Proper Actions and Representation Theory. I
— Discontinuous dual and properness criterion

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Mini-courses of Mini-lectures
AIM Research Community
Representation Theory & Noncommutative Geometry
Organizers: P. Clare, N. Higson, and B. Speh
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Mini · Mini · Mini — Proper actions and representation theory

Some general rules that I try to follow:

- Mini - series

  (possibly loosely related) topics

- Mini - lectures

  (short talks that fit into teatime)

- Minimal prerequizites.

I am going to talk about some aspects of transformation groups
in loose relationship to representation theory,
hopefully somewhat relaxing for teatime/bedtime.

The Calabi–Markus phenomenon (1962)

In contrast to the Bonnet–Myers theorem in Riemannian geometry, global features of pseudo-Riemannian manifolds are quite mysterious:

Theorem 1. (Calabi–Markus, 1962*)
Any de Sitter manifold is non-compact.

de Sitter mfd = Lorentzian manifold with sectional curvature $\equiv 1$

Model space: $\{(x_1, \cdots, x_{n+1}) : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 1\}$ in

$$\mathbb{R}^{n,1} = (\mathbb{R}^{n+1}, dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2)$$

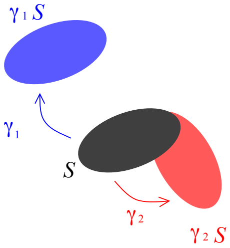
* E. Calabi–L. Markus, Relativistic space forms, Ann. Math., **75**, (1962), 63–76.

Basic notion ··· proper [properly discontinuous, free] action

$$L \overset{\text{action}}{\curvearrowright} X$$

loc. compact group L acts on loc. compact Hausdorff space X

$$\begin{array}{ccc} X & & L \\ \text{subset} \cup & \rightsquigarrow & \cup \\ S & & L_S := \{\gamma \in L : \gamma S \cap S \neq \emptyset\} \end{array}$$



$\Longrightarrow$ $\gamma_1 \notin L_S$, $\gamma_2 \in L_S$

Basic notion ··· proper [properly discontinuous, free] action

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loc. compact group — loc. compact Hausdorff space

$$\begin{array}{ccc} X & & L \\ \text{subset} \cup & \rightsquigarrow & \cup \\ S & & L_S := \{\gamma \in L : \gamma S \cap S \neq \emptyset\} \end{array}$$

$$S = \{x\} \quad \rightsquigarrow \quad L_{\{x\}} \equiv L_x = \text{stabilizer of } x$$

Definition

$L \curvearrowright X$ is proper	$\Longleftrightarrow$	L_S is compact	${}^\forall S$: compact.
$L \curvearrowright X$ is properly discont	$\Longleftrightarrow$	L_S is finite	${}^\forall S$: compact.
$L \curvearrowright X$ is free	$\Longleftrightarrow$	$\# L_x = 1$	$\forall x \in X$.

Covering transformation and properly discontinuous action

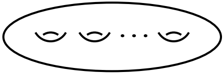
$\Gamma \curvearrowright X$ (X: C^∞ manifold) properly discontinuously and freely

$\Longrightarrow$ The quotient $\Gamma\backslash X$ carries a C^∞-manifold structure such that $X \to \Gamma\backslash X$ is a covering.

Example

$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$

$\downarrow$

$\Sigma_g =$  $(g \geq 2) \simeq \pi_1(\Sigma_g) \backslash \mathbb{H}$ ($\pi_1(\Sigma_g)$: surface group)

Uniformization theorem (Klein–Poincaré–Koebe)

Properly discontinuous actions: Riemannian geometry

(X, g) : a complete Riemannian manifold,

$G = \mathrm{Isom}(X)$: the group of isometries,

$\Gamma \subset G$ subgroup.

Proposition 2 (i) $\iff$ (ii) on Γ

(i) Γ is discrete subgroup in G.

(ii) Γ acts properly discontinuously on X.

