A GENERATING OPERATOR FOR RANKIN–COHEN BRACKETS

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Abstract. Motivated by the classical ideas of generating functions for orthogonal polynomials, we initiate a new line of investigation on "generating operators" for a family of differential operators between two manifolds. We prove a novel formula of the generating operators for the Rankin–Cohen brackets by using higher-dimensional residue calculus. Various results on the generating operators are also explored from the perspective of infinite-dimensional representation theory.

Keywords and phrases: generating operator, symmetry breaking operator, holographic transform, Rankin–Cohen bracket, orthogonal polynomial, branching rule, Hardy space.

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1 Introduction

To any sequence $\{a_\ell\}_{\ell\in\mathbb{N}}$ one may associate a formal power series such as $\sum_{\ell=0}^{\infty} a_\ell t^\ell$ or $\sum_{\ell=0}^{\infty} a_\ell \frac{t^\ell}{\ell!}$. The resulting *generating functions* are fascinating objects providing powerful tools for studying various combinatorial problems when a_ℓ are integers or, more generally, polynomials. One may quantize this construction by considering differential operators as non-commutative analogues of polynomials and may study the resulting "generating operators". Dealing with the sequence of differential operators given by iterated powers of some remarkable operator yields the notion of an operator semigroup which is nowadays a classical tool for the spectral theory of unbounded operators (*e.g.* the Hille–Yosida theory). We explore yet another direction by introducing a sequence of differential operators with a different algebraic structure which is not defined by one single operator anymore.

Let us start with our general setting. Suppose that $\Gamma(X)$ and $\Gamma(Y)$ are the spaces of functions on X and Y, respectively. Given a family of linear operators $R_\ell \colon \Gamma(X) \to \Gamma(Y)$, we consider a formal power series

$$T \equiv T(\{R_\ell\}; t) := \sum_{\ell=0}^{\infty} \frac{R_\ell}{\ell!} t^\ell \in \operatorname{Hom}(\Gamma(X), \Gamma(Y)) \otimes \mathbb{C}[[t]]. \tag{1.1}$$

When $X = \{\text{point}\}$, R_ℓ is identified with an element of $\Gamma(Y)$, and such a formal power series is called a *generating function*, which has been particularly prominent in the classical study of orthogonal polynomials for $\Gamma(Y) = \mathbb{C}[y]$, see *e.g.,* [10, 11].

When $X = Y$, $\operatorname{Hom}(\Gamma(X), \Gamma(Y)) \simeq \operatorname{End}(\Gamma(X))$ has a ring structure and one may take R_ℓ to be the ℓ-th power of a *single* operator R on X. In this case, the operator T in (1.1) may be written as e^{tR} if the summation converges. We note that even if R is a differential operator on a manifold X, the resulting operator $T = e^{tR}$ is not a differential operator any more in general . For example, if $R = \frac{d}{dz}$ acting on $\mathcal{O}(\mathbb{C})$, then $T = e^{t\frac{d}{dz}}$ is the shift operator $f(z) \mapsto f(z+t)$. For a self-adjoint operator R with bounded eigenvalues from the above, the operator T has been intensively studied as the *semigroup* e^{tR} generated by R for $\operatorname{Re} t > 0$: typical examples include

- the heat kernel for $R = \Delta$,
- the Hermite semigroup for $R = \frac{1}{4}(\Delta - |x|^2)$ on $L^2(\mathbb{R}^n)$,
- the Laguerre semigroup for $R = |x|(\frac{\Delta}{4} - 1)$ on $L^2(\mathbb{R}^n, \frac{1}{|x|}dx)$.

Let us consider a more general setting where we allow $X \neq \{\text{point}\}$ and $X \neq Y$. In this generality, we refer to T in (1.1) as the *generating operator* for a family of operators $R_\ell \colon \Gamma(X) \to \Gamma(Y)$.

In the present work we initiate a new line of investigation of "generating operators" in the setting that $(X, Y) = (\mathbb{C}^2, \mathbb{C})$ and that $\{R_\ell\}$ are the Rankin–Cohen brackets [2, 9]. We shall find a closed formula of the generating operator T as an integral operator, through which we explore its basic properties and various aspects.

It is known that covariant differential operators are often obtained as residues of a meromorphic family of integral transformations. For instance, the iterated powers of the Dirac operator are the residues of the meromorphic family of the Knapp–Stein intertwining operators, see *e.g.,* a recent paper [1].

The inverse direction is more involved. In fact, some covariant differential operators cannot be obtained as residues, which are referred to as *sporadic operators*. One of the important applications of the *generating operator* introduced in this article provides us a method to go in the inverse direction, namely, to construct a meromorphic family of non-local symmetry breaking operators out of discrete data. In the subsequent paper [5], we give a toy model which constructs various fundamental operators such as invariant trilinear forms on infinite-dimensional representations, the Fourier and the Poisson transforms on the anti-de Sitter space, and non-local symmetry breaking operators for the fusion rules among others, out of just countable data of the Rankin–Cohen brackets, for which the key of the proof is the explicit formula (2.1) of the *generating operator* proved in this article.

The article is organized as follows. In Section 2 we give an integral expression of the "generating operator" T of the Rankin–Cohen bidifferential operators (Theorem 2.3), and discuss the domain of holomorphy. In Section 3, we introduce a second-order differential operator P on $\mathbb{C}^2$ which plays a key role in the detailed analysis of T (Theorems 3.1 and 4.1). In Section 5 we focus on operators between Hilbert spaces, and prove that T gives rise to a natural decomposition of the completed tensor product of two Hardy spaces (Theorem 5.1). In Section 6 we discuss briefly different perspectives of the generating operator T from the viewpoint of unitary representation theory of real reductive groups, in particular, from that of symmetry breaking operators and holographic operators associated with branching problems (*fusion rules*) for $SL(2,\mathbb{R})$.

Notation. $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

2 Basic properties of the integral operator T

Let D be an open set in $\mathbb{C}$. For a holomorphic function $f(\zeta_1, \zeta_2)$ in $D \times D$, we introduce an integral transform by

(2.1)
$$(Tf)(z,t) := \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_1} \oint_{C_2} \frac{f(\zeta_1,\zeta_2)}{(\zeta_1 - z)(\zeta_2 - z) + t(\zeta_1 - \zeta_2)} d\zeta_1 d\zeta_2,$$

where C_j are contours in D around the point z ($j = 1, 2$). The denominator will be denoted by

$$Q \equiv Q(\zeta_1, \zeta_2; z, t) := (\zeta_1 - z)(\zeta_2 - z) + t(\zeta_1 - \zeta_2). \tag{2.2}$$

We note that the denominator is an irreducible polynomial of ζ_1 and ζ_2 when $t \neq 0$. We shall give closed formulas of the transform $Tf(z,t)$ for a family of meromorphic functions $f(\zeta_1, \zeta_2)$, corresponding to the minimal K-types in representation theory, see Example 3.9.

We begin with general properties of the operator T.

