Bounded domains and the zero sets of Fourier transforms

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Contents

1. Introduction
2. Description of $\mathcal{N}(\Omega)$ in terms of Ω
3. Injectivity of $\Omega \mapsto \mathcal{N}(\Omega)$
4. Characterization of convexity of Ω by means of $\mathcal{N}(\Omega)$
5. Perturbation of Ω and $\mathcal{N}(\Omega)$
6. Asymptotic behavior of the zeros of a certain class of entire functions

1 Introduction

The study of (an asymptotic behavior of) the Fourier transform $\mathcal{F}\chi_\Omega(\zeta)$ of a characteristic function χ_Ω for a (convex) domain Ω is very old and has played an important role in various contexts:

F.John	(1934) homogeneous integral equation.
C.S.Herz	(1962) spectral theory of bounded functions.
È.B.Vinberg	(1963) complex homogeneous domains.
C.A.Berenstein	(1976) the Pompeiu problem.

The purpose of this note is to give an exposition of the study of the relations between the geometry of a given domain Ω and the zero set $\mathcal{N}(\Omega)$ of the Fourier transform $\mathcal{F}\chi_\Omega$. In a special case, these are closely related to the Pompeiu problem which has originated in integral geometry (*[22]*, *[23]*) or a free boundary problem of the Laplace operator called Schiffer's conjecture (*[28]*, Problem 80). We treat in a more general setting the assignment (1.4) from a bounded domain Ω in $\mathbb{R}^n$ to a complex analytic set $\mathcal{N}(\Omega)$ in $\mathbb{C}^n$, which is defined in (i) below. A detailed account is to appear in *[19]*.

Suppose Ω is a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is C^1 diffeomorphic to S^{n-1}. We associate the following three objects to Ω:

Proceeding of the Conference
75 YEARS OF RADON TRANSFORM
 pages 223– 239

i) The null variety $\mathcal{N}(\Omega) := \{\zeta \in \mathbb{C}^n : \mathcal{F}\chi_\Omega(\zeta) = 0\}$ $(\subset \mathbb{C}^n)$. Here

$$\mathcal{F}\chi_\Omega(\zeta) := \int_\Omega e^{\sqrt{-1}(x_1\zeta_1+\ldots+x_n\zeta_n)}\,dx_1\ldots dx_n$$

is the Fourier transform of the characteristic function χ_Ω, which is a holomorphic function of the n variables $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$.

ii) An integral transform $T_\Omega : C(\mathbb{R}^n) \longrightarrow C(M(n))$ defined by $(T_\Omega f)(g) = \int_\Omega f(gx)dx$. Here $M(n) = O(n) \ltimes \mathbb{R}^n$ is the Euclidean motion group.

iii) An overdetermined problem:

$$(N)_\lambda \qquad \begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, \; u \equiv \text{ constant} & \text{on } \partial\Omega. \end{cases}$$

Here $\frac{\partial}{\partial \nu}$ stands for the outward normal vector field on $\partial\Omega$.

In a special case, it is a well known result based on an argument of spectral synthesis of L.Schwartz that these three objects are related with one another:

Fact 1.1: *([7], [26]) In the above setting, the following three conditions on Ω are equivalent:*

(*a*) *There exists $r > 0$ such that $\mathcal{N}(\Omega) \supset S_{\mathbb{C}}(0 : r)$.*

(*b*) *Ker $T_\Omega \neq \{0\}$.*

(*c*) *There exist $\lambda > 0$ and a nontrivial solution u of $(N)_\lambda$.*

Here, we define a complex quadric by

$$S_{\mathbb{C}}(a : r) := \{\zeta \in \mathbb{C}^n : \sum_{j=1}^{n}(\zeta_j - a_j)^2 = r^2\}, \tag{1.2}$$

for $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $r \in \mathbb{R}$. In (a) and (c), we have a relation $\lambda = r^2$.

A ball in $\mathbb{R}^n$ satisfies the three equivalent conditions in Fact (1.1). In fact, denote by $J_\nu(z)$ the ν-th Bessel function which is a solution to $\left((z\frac{d}{dz})^2 + z^2 - \nu^2\right)u = 0$. We fix a positive zero r of $J_{\frac{n}{2}}(r)$ (there exist countably many positive zeros). We define a holomorphic function of $z \in \mathbb{C}$ by $f_\nu(z) := (2\pi)^{\frac{n}{2}}\frac{J_\nu(z)}{z^\nu}$. If Ω is the unit ball in $\mathbb{R}^n$, then we have a formula

$$\mathcal{F}\chi_\Omega(\zeta) = f_{\frac{n}{2}}\left(\sqrt{{\zeta_1}^2 + \ldots + {\zeta_n}^2}\right), \qquad \text{for } \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n. \tag{1.3}$$

Then it is not hard to check (cf. [7]):

$$\mathcal{N}(\Omega) \supset S_{\mathbb{C}}(0 : r), \tag{1.3.1}$$

$$\text{Ker } T_\Omega \ni f_{\frac{n}{2}-1}\left(r\sqrt{x_1{}^2+\ldots+x_n{}^2}\right), \tag{1.3.2}$$

$$f_{\frac{n}{2}-1}\left(r\sqrt{x_1{}^2+\ldots+x_n{}^2}\right) \text{ is a solution to } (N)_{r^2}. \tag{1.3.3}$$

Conversely, it has been a long standing conjecture (the Pompeiu problem, Schiffer's conjecture) that a ball is conjecturally the only domain satisfying one of (therefore, any of) (a) - (c) in Fact (1.1). On the other hand, each of them has its own interesting generalizations and developments, which are not necessarily related to other problems. As for (i), there have been a lot of extensive research on the symmetric property of solutions to a partial differential equation with some symmetry (e.g. *[24]*,*[2]*). As for (ii), the integral transform T_Ω is defined for arbitrary homogeneous space G/H as a G-intertwining operator $T_\Omega \colon C(G/H) \to C(G)$, $f \mapsto \int_\Omega f(gx)\,dx$ where Ω is a fixed, relatively compact subset of G/H (we regard $\mathbb{R}^n \simeq M(n)/O(n)$ in (ii)). The study of the image or the kernel of T_Ω is closely related to non-commutative harmonic analysis on a homogeneous space G/H (e.g. *[8]*, see also *[19]*, Theorem 1.2.17 for a collection of various results in this direction). A survey of another interesting direction of research on T_Ω can be found in *[29]* whose concern is mainly with minimal determining subsets such as a generalization of two-circles theorem of Delsarte. On the other hand, in this paper, we concentrate on the object (iii), that is, we investigate the assignment

$$\begin{array}{cccc} \mathcal{N}\colon \{\text{Bounded measurable sets in } \mathbb{R}^n\} & \longrightarrow & \{\text{Analytic sets in } \mathbb{C}^n\}. \\ \Cup & & \Cup \\ \Omega & \longrightarrow & \mathcal{N}(\Omega) \end{array} \tag{1.4}$$