(ii) $\Rightarrow$ (i) easy.

(i) $\Rightarrow$ (ii) The proof depends heavily on the positivity of g.

Use Ascoli–Arzela to the metric space (X, g).

Calabi–Markus phenomenon (1962) in group language

Riemannian geometry

Actions of discrete subgroups of isometries

$\Longleftrightarrow$ isometric properly discontinuous actions

Lorentzian geometry

Actions of discrete subgroups of isometries

$\not\Longleftrightarrow$ isometric properly discontinuous actions

$\Gamma \underset{\text{discrete}}{\subset} G \underset{\text{isometry}}{\curvearrowright} G/H \simeq \underbrace{\{x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 1\}}_{\text{de Sitter space}} \subset \mathbb{R}^{n,1}$

Theorem 1′.(Calabi–Markus)* Let $(G, H) = (O(n, 1), O(n-1, 1))$.
If a discrete subgroup Γ of G acts on G/H properly discontinuously, then Γ must be a finite group.

* E. Calabi–L. Markus, Relativistic space forms, Ann. Math., **75**, (1962), 63–76.

proper + discrete = properly discontinuous

properly discontinuous action

||

proper action

+

group is discrete

finite

||

compact

+

discrete

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$L \curvearrowright X$ is free	$\iff$	$\# L_x = 1$	$\forall x \in X$.

proper + discrete = properly discontinuous

action | properly discontinuous action

||

action | proper action

+

group is discrete

Definition (discontinuous group for X) For a G-space X, we say Γ is a discontinuous group for X if Γ is a discrete subgroup of G and the Γ-action on X is proper.

Proper actions and proper maps

G: locally compact group

X: locally compact, Hausdorff space

Definition (proper action)

$L \curvearrowright X$ is proper

$\iff$ $L \times X \to X \times X$, $\quad (g, x) \mapsto (x, gx)$ is a proper map.

$\iff$ $L_{S \to T}$ is compact $\quad {}^{\forall}$ compact $S, T \subset X$.

$\iff$ L_S ($\equiv L_{S \to S}$) is compact $\quad {}^{\forall}$ compact $S \subset X$.

$$L_{S \to T} := \{g \in L : gS \cap T \neq \emptyset\} \quad \text{for } S, T \subset X.$$

Definition A continuous map $f : X \to Y$ is proper if $f^{-1}(S)$ is compact for any compact $S \subset Y$.

Proper maps and representation theory

Definition A continuous map $f : X \to Y$ is proper if $f^{-1}(S)$ is compact for any compact $S \subset X$.

cf. Branching problem in rep theory: Study the restriction $\pi|_H$ for

$$H \subset G \overset{\pi}{\to} GL(\mathcal{H}).$$

unitary dual $\widehat{G} \ni \pi \overset{\text{orbit philosophy}}{\longleftarrow \cdots \longrightarrow} \mathcal{O}_\pi \subset \mathfrak{g}^*$ (coadjoint orbit)

$\pi|_H$ is discretely decomposable * $\overset{?}{\therefore}$ $\mathrm{pr}: \mathcal{O}_\pi \hookrightarrow \mathfrak{g}^* \to \mathfrak{h}^*$ is proper.

* T. Kobayashi, Ann. Math. (1998); Duflo–Vargas, Proc. Japan Acad., (2010).

Proper actions and representation theory

$L \curvearrowright X$ is a proper action.
$\iff L_{S \to T}$ is compact $^\forall$ compact $S, T \subset X$.

$$L_{S \to T} := \{g \in L : gS \cap T \neq \emptyset\} \quad \text{for } S, T \subset X.$$

- Geometric viewpoint

  The local to global study of geometries

  When we highlight "homogeneous structure" as a local property, "discontinuous groups" are responsible for the global geometry.

- Analytic viewpoint & Representation theory

  Quantify "properness" of actions (3rd and 4th lectures)

  e.g., asymptotic estimates of volume.