Theorem 2.1.
(1) *There exists an open neighbourhood U of $D \times \{0\}$ in $\mathbb{C}^2$ such that $T\colon \mathcal{O}(D \times D) \to \mathcal{O}(U)$ is well-defined.*
(2) *$Tf(z, 0) = f(z, z)$ for any $z \in D$.*
(3) *For any neighbourhood U of $D \times \{0\}$ in $\mathbb{C}^2$, T is injective.*

Proof of (1) and (2) in Theorem 2.1. (1) For $z \in D$ and $t \in \mathbb{C}$, we define an analytic set by

$$\mathcal{N}_{z,t} := \{(\zeta_1, \zeta_2) \in D \times D : Q(\zeta_1, \zeta_2; z, t) = 0\}.$$

Then there exists a neighbourhood W of $t = 0$ such that $C_1 \times C_2 \subset \mathcal{N}_{z,t}$ for all $t \in W$. The integral (2.1) does not change if we replace $C_1 \times C_2$ by a compact surface S, as far as S belongs to the same second homology class in $D \times D \setminus \mathcal{N}_{z,t}$. We define $d(z) \equiv d(z, \partial D)$ to be the distance from z to the boundary ∂D. We set $d(z) := \infty$ if $\partial D = \emptyset$, namely, if $D = \mathbb{C}$. We claim that $Tf(z,t)$ is well-defined and holomorphic in

$$U_D := \{(z,t) \in D \times \mathbb{C} : 2|t| < d(z, \partial D)\}. \tag{2.3}$$

We fix $z \in D$, and set $R := d(z, \partial D)$. Let $\varepsilon > 0$. If we take $C_1 = C_2$ to be the circle of radius $R(1-\varepsilon)$ centered at z, then $(C_1 \times C_2) \cap \mathcal{N}_{z',t} = \emptyset$

for any (z', t) satisfying $|z' - z| < R\varepsilon$ and $2|t| < R(1 - 3\varepsilon)$ because

$$|(\zeta_1 - z')(\zeta_2 - z')| > R^2(1 - 2\varepsilon)^2 > R^2(1 - \varepsilon)(1 - 3\varepsilon) > |t(\zeta_1 - \zeta_2)|.$$

Therefore, $(C_1 \times C_2) \cap \mathcal{N}_{z',t} = \emptyset$, hence $Tf(z', t)$ is holomorphic in this region. Taking the limit as $\varepsilon \to 0$, we conclude that Tf is well-defined and holomorphic in the open neighbourhood of $\{z\} \times \{t \in \mathbb{C} : 2|t| < d(z, \partial D)\}$ for every $z \in D$, hence it is holomorphic in U_D.
(2) Clear from Cauchy's integral formula. □

Example 2.2. We make explicit two important examples of the domains U_D introduced in (2.3).
(1) $U_D = \mathbb{C} \times \mathbb{C}$ if $D = \mathbb{C}$.
(2) $U_D = \{(z, t) \in \mathbb{C}^2 : 2|t| < \operatorname{Im} z\}$ if D is the upper half plane $\Pi := \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta > 0\}$.

Before giving a proof of the third statement of Theorem 2.1, we show that T is a "generating operator" for the family of the Rankin–Cohen brackets. For $\ell \in \mathbb{N}$ we define $R_\ell \colon \mathcal{O}(D \times D) \to \mathcal{O}(D)$, $f(\zeta_1, \zeta_2) \mapsto (R_\ell f)(z)$ by

$$R_\ell f(z) := \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j}^2 \left.\frac{\partial^\ell f(\zeta_1, \zeta_2)}{\partial \zeta_1^{\ell-j} \partial \zeta_2^j}\right|_{\zeta_1 = \zeta_2 = z}. \tag{2.4}$$

Theorem 2.3 (generating operator of the Rankin–Cohen brackets)**.** *The integral operator T in* (2.1) *is expressed as*

$$Tf(z, t) = \sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} R_\ell f(z) \quad \textit{for any } f \in \mathcal{O}(D \times D).$$

Remark 2.4. *For $f(\zeta_1, \zeta_2) = f_1(\zeta_1) f_2(\zeta_2)$ with some $f_1, f_2 \in \mathcal{O}(D)$, $(R_\ell f)(z)$ takes the form $\sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j}^2 \frac{\partial^{\ell-j} f_1(z)}{\partial z^{\ell-j}} \frac{\partial^j f_2(z)}{\partial z^j}$, which is the Rankin–Cohen bidifferential operator $R_{\lambda', \lambda''}^{\lambda'''}(f_1, f_2)$ at $(\lambda', \lambda'', \lambda''') = (1, 1, 2 + 2\ell)$ with the notation as in* [7, (2.1)].

Proof of Theorem 2.3. By the first statement of Theorem 2.1, one can expand $Tf(z, t)$ into the Taylor series of t:

$$Tf(z, t) = \sum_{\ell=0}^{\infty} t^\ell (T_\ell f)(z)$$

with coefficients $T_\ell f(z) \in \mathcal{O}(D)$. Accordingly, we expand Q^{-1} into the Taylor series of t:

$$\frac{1}{Q} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (\zeta_1 - \zeta_2)^\ell t^\ell}{(\zeta_1 - z)^{\ell+1}(\zeta_2 - z)^{\ell+1}}. \tag{2.5}$$

An iterated use of the Cauchy integral formula gives an explicit formula of $(T_\ell f)(z)$ by

$$\begin{aligned}(T_\ell f)(z) =& \frac{(-1)^\ell}{(2\pi\sqrt{-1})^2} \oint_{C_1}\oint_{C_2} \frac{(\zeta_1 - \zeta_2)^\ell f(\zeta_1, \zeta_2)}{(\zeta_1 - z)^{\ell+1}(\zeta_2 - z)^{\ell+1}} d\zeta_1 d\zeta_2 \\ =& \frac{(-1)^\ell}{2\pi\sqrt{-1}} \oint_{C_2} \frac{(\frac{\partial}{\partial \zeta_1})^\ell|_{\zeta_1 = z}((\zeta_1 - \zeta_2)^\ell f(\zeta_1, \zeta_2))}{\ell!(\zeta_2 - z)^{\ell+1}} d\zeta_2 \\ =& \frac{1}{2\pi\sqrt{-1}} \sum_{j=0}^{\ell} \frac{(-1)^j \ell!}{(j!)((\ell - j)!)^2} \oint_{C_2} \frac{\frac{\partial^{\ell-j} f}{\partial \zeta_1^{\ell-j}}(z, \zeta_2)}{(\zeta_2 - z)^{j+1}} d\zeta_2 \\ =& \sum_{j=0}^{\ell} \frac{(-1)^j \ell!}{(j!(\ell - j)!)^2} \frac{\partial^\ell f(\zeta_1, \zeta_2)}{\partial \zeta_1^{\ell-j} \partial \zeta_2^j}\bigg|_{\zeta_1 = \zeta_2 = z} = \frac{1}{\ell!}(R_\ell f)(z).\end{aligned}$$

Hence the theorem is shown. □

We are ready to prove the third statement of Theorem 2.1, which uses the property that the signature of the coefficients in (2.4) alternates.

Proof of (3) in Theorem 2.1. Suppose $Tf \equiv 0$ for $f \in \mathcal{O}(D \times D)$. We set $a_{i,j}(z) := \frac{\partial^{i+j} f}{\partial \zeta_1^i \partial \zeta_2^j}\Big|_{\zeta_1 = \zeta_2 = z}$. We shall prove $a_{i,j}(z) \equiv 0$ for all i, j by the induction on $k := i + j$. The case $k = 0$ is clear because $a_{0,0}(z) = f(z, z) = (Tf)(z, 0)$. Suppose now that $a_{i,j}(z) \equiv 0$ for all $i + j = k$. Since $\frac{d}{dz} a_{i,j}(z) = a_{i+1,j}(z) + a_{i,j+1}(z)$, one has $a_{k+1-j,j} + a_{k-j,j+1} = 0$ for all $0 \le j \le k$, namely, $a_{k+1-j,j} = (-1)^j a_{k+1,0}$. In turn, $(\frac{\partial}{\partial t})^{k+1}|_{t=0} Tf = (\sum_{j=0}^{k+1} \binom{k+1}{j}^2) a_{k+1,0}$. Hence $a_{k+1,0}(z) \equiv 0$, and thus $a_{i,j}(z) \equiv 0$ for all i, j with $i + j = k + 1$. Therefore, the holomorphic function $f(\zeta_1, \zeta_2)$ must be identically zero. □

