)}N\cdot o$ is tangent to $\partial(b^{-1}\Omega)$ at $\psi_{b^{-1}\Omega,\varepsilon}(e)$ $(b\in K)$. Since $h_{b^{-1}\Omega,\varepsilon}(e)=h_{\Omega,\varepsilon}(b)$ and $\psi_{b^{-1}\Omega,\varepsilon}^{-1}(e)=b^{-1}\psi_{\Omega,\varepsilon}^{-1}(b)$ (Lemma (3.2.19)), one sees that $a_0^{h_{\Omega,\varepsilon}(b)}N\cdot o$ is tangent to $\partial\Omega$ at $\psi_{\Omega,\varepsilon}(b)$. In the preceding proof of the Proposition (3.4.2), we considered the maximum (or minimum) of the coordinate function $y_{n|\partial(b^{-1}\Omega)}$ in a half space model. Owing to Lemma (3.5.2), we see that

$$\Omega\subset\bigcap_{b\in\Omega}\{ba_0^t n\cdot o\in X;\ -\infty<t<h_{\Omega,1}(b),\ n\in N\},$$

and

$$\Omega\subset\bigcap_{b\in\Omega}\{ba_0^t n\cdot o\in X;\ h_{\Omega,-1}(b)<t<+\infty,\ n\in N\}.$$

For the converse inclusion, we make a simple observation: For a fixed element $x\in X\setminus\Omega$, let $q_\varepsilon\in\partial\Omega$ be the nearest (farthest) point from $\partial\Omega$ when $\varepsilon=1$ (resp. $\varepsilon=-1$). Then

$$x \in \{\psi_{\Omega,1}(q_1)a_0^t n\cdot o \in X;\ h_{\Omega,1}(\psi_{\Omega,1}(q_1)) \le t < +\infty,\ n \in N\},$$

and

$$x \in \{\psi_{\Omega,-1}(q_{-1})a_0^t n\cdot o \in X;\ -\infty < t \le h_{\Omega,-1}(\psi_{\Omega,-1}(q_{-1})),\ n \in N\}.$$

Hence the corollary. Q.E.D.

§ 5. Asymptotic behaviour of $\tilde{\chi}_\Omega(\zeta)$

Using the basic properties of a strictly H-convex domain Ω in $SO_0(n, 1)/SO(n)$ which we have prepared, we find that the zeros of $\tilde{\chi}_\Omega(\zeta, bM)$ are distributed with bounded imaginary parts and we obtain the asymptotic behaviour of $\tilde{\chi}_\Omega(\zeta, bM)$ as $\mathrm{Re}\,\zeta \longrightarrow \pm\infty$ just as we did in Euclidian case. The main result in this section is Proposition (3.5.13).

Throughout this section, $X = SO_0(n, 1)/SO(n)$ with constant sectional curvature $-k^2$ $(k>0)$, and Ω is a strictly H-convex domain (Definition (3.2.13)) in X.

In a half space model X_3, the following formula holds:

$$(3.5.1) \qquad y_n(na_0^t\cdot o_3) = e^{kt}, \qquad \text{for } t \in \boldsymbol{R} \text{ and } n \in N.$$

Here, $y_n: X \equiv X_3 \longrightarrow \boldsymbol{R}$ is the n-th coordinate function, $o_3 := {}^t(0, \cdots, 0, 1) \in X_3$ $(\subset \boldsymbol{R}^n)$, and $a_0 = \exp(kH) \in A$ is the positive unit vector defined in (3.1.1).

In fact, with notations in § 3 of this Chapter freely (for instance, $\{N_j\}_{1\le j\le n-1}$ is a basis of the Lie algebra $\mathfrak{n}$ over $\boldsymbol{R}$ as a vector space), let

$$n := \exp\Big(\sum_{j=1}^{n-1} z_j N_j\Big) \in N, \qquad (z_j \in \boldsymbol{R}),$$

and

$$Z := \frac{1}{2}\sum_{j=1}^{n-1} z_j^2 \in \boldsymbol{R}.$$

Then, $y_n(na_0^t\cdot o_3) = y_n(\Phi_{31}\circ\Phi_{13}(na_0^t\cdot o_3))$

$$= y_n\circ\Phi_{31}(na_0^t\cdot\Phi_{13}(o_3))$$

$$= y_n\circ\Phi_{31}\left(\begin{bmatrix} 1 & & 0 & -z_1 & z_1 \\ & \ddots & & \vdots & \vdots \\ 0 & & 1 & -z_{n-1} & z_{n-1} \\ z_1 & \cdots & z_{n-1} & 1-Z & Z \\ z_1 & \cdots & z_{n-1} & -Z & 1+Z \end{bmatrix}\begin{bmatrix} 1 & & 0 & & \\ & \ddots & & & 0 \\ 0 & & 1 & & \\ & 0 & & \mathrm{ch}(kt) & \mathrm{sh}(kt) \\ & & & \mathrm{sh}(kt) & \mathrm{ch}(kt) \end{bmatrix}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$=y_n\circ\Phi_{31}\left(\begin{bmatrix} z_1\exp(-kt) \\ \vdots \\ z_{n-1}\exp(-kt) \\ Z\exp(-kt)+\mathrm{sh}(kt) \\ Z\exp(-kt)+\mathrm{ch}(kt)\end{bmatrix}\right)=y_n\left(\begin{bmatrix} z_1 \\ \vdots \\ z_{n-1} \\ \exp(kt)\end{bmatrix}\right)=\exp(kt),$$

proving (3.5.1).

The formula (3.5.1) says that the horosphere:

$$a_0^tN\cdot o=Na_0^t\cdot o\ \ (\subset X)$$

is realized as a hyperplane:

$$y_n\equiv\exp(kt),$$

in a upper half space model $X_3\subset \boldsymbol{R}^n$.

The next Lemma (3.5.2) helps us to understand the supporting function which we defined in § 2. As usual we sometimes regard a function defined on a homogeneous space G/H as a right H-invariant function on G.

LEMMA (3.5.2). *With notation as above, suppose Ω be a strictly H-convex domain in $X=SO_0(n,1)/SO(n)$. Let $h_\varepsilon\equiv h_{\Omega,\varepsilon}:G/P\longrightarrow\boldsymbol{R}$, $(\varepsilon=\pm1)$ be the supporting function (Definition (3.2.16)) and $\phi_\varepsilon\equiv\phi_{\Omega,\varepsilon}:\partial\Omega\longrightarrow G/P$ be the Gauss map (Definition (3.2.11)). Then the horosphere $a_0^tN\cdot o$ is tangent to $\partial\Omega$ if and only if $t=h_\varepsilon(e)$, $(\varepsilon=\pm1)$. In this case, the point of tangency is $\phi_{\pm1}^{-1}(e)$, where the upper or lower sign is according as $\varepsilon=1$ or $\varepsilon=-1$.*