Proper actions and representation theory

Plan

1. Discontinuous dual and properness criterion (4/25)
2. The Mackey analogy and proper actions (5/2)
3. Tempered subgroups (5/9)
4. Tempered homogeneous spaces (5/16)

Elementary consequences of proper actions

L : locally compact group.

X : locally compact, Hausdorff space.

<u>Proposition</u> If L acts properly on X, then one has
(1) L/X is Hausdorff in the quotient topology;
(2) Any orbit $L \cdot x$ is closed in X;
(3) Any isotropy subgroup L_x is compact.

- Hausdorff $\Longrightarrow$ T_1 Trivial

  $(1) \Longrightarrow (2)$
  global   local

- (2) and (3) are easily verified.

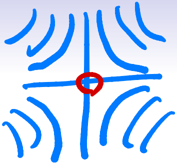
Delicate examples

$$L \curvearrowright X \quad \text{manifold}$$

(A)	free action	$\overset{?}{\Longrightarrow}$	proper action
(B)	any orbit is closed	$\overset{?}{\Longrightarrow}$	$L\backslash X$ is Hausdorff

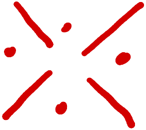
Shall see counterexamples to (A) and (B).

Delicate examples



$$a \in \mathbb{R}_{>0} \curvearrowright X = \mathbb{R}^2 \setminus \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}, \quad \begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}ax\\ \frac{1}{a}y\end{pmatrix}$$

This action is free, and any orbit is closed.
But the action is not proper, and $\mathbb{R}_{>0}\backslash X$ is not Hausdorff.



<u>Interpretation</u> in group language

$$A = \left\{\begin{pmatrix}a & 0\\0 & a^{-1}\end{pmatrix} : a > 0\right\} \subset G = SL(2,\mathbb{R}) \supset N = \left\{\begin{pmatrix}1 & n\\0 & 1\end{pmatrix} : n \in \mathbb{R}\right\}$$

$$\mathbb{R}_{>0} \simeq A \curvearrowright G/N \simeq X = \mathbb{R}^2 \setminus \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$$

$A \curvearrowright G/N$ non-proper $\iff$ $N \curvearrowright G/A$ non-proper (Lorentz isometry)

Lipsman's conjecture (1995)

Setting $X = G/H$ where $\underset{\text{closed subgp}}{L} \subset \underset{\text{Lie gp}}{G} \supset \underset{\text{closed subgp}}{H}$

Lipsman's conjecture(1995)* G: 1-conn nilpotent Lie group

$$L \curvearrowright X \text{ free} \overset{?}{\Longleftrightarrow} L \curvearrowright X \text{ proper}$$

True : G: 2-step nilpotent Lie group (Nasrin '01)

G: 3-step nilpotent Lie group (Baklouti '05, Yoshino '07)**

* R. Lipsman, Proper actions and a cocompactness condition, J. Lie Theory **5** (1995), 25–39.

** A. Baklouti, Internat. J. Math. **16** (2005); T. Yoshino, Internat. J. Math. **18** (2007).

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G: 3-step nilpotent Lie group (Baklouti '05, Yoshino '07)**

False : G: 4-step nilpotent Lie group (Yoshino)***

$$L \simeq \mathbb{R}^2 \curvearrowright X \simeq \mathbb{R}^5$$ (nilmanifold)

* R. Lipsman, Proper actions and a cocompactness condition, J. Lie Theory **5** (1995), 25–39.