3 Differential operator P and the generating operator

The following differential operator on $\mathbb{C}^2$ plays a key role in the analysis of the generating operator T.

$$P := (\zeta_1 - \zeta_2)^2 \frac{\partial^2}{\partial \zeta_1 \partial \zeta_2} - (\zeta_1 - \zeta_2)(\frac{\partial}{\partial \zeta_1} - \frac{\partial}{\partial \zeta_2}). \tag{3.1}$$

The goal of this section is to prove the following:

Theorem 3.1. *Let D be an open set in $\mathbb{C}$. For any $f \in \mathcal{O}(D \times D)$,*

$$T(Pf)(z,t) = -(t\frac{\partial}{\partial t})(t\frac{\partial}{\partial t} + 1)Tf(z,t).$$

One derives from Theorem 3.1 that the set of eigenvalues of P is discrete:

Corollary 3.2 (Eigenvalues of P)**.** *Let D be a connected open set in $\mathbb{C}$. If there is a non-zero function $f \in \mathcal{O}(D \times D)$ satisfying $Pf = \lambda f$ for some $\lambda \in \mathbb{C}$, then λ is of the form $-\ell(\ell+1)$ for some $\ell \in \mathbb{N}$.*

For $\ell \in \mathbb{N}$, we consider the space of all eigenfunctions:

$$\mathcal{S}ol(D \times D)_\ell := \{f \in \mathcal{O}(D \times D) : Pf = -\ell(\ell+1)f\}. \tag{3.2}$$

We shall see in Corollary 4.2 that $\mathcal{S}ol(D \times D)_\ell$ is infinite-dimensional for any $\ell \in \mathbb{N}$ and for any non-empty open subset D.

Remark 3.3. (1) *In Theorem 5.1, we shall prove that P defines a self-adjoint operator on the completed tensor product of two Hardy spaces.*
(2) *Taking this opportunity, we would like to point out that the first term of the differential operator $P_{\lambda',\lambda''}$ in* [7, (2.31)] *was wrongly stated: the correct formula is*

$$P_{\lambda',\lambda''} = (\zeta_1 - \zeta_2)^2 \frac{\partial^2}{\partial \zeta_1 \partial \zeta_2} - (\zeta_1 - \zeta_2)(\lambda''\frac{\partial}{\partial \zeta_1} - \lambda'\frac{\partial}{\partial \zeta_2}).$$

All the theorems involving $P_{\lambda',\lambda''}$ valid with this definition.

Proof of Corollary 3.2. Suppose $Pf = \lambda f$ with $f \not\equiv 0$. Then $Tf \not\equiv 0$ because T is injective by Theorem 2.1. By Theorem 3.1, one has

$$\vartheta_t(\vartheta_t + 1)Tf = -T(Pf) = -\lambda Tf,$$

where ϑ_t denotes the Euler homogeneity operator $t\frac{\partial}{\partial t}$. We observe that $\vartheta_t(\vartheta_t + 1)t^\ell = \ell(\ell+1)t^\ell$ for every $\ell \in \mathbb{N}$. Since $Tf(z,t)$ is holomorphic in a neighbourhood of $t = 0$, possible eigenvalues of $\vartheta_t(\vartheta_t + 1)$ are of the form $\ell(\ell+1)$ for some $\ell \in \mathbb{N}$, and the corresponding eigenfunctions are of the form $t^\ell \varphi(z)$ for some holomorphic function $\varphi(z) \in \mathcal{O}(D)$. Thus the corollary is proved. $\square$

The following statement is clear from the above proof.

Corollary 3.4. *Let $\ell \in \mathbb{N}$. Then the following two conditions on $f \in \mathcal{O}(D \times D)$ are equivalent:*
(i) $f \in \mathcal{S}ol(D \times D)_\ell$,
(ii) $Tf(z,t)$ *is of the form* $t^\ell \varphi(z)$ *for some* $\varphi \in \mathcal{O}(D)$.

The rest of the section is devoted to the proof of Theorem 3.1 by comparing the integral expressions of $T(Pf)$ and $\vartheta_t(\vartheta_t + 1)Tf$.

The following formula for $\vartheta_t(\vartheta_t + 1)Tf$ is an immediate consequence of the definition (2.1) of the generating operator T. For the rest of the paper, we omit writing the contours C_1 and C_2 in the integrals for simplicity.

Lemma 3.5. *For any $f \in \mathcal{O}(D \times D)$, one has*

$$\vartheta_t(\vartheta_t+1)Tf(z,t) = \frac{-2}{(2\pi\sqrt{-1})^2} \oint \oint \frac{(\zeta_1 - \zeta_2)(\zeta_1 - z)(\zeta_2 - z)t}{Q^3} f d\zeta_1 d\zeta_2.$$

It is more involved to find the integral expression of $T(Pf)$. For this, we set

$$I_j(f) := \frac{1}{(2\pi\sqrt{-1})^2} \oint \oint \frac{\zeta_j^2 - \zeta_1\zeta_2}{Q} \frac{\partial^2 f}{\partial\zeta_1 \partial\zeta_2} d\zeta_1 d\zeta_2 \quad \text{for } j = 1, 2,$$

$$I_3(f) := -\frac{1}{(2\pi\sqrt{-1})^2} \oint \oint \frac{(\zeta_1 - \zeta_2)(\frac{\partial}{\partial\zeta_1} - \frac{\partial}{\partial\zeta_2})f}{Q} d\zeta_1 d\zeta_2.$$

By the definition (3.1) of P, one has

$$T(Pf) = I_1(f) + I_2(f) + I_3(f).$$

In view of Lemma 3.5, Theorem 3.1 will be derived from the following two propositions.

Proposition 3.6. *Let $\varepsilon(1) := -1$ and $\varepsilon(2) = 1$. For $j = 1, 2$, one has*
(3.3)

$$I_j(f) = \frac{1}{(2\pi\sqrt{-1})^2} \oint \oint \frac{\varepsilon(j)(\zeta_j - z)^2 + 2t(\zeta_1 - z)(\zeta_2 - z)\zeta_j}{Q^2} f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2.$$

Proposition 3.7.

$$I_3(f) = \frac{1}{(2\pi\sqrt{-1})^2}\oint\oint \frac{(\zeta_1 - z)^2 + (\zeta_2 - z)^2}{Q^2} f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2.$$

For the proof of Propositions 3.6 and 3.7, we need some preparations. We define $\xi_1 \equiv \xi_1(\zeta_2)$ and $\xi_2 \equiv \xi_2(\zeta_1)$ by

$$\xi_1 := \frac{(\zeta_2 - z)z + t\zeta_2}{\zeta_2 - z + t}, \qquad \xi_2 := \frac{(\zeta_1 - z)z - t\zeta_1}{\zeta_1 - z - t}. \tag{3.4}$$

Then one has $Q(\xi_1, \zeta_2) = Q(\zeta_1, \xi_2) = 0$, where we recall

$$Q \equiv Q(\zeta_1, \zeta_2) = (\zeta_1 - z)(\zeta_2 - z) + t(\zeta_1 - \zeta_2).$$

We set $\widetilde{\zeta_1} := \zeta_1 - z - t = \frac{\partial Q}{\partial \zeta_2}$ and $\widetilde{\zeta_2} := \zeta_2 - z + t = \frac{\partial Q}{\partial \zeta_1}$. One has

$$Q = \widetilde{\zeta_1}\widetilde{\zeta_2} + t^2 = \widetilde{\zeta_2}(\zeta_1 - \xi_1) = \widetilde{\zeta_1}(\zeta_2 - \xi_2).$$