PROOF. Suppose that the horosphere $a_0^tN\cdot o$ be tangent to the boundary $\partial\Omega$ at x for some $t\in\boldsymbol{R}$. Then in a half space model, the outer unit normal vector of $\partial\Omega$ is given by,

$$\nu(x)=\varepsilon k y_n(x)\frac{\partial}{\partial y_n}\bigg|_x,$$

with $\varepsilon=1$ or -1.
Then,

$$\phi_\varepsilon(x)=\infty=P,\quad (\text{via } \boldsymbol{R}^{n-1}\cup\{\infty\}\simeq G/P). \tag{3.5.3}$$

Let $gM\in G/M$ be the corresponding element of $(x,\nu(x))=(\phi_\varepsilon^{-1}(\infty),\ \nu(\phi_\varepsilon^{-1}(\infty)))\in S(TX)$, by the isomorphism

$$G/M \simeq S(TX) \text{ (tangent sphere bundle)} \qquad ((3.2.4)).$$

From Lemma (3.2.8), $gm_\delta N\cdot o$ ($\delta=\pm 1$) are all the horospheres that are tangent to $\partial\Omega$ at x. In order that $a_0^t N\cdot o$ and $gm_\delta N\cdot o$ are coincident, it is necessary from (3.5.3) and Remark (3.2.22) that

$$\varepsilon=\delta$$

and from (3.2.3) it is also necessary that

$$a_0^t MN = gm_\delta MN.$$

Thus $NMa_0^{-t}=NMm_\varepsilon^{-1}g^{-1}$.
Since the Iwasawa projection $A: G=NAK \longrightarrow \mathfrak{a}$ is left NM invariant, we get

$$A(a_0^{-t})=A(m_\varepsilon^{-1}g^{-1}).$$

Hence we have $t\log(a_0)=-A(m_\varepsilon^{-1}g^{-1})$. In terms of our definition of the supporting function h_ε (Definition (3.2.16)), this is equivalent to $t=h_\varepsilon(P)$.
The converse statement is similar. Q.E.D.

Identifying $\mathfrak{a}_C^*$ with $\boldsymbol{C}$ by

$$\mathfrak{a}_C^* \ni \zeta\alpha \longleftrightarrow \zeta \in \boldsymbol{C}, \qquad ((3.3.1))$$

we regard the Fourier transform (Definition (3.1.4)) on X as a map

$$\mathcal{F}: \mathcal{E}'(X) \longrightarrow \mathcal{A}(\boldsymbol{C}\times K/M).$$

For $\zeta\in\boldsymbol{C}\simeq\mathfrak{a}_C^*$ and $b\in K$,

$$\begin{aligned}
\mathcal{F}\chi_\Omega(\zeta, bM) &= \int_X \chi_\Omega(x)\exp(\langle i\zeta+\rho,\ A(b^{-1}x)\rangle)dx \\
&= \int_X \chi_\Omega(bx)\exp(\langle i\zeta+\rho,\ A(x)\rangle)dx \\
&= \int_A\int_N \chi_\Omega(ban\cdot o)\exp(\langle i\zeta+\rho,\ A(an)\rangle)dnda \\
&= \int_{-\infty}^{+\infty}\int_N \chi_\Omega(ba_0^t n\cdot o)\exp(\langle i\zeta+\rho,\ A((a_0^t n a_0^{-t})a_0^t)\rangle)dndt \\
&= \int_{-\infty}^{+\infty}\int_N \chi_\Omega(ba_0^t n\cdot o)\exp(\langle i\zeta+\rho,\ A(a_0^t)\rangle)dndt \\
&= \int_{-\infty}^{+\infty} S_\Omega(bM, t)\exp kt\Big(i\zeta+\frac{n-1}{2}\Big)dt. \qquad (3.5.4)
\end{aligned}$$

Here, $S\equiv S_\Omega$: $K/M\times\boldsymbol{R}\longrightarrow\boldsymbol{R}$ is given by,

$$(3.5.5) \qquad S_{\Omega}(bM, t) := \int_N \chi_{\Omega}(ba_0^t n\cdot o)dn = \int_N \chi_{b^{-1}\Omega}(a_0^t n\cdot o)dn, \qquad (b\in K, t\in \boldsymbol{R}).$$

The following lemma corresponds to Lemma (2.2.8) in $\boldsymbol{R}^n$ case.

LEMMA (3.5.6). *Suppose Ω be strictly H-convex and retain notations as above. Then, $S\equiv S_{\Omega}: K/M\times\boldsymbol{R}\longrightarrow\boldsymbol{R}$ is a continuous function with compact support $\bar{V}$, and $S_{\Omega}|_V\in C^{\infty}(V)$. Here, V is an open set in $K/M\times\boldsymbol{R}$ given by,*

$$V := \{(bM, p)\in K/M\times\boldsymbol{R};\ bM\in K/M,\ h_{-1}(bM)<p<h_1(bM)\},$$

and $\bar{V}$ denotes its closure. More precisely, for a sufficiently small $\delta>0$, there are two C^{∞} functions: $S_{\varepsilon}\equiv S_{\Omega,\varepsilon}: K/M\times(-\delta,\delta)\ni(bM,t)\longmapsto S_{\varepsilon}(bM,t)\in\boldsymbol{R}$, $(\varepsilon=\pm1)$, such that $S(bM,t)$ are represented in the neighbourhood of ∂V as follows:

$$(3.5.7) \quad S(bM,t) = \frac{(2\pi)^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}(K_{-1}\circ\nu_{-1}^{-1}(b))^{-1/2}(t-h_{-1}(b))_+^{(n-1)/2}(1+S_{-1}(b, t-h_{-1}(b))(t-h_{-1}(b))),$$
$$\text{for } |t-h_{-1}(b)|<\delta,$$
$$= \frac{(2\pi)^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}(K_1\circ\nu_1^{-1}(b))^{-1/2}(t-h_1(b))_-^{(n-1)/2}(1+S_1(b, t-h_1(b))(t-h_1(b))),$$
$$\text{for } |t-h_1(b)|<\delta.$$

PROOF. Smoothness of S_{ε} can be proved in the same way as in Lemma (2.2.8) (Euclidean case). Let us obtain the first approximation of $S(bM,t)$ as $t\longrightarrow h_{\varepsilon}(bM)$.

Fix $bM\in K/M$ and $\varepsilon\in\{\pm1\}$. Let $\{y_1,\cdots,y_n\}$ be the (global) coordinate in a half space model X_3 as usual (notation § 3). Then from (3.3.8), the volume element in X is given by,

$$(\det(g_{ij}))^{1/2}dy_1\cdots dy_n = (ky_n)^{-n}dy_1\cdots dy_n.$$

The Haar measure dn of the unipotent Lie group N induced from the Riemannian metric on X (§ 1) is given by,

$$dn = (ky_n)^{-n+1}dy_1\cdots dy_{n-1}.$$

Therefore from (3.5.5), we have

$$S_\Omega(bM, t)=\int_{\substack{y_n=\exp(kt)\\(y_1,\cdots,y_n)\in b^{-1}\Omega}} \frac{dy_1\cdots dy_{n-1}}{(ky_n)^{n-1}}. \tag{3.5.8}$$

Set

$$p_\varepsilon := b^{-1}\psi_{\Omega,\varepsilon}^{-1}(b)=\psi_{b^{-1}\Omega,\varepsilon}(e)\in b^{-1}(\partial\Omega), \tag{3.5.9}$$

where we regard $\psi_{\Omega,\varepsilon}^{-1}$: $G/P\longrightarrow\partial\Omega$ as a map from G to $\partial\Omega$.