** A. Baklouti, Internat. J. Math. **16** (2005); T. Yoshino, Internat. J. Math. **18** (2007).

*** T. Yoshino, A counterexample to Lipsman's conjecture, Internat. J. Math. **16** (2005), pp. 561–566.

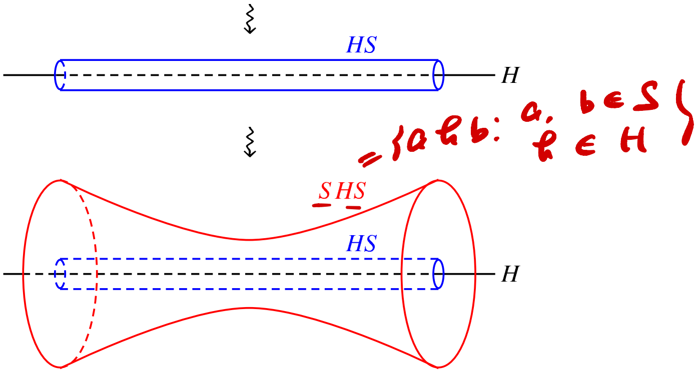
Proper actions — three directions

- Give a “handy” criterion to detect proper actions
  $\rightsquigarrow$ geometric applications.
- Relax the definition of proper actions, *e.g.* “measurably proper”
  $\rightsquigarrow$ connection to representation theory.
- Quantify “proper actions”
  $\rightsquigarrow$ connection to global analysis.

Expanding H by compact set S

$$G \supset H$$

S : compact subset



$\pitchfork$ and $\sim$ for locally compact group G

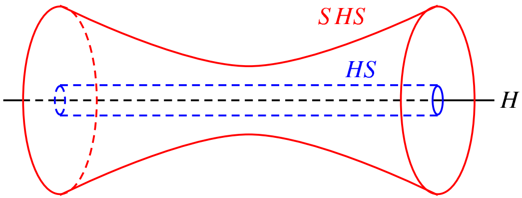
$$L \quad \subset \quad G \quad \supset \quad H$$

Idea: forget even that L and H are subgroups

Definition

1) $L \pitchfork H \iff \overline{L \cap SHS}$ is compact
for any compact subset $S \subset G$

2) $L \sim H \iff {}^{\exists}$ compact subset $S \subset G$.
such that $L \subset SHS$ and $H \subset SLS$.



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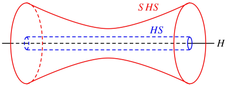
Ex. $G = \mathbb{R}^n$; L, H subspaces

$L \pitchfork H \iff L \cap H = \{0\}$.

$L \sim H \iff L = H$.



$\pitchfork$ and $\sim$ (meaning)

$$L \quad \subset \quad \underset{\text{loc compact group}}{G} \quad \supset \quad H$$

Meaning of $\pitchfork$: If both L and H are closed subgroups, then

$$\begin{array}{ccc} L \pitchfork H & \Longleftrightarrow & L \curvearrowright G/H \text{ proper action} \\ \Updownarrow & & \Updownarrow \text{ duality} \\ H \pitchfork L & \Longleftrightarrow & H \curvearrowright G/L \text{ proper action} \end{array}$$

$\sim$ defines an equivalence relation suitable for $\pitchfork$

$$H \sim H' \Longrightarrow \boxed{H \pitchfork L \Longleftrightarrow H' \pitchfork L}$$

Discontinuous duality theorem

G: locally compact topological group, separable

$G \supset H$ subset

$\rightsquigarrow \pitchfork (H : G) := \{L : L \pitchfork H\}$ discontinuous dual

Theorem 3 (Yoshino (2007) *, discontinuous duality theorem)**
Any subset H is determined uniquely by $\pitchfork (H : G)$ up to $\sim$.

cf. $G \rightsquigarrow \widehat{G}$ (unitary dual)

Fact (Pontrjagin–Tannaka–Tatsuuma duality theorem)
G is recovered from the unitary dual $\widehat{G}$.