We list some convenient formulæ which are direct from the definition.

$$\xi_1 - \zeta_2 = \frac{-(\zeta_2 - z)^2}{\widetilde{\zeta_2}}, \quad \zeta_1 - \xi_1 = \frac{Q}{\widetilde{\zeta_2}}, \quad \xi_1 - z = \frac{t(\zeta_2 - z)}{\widetilde{\zeta_2}}. \tag{3.5}$$

$$\xi_2 - \zeta_1 = -\frac{(\zeta_1 - z)^2}{\widetilde{\zeta_1}}, \quad \zeta_2 - \xi_2 = \frac{Q}{\widetilde{\zeta_1}}, \quad \xi_2 - z = \frac{-t(\zeta_1 - z)}{\widetilde{\zeta_1}}. \tag{3.6}$$

In the one-variable case, the Cauchy integral formula implies

$$\frac{1}{\ell!}\oint \frac{\varphi^{(k)}(\zeta)}{(\zeta - z)^{\ell+1}} d\zeta = \frac{1}{(\ell + k)!}\oint \frac{\varphi(\zeta)}{(\zeta - z)^{\ell+k+1}} d\zeta$$

for a holomorphic function $\varphi(\zeta)$ and for any $\ell, k \in \mathbb{N}$. However, in our setting, since Q is an irreducible polynomial of the two variables ζ_1 and ζ_2 for $t \neq 0$, the integration formulæ for derivatives of a holomorphic function $F(\zeta_1, \zeta_2)$ against the integral kernel Q^{-1} or its power are not so simple as in the one-variable case. We establish such formulæ for derivatives against the integral kernel $\zeta_j^a Q^{-b}$ $(a, b \in \mathbb{N})$ as below.

For $a, b \in \mathbb{N}$, we define functions $H_{a,b}(\zeta_1, \zeta_2)$ inductively by the following recurrence relation

$$H_{a,b} := (\xi_1 - z)^a H_{0,b} + \widetilde{\zeta_2}^{-1} Q \sum_{i=0}^{a-1} (\xi_1 - z)^i H_{a-1-i,b-1}, \tag{3.7}$$

with initial terms

$$H_{a,0} := 0 \quad \text{and} \quad H_{0,b} := b\widetilde{\zeta_2}. \tag{3.8}$$

Lemma 3.8. *For any $a, b \in \mathbb{N}$, one has*

$$\oint \frac{(\zeta_1 - z)^a}{Q^b} \frac{\partial F}{\partial \zeta_1} d\zeta_1 = \oint \frac{H_{a,b}}{Q^{b+1}} F d\zeta_1. \tag{3.9}$$

Analogous formulæ to (3.9) hold if we replace ζ_1 by ζ_2 and t by $-t$.

Proof. We begin with the case $a = 0$. Since $\xi_1 \equiv \xi_1(\zeta_2)$ is independent of the variable ζ_1, one has

$$\oint \frac{1}{(\zeta_1 - \xi_1)^b} \frac{\partial F}{\partial \zeta_1} d\zeta_1 = \frac{2\pi\sqrt{-1}}{(b-1)!} \frac{\partial^b F}{\partial \zeta_1^b}(\xi_1, \zeta_2) = b \oint \frac{F}{(\zeta_1 - \xi_1)^{b+1}} d\zeta_1.$$

Since $Q = \widetilde{\zeta_2}(\zeta_1 - \xi_1)$, the identity (3.9) holds for $a = 0$ with $H_{0,b} = b\widetilde{\zeta_2}$.

For $a \geq 1$, we proceed by induction on b. Obviously, (3.9) holds for $b = 0$ with $H_{a,0} = 0$. Suppose $a, b \geq 1$. By (3.5), one has

$$\frac{(\zeta_1 - z)^a}{Q^b} = \frac{(\xi_1 - z)^a}{Q^b} + \frac{1}{\widetilde{\zeta_2}} \sum_{i=0}^{a-1} \frac{(\xi_1 - z)^i (\zeta_1 - z)^{a-1-i}}{Q^{b-1}}.$$

Since both ξ_1 and $\widetilde{\zeta_2}$ are independent of the variable ζ_1, the induction step for the identity (3.9) is justified by the recurrence relation (3.7) defining $H_{a,b}$. □

Here are the first two examples of the family $H_{a,b}$ for $b = 1$ and 2.

$$H_{a,1} = t^a (\zeta_2 - z)^a \widetilde{\zeta_2}^{1-a}, \tag{3.10}$$

$$H_{a,2} = t^{a-1} (\zeta_2 - z)^{a-1} \widetilde{\zeta_2}^{1-a} (2t(\zeta_2 - z) + aQ). \tag{3.11}$$

The proof for Proposition 3.6 uses a special case of the formulæ (3.9):

$$\oint \frac{(\zeta_1 - z)^2}{Q^2} \frac{\partial F}{\partial \zeta_1} d\zeta_1 = \oint \frac{2t(\zeta_1 - z)(\zeta_2 - z)F}{Q^3} d\zeta_1, \tag{3.12}$$

$$\oint \frac{\zeta_j}{Q} \frac{\partial F}{\partial \zeta_j} d\zeta_j = -(Q - \frac{\partial Q}{\partial \zeta_j}\zeta_j) \oint \frac{F}{Q^2} d\zeta_j \qquad \text{for } j = 1, 2. \tag{3.13}$$

Proof of Proposition 3.6. By the definition (3.4) of ξ_1 and ξ_2, a direct computation shows

$$\widetilde{\zeta_1}\widetilde{\zeta_2}\xi_1 = (z + t)Q - t^2\zeta_1, \qquad \widetilde{\zeta_1}\widetilde{\zeta_2}\xi_2 = (z - t)Q - t^2\zeta_2. \tag{3.14}$$

By (3.9) and (3.10), one has

$$\oint \frac{-\zeta_1\zeta_2+\zeta_2^2}{Q}\frac{\partial F}{\partial \zeta_2}d\zeta_2 = \xi_2(\xi_2-\zeta_1)\widetilde{\zeta_1}\oint \frac{F}{Q^2}d\zeta_2.$$

By (3.6) and (3.14), the right-hand side equals

$$-\xi_2(\zeta_1-z)^2\oint \frac{F}{Q^2}d\zeta_2 = -\oint \frac{((z-t)Q-t^2\zeta_2)(\zeta_1-z)^2}{\widetilde{\zeta_1}\widetilde{\zeta_2}Q^2}Fd\zeta_2.$$

Applying this formula to $F := \frac{\partial f}{\partial \zeta_1}$, one has

(3.15)
$$(2\pi\sqrt{-1})^2 I_2(f) = -\oint\oint \frac{((z-t)Q-t^2\zeta_2)(\zeta_1-z)^2}{\widetilde{\zeta_1}\widetilde{\zeta_2}Q^2}\frac{\partial f}{\partial \zeta_1}d\zeta_1 d\zeta_2.$$

Since $-\widetilde{\zeta_1}\widetilde{\zeta_2}(Q+t^2)+Q^2=t^4$, one has

$$\frac{1}{\widetilde{\zeta_2}Q^2} = -\frac{\widetilde{\zeta_1}}{t^4Q^2}(Q+t^2)+\frac{1}{t^4\widetilde{\zeta_2}}.$$