From Lemma (3.2.19), we have

$$\psi_{b^{-1}\Omega,\varepsilon}(p_\varepsilon)=b^{-1}\psi_{\Omega,\varepsilon}(b\cdot p_\varepsilon)=P\in G/P,$$

and

$$h_{b^{-1}\Omega,\varepsilon}(e)=h_{\Omega,\varepsilon}(b).$$

From Lemma (3.5.2), the horosphere $a_0^{h_{\Omega,\varepsilon}(b)}N\cdot o$ (i.e. the hyperplane $y_n\equiv\exp(kh_{\Omega,\varepsilon}(b))$ in a half space model X_3) is tangent to $b^{-1}(\partial\Omega)$ at p_ε.

Let f be a function which represents $b^{-1}(\partial\Omega)$ $(\subset X_3\subset \boldsymbol{R}^n)$ as a graph near p_ε. Then the function f satisfies the assumption (3.3.11) in Lemma (3.3.10). Let $B_l\equiv B_l(p_\varepsilon)$ $(1\le l\le n-1)$ be the eigenvalues of $\left(\frac{\partial^2 f}{\partial y_i\partial y_j}(p_\varepsilon)\right)_{1\le i,j\le n-1}$. Then by applying Lemma (3.3.10), the principal curvatures λ_l of $\partial\Omega$ satisfies:

$$\lambda_l(p_\varepsilon)=-\varepsilon k(1+y_n(p_\varepsilon)B_l), \qquad (1\le l\le n-1). \tag{3.5.10}$$

By an orthogonal transformation on $\boldsymbol{R}^{n-1}$, we may assume that $y_1, \cdots, y_{n-1}$ are in the principal directions at p_ε. Then,

$$\begin{aligned}
&f(y_1, \cdots, y_{n-1})\\
&=\exp(kh_\varepsilon(b))+\frac{1}{2}\sum_{l=1}^{n-1}B_l(y_l-y_l(p_\varepsilon))^2+O\Big(\sum_{l=1}^{n-1}|y_l-y_l(p_\varepsilon)|^3\Big)\\
&=\exp(kh_\varepsilon(b))\Big(1+\frac{1}{2}\exp(-kh_\varepsilon(b))\sum_{l=1}^{n-1}B_l(y_l-y_l(p_\varepsilon))^2+O\Big(\sum_{l=1}^{n-1}|y_l-y_l(p_\varepsilon)|^3\Big)\Big)\\
&=\exp(kh_\varepsilon(b))\exp\Big(\frac{1}{2}\exp(-kh_\varepsilon(b))\sum_{l=1}^{n-1}B_l(y_l-y_l(p_\varepsilon))^2+O\Big(\sum_{l=1}^{n-1}|y_l-y_l(p_\varepsilon)|^3\Big)\Big)\\
&=\exp\Big(kh_\varepsilon(b)+\frac{1}{2}\exp(-kh_\varepsilon(b))\sum_{l=1}^{n-1}B_l(y_l-y_l(p_\varepsilon))^2+O\Big(\sum_{l=1}^{n-1}|y_l-y_l(p_\varepsilon)|^3\Big)\Big)
\end{aligned}$$

$$=\exp\Big(kh_\varepsilon(b)-\frac{\varepsilon}{2k}\exp(-2kh_\varepsilon(b))\sum_{l=1}^{n-1}(\lambda_l+\varepsilon k)(y_l-y_l(p_\varepsilon))^2 \\ +O\Big(\sum_{l=1}^{n-1}|y_l-y_l(p_\varepsilon)|^3\Big)\Big).$$

By changing the coordinate $(y_1, \cdots, y_{n-1}, y_n)\longrightarrow(y_1, \cdots, y_{n-1}, t)$ (here, $y_n=\exp(kt)$), $b^{-1}(\partial\Omega)$ is represented as a graph:

$$t=h_\varepsilon(b)-\frac{\varepsilon}{2k^2}\exp(-2kh_\varepsilon(b))\sum_{l=1}^{n-1}(\lambda_l+\varepsilon k)(y_l-y_l(p_\varepsilon))^2 \\ +O\Big(\sum_{l=1}^{n-1}|y_l-y_l(p_\varepsilon)|^3\Big). \tag{3.5.11}$$

Therefore from (3.5.8), the first approximation of

$$S_\Omega(bM, t)=\int_{\substack{y_n=\exp(kt)\\(y_1,\cdots,y_n)\in b^{-1}\Omega}}\frac{dy_1\cdots dy_{n-1}}{(ky_n)^{n-1}} \qquad ((3.5.8))$$

is obtained just the same as in Lemma (2.2.8) of Chapter 2 as follows:

$$(k\exp(kh_\varepsilon(b))^{-n+1}(2\pi)^{(n-1)/2}\Gamma\Big(\frac{n+1}{2}\Big)^{-1}\Big(\prod_{l=1}^{n-1}k^{-2}\exp(-2kh_\varepsilon(b))(\lambda_l+\varepsilon k)\Big)^{-1/2} \\ \times(t-h_\varepsilon(b))_{-\varepsilon}^{(n-1)/2} \\ =(2\pi)^{(n-1)/2}\Gamma\Big(\frac{n+1}{2}\Big)^{-1}\Big(\prod_{l=1}^{n-1}(\lambda_l+\varepsilon k)\Big)^{-1/2}(t-h_\varepsilon(b))_{-\varepsilon}^{(n-1)/2} \\ =(2\pi)^{(n-1)/2}\Gamma\Big(\frac{n+1}{2}\Big)^{-1}(K_{b^{-1}\Omega,\varepsilon}\circ\phi_{b^{-1}\Omega,\varepsilon}^{-1}(e))^{-1/2}(t-h_\varepsilon(b))_{-\varepsilon}^{(n-1)/2} \\ =(2\pi)^{(n-1)/2}\Gamma\Big(\frac{n+1}{2}\Big)^{-1}(K_{\Omega,\varepsilon}\circ\phi_{\Omega,\varepsilon}^{-1}(b))^{-1/2}(t-h_\varepsilon(b))_{-\varepsilon}^{(n-1)/2},$$

proving the lemma. Q.E.D.

Let Ω be a strictly convex domain. Introduce a function $d\equiv d_\Omega: K/M=G/P\longrightarrow \boldsymbol{R}$ (cf. (2.1.16)) by,

$$d(bM)=\frac{-\log K_1\circ\phi_1^{-1}(b)+\log K_{-1}\circ\phi_{-1}^{-1}(b)}{2H(b)}, \quad (bM\in K/M). \tag{3.5.12}$$

PROPOSITION (3.5.13). *Suppose Ω be strictly convex and retain notations as above. Then,*
1) *$\tilde{\chi}_\Omega(\zeta, bM)\in\mathcal{A}(\boldsymbol{C}\times K/M)$, and $\tilde{\chi}_\Omega(\zeta, bM)$ is an entire function of $\zeta\in\boldsymbol{C}$ for each fixed $bM\in K/M$.*

2) *For any* $bM\in K/M$ *and any* $\xi, \eta\in \boldsymbol{R}$,

$$\tilde{\chi}_{\Omega}(\xi+i\eta, bM)=\overline{\tilde{\chi}_{\Omega}(-\xi+i\eta, bM)}, \tag{3.5.14}$$

3) $\sup\{|\operatorname{Im}\zeta|\in \boldsymbol{R};\ \tilde{\chi}_{\Omega}(\zeta, bM)=0,\ \zeta\in \boldsymbol{C},\ bM\in K/M\}<\infty$.