Many of the basic questions concerning the assignment (1.4) have not found a final answer. In this note we give an exposition of some partial results of the following naive questions:

Question 1.5

1) *Describe $\mathcal{N}(\Omega)$ in terms of geometric quantities of Ω.*
2) *Study the injectivity of the assignment $\Omega \mapsto \mathcal{N}(\Omega)$. That is, does the null variety $\mathcal{N}(\Omega)$ determine the original domain Ω ?*
3) *How a perturbation of Ω affects the null variety $\mathcal{N}(\Omega)$ when Ω satisfies the properties in Fact (1.1) ?*

For visualization of $\mathcal{N}(\Omega)$ in the case $n=2$, we define the real points of $\mathcal{N}(\Omega)$ by

$$\mathcal{N}(\Omega)_{\mathbb{R}} := \mathcal{N}(\Omega) \cap \mathbb{R}^n.$$

Here are examples of the null variety $\mathcal{N}(\Omega)_{\mathbb{R}} \subset \mathbb{R}^2$ for typical bounded domains $\Omega \subset \mathbb{R}^2$ (later, we shall look at $\mathcal{N}(\Omega) \cap S$ (see (2.1.1)), however, if Ω is centrally symmetric then $\mathcal{N}(\Omega)_{\mathbb{R}}$ plays the same role as $\mathcal{N}(\Omega) \cap S$):

(1.6)(*a*)	(1.6)(*b*)	(1.6)(*c*)
Ω: unit disk in $\mathbb{R}^2$	Ω: square in $\mathbb{R}^2$	Ω: regular hexagon in $\mathbb{R}^2$

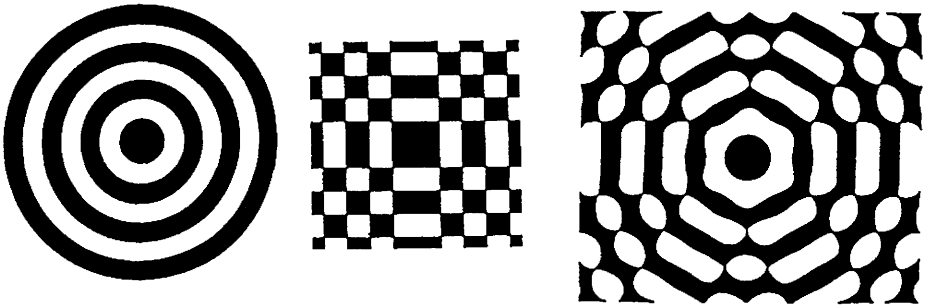

Figure 1.6

In the figures above, the black parts mean $\{(\zeta_1, \zeta_2) \in \mathbb{R}^2 \colon \mathcal{F}\chi_\Omega(\zeta_1, \zeta_2) > 0\}$ and the white ones mean $\{(\zeta_1, \zeta_2) \in \mathbb{R}^2 \colon \mathcal{F}\chi_\Omega(\zeta_1, \zeta_2) < 0\}$. The null variety $\mathcal{N}(\Omega)_{\mathbb{R}}$ is the boundary of the black parts and white ones. We remark that the 'first' zero point set of $\mathcal{F}\chi_\Omega(\zeta_1, \zeta_2)$ in (1.6)(c) looks like a circle, but is not actually a circle thanks to *[7]*, Theorem 5.7.

Observation 1.7 *Let us give some very elementary observations of the Figure (1.6), which lead us to suitable formulations for Question (1.5) on the study of the null variety* $\mathcal{N}(\Omega)$.

a) All of the domains Ω *in (1.6) are centrally symmetric.*

b) All of the domains Ω *in (1.6) are convex and only the domain in (1.6)(a) (a ball) is strictly convex.*

c) The null variety $\mathcal{N}(\Omega)_{\mathbb{R}}$ *for a ball (1.6)(a) consists of infinitely many connected components, any of which is compact.*

d) The null variety $\mathcal{N}(\Omega)_{\mathbb{R}}$ *for a cubic domain (1.6)(b) is noncompact and connected.*

Another interesting observation due to B.Ørsted is that it looks much easier to distinguish the shapes of three null varieties $\mathcal{N}(\Omega)_{\mathbb{R}}$ in Figure (1.6) than to distinguish those of the original domains Ω. From the viewpoint of computer science (as another aspect of Question (1.5)(2)), we might expect a new method of 'recognition of shape' ('shape' could involve some quantities in differential geometry) in some family of domains by using the null varieties $\mathcal{N}(\Omega)$.

2 Description of $\mathcal{N}(\Omega)$ in terms of Ω

In this section, as a simplest case, we shall generalize the feature (1.7)(c) about the real points $\mathcal{N}(\Omega)_{\mathbb{R}}$ for a ball to general strictly convex domains Ω in $\mathbb{R}^n$. The results here can be generalized to horospherically convex domains in a hyperbolic space $SO_0(n,1)/SO(n)$ by using Radon-Fourier transforms for Riemannian symmetric spaces introduced by Helgason (ref. *[14]*). We define an $n+1$ dimensional manifold

$$S: = S^{n-1} \underset{\mathbb{Z}_2}{\to} \times \mathbb{C}^{\times} = \{\zeta \cdot \omega : \zeta \in \mathbb{C}^{\times}, \omega \in S^{n-1}\} \subset \mathbb{C}^n. \tag{2.1.1}$$

Then S contains $\mathbb{R}^{n-1} \setminus \{0\}$ as a hypersurface and so $\mathcal{N}(\Omega)_{\mathbb{R}} \subset \mathcal{N}(\Omega) \cap S \subset \mathcal{N}(\Omega)$.