Thus the function

$$\frac{((z-t)Q-t^2\zeta_2)(\zeta_1-z)^2}{\widetilde{\zeta_1}}\cdot\frac{1}{t^4\widetilde{\zeta_2}}\frac{\partial f}{\partial \zeta_1} = \frac{\xi_2(\zeta_1-z)^2}{t^4}\frac{\partial f}{\partial \zeta_1}$$

is holomorphic function in ζ_1, and does not contribute to the integral in (3.15), which reduces therefore to

$$\begin{aligned}
&\oint\oint \frac{(Q+t^2)((z-t)Q-t^2\zeta_2)(\zeta_1-z)^2}{t^4Q^2}\frac{\partial f}{\partial \zeta_1}d\zeta_1 d\zeta_2\\
&=\oint\oint \frac{(t^2Q(z-t-\zeta_2)-t^4\zeta_2)(\zeta_1-z)^2}{t^4Q^2}\frac{\partial f}{\partial \zeta_1}d\zeta_1 d\zeta_2\\
&=-\oint\oint \frac{\widetilde{\zeta_2}(\zeta_1-z)^2}{t^2Q}\frac{\partial f}{\partial \zeta_1}d\zeta_1 d\zeta_2-\oint\oint \frac{\zeta_2(\zeta_1-z)^2}{Q^2}\frac{\partial f}{\partial \zeta_1}d\zeta_1 d\zeta_2\\
&=-\oint\oint \frac{(\zeta_2-z)^2}{Q^2}f d\zeta_1 d\zeta_2-\oint\oint \frac{2t(\zeta_1-z)(\zeta_2-z)\zeta_2}{Q^3}f d\zeta_1 d\zeta_2.
\end{aligned}$$

In the first equality, we have used the fact that the integral involving Q^2 in the numerator vanishes. The last equality follows from Lemma 3.8, or more precisely, from (3.12) and (3.13). Hence the formula for $I_2(f)$ is proved. The proof for $I_1(f)$ is similar. □

Proof of Proposition 3.7. By (3.13),

$$\oint \frac{(\zeta_1-\zeta_2)\frac{\partial f}{\partial \zeta_1}}{Q} d\zeta_1 =(-Q+\frac{\partial Q}{\partial \zeta_1}\zeta_1 - \frac{\partial Q}{\partial \zeta_1}\zeta_2)\oint \frac{f}{Q^2} d\zeta_1$$

$$\oint \frac{(\zeta_1-\zeta_2)\frac{\partial f}{\partial \zeta_2}}{Q} d\zeta_2 =(\frac{\partial Q}{\partial \zeta_2}\zeta_1 + Q - \frac{\partial Q}{\partial \zeta_2}\zeta_2)\oint \frac{f}{Q^2} d\zeta_2.$$

Therefore one obtains

$$I_3(f) = \frac{1}{(2\pi\sqrt{-1})^2}\oint\oint \frac{-(\zeta_1-z)^2-(\zeta_2-z)^2}{Q^2} f d\zeta_1 d\zeta_2$$

because $(-Q+\widetilde{\zeta_2}\zeta_1-\widetilde{\zeta_2}\zeta_2)-(\widetilde{\zeta_1}\zeta_1+Q-\widetilde{\zeta_1}\zeta_2) = -(\zeta_1-z)^2-(\zeta_2-z)^2$. □

By Propositions 3.6 and 3.7, the proof of Theorem 3.1 is now complete.

We end this section by providing an example of closed formulæ for $Tf(z,t)$ for a specific family of functions $f \in \mathcal{O}(\Pi\times\Pi)$, where Π is the upper half plane. The family $\{f_\ell\}_{\ell\in\mathbb{N}}$ below gives the complete set of "singular vectors" in the tensor product of the two Hardy space, see Section 6.3:

Example 3.9. For $\ell \in \mathbb{N}$, we set

(3.16) $$f_\ell(\zeta_1,\zeta_2) := (\zeta_1-\zeta_2)^\ell(\zeta_1+\sqrt{-1})^{-\ell-1}(\zeta_2+\sqrt{-1})^{-\ell-1}.$$

Then one has the following:

(1) $Pf_\ell = -\ell(\ell+1)f_\ell$.

(2) $(Tf_\ell)(z,t) = \binom{2\ell}{\ell} t^\ell (z+\sqrt{-1})^{-2\ell-2}$.

Proof. (1) We set $[\ell,b,c] := (\zeta_1-\zeta_2)^\ell(\zeta_1+i)^{-b}(\zeta_2+i)^{-c}$. By a direct computation from the definition (3.1) of P, one has

$$P[\ell,b,c] = -\ell(\ell+1)[\ell,b,c] + bc[\ell+2,b+1,c+1] + (\ell+1)b[\ell+1,b+1,c] - (\ell+1)c[\ell+1,b,c+1].$$

Since $[\ell+1,b+1,c]-[\ell+1,b,c+1] = -[\ell+2,b+1,c+1]$, we have $P[\ell,\ell+1,\ell+1] = -\ell(\ell+1)[\ell,\ell+1,\ell+1]$.

(2) By Corollary 3.4 and Theorem 2.3, one has

$$(Tf_\ell)(z,t) = \frac{1}{\ell!}t^\ell(R_\ell f_\ell)(z).$$

By the definition (2.4) of the Rankin–Cohen bracket R_ℓ, one has

$$\begin{aligned}(R_\ell f_\ell)(z) =&(R_\ell(\zeta_1-\zeta_2)^\ell)(z+\sqrt{-1})^{-2\ell-2}\\ =&\frac{(2\ell)!}{\ell!}(z+\sqrt{-1})^{-2\ell-2},\end{aligned}$$

where the second equation follows from the formula

$$\sum_{j=0}^{\ell}\binom{\ell}{j}^2=\frac{(2\ell)!}{\ell!\ell!}.$$

Thus the second assertion is verified. □

4 Generating operators and holographic operators

Throughout this section, we assume that D is a convex domain in $\mathbb{C}$. Then any two elements $\zeta_1, \zeta_2 \in D$ can be joined by a line segment contained in D. For $\ell \in \mathbb{N}$, we consider a weighted average of $g \in \mathcal{O}(D)$ along the line segment between ζ_1 and ζ_2 given by

$$(\Psi_\ell g)(\zeta_1,\zeta_2) := (\zeta_1-\zeta_2)^\ell\int_{-1}^{1} g(\frac{(\zeta_2-\zeta_1)v+(\zeta_1+\zeta_2)}{2})(1-v^2)^\ell dv.$$

We investigate the "generating operator" T in connection with Ψ_ℓ. Recall from Corollary 3.4 that if $f \in \mathcal{S}ol(D\times D)_\ell$, namely, if $Pf = -\ell(\ell+1)f$, then $t^{-\ell}(Tf)(z,t)$ is independent of t, which we shall simply denote by $(t^{-\ell}Tf)(z)$.

Theorem 4.1. *Let* $\ell \in \mathbb{N}$.
(1) $t^{-\ell}T\colon \mathcal{S}ol(D\times D)_\ell \xrightarrow{\sim} \mathcal{O}(D)$ *is a bijection.*
(2) *The inverse of* $t^{-\ell}T$ *is given by the integral operator* Ψ_ℓ*, namely,* $\Psi_\ell\colon \mathcal{O}(D) \xrightarrow{\sim} \mathcal{S}ol(D\times D)_\ell$ *is a bijection and* $t^{-\ell}T\circ\Psi_\ell = \frac{2^{2\ell+1}}{2\ell+1}\,\mathrm{id}$.