4) *When* $|\eta|<C$ (*C is any constant*), $\tilde{\chi}_{\Omega}$ *has the following asymptotics*:

$$\begin{aligned}(3.5.15)\quad &\tilde{\chi}_{\Omega}(\xi+i\eta, bM)\\ &\sim(2\pi)^{(n-1)/2}k^{-(n+1)/2}e^{\pi i(n+1)/4}(K_{+1}\circ\phi_{+1}^{-1}(bM))^{-1/2}e^{kh_{+1}(bM)(i\zeta+(n-1)/2)}\\ &\times\left(\exp\left(-H(bM)\left(k\left(i\zeta+\frac{n-1}{2}\right)+d(bM)\right)\right)+\exp\left(-\frac{\pi i(n+1)}{2}\right)\right)|\xi|^{-(n+1)/2}\\ &+O(|\xi|^{-(n+3)/2}),\qquad as\ \xi\longrightarrow+\infty.\end{aligned}$$

The estimate is uniform with respect to $bM\in K/M$.

PROOF. 1) and 2) are followed immediately from that $\chi_{\Omega}(x)$ is a real valued function with compact support.
To prove 3) and 4), first let us recall (3.5.4):

$$\tilde{\chi}_{\Omega}(\zeta, bM)=\int_{-\infty}^{+\infty}S_{\Omega}(bM, t)\exp kt\left(i\zeta+\frac{n-1}{2}\right)dt.$$

Then 3) is derived from Proposition (2.2.16) and Lemma (3.5.6). For 4), we obtain the asymptotic behaviour (3.5.15) from Lemma (3.5.6) in the same way as we obtained Proposition (2.2.32) from Lemma (2.2.8) in $\boldsymbol{R}^n$ case. ($h_{-1}(bM)$ in place of $-h(-\omega)$, $h_{+1}(bM)$ in place of $h(\omega)$, $k\left(\zeta-i\frac{n-1}{2}\right)$ in place of ζ, etc.) Q.E.D.

§ 6. Main theorem

In this section, we shall state our main theorem for $SO_0(n, 1)/SO(n)$ case.

Let Ω be a strictly H-convex domain (Definition (3.2.13)) in $SO_0(n, 1)/SO(n)$, which has constant sectional curvature $-k^2$ ($k>0$). Let us recall some notations.

$\phi_{\varepsilon}\equiv\phi_{\Omega,\varepsilon}$: $\partial\Omega \xrightarrow{\sim} G/P=K/M$, ($\varepsilon=\pm 1$) the 'Gauss map' ((3.2.11)).
$K_{\varepsilon}\equiv K_{\Omega,\varepsilon}$: $\partial\Omega\longrightarrow \boldsymbol{R}$, ($\varepsilon=\pm 1$), the 'Gauss curvature' ((3.4.1)).
$H_{\varepsilon}\equiv H_{\Omega,\varepsilon}$: $G/P\longrightarrow \boldsymbol{R}_{+}$, ($\varepsilon=\pm 1$), the breadth function ((3.2.18)).
$d\equiv d_{\Omega}$: $G/P=k/M\ni bM\longmapsto\dfrac{-\log K_1\circ\phi_1^{-1}(b)+\log K_{-1}\circ\phi_{-1}^{-1}(b)}{2H(b)}\in \boldsymbol{R}$, ((3.5.12))

Set a C^∞ function $f\equiv f_\Omega : K/M\times \boldsymbol{R}\longrightarrow \boldsymbol{C}$ by,

$$f(bM, m)\equiv f_m(bM):=\frac{1}{k}\left(\frac{\pi(4m+n-1)}{2H(bM)}+i\ d(bM)\right)+i\frac{n-1}{2}. \tag{3.6.1}$$

Let $\{\alpha\}=\Sigma^+(\mathfrak{g}, \mathfrak{a})$ and identify $\mathfrak{a}_C^*$ with $\boldsymbol{C}$ by,

$$\mathfrak{a}_C^* \ni \zeta\alpha \longleftrightarrow \zeta \in \boldsymbol{C}. \tag{3.6.2}$$

Using this identification, the Fourier transform is regarded as

$$\mathcal{F} : \mathcal{E}'(X)\longrightarrow \mathcal{A}(\boldsymbol{C}\times K/M).$$

For a bounded measurable set Ω in X, we put

$$\mathcal{N}(\Omega):=\{(\zeta, bM)\in \boldsymbol{C}\times K/M; \mathcal{F}\chi_\Omega(\zeta, bM)=0\}. \tag{3.6.3}$$

Then the following theorem is our main theorem in this chapter which describes the asymptotic behaviour of $\mathcal{N}(\Omega)$ when Ω is strictly H-convex domain in X.

THEOREM (3.6.4). *Suppose Ω be a strictly H-convex domain in $X=SO_0(n, 1)/SO(n)$, and retain notations as above. There is an integer $m_0\equiv m_0(\Omega)$ dependent only on Ω. Then*

$$\mathcal{N}(\Omega)=\left(\coprod_{m=m_0}^{\infty}(\mathcal{N}_m^+ \amalg \mathcal{N}_m^-)\right) \amalg \text{(compact set)}, \quad (\textit{disjoint union}),$$

where for each integer $m\geq m_0$, $\mathcal{N}_m^\pm$ is a regular submanifold in $\boldsymbol{C}\times K/M$, and is analytically diffeomorphic to K/M $(\simeq S^{n-1})$.

More precisely, for each integer $m\geq m_0$, there exist analytic maps $F_m^\varepsilon : S^{n-1}\longrightarrow \boldsymbol{C}$, $(\varepsilon=\pm 1)$ such that the following three conditions are satisfied for $m\geq m_0$:

$$\mathcal{N}_m^\pm=\{(F_m^\pm(bM), bM)\in \boldsymbol{C}\times K/M;\ bM\in K/M\}.$$

And for any element bM of K/M,

$$F_m^+(bM)=f_m(bM)+O(m^{-1}) \qquad as\ N\ni m\longrightarrow\infty, \tag{3.6.5}$$

and,

$$F_m^+(bM)=-\overline{F_m^-(bM)}. \tag{3.6.6}$$

In (3.6.5), *the estimate is uniform with respect to $bM\in K/M$.*

Example (3.6.7). Let $\Omega=B(R)$ a ball with radius R $(0<R<1)$, $r:=\dfrac{\exp(kR)+1}{\exp(kR)-1}$ (cleary we have $0<r<1$).