** T. Yoshino, Discontinuous duality theorem, Internat. J. Math. **18** (2007), pp. 887–893. $\cdots$ loc. compact gp

Properness criterion for reductive groups

G: real reductive Lie group

Want to find a handy criterion for two subsets $L, H \subset G$ such that

$$L \pitchfork H,$$

or

$$L \sim H.$$

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Want to find a handy criterion for two subsets $L, H \subset G$ such that
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$G = K \exp(\mathfrak{a}) K$: Cartan decomposition

$\mu: G \to \mathfrak{a}/W$: Cartan projection $(W \equiv W(\Sigma(\mathfrak{g}, \mathfrak{a}))$: Weyl gp.)

E.g. $\mu: \quad GL(n, \mathbb{R}) \to \mathbb{R}^n/\mathfrak{S}_n$

$$g \mapsto \frac{1}{2}(\log \lambda_1, \cdots, \log \lambda_n)$$

Here, $\lambda_1 \geq \cdots \geq \lambda_n\ (> 0)$ are the eigenvalues of ${}^t g g$.

$$G = GL(n, \mathbb{R})$$
$$K = O(n)$$
$$\mathfrak{a} \simeq \mathbb{R}^n$$
$$\text{Weyl group} \simeq \mathcal{S}_n$$

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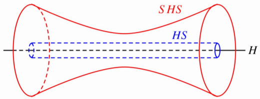
Theorem 4 *

(1) $L \sim H$ in G $\Longleftrightarrow$ $\mu(L) \sim \mu(H)$ in $\mathfrak{a}$.

(2) $L \pitchfork H$ in G $\Longleftrightarrow$ $\mu(L) \pitchfork \mu(H)$ in $\mathfrak{a}$.

abelian



* T. Kobayashi, Math. Ann. (1989); J. Lie Theory **6** (1996) 147–163. ; Y. Benoist, Ann. Math., **144** (1996) 315–347.

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(2) $L \pitchfork H$ in G $\iff$ $\mu(L) \pitchfork \mu(H)$ in $\mathfrak{a}$.

abelian

Special cases include

$\Rightarrow$ in (1): Uniform error estimates of eigenvalues when a matrix is perturbed.

$\Leftrightarrow$ in (2): Criterion for proper actions.

- Quantitative estimate for properness (3rd lecture)

* T. Kobayashi, Math. Ann. (1989); J. Lie Theory **6** (1996) 147–163. ; Y. Benoist, Ann. Math., **144** (1996) 315–347.

Properness criterion — special case (H, L reductive)

Give a flavor of proof in a special case.

For a reductive subgroup G' in G, the Cartan projection of G' takes the form $\mu(G') = W \cdot \mathfrak{a}_{G'}$ in $\mathfrak{a}$ (after conjugation of G' in G):

$$\begin{array}{ccccccccc} \mathfrak{g} & = & \mathfrak{k} & + & \mathfrak{p} & \supset & \mathfrak{p} & \supset & \mathfrak{a} \\ & & & & & & & \text{max abelian} & \\ \cup & & \cup & & \cup & & \cup & & \cup \\ \mathfrak{g}' & = & \mathfrak{k}' & + & \mathfrak{p}' & \supset & \mathfrak{p}' & \supset & \mathfrak{a}_{G'} := \mathfrak{a} \cap \mathfrak{g}'. \\ & & & & & & & \text{max abelian} & \end{array}$$

A special case of Theorem 4 includes:

Theorem 5* Assume $H, L \subset G$ are reductive subgroups.
$L \curvearrowright G/H$ proper $\iff$ $\mathfrak{a}_H \cap W \cdot \mathfrak{a}_L = \{0\}$ in $\mathfrak{a}$.

Remark easy to see $\mu(H) \pitchfork \mu(L)$ in $\mathfrak{a}$ $\iff$ $\mathfrak{a}_H \cap W \cdot \mathfrak{a}_L = \{0\}$.

* Kobayashi, Proper action on homogeneous spaces of reductive type, Math. Ann. (1989).