Suppose Ω is a convex domain in $\mathbb{R}^n$. We equip $\mathbb{R}^n$ with the standard inner product $(\ ,\ )$ and denote the unit sphere by S^{n-1}. The supporting function and the breadth function of Ω are given by

$$h \equiv h_\Omega : S^{n-1} \longrightarrow \mathbb{R}, \quad \omega \longmapsto \sup_{x\in\Omega}(x,\omega), \tag{2.1.2}$$

$$H \equiv H_\Omega : S^{n-1} \longrightarrow \mathbb{R}_+, \quad \omega \longmapsto h(\omega) + h(-\omega). \tag{2.1.3}$$

We define the Gauss map $\nu \equiv \nu_\Omega : \partial\Omega \to S^{n-1}$ by its outer normal vector field, and define the Gauss-Kronecker curvature $K \equiv K_\Omega : \partial\Omega \to \mathbb{R}$ by the Jacobian of $d\nu$ with respect to the induced metric from $\mathbb{R}^n$. Here we choose an orientation of $\partial\Omega$ so that ν preserves an orientation. In particular, K is everywhere positive if Ω is a ball. If Ω is strictly convex, we put

$$\varkappa \equiv \varkappa_\Omega : S^{n-1} \to \mathbb{R}_+, \quad \omega \longmapsto K_\Omega \circ \nu_\Omega^{-1}(\omega), \tag{2.1.4}$$

$$d \equiv d_\Omega : S^{n-1} \to \mathbb{R}, \quad \omega \longmapsto \frac{\log \varkappa_\Omega(-\omega) - \log \varkappa_\Omega(\omega)}{2H_\Omega(\omega)}. \tag{2.1.5}$$

Theorem 2.2: *([18]). Let Ω be a strictly convex domain in $\mathbb{R}^n$. Retain notation as above. Then there exists an integer $m_0 \equiv m_0(\Omega)$ depending only on Ω such that we have a disjoint union*

$$\mathcal{N}(\Omega) \cap S = \left(\coprod_{m=m_0}^{\infty} \mathcal{N}_m \right) \coprod \ \textit{(compact set)}. \tag{2.2.1}$$

Here for each integer $m \geq m_0$, $\mathcal{N}_m$ is a regular submanifold in $S (\subset \mathbb{C}^n)$, and is analytically diffeomorphic to S^{n-1}. More precisely, $\mathcal{N}_m$ has the following asymptotic behavior: There is a family of analytic maps $F_m : S^{n-1} \longrightarrow \mathbb{C}$ $(m \in \mathbb{N},\ m \geq m_0)$ such that

$$F_m(\omega) = \frac{2\pi m}{H(\omega)} + \left(\frac{\pi(n-1)}{2H(\omega)} + \sqrt{-1}d(\omega) \right) + O(m^{-1}), \quad \textit{as } m \to \infty. \tag{2.2.2)(a}$$

$$F_m(\omega) = \overline{F_m(-\omega)}. \tag{2.2.2)(b}$$

$$\mathcal{N}_m = \{F_m(\omega) \cdot \omega : \omega \in S^{n-1}\} \ (\subset S \subset \mathbb{C}^n). \tag{2.2.2)(c}$$

In (2.2.2)(a) the estimate of the error terms is uniform with respect to $\omega \in S^{n-1}$.

Conjecture 2.3: *In the setting of Theorem (2.2), we have conjecturally a disjoint union of countably many regular submanifolds:*

$$\mathcal{N}(\Omega) \cap S \simeq \coprod_{m=1}^{\infty} \mathcal{N}_m. \tag{2.3.1}$$

This conjecture involves:
'compact set' in (2.2.1) would be removed, *(2.3.2)(a)*
$m_0(\Omega) = 1$ *(the phase principle).* *(2.3.2)(b)*

As we have seen at the beginning of Introduction, Theorem (2.2) (at least except for a formulation) has been essentially obtained in various contexts of classical works. Here we only give two comments on the proof:

It is a classical geometric point of view that a Fourier transform of n variables can be factorized by the Radon transform and a Fourier transform of one variable. This is the method of F.John *[16]* in his calculation of the asymptotic behavior of $\mathcal{F}\chi_\Omega(\zeta)$ ($\zeta \in \mathbb{R}^n$) where Ω is centrally symmetric and strictly convex domains. In Herz's paper *[15]* (see also *[3]*) he obtained the asymptotic behavior of $\mathcal{F}\chi_\Omega(\zeta)$ ($\zeta \in \mathbb{R}^n$) by using the saddle point method. If we apply the method of F.John to our more general case for $\zeta \in S$ ($\subset \mathbb{C}^n$), then it gives an alternative and simple proof of Theorem 3(ii) in *[15]* with an error estimate, where the 'most difficult part' was to improve error terms (see page 83, line 1 in *[15]*).

The above consideration reduces the problem to the zero set of a holomorphic function of certain type. As a function of z, it is well known that the triangular function $\sin z$, $\frac{1}{\Gamma(a+z)\Gamma(b-z)}$, the Bessel function $J_\lambda(z)$, the associated Legendre function of the first kind $P_z^\mu(x)$ and so on have countably many zeros which are distributed in a regular fashion with bounded imaginary parts. This can be explained by the fact that these functions are essentially the Fourier transform of compactly supported functions φ with two singularities at $x = x_0, x_1$ such that $\varphi(x) \sim c_0(x - x_0)_+^\lambda, c_1(x - x_1)_-^\lambda$. As an appendix, we give a short explanation of it in §6.

Corollary 2.4: *Suppose that* Ω *is strictly convex domain in* $\mathbb{R}^n$. *Then the following conditions are equivalent:*

(*a*) Ω *is centrally symmetric with respect to the center of gravity.*

(*b*) $\mathcal{N}(\Omega)_{\mathbb{R}}$ *contains countably many hypersurfaces in* $\mathbb{R}^n$ *as connected components.*

The non-trivial implication (b) $\Rightarrow$ (a) is followed by Theorem (2.2) and the uniqueness of the Minkowski problem (e.g. *[9]*).