First note that

$$\mathcal{N}(g\cdot B(R))=\mathcal{N}(B(R)), \qquad \text{for any } g\in G=SO_0(n,1). \tag{3.6.8}$$

Indeed, fix $g\in G$, and $b\in K$. Let $b^{-1}g=n'a'k'$ ($n'\in N$, $a'\in A$, and $k'\in K$) be the Iwasawa decomposition of $b^{-1}g$. From [14] Ch. 1 Theorem 5.8 we have,

$$\begin{aligned}\tilde{\chi}_{g\cdot B(R)}(\zeta, bM)&=\int\chi_{g\cdot B(R)}(x)\exp\langle i\zeta+\rho, A(b^{-1}x)\rangle dx\\ &=\int_X \chi_{B(R)}(x)\exp\langle i\zeta+\rho, A(b^{-1}gx)\rangle dx\\ &=\int_K\int_A \chi_{B(R)}(ka\cdot o)\exp\langle i\zeta+\rho, A(b^{-1}gka)\rangle Q(a)dkda\\ &=\int_K\int_A \chi_{B(R)}(ka\cdot o)\exp\langle i\zeta+\rho, A(a')+A(k'ka)\rangle Q(a)dkda.\end{aligned}$$

Since $B(R)$ is K-stable,

$$\begin{aligned}&=\int_K\int_A \chi_{B(R)}(ka\cdot o)\exp\langle i\zeta+\rho, A(a')+A(ka)\rangle Q(a)dkda\\ &=\exp\langle i\zeta+\rho, A(a')\rangle\int_K\int_A \chi_{B(R)}(ka\cdot o)\exp\langle i\zeta+\rho, A(ka)\rangle Q(a)dkda.\end{aligned}$$

This implies (3.6.8).

The Fourier transform of the characteristic function $\chi_{B(R)}$ of a ball with radius R is obtained in [3] in a rank one symmetric space. After reviewing it for the reader's convenience, the asymptotics of the zeros of $\tilde{\chi}_{B(R)}(\zeta, bM)$ is found on the basis of the classical result of the asymptotic behaviour of the hypergeometric function $\mathrm{F}(a, b, c; x)$ as $|a-b|$ tends to infinity with $|a+b|$ bounded.

$$\begin{aligned}&\tilde{\chi}_{B(R)}(\zeta, bM)\\ &=\left(\frac{2}{k}\right)^n \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\int_0^r\int_0^\pi \frac{s^{n-1}}{(1-s^2)^n}\left(\frac{1-s^2}{1-2s\cos\theta+s^2}\right)^{i\zeta+(n-1)/2} dsd\theta\\ &=\left(\frac{2}{k}\right)^n \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^r s^{n-1}(1-s^2)^{i\zeta-(n+1)/2}\mathrm{F}\left(i\zeta+\frac{1}{2}, i\zeta+\frac{n-1}{2}, \frac{n}{2}, s^2\right)ds\\ &=\left(\frac{2}{k}\right)^n \frac{\pi^{n/2}}{\Gamma(n/2+1)}r^n(1-r^2)^{i\zeta-(n-1)/2}\mathrm{F}\left(i\zeta+\frac{1}{2}, i\zeta+\frac{n-1}{2}, \frac{n}{2}, r^2\right)\end{aligned}$$

$$=\left(\frac{2}{k}\right)^n \frac{\pi^{n/2}}{\Gamma(n/2+1)} r^n(1-r^2)^{-n/2}\mathrm{F}\left(i\zeta+\frac{1}{2},\ -i\zeta+\frac{1}{2}, 1+\frac{n}{2}, r^2(r^2-1)^{-1}\right)$$

$$=\frac{(2\pi)^{n/2}}{k^n}\ (\sinh kR)^{n/2} P^{-n/2}_{-i\zeta-1/2}\ (\cosh kR).$$

On the other hand, we have the following asymptotics (cf. [7]).

$$\mathrm{F}\left(i\zeta+\frac{1}{2}, -i\zeta+\frac{1}{2}, 1+\frac{n}{2}, \frac{1-\cosh kR}{2}\right)$$

$$\sim\frac{(2\pi)^{1/2}\Gamma(1+n/2)(i\zeta)^{(n+1)/2}(1-e^{-kR})^{-(n+1)/2}(1+e^{-kR})^{(n-1)/2}}{\Gamma(i\zeta+1/2)\Gamma(-i\zeta+1/2)\Gamma(i\zeta+(n+1)/2)\Gamma(-i\zeta+(n+1)/2)}$$

$$\times\{\exp(-(i\zeta+1/2)kR+\exp(\pm(n+1)/2)+(-i\zeta+1/2)kR)\}(1+O(|\zeta|^{-1})).$$

Here the upper or lower sign is according as $-\pi+\delta<\arg\zeta<-\delta$ or $\delta<\arg\zeta<\pi-\delta$, with $\delta>0$.

Thus the zeros of $\tilde{\chi}_\Omega(\zeta, bM)$ have the following asymptotic behaviour:

$$\zeta=\frac{4m+(n-1)}{4kR}\pi+O(m^{-1}) \qquad (m\in \boldsymbol{Z},\ m\gg 0).$$

$$\zeta=\frac{4m-(n-1)}{4kR}\pi+O(m^{-1}) \qquad (m\in \boldsymbol{Z},\ m\ll 0).$$

Of course this coincides with what is obtained by applying Theorem (3.6.4). In fact, one sees that $H_\Omega(bM)\equiv 2R$ and $d_\Omega(bM)\equiv -\frac{k(n-1)}{2}$ from Example (3.3.21), which implies $f_m(bM)=\frac{4m+(n-1)}{4kR}\pi$ modulo $\boldsymbol{Z}/kR$, (Definition (3.6.1)).

§ 7. Special case

Throughout this section, $X=G/K=SO_0(2,1)/SO(2)$, which has constant sectional curvature $-k^2$ $(k>0)$.

We identify $K\simeq SO(2)$ with $S^1=\boldsymbol{R}/2\pi\boldsymbol{Z}$ by,

(3.7.1) $$S^1\ni\theta\longmapsto b(\theta):=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\in K.$$

Now that $M\equiv Z_K(\mathfrak{a})=\{1\}$, we also identify $G/P\simeq K/M$ with S^1 via

(3.7.1). In this section we use a global coordinate (x, y) instead of usual (y_1, y_2) in a half space model X_3 (notation in § 3). The S^1 ($\subset G$) action on X_3 is given by,

$$
\begin{aligned}
b(\theta)\cdot(x, y) &= \Phi_{31}(b(\theta)\cdot\Phi_{13}(x, y)) \\
&= \Phi_{31}\left(\frac{1}{2y}\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2x\\ x^2+y^2-1\\ x^2+y^2+1\end{pmatrix}\right) \\
&= \Phi_{31}\left(\frac{1}{2y}\begin{pmatrix}2x\cos\theta-(x^2+y^2-1)\sin\theta\\ 2x\sin\theta+(x^2+y^2-1)\cos\theta\\ x^2+y^2+1\end{pmatrix}\right) \\
&= \frac{2y}{(x^2+y^2)(1-\cos\theta)+1+\cos\theta-2x\sin\theta} \\
&\quad\times\begin{pmatrix}2x\cos\theta-(x^2+y^2-1)\sin\theta\\ 1\end{pmatrix}.
\end{aligned}
\tag{3.7.2}
$$