Reduction of properness criterion to abelian subgps

$$H \sim A_H$$
$$L \sim A_L$$

Theorem 4 (properness criterion) $\supset$ Theorem 5 (H, L reductive) $\supset$ Theorem 5' (H, L abelian)

$G = K\exp(\mathfrak{a})K$ Cartan decomposition

W : Weyl group of $\Sigma(\mathfrak{g}, \mathfrak{a})$

Suppose $\mathfrak{l}, \mathfrak{h} \subset \mathfrak{a}$ abelian subspaces, $L = \exp \mathfrak{l}$, $H = \exp \mathfrak{h}$.

Theorem 5′ $L \curvearrowright G/H$ proper $\iff$ $\mathfrak{l} \cap W\mathfrak{h} = \{0\}$.

$\Longleftarrow$ non-trivial.

Proof of Theorem 5' for abelian $H, L \subset G$: Step 1

Suppose $\mathfrak{l}, \mathfrak{h} \subset \mathfrak{a}$.

Want to prove: $L \curvearrowright G/H$ not proper $\Rightarrow \mathfrak{l} \cap W\mathfrak{h} \neq \{0\}$.

Assume $L \cap SHS$ is non-compact for some compact subset $S \subset G$. One can find sequences

$$\begin{cases} \exp(t_n \underset{\mathfrak{l}}{\underset{\cap}{Y_n}}) = \underset{S}{\underset{\cap}{c_n}} \exp(t'_n \underset{\mathfrak{h}}{\underset{\cap}{Z_n}}) \underset{S}{\underset{\cap}{d_n}} \quad \text{in } G. \\ 0 < t_n \uparrow \infty \\ c_n \to c, \ d_n \to d \text{ in } S \\ Y_n \to Y(\neq 0) \in \mathfrak{l}, \ Z_n \to Z(\neq 0) \in \mathfrak{h}. \end{cases}$$

By taking subsequences, renormalizing, and replacing $\mathfrak{h} \rightsquigarrow w \cdot \mathfrak{h}$ $(w \in W)$, $S \rightsquigarrow KSK$, we may assume $t'_n = t_n$ and $Y, Z \in \overline{\mathfrak{a}_+}$.

Proof of Theorem 5' for abelian $H, L \subset G$: Step 2

Plan $L \pitchfork H \Rightarrow Y = Z \Rightarrow \mathfrak{l} \cap W\mathfrak{h} \neq \{0\}$.

Have seen, if $L \pitchfork H$, one finds sequences (after replacing $\mathfrak{h}$ by $w \cdot \mathfrak{h}$ for some $w \in W$):

$$\begin{cases} c_n = \exp(t_n Y_n) d_n^{-1} \exp(-t_n Z_n) & (1) \\ t_n \uparrow \infty; c_n \to c,\ d_n \to d \text{ in } G. \\ Y_n \to Y \in \mathfrak{l} \cap \overline{\mathfrak{a}_+},\ Z_n \to Z \in \mathfrak{h} \cap \overline{\mathfrak{a}_+}. \end{cases}$$

Sketch Apply (1) to $G \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{g} = \bigoplus_{\alpha \in \Sigma(\mathfrak{g};\mathfrak{a}) \cup \{0\}} \mathfrak{g}_\alpha$

$\rightsquigarrow$

$$\mathrm{Ad}(c)\mathfrak{g}_\alpha = \lim_{n\to\infty} \bigoplus_{\beta \in \Sigma(\mathfrak{g},\mathfrak{a}) \cup \{0\}} e^{t_n(\beta(Y_n) - \alpha(Z_n))} \mathrm{pr}_\beta(\mathrm{Ad}(d_n^{-1})\mathfrak{g}_\alpha) \subset \bigoplus_{\beta(Y) \geq \alpha(Z)} \mathfrak{g}_\beta.$$

This argument leads us to $Y = Z \in \mathfrak{l} \cap \mathfrak{h}$. $\Longrightarrow \mathfrak{l} \cap W \cdot \mathfrak{h} \neq \{0\}$. □

Criterion for the Calabi–Markus phenomenon

Corollary 6 (criterion of Calabi–Markus phenomenon) *

$G \supset H$ pair of real reductive Lie groups.