Corollary 2.5: *(see [3], [4], [17], [5], [18]) Suppose that Ω is a strictly convex domain in $\mathbb{R}^n$. Then the following conditions on Ω are equivalent:*

(*a*) *Ω is a ball in $\mathbb{R}^n$.*

(*b*) *$\mathcal{N}(\Omega)_{\mathbb{R}}$ contains countably many hypersurfaces which approximate hyperspheres asymptotically.*

(*c*) *There exist countably many eigenvalues for the overdetermined Neumann problem $(N)_\lambda$.*

We write $S(a : r) := S_{\mathbb{C}}(a : r) \cap \mathbb{R}^n$ (see (1.2) for the definition). In (b), 'asymptotically' means that there exist a sequence of $a(j) \in \mathbb{R}^n$ $(j \in \mathbb{N}_+)$, an increasing sequence $\mathbb{R} \ni r(j) \uparrow \infty$ (as $j \to \infty$), a constant $C > 0$, a constant $0 < \epsilon \leq 1$ and a sequence of hypersurfaces $X_j \subset \mathcal{N}(\Omega)_{\mathbb{R}}$ such that

$$dist(X_j, S(a(j) : r(j))) \leq Cr(j)^{-\epsilon},$$

for any $j \in \mathbb{N}_+$. Here for closed subsets $S, T \subset \mathbb{R}^n$, a distance between S and T (introduced by D.Pompeiu, 1905) is given by $dist(S,T) := \max_{x\in S}\min_{y\in T}|x-y| + \max_{y\in T}\min_{x\in S}|x-y|$.

The first contribution in the direction of Corollary (2.5) is due to *[3]*. We remark that the centers of hyperspheres in (2.5)(b) are not necessarily the origin. If we replace the condition (2.5)(b) by that $\mathcal{N}(\Omega)_{\mathbb{R}}$ contains infinitely many hyperspheres, then we can drop the assumption of convexity of Ω and only assume that $\partial\Omega$ is connected. This is obtained in *[5]*.

3 Injectivity of $\Omega \mapsto \mathcal{N}(\Omega)$

In this section we deal with the injectivity problem of the assignment $\Omega \mapsto \mathcal{N}(\Omega)$ given in (1.4). We begin with some remarks about a formulation of the injectivity:

Remark 3.1:

(1) The injectivity should be interpreted up to parallel displacements. That is, if Ω and Ω' differs only by a parallel displacement then the Fourier transform of χ_Ω differs from that of $\chi_{\Omega'}$ only by the multiplication by a non-zero function and so $\mathcal{N}(\Omega) = \mathcal{N}(\Omega')$.

(2) The injectivity of $\mathcal{N}$ does not hold if we allow Ω to be disconnected. That is, we can find two non-connected domains Ω_1 and Ω_2 in $\mathbb{R}^n$ $(n \geq 1)$ such that $\mathcal{N}(\Omega_1) = \mathcal{N}(\Omega_2)$ (see *[18]* Example (1.3)).

(3) The injectivity of $\Omega \mapsto \mathcal{N}(\Omega)_{\mathbb{R}}$ does not hold if Ω is not necessarily centrally symmetric domain (that is, the real points of $\mathcal{N}(\Omega)$ are too small to determine Ω in general) (see *[18]* Example (1.5)).

However, we have an obvious affirmative example in the case $n = 1$. That is, if Ω is an interval in $\mathbb{R}^1$ with a length A, then $\mathcal{N}(\Omega) = \{2A^{-1}n\pi : n \in \mathbb{Z}, n \neq 0\}$ ($\subset \mathbb{C}^1$). Thus, the period of $\mathcal{N}(\Omega)$ determines the length A of a given interval Ω. For a higher dimension, we have the following affirmative results:

Corollary 3.2: *(see [16]) The correspondence $\Omega \mapsto \mathcal{N}(\Omega)_{\mathbb{R}}$ is injective from {strictly convex and centrally symmetric domains in $\mathbb{R}^n$} to {real analytic varieties} up to parallel translations.*

Corollary 3.3: *The correspondence $\Omega \mapsto \mathcal{N}(\Omega)$ is injective from {strictly convex domains in $\mathbb{R}^2$} to {complex analytic varieties} up to parallel translations.*

The idea of injectivity of Corollary (3.3) is based on the following:

Problem 3.4: Recover a convex domain Ω from the null variety $\mathcal{N}(\Omega)$ in the following procedure by showing the uniqueness of the solution to (3.4.1) (the step (e) $\Rightarrow$ (f)):

(a) Ω : a strictly convex domain in $\mathbb{R}^n$.

$\downarrow$ $\Leftarrow$ Definition in §1

(b) $\mathcal{N}(\Omega)$: the null variety in $\mathbb{C}^n$.

$\downarrow$

(c) An asymptotic behavior of $\mathcal{N}(\Omega) \cap S$.

$\downarrow$ $\Leftarrow$ Theorem (2.2)

(d) $H_\Omega : S^{n-1} \to \mathbb{R}$ and $d_\Omega : S^{n-1} \to \mathbb{R}$.

$\downarrow$ $\Leftarrow$ A curvature formula in terms of h_Ω

(e) The supporting function h_Ω satisfies a single differential equation (3.4.1) on S^{n-1}.

$\downarrow$

(f) The uniqueness of a solution h_Ω satisfying (3.4.1) up to linear functions on S^{n-1}.

Here h_Ω in (e) satisfies the following differential equation of second order:

$$\det(D^2 h_\Omega + h_\Omega)(\omega) = A(\omega)\det(D^2(B - h_\Omega) + B - h_\Omega)(\omega), \qquad (3.4.1)$$

where D^2 denotes the Hessian on the unit sphere S^{n-1} and A, $B \in C^\infty(S^{n-1})$ are determined by $\mathcal{N}(\Omega)$.

In the case $n = 2$, the differential equation (3.4.1) is linear. We have an explicit formula of the inverse of the assignment $\Omega \mapsto \mathcal{N}(\Omega)$, which proves Corollary (3.3).

On the other hand, in the hyperbolic space $SO_0(n,1)/SO(n)$, the simplest case ($n=2$) involves a *non-linear* ordinary differential equation, which we can reduce to the Duffing equation:

$$f'' = -\frac{1}{4}(f - f^{-3}), \tag{3.5}$$

after a change of variables (see *[18]*, §3.7).