As an immediate consequence of Theorem 4.1 (2), one has the following:

Corollary 4.2. *For any* $\ell \in \mathbb{N}$, $\mathcal{S}ol(\mathbb{C}\times\mathbb{C})_\ell$ *is infinite-dimensional.*

Proof of Theorem 4.1. First, we prove Image $\Psi_\ell \subset \mathcal{S}ol(D \times D)_\ell$. Recall from (3.1) the definition of P. A direct computation shows

$$
\begin{aligned}
&P(\Psi_\ell g) + \ell(\ell+1)\Psi_\ell g \\
=& -\frac{1}{2}(\zeta_1 - \zeta_2)^{\ell+1} \int_{-1}^{1} \frac{\partial}{\partial v}(g'(\frac{(\zeta_2 - \zeta_1)v + (\zeta_1 + \zeta_2)}{2})(1 - v^2)^{\ell+1})dv,
\end{aligned}
$$

which vanishes for any $g \in \mathcal{O}(D)$. Hence $\Psi_\ell g$ is an eigenfunction of P for the eigenvalue $-\ell(\ell+1)$.

Second, we prove that the "generating operator" T gives the inverse of Ψ_ℓ up to scalar multiplication, that is,

$$
T(\Psi_\ell g)(z,t) = \frac{2^{2\ell+1}}{2\ell+1} t^\ell g(z) \qquad \text{for any } g \in \mathcal{O}(D). \tag{4.1}
$$

To see (4.1), we observe from Corollary 3.4 that $t^{-\ell}(T\Psi_\ell g)(z,t)$ does not depend on the variable t because $\Psi_\ell g \in \mathcal{S}ol(D \times D)_\ell$. On the other hand, it follows from the expansion (2.5) that the coefficient of t^ℓ in $(Tf)(z,t)$ is given by

$$
\frac{1}{(2\pi\sqrt{-1})^2} \oint \oint \frac{(-1)^\ell (\zeta_1 - \zeta_2)^\ell f(\zeta_1, \zeta_2)}{(\zeta_1 - z)^{\ell+1}(\zeta_2 - z)^{\ell+1}} d\zeta_1 d\zeta_2.
$$

Applying this to $f = \Psi_\ell g$, one sees that $t^{-\ell}(T\Psi_\ell g)(z,t)$ is equal to

$$
\begin{aligned}
&\frac{(-1)^\ell}{(2\pi\sqrt{-1})^2} \oint \oint \frac{(\zeta_1 - \zeta_2)^{2\ell} \int_{-1}^{1} g(\frac{(\zeta_2-\zeta_1)v + (\zeta_1+\zeta_2)}{2})(1-v^2)^\ell dv}{(\zeta_1 - z)^{\ell+1}(\zeta_2 - z)^{\ell+1}} d\zeta_1 d\zeta_2 \\
=& \frac{(-1)^\ell}{(\ell!)^2} \left.\frac{\partial^{2\ell}}{\partial\zeta_1^\ell \partial\zeta_2^\ell}\right|_{\zeta_1=\zeta_2=z} ((\zeta_1 - \zeta_2)^{2\ell} \int_{-1}^{1} g(\frac{(\zeta_2 - \zeta_1)v + (\zeta_1 + \zeta_2)}{2})(1 - v^2)^\ell dv).
\end{aligned}
$$

An iterated use of the Leibniz rule develops the right-hand side as a sum of various derivatives, among which the only non-vanishing term is

$$
g(z)\frac{(2\ell)!}{(\ell!)^2} \int_{-1}^{1} (1 - v^2)^\ell dv = \frac{2^{2\ell+1}}{2\ell+1} g(z).
$$

Thus we have shown (4.1), hence the injective morphism $t^{-\ell}T \colon \mathcal{S}ol(D \times D)_\ell \to \mathcal{O}(D)$ is also surjective.

Finally, let us show the surjectivity of Ψ_ℓ. For any $f \in \mathcal{S}ol(D \times D)_\ell$, there exists $g \in \mathcal{O}(D)$ such that $(Tf)(z,t) = t^\ell g(z)$ by Corollary 3.4. Since the right-hand side equals $\frac{2\ell+1}{2^{2\ell+1}} T\Psi_\ell(g)$ by (4.1), one has

$f = \frac{2\ell+1}{2^{2\ell+1}}\Psi_\ell g$ because T is injective. Thus the surjectivity of Ψ_ℓ is shown. □

Remark 4.3. *When D is the upper half plane Π, the integral operator Ψ_ℓ appeared in the study of the holographic transforms for the branching problem of infinite-dimensional representations of $SL(2,\mathbb{R})$. In this case, the bijectivity of Ψ_ℓ was shown in* [7] *by a different approach based on the representation theory. See Section 6.*

5 The generating operator T and the Hardy space

Let Π be the upper half plane. As we have seen in Example 2.2, the "generating operator" $T\colon \mathcal{O}(\Pi\times\Pi) \to \mathcal{O}(U_\Pi)$ is well-defined where $U_\Pi = \{(z,t) \in \mathbb{C}^2 : 2|t| < \operatorname{Im} z\}$. This section discusses how the generating operator T acts on the tensor product of two Hardy spaces.

We recall that the Hardy space on Π is a Hilbert space defined by

$$\mathbf{H}(\Pi) = \{h \in \mathcal{O}(\Pi) : \|h\|^2_{\mathbf{H}(\Pi)} := \sup_{y>0}\int_{-\infty}^{\infty} |h(x+\sqrt{-1}y)|^2 dx < \infty\}.$$

Let $\mathbf{H}(\Pi\times\Pi)$ be the Hilbert completion $\mathbf{H}(\Pi)\widehat{\otimes}\mathbf{H}(\Pi)$ of the tensor product of two Hardy spaces $\mathbf{H}(\Pi)$. Any holomorphic differential operator P acting on $\mathcal{O}(\Pi\times\Pi)$ induces a continuous operator on this Hilbert space. In turn, the eigenspace $\mathbf{H}(\Pi\times\Pi)_\ell := \mathcal{S}ol(\Pi\times\Pi)_\ell \cap \mathbf{H}(\Pi\times\Pi)$ is a Hilbert subspace for every $\ell\in\mathbb{N}$.

Theorem 5.1. *Let P be the differential operator given in* (3.1)*.*
(1) *The differential operator P defines a self-adjoint operator on the Hilbert space $\mathbf{H}(\Pi\times\Pi)$.*
(2) (Eigenspace decomposition) *$\mathbf{H}(\Pi\times\Pi)$ decomposes into the discrete Hilbert sum of eigenspaces $\mathbf{H}(\Pi\times\Pi)_\ell$ of P where ℓ runs over $\mathbb{N}$.*
(3) *The generating operator T induces a family of linear operators*

$$t^{-\ell}T\colon \mathbf{H}(\Pi\times\Pi)_\ell \xrightarrow{\sim} \mathcal{O}(\Pi)\cap L^2(\Pi, y^{2\ell}dxdy)$$

which are unitary up to rescaling:

$$\|t^{-\ell}Tf\|^2_{L^2(\Pi,y^{2\ell+2}dxdy)} = b_\ell\|f\|^2_{\mathbf{H}(\Pi\times\Pi)} \quad \textit{for any } f\in\mathbf{H}(\Pi\times\Pi)_\ell \tag{5.1}$$

where we set

$$b_\ell := \frac{(2\ell)!}{2^{2\ell+2}\pi(2\ell+1)(\ell!)^2} = \frac{(2\ell-1)!!}{4\pi(2\ell+1)(2\ell)!!}. \tag{5.2}$$

For the proof of Theorem 5.1, we use the double Fourier–Laplace transform $\mathcal{F}$ defined by

$$F(x,y) \mapsto (\mathcal{F}F)(\zeta_1,\zeta_2) := \int_0^\infty \int_0^\infty F(x,y)e^{\sqrt{-1}(x\zeta_1+y\zeta_2)}dxdy.$$

According to the Payley–Wiener theorem, the Fourier–Laplace transform $\mathcal{F}$ establishes a bijection from $L^2(\mathbb{R}_+\times\mathbb{R}_+)$ onto $\mathbf{H}(\Pi\times\Pi)$, and satisfies $\|\mathcal{F}F\|^2_{\mathbf{H}(\Pi\times\Pi)} = (2\pi)^2\|F\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}_+)}$ for all $F \in L^2(\mathbb{R}_+\times\mathbb{R}_+)$. The inverse $\mathcal{F}^{-1}\colon \mathbf{H}(\Pi\times\Pi) \to L^2(\mathbb{R}_+\times\mathbb{R}_+)$ is given by

$$(\mathcal{F}^{-1}f)(x,y) = \lim_{\eta_1\downarrow 0}\lim_{\eta_2\downarrow 0}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} f(\zeta_1,\zeta_2)e^{-\sqrt{-1}(\zeta_1 x+\zeta_2 y)}d\xi_1 d\xi_2,$$

where we write $\zeta_j = \xi_j + \sqrt{-1}\eta_j$.