Fix $\varepsilon\in\{\pm 1\}$. Let a strictly convex domain Ω be realized in a half space model X_3, and put

$$p=\psi_\varepsilon^{-1}(e)\in\partial\Omega,$$

where $\psi_\varepsilon:\partial\Omega\longrightarrow G/P$ is a Gauss map defined in (3.2.11). Then $\partial\Omega$ is locally represented as a graph:

$$y=f(x),$$

where $f(x)$ is defined in a neighbourhood U ($\subset \boldsymbol{R}$) containing $x(p)\in\boldsymbol{R}$. Since $p\in\partial\Omega$ is a critical point of the second coordinate function $y_{|\partial\Omega}$, we have the following equations from Lemma (3.3.10) and from our definition of K_ε,

$$K_\varepsilon(p)=-\varepsilon k y(p)\frac{d^2f}{dx^2}(x(p)), \tag{3.7.3}$$

as well as

$$\frac{df}{dx}(x(p))=0. \tag{3.7.4}$$

Set a new function $A: U\times\boldsymbol{R}\longrightarrow\boldsymbol{R}$ by,

$$A(x,\theta)=(x^2+f(x)^2)(1-\cos\theta)+1+\cos\theta-2x\sin\theta. \tag{3.7.5}$$

Then simple calculation yields:

$$
\left\{
\begin{array}{ll}
A(x, 0)=2, & \dfrac{\partial^2 A}{\partial X^2}(x, 0)=0, \\[2ex]
\dfrac{\partial A}{\partial x}(x, 0)=0, & \dfrac{\partial^2 A}{\partial x \partial \theta}(x, 0)=2, \\[2ex]
\dfrac{\partial A}{\partial \theta}(x, 0)=2x, & \dfrac{\partial^2 A}{\partial \theta^2}(x, 0)=x^2+f(x)^2-1.
\end{array}
\right. \tag{3.7.6}
$$

From (3.7.2) and (3.7.5), the function $y_{|b(\theta)^{-1}\partial\Omega}$ equals

$$
=\frac{2f(x)}{A(x, \theta)}. \tag{3.7.7}
$$

Let us find the critical point of the function $y_{|b(\theta)^{-1}\partial\Omega}$. Put $\dfrac{\partial}{\partial x} \dfrac{2f(x)}{A(x, \theta)}=0$. Then we have

$$
\frac{df}{dx}(x)A(x, \theta)-f(x)\frac{\partial A}{\partial x}(x, \theta)=0. \tag{3.7.8}
$$

From the implicit function theorem, (3.7.8) is locally solved by a smooth function $x=\varphi(\theta)$: that is,

$$
\varphi(0)=x(p),
$$

and

$$
\frac{df}{dx}(\varphi(\theta))A(\varphi(\theta), \theta)-f(\varphi(\theta))\frac{\partial A}{\partial x}(\varphi(\theta), \theta)\equiv 0. \tag{3.7.9}
$$

Put

$$
F:=f(\varphi(0))=\exp kh_{\Omega,\varepsilon}(0) \in \boldsymbol{R}, \tag{3.7.10}
$$

$$
G:=\frac{d^2 f}{dx^2}(\varphi(0))=-\frac{K_\varepsilon(p)}{\varepsilon kF} \in \boldsymbol{R}, \tag{3.7.11}
$$

and

$$
X:=\varphi(0) \in \boldsymbol{R}. \tag{3.7.12}
$$

Differentiating the identity (3.7.9) with respect to θ, we have,

$$
\frac{d^2 f}{dx^2}(\varphi(\theta))\frac{d\varphi}{d\theta}(\theta)A(\varphi(\theta), \theta)+\frac{df}{dx}(\varphi(\theta))\frac{\partial A}{\partial \theta}(\varphi(\theta), \theta)
$$

$$-f(\varphi(\theta))\frac{d\varphi}{d\theta}(\theta)\frac{\partial^2 A}{\partial x^2}(\varphi(\theta),\theta)-f(\varphi(\theta))\frac{\partial^2 A}{\partial x\partial\theta}(\varphi(\theta),\theta)=0.$$

By substituting (3.7.4) and (3.7.6) into the above equation,

$$2G\frac{d\varphi}{d\theta}(0)-2F=0.$$

Hence,

$$\frac{d\varphi}{d\theta}(0)=FG^{-1}. \tag{3.7.13}$$

Set $a(\theta):=A(\varphi(\theta),\theta)$, and $c(\theta):=f(\varphi(\theta))$. Then by simple calculations, we get

$$\begin{cases} a(0)=2, \\ \dfrac{da}{d\theta}(0)=2X, \\ \dfrac{d^2a}{d\theta^2}(0)=4FG^{-1}+X^2+F^2-1, \end{cases} \tag{3.7.14}$$

and

$$\begin{cases} c(0)=F, \\ \dfrac{dc}{d\theta}(0)=0, \\ \dfrac{d^2c}{d\theta^2}(0)=F^2G^{-1}. \end{cases} \tag{3.7.15}$$

Now, using the K-equivariance of the supporting function (Lemma (3.2.19)), we have,

$$\exp kh_{\Omega,\varepsilon}(\theta)\equiv\exp kh_{\Omega,\varepsilon}(b(\theta))=\exp kh_{b(\theta)^{-1}\Omega,\varepsilon}(0).$$

From Lemma (3.4.2) and (3.7.7),

$$\exp kh_{b(\theta)^{-1}\Omega,\varepsilon}(0)=\frac{2f(\varphi(\theta))}{A(\varphi(\theta),\theta)}=\frac{2c(\theta)}{a(\theta)}.$$

Hence,

$$\exp kh_{\Omega,\varepsilon}(\theta)=\frac{2c(\theta)}{a(\theta)}. \tag{3.7.16}$$

Differentiating (3.7.16) with respect to θ, and then evaluating at $\theta=0$, we have,

$$\begin{aligned}\frac{d}{d\theta}\Big|_{\theta=0}\exp kh_{\Omega,\varepsilon}(\theta)&=2a^{-2}(\theta)\Big(\frac{dc(\theta)}{d\theta}a(\theta)-c(\theta)\frac{da(\theta)}{d\theta}\Big)\Big|_{\theta=0}\\&=2^{-1}(-2FX)=-FX.\end{aligned} \tag{3.7.17}$$