Remark 3.6: We are interested not only in the injectivity of $\Omega \mapsto \mathcal{N}(\Omega)$ but also in a characterization of the image $\mathcal{N}(\Omega)$ in a suitable sense. In an asymptotic sense, this problem corresponds to the existence part of the Minkowski problem in our formulation of (3.4) (see (3.4.1)) in a special case where Ω is centrally symmetric and strictly convex. In the case $n=2$, a characterization of the image $\mathcal{N}(\Omega)$ (in an asymptotic sense) is given in terms of a Dirichlet series determined by the null variety $\mathcal{N}(\Omega)$ (see *[18]* Proposition (2.3.20)).

4 Characterization of convexity of Ω by means of $\mathcal{N}(\Omega)$

So far we have treated convex domains. Conversely, in this section, we treat a characterization of convexity in terms of an asymptotic behavior of $\mathcal{N}(\Omega)$ in the case of $n=2$. Recall that $S = S^1 \underset{\mathbb{Z}_2}{\to} \times BbbC^\times = \left\{\zeta \cdot \omega : \zeta \in \mathbb{C}, \omega \in S^1 \subset \mathbb{R}^2\right\} \subset \mathbb{C}^2$ (see (2.1.1)). Then the asymptotic behavior of $\mathcal{N}(\Omega)$ in Theorem (2.2) characterizes the convexity of Ω:

Theorem 4.1: *(see [19]) Suppose that Ω is a bounded multiply-connected domain in $\mathbb{R}^2$ with finitely many analytic boundaries. If $\mathcal{N}(\Omega)$ has the following asymptotic behavior (4.1.1), then Ω is a strictly convex domain (in particular, $\partial\Omega$ is connected). **(4.1.1)** There exist $m_0 \in \mathbb{N}$, continuous functions $H : S^1 \to \mathbb{R}_+$, $d : S^1 \to \mathbb{R}$ and $F_m : S^1 \to \mathbb{C}$ ($\mathbb{N} \ni m \geq m_0$) such that*

$$F_m(\omega) = \frac{2m\pi}{H(\omega)} + \frac{\pi}{2H(\omega)} + \sqrt{-1}\, d(\omega) + O(m^{-1}) \quad \textit{as } m \to \infty, \tag{4.1.1)(a}$$

$$\mathcal{N}(\Omega) \cap S = \left(\coprod_{m \geq m_0} \mathcal{N}_m\right) \coprod \textit{(compact set)} \qquad \textit{(disjoint union)}. \tag{4.1.1)(b}$$

Here we put

$$\mathcal{N}_m := \left\{F_m(\omega) \cdot \omega : \omega \in S^1\right\} \subset \mathbb{C}^2.$$

Moreover, we have $H(\omega) = H_\Omega(\omega)$ (see (2.1.3)) and $d(\omega) = d_\Omega(\omega)$ (see (2.1.5)).

Remark 4.2: In the condition (4.1.1)(a), we can replace $O(m^{-1})$ by $o(1)$.

5 Perturbation of Ω and $\mathcal{N}(\Omega)$

If Ω is a ball in $\mathbb{R}^n$, then every connected component of $\mathcal{N}(\Omega)$ is of the form $S_{\mathbb{C}}(0:\alpha)$ for some $\alpha > 0$ (see (1.3)). If Ω is a convex domain in $\mathbb{R}^2$ and $\mathcal{N}(\Omega)$ contains $S_{\mathbb{C}}(0:\alpha)$ for some $\alpha \in \mathbb{C}$, then Ω is *close* to a ball in the following sense:

Theorem 5.1: *([6]). Suppose Ω is a convex domain in $\mathbb{R}^2$. If $\mathcal{N}(\Omega) \supset S_{\mathbb{C}}(0:\alpha)$ for some $\alpha \in \mathbb{C}$, then a breadth function must satisfy*

$$2 \min_{\omega \in S^1} H_\Omega(\omega) > \max_{\omega \in S^1} H_\Omega(\omega).$$

Loosely speaking, the result (5.1) of Brown and Kahane asserts that a long thin convex domain in $\mathbb{R}^2$ ('far from' being a ball) never satisfies the conditions (1) - (3) in Fact (1.1). Conversely, we shall treat the case where Ω is sufficiently 'close to' a ball in this section.

In order to define the 'closeness' and to give a precise (quantitative) estimate on how perturbations of a ball affect the null variety $\mathcal{N}(\Omega)$, we consider the following deformation of domains: Given $0 < T$ and a continuous map $g : [0,T] \times S^{n-1} \to \mathbb{R}_+$ we define a family of star-shaped domains $\{\Omega(g(t,\cdot))\}_{0 \le t \le T} \equiv \{\Omega_t\}_{0 \le t \le T}$ in $\mathbb{R}^n$ by

$$\Omega(g(t,\cdot)) \equiv \Omega_t := \left\{\rho \cdot \eta \in \mathbb{R}^n : \eta \in S^{n-1}, 0 \le \rho \le g(t,\eta)\right\}.$$

From definition,

$$\Omega_0 \text{ is the unit ball} \quad \Leftrightarrow \quad g(0,\eta) \equiv 1.$$

In view of the fact that parallel translations and similarity transformations of Ω do not affect the properties in Fact (1.1), we introduce a notion of *unessential perturbation* as follows. We call $\{\Omega_t\}$ *unessential* if there exist $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$ such that $g_t(0,\eta) = a + (b,\eta)$ ($g_t := \frac{\partial g}{\partial t}$). This means that the deformation $\{\Omega_t\}$ is degenerate at $t = 0$ up to similarity transformations and parallel translations.