The change of variables $(x,y) = (\frac{s}{2}(1-v), \frac{s}{2}(1+v))$ yields a unitary map $L^2(\mathbb{R}_+\times(-1,1), sdsdv) \xrightarrow{\sim} L^2(\mathbb{R}_+\times\mathbb{R}_+, 2dxdy)$. We denote its composition with $\mathcal{F}$ by

$$\widetilde{\mathcal{F}}\colon L^2(\mathbb{R}_+\times(-1,1), sdsdv) \to \mathbf{H}(\Pi\times\Pi).$$

The inverse is given by $(\widetilde{\mathcal{F}}^{-1}f)(s,v) = (\mathcal{F}^{-1}f)(\frac{s}{2}(1-v), \frac{s}{2}(1-v))$.

Proposition 5.2. (1) $\widetilde{\mathcal{F}}\colon L^2(\mathbb{R}_+\times(-1,1), sdsdv) \xrightarrow{\sim} \mathbf{H}(\Pi\times\Pi)$ *is a unitary map up to a scalar multiplication, namely,*

$$\|f\|^2_{\mathbf{H}^2(\Pi\times\Pi)} = 2\pi^2\|(\widetilde{\mathcal{F}}^{-1}f)(s,v)\|^2_{L^2(\mathbb{R}_+\times(-1,1),sdsdv)} \quad \textit{for } f\in\mathbf{H}(\Pi\times\Pi).$$

(2) *The operator* $\widetilde{P} := \widetilde{\mathcal{F}}^{-1}\circ P\circ\mathcal{F}$ *takes the following form:*

$$\widetilde{P} = (1-v^2)\partial_v^2 - 2v\partial_v. \tag{5.3}$$

Proof. (1) For any $F(x,y)$, one has $\|\mathcal{F}F\|^2_{\mathbf{H}(\Pi\times\Pi)} = (2\pi)^2\|F\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}_+,dxdy)} = 2\pi^2\|F(\frac{s}{2}(1-v), \frac{s}{2}(1+v))\|^2_{L^2(\mathbb{R}_+\times(-1,1),sdsdv)}$.
(2) The Fourier transform $\mathcal{F}$ induces an isomorphism between the two Weyl algebras $\mathbb{C}[\zeta_1,\zeta_2,\frac{\partial}{\partial\zeta_1},\frac{\partial}{\partial\zeta_2}]$ and $\mathbb{C}[x,y,\frac{\partial}{\partial x},\frac{\partial}{\partial y}]$ by sending $\frac{\partial}{\partial\zeta_1}$, $\frac{\partial}{\partial\zeta_2}$, ζ_1, and ζ_2 to $\sqrt{-1}x$, $\sqrt{-1}y$, $\sqrt{-1}\frac{\partial}{\partial x}$, and $\sqrt{-1}\frac{\partial}{\partial y}$, respectively. In particular, the holomorphic differential operator P in (3.1) is transformed into the operator $\widehat{P} = (\partial_x-\partial_y)^2(xy) + (\partial_x-\partial_y)(x-y)$.

By the change of variables $(x,y) = (\frac{s}{2}(1-v), \frac{s}{2}(1+v))$, one has

$$x - y = -sv, \qquad xy = \frac{s^2}{4}(1-v^2), \qquad \partial_x - \partial_y = -\frac{2}{s}\frac{\partial}{\partial v}.$$

Hence the differential operator $\widehat{P}$ is transformed into $\widetilde{P}$. □

Proof of (1) and (2) in Theorem 5.1. (1) By Proposition 5.2, the differential operator P is equivalent via $\widetilde{\mathcal{F}}$ to the Legendre differential operator $\widetilde{P}$ which does not involve the variable s. Since $\widetilde{P}$ defines a self-adjoint operator on $L^2(\mathbb{R}_+ \times (-1,1), sdsdv)$, see Fact 7.1 (2) in Appendix, so does P on $\mathbf{H}(\Pi \times \Pi)$ via $\widetilde{\mathcal{F}}$.
(2) By (5.3), $Pf = \lambda f$ if and only if $\widetilde{P}(\widetilde{\mathcal{F}}^{-1}f) = \lambda(\widetilde{\mathcal{F}}^{-1}f)$. Hence $\widetilde{\mathcal{F}}$ induces an isomorphism $L^2(\mathbb{R}_+, sds) \otimes \mathbb{C}P_\ell(v) \xrightarrow{\sim} \mathbf{H}(\Pi \times \Pi)_\ell$ for every $\ell \in \mathbb{N}$, where $P_\ell(v)$ is the ℓ-th Legendre polynomial. Therefore the proof of the second statement is reduced to the classical theorem that $\{P_\ell\}_{\ell \in \mathbb{N}}$ forms an orthogonal basis in $L^2((-1,1), dv)$, see Fact 7.1 (1). □

To prove the third statement of Theorem 5.1, we apply the "generating operator" T to the diagram below:

$$\begin{array}{cccc} \mathcal{O}(\Pi \times \Pi) & \supset \mathbf{H}(\Pi \times \Pi) & \underset{\widetilde{\mathcal{F}}}{\xleftarrow{\sim}} & L^2(\mathbb{R}_+, sds)\widehat{\otimes} L^2(-1,1) \\ \cup & \cup & & \cup \\ \mathcal{S}ol(\Pi \times \Pi)_\ell & \supset \mathbf{H}(\Pi \times \Pi)_\ell & \underset{\widetilde{\mathcal{F}}}{\xleftarrow{\sim}} & L^2(\mathbb{R}_+, sds) \otimes \mathbb{C}P_\ell(v). \end{array} \tag{5.4}$$

We recall that the weighted Bergman space is defined by

$$\mathbf{H}^2(\Pi)_\lambda := \mathcal{O}(\Pi) \cap L^2(\Pi, y^{\lambda-2}dxdy)$$

for $\lambda > 1$. We also recall some basic properties of the Fourier–Laplace transform of one variable $\varphi(\xi) \mapsto (\mathcal{F}_\mathbb{R}\varphi)(z) := \int_0^\infty \varphi(\xi) e^{\sqrt{-1}z\xi} d\xi$. By the Plancherel formula, one has

$$\int_\mathbb{R} |\mathcal{F}_\mathbb{R}\varphi(x + \sqrt{-1}y)|^2 dx = 2\pi \int_0^\infty |\varphi(\xi)|^2 e^{-2y\xi} d\xi.$$

Integrating the both-hand sides against the measure $y^{\lambda-2}dy$, one obtains

$$\|\mathcal{F}_\mathbb{R}\varphi\|^2_{\mathbf{H}^2(\Pi)_\lambda} = 2^{2-\lambda}\pi\Gamma(\lambda-1)\|\varphi\|^2_{L^2(\mathbb{R}_+, \xi^{1-\lambda}d\xi)}. \tag{5.5}$$

Thus $\mathcal{F}_{\mathbb{R}}$ gives a bijection from $L^2(\mathbb{R}_+, \xi^{1-\lambda}d\xi)$ onto $\mathbf{H}^2(\Pi)_\lambda$, see [3, Thm. XIII.1.1] for details.