Then (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv).

(i) G/H admits a discontinuous group $\Gamma \simeq \mathbb{Z}$.

(ii) G/H admits an infinite discontinuous group Γ.

(iii) $G \not\sim H$.

(iv) $\operatorname{rank}_{\mathbb{R}} G > \operatorname{rank}_{\mathbb{R}} H$.

(i) $\underset{\text{clear}}{\Longrightarrow}$ (ii) $\underset{\substack{\Gamma \pitchfork G \\ \Gamma \not\pitchfork H}}{\Longrightarrow}$ (iii) $\underset{\text{Cartan decomposition}}{\Longrightarrow}$ (iv) $\underset{\text{Theorem 5}}{\Longrightarrow}$ (i)

Theorem 1' (Calabi–Markus, 1962)** $(G, H) = (O(n,1), O(n-1,1))$.

G/H does not admit an infinite discontinuous group.

* Kobayashi, Proper action on homogeneous spaces of reductive type, Math. Ann. (1989).

** E. Calabi–L. Markus, Relativistic space forms, Ann. Math., **75**, (1962), 63–76.

Example $G/H = SL(n,\mathbb{R})/SL(m,\mathbb{R})$ $(n > m)$

<u>Ex.</u> $^\exists$ proper action of $SL(2,\mathbb{R})$ on $SL(n,\mathbb{R})/SL(m,\mathbb{R})$ if n is even.

- Cartan projection $\mu\colon G \to \mathfrak{a}/\mathfrak{S}_n$ for $G = SL(n,\mathbb{R})$.

$$W \simeq \mathfrak{S}_n \curvearrowright \mathfrak{a} := \{(a_1,\cdots,a_n) : \sum_{j=1}^{n} a_j = 0\} \underset{\text{diag}}{\hookrightarrow} \mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$$

- For $H = SL(m,\mathbb{R})$ $(m < n)$,

$$\therefore \quad \mu(H) = \mathfrak{S}_n \cdot \mathfrak{a}_H = \mathfrak{S}_n \cdot \{(b_1,\cdots,b_m,0,\cdots,0) : \sum_{j=1}^{m} b_j = 0\}.$$

- For $L := \varphi(SL(2,\mathbb{R}))$, where $\varphi\colon SL(2,\mathbb{R}) \to SL(n,\mathbb{R})$ is an irreducible n-dimensional rep,

$$\mu(L) = \mathfrak{S}_n \cdot \mathfrak{a}_L = \mathfrak{S}_n \cdot \mathbb{R}(n-1, n-3, \cdots, 1-n).$$

$$\therefore \ L \curvearrowright G/H \text{ proper} \iff \mu(L) \pitchfork \mu(H) = \{0\}$$
$$\iff n \text{ is even or } n - m \geq 2.$$

Properly discontinuous action of surface group

$\pi_1(\Sigma_g) \cdots$ surface group $\quad (g \geq 2)$

<u>Theorem 7</u>

If G/H is a reductive symmetric space then (i) $\iff$ (ii) $\iff$ (iii).

(i) G/H admits a discontinuous group $\Gamma \simeq \mathbb{Z}$ generated by a unipotent element.

(ii) G/H admits a proper action of a subgroup L which is locally isomoprhic to $SL(2,\mathbb{R})$.

(iii) G/H admits a discontinuous group $\Gamma \simeq$ surface group.

For a pair of real reductive Lie groups $G \supset H$, (i) $\iff$ (ii) $\Longrightarrow$ (iii).

T. Okuda, Classification of semisimple symmetric spaces with proper $SL(2,\mathbb{R})$-actions, J. Differential Geom. **94** (2013), 301–342.

Proper actions and representation theory

Plan

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