We introduce a family of seminorms $|\cdot|'_r$ on $L^2(S^{n-1})$ parameterized by $r > 0$ by

$$|h|'_r := \left\{\sum_{k=1}^{\infty} \|h_k\|^2_{L^2(S^{n-1})} J_{k+\frac{n}{2}-1}(r)^2\right\}^{\frac{1}{2}}$$

if $h = \sum_{k=0}^{\infty} h_k \in L^2(S^{n-1})$ is a decomposition of h into spherical harmonics h_k of degree k. Let $j(\nu,k) (k \in \mathbb{N}_+)$ be the positive zeros of $J_\nu(z)$ arranged in ascending order. For $R \ge j(\frac{n}{2},1)$ we denote by $k_R \in \mathbb{N}_+$ the integer such that $0 < j\left(\frac{n}{2},1\right) < j\left(\frac{n}{2},2\right) < \ldots < j\left(\frac{n}{2},k_R\right) \le R < j\left(\frac{n}{2},k_R+1\right)$. For $h \in L^2(S^{n-1})$, we define

$$|h|_R := \min_{1 \le k \le k_R} |h|'_{j(\frac{n}{2},k)}. \tag{5.3.1}$$

Given $0 < T$ and a C^2 function $g : [0,T] \times S^{n-1} \longrightarrow \mathbb{R}_+$, we define

$$[g]_R := \frac{|g_t(0,\cdot)|_R}{||g||_{C^2([0,T]\times S^{n-1})}} \ (\geq 0), \tag{5.3.2}$$

Then $[g]_R$ is a non-increasing function of R with the following property:

$$[g]_R = 0 \quad \Leftrightarrow \quad \{\Omega(g(t,\cdot))\} \text{ is unessential.} \tag{5.3.3}$$

Theorem 5.4: *([17], [20]) Let $R \gg 0$. There exists a constant $C(n,R) > 0$ with the following property: Suppose $0 < T$ and that $\Omega_t \equiv \Omega(g(t,\cdot))$ $(0 \leq t \leq T)$ is a family of domains in $\mathbb{R}^n$ given by a C^2 map $g : [0,T] \times S^{n-1} \longrightarrow \mathbb{R}_+$ satisfying $g(0,\eta) \equiv 1$ and $|g_t(0,\eta)| \leq 1$ $(\eta \in S^{n-1})$. If there exist $t_0 \in \mathbb{R}, x \in \mathbb{R}^n$ and $r > 0$ such that*

$$\begin{gathered} ||x|| + r < R, \\ 0 \leq t_0 < \min\,(T, C(n,R)[g]_R), \\ \mathcal{N}(\Omega_{t_0})_{\mathbb{R}} \cap B(0:R) \supset S(x:r), \end{gathered}$$

then $t_0 = 0$ and so Ω_{t_0} is a ball.

Corollary 5.5: *Let $R \gg 0$ and $C(n,R) > 0$ the constant in Theorem (5.4). Suppose $0 < T$ and that $\Omega_t \equiv \Omega(g(t,\cdot))$ $(0 \leq t \leq T)$ is a family of domains in $\mathbb{R}^n$ given by a C^2 map $g : [0,T] \times S^{n-1} \longrightarrow \mathbb{R}_+$ satisfying $g(0,\eta) \equiv 1$ and $|g_t(0,\eta)| \leq 1$ $(\eta \in S^{n-1})$. Assume that there exist $\lambda_0, t_0 \in \mathbb{R}$ and $u \in C^2(\Omega_{t_0}) \cap C^1(\overline{\Omega_{t_0}})$ such that*

$$\begin{gathered} 0 < \lambda_0 < R^2, \\ 0 \leq t_0 < \min\,(T, C(n,R)[g]_R), \\ u \neq 0 \text{ is a solution of } (N)_{\lambda_0}. \end{gathered}$$

Then $t_0 = 0$ and Ω_{t_0} is the unit ball.

Remark 5.6: The above results hold for a $C^{1,\alpha}$ map $g\colon [0,T] \times S^{n-1} \longrightarrow \mathbb{R}_+$ with some $0 < \alpha \leq 1$ (see *[20]*). Recently, Agranovsky *[1]* obtained a similar result to Corollary (5.5) assuming that the dimension $n = 2$ and and assuming the existence of a solution to $(N)_{\lambda_t}$ for *all* t with the condition that both the boundary $\partial\Omega_t$ and the eigenvalues λ_t depend analytically on the parameter t. His approach is quite different from ours and uses Riemann's mapping theorem for $\mathbb{C} \simeq \mathbb{R}^2$.

6 Asymptotic behavior of the zeros of a certain class of entire functions

As a function of z, it is a classical result that some of special functions have countably many zeros which are distributed in a regular fashion with bounded imaginary parts:

$F(z)$	$\{z \in \mathbb{C} : F(z) = 0\}$
$\sin z$	$2\pi n \quad (n \in \mathbb{Z})$
$\frac{1}{\Gamma(a+z)\Gamma(b-z)}$	$-a-n,\ b+n \quad (n \in \mathbb{N})$
$J_\lambda(z)$	$\pm\frac{(4n+2\lambda-1)\pi}{4} + O(n^{-1})$ as $n \to \infty$
$P_z^\mu(x)$	$\pm\frac{(4n-2\mu-1)\pi}{4\cos^{-1}(x)} + O(n^{-1})$ as $n \to \infty$

In this section we give an explanation based on the fact that these functions are essentially the Fourier transforms of compactly supported functions $f \in \mathcal{C}^2(\lambda)$ (see (6.3) for definition).

For a non-zero, bounded, compactly supported function f on $\mathbb{R}$, the Fourier transform $\mathcal{F}f(\zeta) = \int_{-\infty}^{\infty} f(x)e^{\sqrt{-1}x\zeta}\,dx$ is a holomorphic function of $\zeta \in \mathbb{C}$. We define a discrete subset of $\mathbb{C}$ by:

$$\mathcal{N}(f) := \{\zeta \in \mathbb{C} : \mathcal{F}f(\zeta) = 0\}. \tag{6.1}$$

Given $\delta > 0$, we define a class of functions:

$$\begin{aligned}\Psi(\delta) := \{\, \varphi \in C^\infty(\mathbb{R})\colon &\varphi(x) \in \mathbb{R}, \varphi(x) = \varphi(-x),\\ &\operatorname{supp}\varphi \subset [-2\delta, 2\delta],\ \varphi(x) \equiv 1 \text{ if } x \in [-\delta, \delta]\}.\end{aligned} \tag{6.2}$$