Since the left hand side is equal to

$$k\frac{dh_{\Omega,\varepsilon}}{d\theta}(0)\exp kh_{\Omega,\varepsilon}(0)=kF\frac{dh_{\Omega,\varepsilon}}{d\theta}(0),$$

we have obtained

$$\frac{dh_{\Omega,\varepsilon}}{d\theta}(0)=-k^{-1}X. \tag{3.7.18}$$

Differentiating (3.7.16) by two times with respect to θ, and then evaluating at $\theta=0$, we have,

$$\begin{aligned}&\frac{d^2}{d\theta^2}\exp kh_{\Omega}(\theta)_{|\theta=0}\\&=2a^{-3}\Big(\frac{d^2c}{d\theta^2}a^2-ca\frac{d^2a}{d\theta^2}-2a\frac{dc}{d\theta}\frac{da}{d\theta}+2c\Big(\frac{da}{d\theta}\Big)^2\Big)\Big|_{\theta=0}\\&=4^{-1}(4F^2G^{-1}-2F(4FG^{-1}+X^2+F^2-1)+8FX^2)\\&=2^{-1}(-2F^2G^{-1}+3FX^2-F^3+F).\end{aligned} \tag{3.7.19}$$

Since the left hand side equals

$$\begin{aligned}&\Big(k\frac{d^2h_{\Omega,\varepsilon}}{d\theta^2}(0)+k^2\Big(\frac{dh_{\Omega,\varepsilon}}{d\theta}(0)\Big)^2\Big)\exp kh_{\Omega,\varepsilon}(0)\\&=\Big(k\frac{d^2h_{\Omega,\varepsilon}}{d\theta^2}(0)+k^2\Big(\frac{dh_{\Omega,\varepsilon}}{d\theta}(0)\Big)^2\Big)F,\end{aligned}$$

the equation (3.7.19) is equivalent to

$$2k\frac{d^2h_{\Omega,\varepsilon}}{d\theta^2}(0)+2k^2\Big(\frac{dh_{\Omega,\varepsilon}}{d\theta}(0)\Big)^2=-2FG^{-1}+3X^2-F^2+1. \tag{3.7.20}$$

Substituting (3.7.18) into the right hand side of (3.7.20),

$$=-2FG^{-1}+3k^2\left(\frac{dh_\Omega}{d\theta}(0)\right)^2-F^2+1.$$

Therefore,

$$2k\frac{d^2h_{\Omega,\varepsilon}}{d\theta^2}(0)-k^2\left(\frac{dh_{\Omega,\varepsilon}}{d\theta}(0)\right)^2+F^2-1=-2FG^{-1}.$$

From (3.7.3),

$$G^{-1}=\left(\frac{d^2f}{dx^2}(x(p))\right)^{-1}=-\varepsilon kFK_{\Omega,\varepsilon}(p)^{-1}.$$

Hence we have obtained the following formula.

$$\frac{1}{K_{\Omega,\varepsilon}\circ\phi_{\Omega,\varepsilon}^{-1}(e)}=\frac{2k\frac{d^2h_{\Omega,\varepsilon}}{d\theta^2}(0)-k^2\left(\frac{dh_{\Omega,\varepsilon}}{d\theta}(0)\right)^2+\exp 2kh_{\Omega,\varepsilon}(0)-1}{2\varepsilon k\exp 2kh_{\Omega,\varepsilon}(0)}.$$

Noting that for any $b\in K$,

$$K_{b^{-1}\Omega,\varepsilon}\circ\phi_{b^{-1}\Omega,\varepsilon}^{-1}(P)=K_{\Omega,\varepsilon}\circ\phi_{\Omega,\varepsilon}^{-1}(bP),$$

and

$$h_{b^{-1}\Omega,\varepsilon}(P)=h_{\Omega,\varepsilon}(bP),\qquad\qquad \text{(Lemma (3.2.19))},$$

we have proved the next proposition.

PROPOSITION (3.7.21) (cf. (2.5.2)). *Let Ω be a strictly H-convex domain in $X=SO_0(3,1)/SO(2)$. Identifying $G/P\simeq K/M$ with $\boldsymbol{R}/2\pi\boldsymbol{Z}$ by* (3.7.1), *we regard the supporting function $h_{\Omega,\varepsilon}$ as a map from $\boldsymbol{R}/2\pi\boldsymbol{Z}$ to $\boldsymbol{R}$ and the generalized Gauss-Kronecker curvature $K_{\Omega,\varepsilon}\circ\phi_{\Omega,\varepsilon}^{-1}$ as a map from $\boldsymbol{R}/2\pi\boldsymbol{Z}$ to $\boldsymbol{R}$ ($\varepsilon=\pm1$). Then $h_{\Omega,\varepsilon}$ and $K_{\Omega,\varepsilon}\circ\phi_{\Omega,\varepsilon}^{-1}$ are related by the following differential equations (cf.* (2.5.2)):

$$\text{(3.7.22)}\quad \frac{1}{K_{\Omega,\varepsilon}\circ\phi_{\Omega,\varepsilon}^{-1}(\theta)}=\frac{2k\frac{d^2h_{\Omega,\varepsilon}}{d\theta^2}(\theta)-k^2\left(\frac{dh_{\Omega,\varepsilon}}{d\theta}(\theta)\right)^2+\exp 2kh_{\Omega,\varepsilon}(\theta)-1}{2\varepsilon k\exp 2kh_{\Omega,\varepsilon}(\theta)}$$

for $\theta\in\boldsymbol{R}/2\pi\boldsymbol{Z}$ ($\varepsilon=\pm1$).

In Euclidean space, a strictly convex domain Ω is recovered by its supporting function h in two ways (Lemma (2.1.11)). That is,

$$\Omega=\bigcap_{\omega\in S^{n-1}}\{x\in \boldsymbol{R}^n;\ (x,\omega)<h(\omega)\},$$

or

$$\nu_\Omega^{-1}(\omega)=\left(\frac{\partial\tilde{h}}{\partial x^i}(\omega)\right)_{i=1,\cdots,n}\qquad(\omega\in S^{n-1}).$$

Here, $\nu_\Omega:\partial\Omega\longrightarrow S^{n-1}$ is the Gauss map and $\tilde{h}:\boldsymbol{R}^n\longrightarrow\boldsymbol{R}$ be a linear extension of h.

In $X=SO_0(n,1)/SO(n)$, the analogy of the first formula is obtained in Corollary (3.4.12). Now we give the analogy of the second formula:

PROPOSITION (3.7.23). *Retain notations in Proposition* (3.7.21). *Then Ω is recovered by one of its supporting functions $h_{\Omega,\varepsilon}$ as follows*:

$$\text{(3.7.24)}\qquad \phi_{\Omega,\varepsilon}^{-1}(\theta)=\frac{1}{k^2(h'_{\Omega,\varepsilon}(\theta))^2+(1+\exp(kh_{\Omega,\varepsilon}(\theta)))^2}\times\begin{pmatrix}-2kh'_{\Omega,\varepsilon}(\theta)\cos\theta-(k^2(h'_{\Omega,\varepsilon}(\theta))^2+\exp(2kh_{\Omega,\varepsilon}(\theta))-1)\sin\theta\\ -2kh'_{\Omega,\varepsilon}(\theta)\sin\theta+(k^2(h'_{\Omega,\varepsilon}(\theta))^2+\exp(2kh_{\Omega,\varepsilon}(\theta))-1)\cos\theta\end{pmatrix}\in X_2.$$

Note that $\theta=0$ corresponds to $v_2=(0,1)\in\partial X_2$ $(\subset\boldsymbol{R}^2)$ in a unit disk model. If you prefer the usual angle in $\partial X_2\simeq S^1$, you should replace θ by $\varphi=\theta+\dfrac{\pi}{2}$.