For $\lambda \in \mathbb{C}$ and $N \in \mathbb{N}$ such that $\operatorname{Re}\lambda + N \geq 0$, we introduce a class of functions $\mathcal{C}^N(\lambda)$. Here a complex valued function $f\colon \mathbb{R} \to \mathbb{C}$ belongs to $\mathcal{C}^N(\lambda)$ if and only if: **(6.3)** there exist $-\infty < \alpha < \beta < \infty$, $a_j, b_j \in \mathbb{C}$ $(j = 0, 1, \ldots, N)$, $0 < \delta < \frac{1}{4}(\beta - \alpha)$, $\varphi \in \Psi(\delta)$ such that the following three conditions hold.

$$a_0 \neq 0, \quad b_0 \neq 0, \tag{6.3.1}$$

$$f(x) = 0 \quad \text{if} \quad x < \alpha \quad \text{or} \quad \beta < x, \tag{6.3.2}$$

$$F_N(f,\varphi)(x) \equiv$$
$$f(x) - \sum_{j=0}^{N}\Big(a_j(x-\alpha)_+^{\lambda+j}\varphi(x-\alpha) + b_j(x-\beta)_-^{\lambda+j}\varphi(x-\beta)\Big)$$
is in $C^{[\Re\lambda]+N,1}(\mathbb{R})$.

From definition, we have a natural inclusion $\cdots \supset \mathcal{C}^N(\lambda) \supset \mathcal{C}^{N+1}(\lambda) \supset \cdots$. The complex numbers a_j, b_j $(0 \leq j \leq N)$ are obviously independent of the choice of $\varphi \in \Psi(\delta)$ and determined by f. So we write $a_j = a_j(f)$, $b_j = b_j(f)$ if we want to emphasize the dependence on f. Similarly we write $\alpha = \alpha(f)$, $\beta = \beta(f)$. Put $A \equiv A(f) := \beta(f) - \alpha(f)$. Then the maps

$$\begin{aligned} a_j, b_j:&\quad \mathcal{C}^N(\lambda) \longrightarrow \mathbb{C}, \qquad (0 \leq j \leq N),\\ A, \alpha, \beta:&\quad \mathcal{C}^N(\lambda) \longrightarrow \mathbb{R}.\end{aligned}$$

are clearly compatible with the inclusion map $\mathcal{C}^{N+1}(\lambda) \hookrightarrow \mathcal{C}^N(\lambda)$. We fix a $\psi \in \Psi(1)$ (Notation (6.2)) once and for all. We introduce a norm $\|\cdot\|_{\mathcal{C}^N(\lambda)}$ on $\mathcal{C}^N(\lambda)$ as follows: For $f \in \mathcal{C}^N(\lambda)$,

$$(6.4.1) \|f\|_{\mathcal{C}^N(\lambda)} := \sum_{j=0}^{N} A(f)^{\mathrm{Re}\,\lambda+j}(|a_j(f)| + |b_j(f)|)$$
$$+A(f)^{[\mathrm{Re}\,\lambda]+N+1} \sup_{\alpha\le x\le\beta}\left|\left(\frac{d}{dx}\right)^{[\mathrm{Re}\,\lambda]+N+1} F_N(f,\varphi)(x)\right|$$

where we put $\varphi(x) := \psi\left(\frac{5x}{A(f)}\right)$ (see also (6.3.3) for $F_N(f,\varphi)$). The point here is that the definition (6.4.1) is invariant under the affine transform of $\mathbb{R}$. That is, $\|f\|_{\mathcal{C}^N(\lambda)} = \|f_{p,q}\|_{\mathcal{C}^N(\lambda)}$ for any $p > 0$, $q \in \mathbb{R}$ if we put $f_{p,q}(x) = f(px+q)$. For $f \in \mathcal{C}^2(\lambda)$, we fix a branch of $\log \frac{a_0(f)}{b_0(f)}$ denoted by $r(f)$ and we define

$$\langle f\rangle := \frac{\left(1+|r(f)|^{[\mathrm{Re}\lambda]+3}\right)\|f\|_{\mathcal{C}^2(\lambda)}}{A(f)^{\mathrm{Re}\lambda}\min(|a_0(f)|,|b_0(f)|)} \ (\ge 2). \tag{6.4.2}$$

We set $B(a:r) := \{\zeta \in \mathbb{C} : |\zeta - a| < r\}$ for $a \in \mathbb{C}, r > 0$ and recall $\mathcal{N}(f) = \{\zeta \in \mathbb{C} : \mathcal{F}f(\zeta) = 0\}$.

Theorem 6.5: *(see [19]). Suppose $\lambda \in \mathbb{C}$ satisfies $Re\lambda > -1$. Then there exist constants $B(\lambda) > 0$ and $D(\lambda) > 0$ with the following properties:*
We put $n_1 \equiv n_1(\lambda, f) := \left[\frac{D(\lambda)\langle f\rangle}{2\pi}\right] - \left[\frac{Re\lambda}{2}\right]$ for $f \in \mathcal{C}^2(\lambda)$,
and $B_{n,\varepsilon} := B\left(\frac{\varepsilon(2n+\lambda)\pi - \sqrt{-1}\,r(f)}{A(f)} : \frac{D(\lambda)\langle f\rangle}{12A(f)n}\right)$ for $\varepsilon = \pm 1$, $n \in \mathbb{N}_+$.
Then there exists a finite set $S(f) \subset \mathbb{C}$ such that the following three conditions are satisfied:

$$\mathcal{N}(f) = S(f) \cup \coprod_{n=n_1}^{\infty} \coprod_{\varepsilon=\pm1} (B_{n,\varepsilon} \cap \mathcal{N}(f)),$$
$$S(f) \subset B\left(\frac{-\sqrt{-1}\,r(f)}{A(f)} : \frac{\sqrt{2}D(\lambda)\langle f\rangle}{A(f)}\right),$$
$$\#S(f) \le \exp(B(\lambda)\langle f\rangle), \quad \#(B_{n,\varepsilon} \cap \mathcal{N}(f)) = 1, \quad \textit{counted with multiplicity.}$$

Next, let $f \in \mathcal{C}^N(\lambda)$ and we put

$$f^\vee(x) := f(-x + \alpha(f) + \beta(f)), \tag{6.6.1}$$
$$\bar{f}(x) := \overline{f(x)}. \tag{6.6.2}$$

Then it is clear from the definition (6.3) that $f^\vee \in \mathcal{C}^N(\lambda)$ and $\bar{f} \in \mathcal{C}^N(\bar{\lambda})$. We say f is *symmetric* if $f^\vee = f$ and f is *real* if $\bar{f} = f$. It follows from the definition that if f is symmetric then $a_j(f) = b_j(f)$ for $0 \le j \le N$ with the notation (6.3) and that if f is real then λ is also real.

Corollary 6.7: *Suppose $\lambda \in \mathbb{C}$ satisfies $Re\lambda > -1$. Assume $f \in \mathcal{C}^2(\lambda)$ is symmetric and real. Then we have*

$$\#\mathcal{N}(f) = \infty, \quad \#(\mathcal{N}(f) \setminus \mathbb{R}) < \infty.$$

More precisely, we have an estimate of the number of exceptional zeros:

$$\#(\mathcal{N}(f) \setminus \mathbb{R}) \leq \exp(B(\lambda)\langle f \rangle),$$

where $B(\lambda)$ is the constant in Theorem (6.5) and $\langle f \rangle$ is defined in (6.4.2).

Remark 6.8: Corollary (6.7) is a kind of generalization of classical results that assert some 'special functions' with real parameter (e.g. Bessel function $J_\lambda(\zeta)$ with $\lambda \in \mathbb{R}$, $\lambda > -1$) have countably many real zeros, and have no non-real zeros. However, there may exist finite number of non-real zeros in our general setting. In fact, for any $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > -1$, we can find a sequence of functions $f_k \in \mathcal{C}^2(\lambda)$ ($k = 1, 2, \dots$) such that $\lim_{k \to \infty} \#(\mathcal{N}(f_k) \setminus \mathbb{R}) = \infty$. The following example is suggested by H.Ochiai. Let fix $f \in \mathcal{C}^2(\lambda)$ which is real and symmetric and choose a sequence of positive integers $r_j > 0$ ($j = 1, 2, \dots$). For each integer $k \in \mathbb{N}$ we define

$$f_k(x) := \prod_{j=1}^{k} \left(\frac{d^2}{dx^2} + r_j \right) \left(f(x)(x - \alpha(f))_+^k (x - \beta(f))_-^k \right).$$

Then $f_k \in \mathcal{C}^2(\lambda)$ is real and symmetric and $\#(\mathcal{N}(f_k) \setminus \mathbb{R}) \geq 2k$. In particular, $\lim_{k \to \infty} \#(\mathcal{N}(f_k) \setminus \mathbb{R}) = \infty$.

Example 6.9: For $\mathrm{Re}\,\lambda > -1$, we set

$$f_\lambda(x) := \begin{cases} (1 - x^2)^\lambda & \text{if } |x| < 1. \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Then we have

$$f_\lambda \in \mathcal{C}^\infty(\lambda) := \bigcap_{\substack{N \in \mathbb{N} \\ N \geq -\mathrm{Re}\,\lambda}} \mathcal{C}^N(\lambda)$$

and

$$\alpha(f_\lambda) = -1, \beta(f_\lambda) = 1, A(f_\lambda) = 2, a_0(f_\lambda) = b_0(f_\lambda) = 2^\lambda.$$

Then Theorem (6.5) says that up to a finite number of zeros (this is in fact empty: the phase principle) the zeros of $\mathcal{F} f_\lambda$ are parameterized by $n \in \mathbb{N}_+$ with the asymptotic behavior

$$\pm \left(n + \frac{\lambda}{2} \right) \pi + O(n^{-1}) \quad \text{as } n \to \infty.$$

From the formula $\mathcal{F}f_\lambda(\zeta) = \sqrt{\pi}\,\Gamma(\lambda+1)\left(\frac{\zeta}{2}\right)^{-\lambda-\frac{1}{2}} J_{\lambda+\frac{1}{2}}(\zeta)$, this gives a well-known asymptotic behavior of the zeros of the Bessel function.

Example 6.10: For $\operatorname{Re}\lambda > -1$ and $0 < \varphi < \pi$, we set

$$f_{\lambda,\varphi}(x) := \begin{cases} (\cos x - \cos\varphi)^\lambda & \text{if } |x| < \varphi, \\ 0 & \text{if } |x| \geq \varphi. \end{cases} \qquad (6.10.1)$$

Then $f_{\lambda,\varphi} \in \mathcal{C}^\infty(\lambda)$ and $\alpha(f_{\lambda,\varphi}) = -\varphi$, $\beta(f_{\lambda,\varphi}) = \varphi$, $A(f_{\lambda,\varphi}) = 2\varphi$, $a_0(f_{\lambda,\varphi}) = b_0(f_{\lambda,\varphi}) = (\sin\varphi)^\lambda$. Then Theorem (6.5) says that the zeros of $\mathcal{F}f_{\lambda,\varphi}$ have the asymptotic behavior

$$\pm\frac{(2n+\lambda)\pi}{2\varphi} + O(n^{-1}) \quad \text{as } n \to \infty.$$

From the formula (8.714)(1) in *[13]*, we have

$$\mathcal{F}f_{\lambda,\varphi}(\zeta) = \sqrt{2\pi}\,\Gamma(\lambda+1)(\sin\varphi)^{\lambda+\frac{1}{2}} P_{\zeta-\frac{1}{2}}^{-\lambda-\frac{1}{2}}(\cos\varphi),$$

where $P_\nu^\mu(z)$ denote the associated Legendre function of the first kind, which is a solution to the differential equation $(1-z^2)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + \left(\frac{\mu^2}{1-z^2}\right)u = 0$. We note that it is elementary to write all zeros down in some special cases such as $\mathcal{F}f_{\lambda,\frac{\pi}{2}}(\zeta) = \frac{\pi\Gamma(\lambda+1)2^{-\lambda}}{\Gamma(\frac{\lambda+\zeta+2}{2})\Gamma(\frac{\lambda-\zeta+2}{2})}$, $\mathcal{F}f_{0,\varphi}(\zeta) = \frac{2\sin(\zeta\varphi)}{\zeta}$. It is known that $P_{\zeta-\frac{1}{2}}^{-\lambda-\frac{1}{2}}(\cos\varphi)$, considered as a function ζ, has infinitely many zeros for $\lambda \geq -\frac{1}{2}$. These are all simple and real. They are symmetric with respect to the origin (see *[13]*, (8.781)).

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