PROOF. From (3.7.10), (3.7.12) and (3.7.18), we have

$$\phi_{\Omega,\varepsilon}^{-1}(0)=\begin{pmatrix}X\\ F\end{pmatrix}=\begin{pmatrix}-kh'_{\Omega,\varepsilon}(0)\\ \exp(kh_{\Omega,\varepsilon}(0))\end{pmatrix}\in X_3.$$

$$=\frac{1}{k^2(h'_{\Omega,\varepsilon}(0))^2+(1+\exp(kh_{\Omega,\varepsilon}(0)))^2}\begin{pmatrix}-2kh'_{\Omega,\varepsilon}(0)\\ k^2(h'_{\Omega,\varepsilon}(0))^2+\exp(2kh_{\Omega,\varepsilon}(0))-1\end{pmatrix}\in X_2.$$

Now, this yields (3.7.24) in view of Lemma (3.2.19). Q.E.D.

Suppose $\mathcal{N}(\Omega)$ be given. Then applying Theorem (3.6.4), we can read two functions $A(\theta)\equiv A_{\mathcal{N}(\Omega)}(\theta)$ and $B(\theta)\equiv B_{\mathcal{N}(\Omega)}(\theta)\in C^\infty(S^1)$ from the asymptotic data of $\mathcal{N}(\Omega)$, such that

$$\text{(3.7.25)}\qquad \frac{K_{\Omega,1}\circ\phi_{\Omega,1}^{-1}(\theta)}{K_{\Omega,-1}\circ\phi_{\Omega,-1}^{-1}(\theta)}=A_{\mathcal{N}(\Omega)}(\theta),$$

and

$$H_\Omega(\theta)=B_{\mathcal{N}(\Omega)}(\theta). \tag{3.7.26}$$

Using the differential equations (3.7.22), we can obtain a single differential equation of $h_{\Omega,-1}(\theta)$. We will illustrate how this differential equation determines Ω in the proof of Proposition (3.7.30). Before doing this, we give an example:

Example (3.7.27). Let $\Omega=B(R)$ (a ball with radius R with its center the origin) and set $r:=\dfrac{\exp(kR)+1}{\exp(kR)-1}$. Then as we saw in Example (3.3.21) and Example (3.6.7), $h_{\Omega,\varepsilon}(\theta)\equiv\varepsilon R$, $H_\Omega(\theta)\equiv 2R$, $K_{\Omega,\varepsilon}(\theta)\equiv\dfrac{k(1+\varepsilon r)^2}{2r}$, and $d_\Omega(\theta)\equiv -\dfrac{k}{2}$. Hence,

$$A(\theta)=K_1/K_{-1}=\left(\frac{1+r}{1-r}\right)^2=\exp kB(\theta), \tag{3.7.28}$$

and

$$B(\theta)=2R. \tag{3.7.29}$$

Finally as an illustration of the use of the differential equation of the supporting functions (Definition (3.2.16)) for the injectivity problem of $\mathcal{N}$ (Problem (B.4)), we prove the following proposition.

PROPOSITION (3.7.30) (Berenstein-Yang [2]). *Let Ω be a strictly H-convex domain in $X=SO_0(2,1)/SO(2)$. If $\mathcal{N}(\Omega)=\mathcal{N}(B(R))$, then $\Omega=g\cdot B(R)$ with some $g\in SO_0(2,1)$.*

PROOF. Put $h(\theta):=h_{\Omega,-1}(\theta)$. From (3.7.26), we have $h_{\Omega,1}(\theta)=h_{\Omega,-1}(\theta)+H_\Omega(\theta)=h(\theta)+B_{\mathcal{N}(\Omega)}(\theta)$. The assumption $\mathcal{N}(\Omega)=\mathcal{N}(B(R))$ and Example (3.7.27) imply

$$A_{\mathcal{N}(\Omega)}(\theta)\equiv A_{\mathcal{N}(B(R))}(\theta)=\exp(2kR), \tag{3.7.31}$$

and

$$B_{\mathcal{N}(\Omega)}(\theta)\equiv B_{\mathcal{N}(B(R))}(\theta)=2R. \tag{3.7.32}$$

Therefore by Lemma (3.7.21), we have the following differential equation for $h(\theta)$:

$$\exp(-2kR)=A^{-1}_{\pi(\Omega)}(\theta)=\left(\frac{K_{\Omega,1}\circ\phi^{-1}_{\Omega,1}(\theta)}{K_{\Omega,-1}\circ\phi^{-1}_{\Omega,-1}(\theta)}\right)^{-1}$$
$$=-\frac{2k(h+B)''-k^2((h+B)')^2-1+\exp 2k(h+B)}{2kh''-k^2(h')^2-1+\exp(2kh)}\times\exp 2k(h_{-1}-h_1).$$

From (3.7.19) and (3.7.25),

$$=-\left(1+\frac{\exp(2kh)(\exp(+4kR)-1)}{2kh''-k^2(h')^2-1+\exp(2kh)}\right)\times\exp(-4kR).$$

Hence,

$$(\exp(2kR)+1)(2kh''-k^2(h')^2-1+\exp(2kh))=-\exp(2kR)(\exp(4kR)-1)$$

Therefore $h(\theta)$ satisfies

(3.7.33) $$2kh''-k^2(h')^2-1+\exp(2k(h+R)).$$

Set

(3.7.34) $$f(\theta):=\exp(-k(h(\theta)+R)/2)\in C^\infty(S^1).$$

Then $f(\theta)$ satisfies the following differential equation:

(3.7.35) $$f''(\theta)=-\frac{1}{4}(f(\theta)-f^{-3}(\theta)).$$

This nonlinear differential equation is solved by,

(3.7.36) $$f(\theta)=(C-\sqrt{C^2-1}\cos(\theta+\theta_0))^{1/2},$$

with constants $C\geq 1$ and $\theta_0\in[0,2\pi)$. (From (3.7.35), we have $4(f')^2+f^2+f^{-2}=2C$ with some constant $C\geq 1$. Then $\frac{d\theta}{df}$ is represented by algebraic functions of f, whose indefinite integral is given by elementary functions and (3.7.36) follows.)

From (3.7.34), we get

$$h_{\Omega,-1}(\theta)=h(\theta)=-R-\frac{1}{k}\log(C-\sqrt{C^2-1}\cos(\theta+\theta_0)).$$

Now Ω is uniquely determined by $h_{\Omega,-1}(\theta)$ by using Proposition (3.7.23). It is shown by simple calculations that the constants C and θ_0 correspond to the translation of $B(R)$ by an element of A and K respectively. Thus the proposition is proved. Q.E.D.

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Note added in proof. Prof. Berenstein kindly informed me of [22] generalizing [1], [2] (cf. Prop. (1.11), Cor. (2.3.11)(2) and Prop. (3.7.30)). [22] Berenstein, C. A. and P. C. Yang, An inverse Neumann problem, J. Reine Angew. Math. **382** (1987), 1–21.

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