We show the following:

Proposition 5.3. *Let* $c_\ell := \frac{(-1)^{\frac{3}{2}\ell}}{(2\ell+1)\ell!}$ *and*

$$T^{\mathcal{F}}(h(z)P_\ell(v)) := c_\ell\, h(\xi)\, \xi^{\ell+1}\, t^\ell.$$

Then the following diagram commutes.

$$\begin{array}{ccc} \mathbf{H}(\Pi\times\Pi)_\ell & \underset{\widetilde{\mathcal{F}}}{\overset{\sim}{\leftarrow}} & L^2(\mathbb{R}_+, sds)\otimes\mathbb{C}P_\ell(v) \\ \overset{\sim}{\rightarrow} & & \overset{\sim}{\rightarrow} \\ T & & T^{\mathcal{F}} \\ \mathbf{H}^2(\Pi)_{2+2\ell}\otimes\mathbb{C}t^\ell & \underset{\mathcal{F}_{\mathbb{R}}\otimes\mathrm{id}}{\overset{\sim}{\leftarrow}} & L^2(\mathbb{R}_+, \xi^{-1-2\ell}d\xi)\otimes\mathbb{C}t^\ell \end{array}$$

Proof. Take any $h \in L^2(\mathbb{R}_+, sds)$. Since $\widetilde{\mathcal{F}}(hP_\ell) \in \mathbf{H}(\Pi\times\Pi)_\ell$ by Proposition 5.2 and its proof, one has

$$\ell! T(\widetilde{\mathcal{F}}(hP_\ell))(z,t) = t^\ell R_\ell \widetilde{\mathcal{F}}(hP_\ell)(z)$$

by Theorem 2.3 and Corollary 3.4. By the definition of $\widetilde{\mathcal{F}}$, one has

$$\widetilde{\mathcal{F}}(hP_\ell)(\zeta_1,\zeta_2) = \frac{1}{2}\int_0^\infty \int_{-1}^1 h(z)P_\ell(v)G(s,v;\zeta_1,\zeta_2)sdsdv,$$

where $G(s,v;\zeta_1,\zeta_2) := e^{\sqrt{-1}\frac{s}{2}((1-v)\zeta_1+(1+v)\zeta_2)}$. The Legendre polynomials $P_\ell(v)$ and the Rankin–Cohen bidifferential operators $R_\ell\colon \mathcal{O}(\mathbb{C}^2)\to\mathcal{O}(\mathbb{C})$ are related to each other via the function G as follows:

$$\begin{aligned} R_\ell G(s,v;\zeta_1,\zeta_2) =& \sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}^2 \left.\frac{\partial^\ell}{\partial\zeta_1^{\ell-j}\partial\zeta_2^j}\right|_{\zeta_1=\zeta_2=z} G(s,v,\zeta_1,\zeta_2) \\ =&(-1)^{\frac{3}{2}\ell}e^{\sqrt{-1}zs}s^\ell P_\ell(v) \end{aligned}$$

for all $\ell\in\mathbb{N}$. Here we have used the Rodrigues formula (7.1) for the second equality.

By using the formula for the L^2-norm P_ℓ (see Fact 7.1 (1)), one has

$$\begin{aligned} R_\ell\widetilde{\mathcal{F}}(hP_\ell)(z) =& \frac{(-1)^{\frac{3}{2}\ell}}{2}\int_0^\infty\int_{-1}^1 h(s)P_\ell(v)^2 e^{\sqrt{-1}zs}s^{\ell+1}dsdv \\ =& \frac{(-1)^{\frac{3}{2}\ell}}{2\ell+1}\mathcal{F}_{\mathbb{R}}(h(s)s^{\ell+1})(z). \end{aligned}$$

Hence Proposition 5.3 is proved. $\square$

Proof of (3) in Theorem 5.1. In light of the isomorphism

$$\widetilde{\mathcal{F}}\colon L^2(\mathbb{R}_+, sds) \otimes \mathbb{C}P_\ell(v) \xrightarrow{\sim} \mathbf{H}(\Pi \times \Pi)_\ell,$$

we take $h \in L^2(\mathbb{R}_+, sds)$ and set $f := \widetilde{\mathcal{F}}(hP_\ell) \in \mathbf{H}(\Pi \times \Pi)_\ell$. By Proposition 5.2, one has

$$\|f\|^2_{\mathbf{H}(\Pi\times\Pi)} = 2\pi^2 \|h\|^2_{L^2(\mathbb{R}_+,sds)} \|P_\ell\|^2_{L^2(-1,1)} = \frac{4\pi^2}{2\ell+1}\|h\|^2_{L^2(\mathbb{R}_+,sds)}. \tag{5.6}$$

Applying (5.5) to $\varphi := t^{-\ell}T^{\mathcal{F}}(hP_\ell)$ with $\lambda = 2\ell+2$, one has from Proposition 5.3 that

$$\begin{aligned}\|t^{-\ell}Tf\|^2_{\mathbf{H}^2(\Pi)_{2+2\ell}} =& 2^{-2\ell}\pi(2\ell)!\|t^{-\ell}T^{\mathcal{F}}(hP_\ell)\|^2_{L^2(\mathbb{R}_+,\xi^{-1-2\ell}d\xi)} \\ =&\frac{\pi(2\ell)!}{2^{2\ell}(2\ell+1)^2(\ell!)^2}\|h\|^2_{L^2(\mathbb{R}_+,sds)}.\end{aligned} \tag{5.7}$$

It follows from (5.2), (5.6) and (5.7) that

$$\|t^{-\ell}Tf\|^2_{\mathbf{H}^2(\Pi)_{2+2\ell}} = \frac{(2\ell-1)!!}{4\pi(2\ell+1)(2\ell)!!}\|f\|^2_{\mathbf{H}(\Pi\times\Pi)} = b_\ell\|f\|^2_{\mathbf{H}(\Pi\times\Pi)}.$$

Hence the third statement of Theorem 5.1 is proved. □

6 Representation theory and the generating operator T

If D is simply connected, then the group $\mathrm{Aut}(D)$ of biholomorphic diffeomorphisms acts transitively on D. This section discusses different perspectives of our generating operator T from the viewpoint of the automorphism group of the domain, in particular, from the (infinite-dimensional) representation theory of real reductive groups. Lie theory reveals structures of the generating operator T that are not otherwise evident.

6.1. Normal derivatives and the generating operator T.

Let π be an irreducible representation of a group G, and G' a subgroup. The G-module π may be seen as a G'-module by restriction, for which we write $\pi|_{G'}$. For an irreducible representation ρ of the subgroup G', a *symmetry breaking operator* (SBO for short) is an intertwining operator from $\pi|_{G'}$ to ρ, whereas a *holographic operator* is an intertwining operator from ρ to $\pi|_{G'}$. Suppose that the representations π and ρ are geometrically defined, *e.g.,* they are realized in the spaces $\Gamma(X)$ and

$\Gamma(Y)$ of functions on a G-manifold X and its G'-submanifold Y, respectively, or more generally, in the spaces of sections for some equivariant vector bundles.

When the restriction $\pi|_{G'}$ is discretely decomposable [4], one may expect that taking "normal derivatives" with respect to the submanifold $Y \hookrightarrow X$ would yield SBOs. However, this is not the case even for the irreducible decomposition (*fusion rule*) of the tensor product of two representations of $SL(2, \mathbb{R})$. See [6, Thm. 5.3] for more general cases. The underlying geometry for the fusion rule of the Hardy spaces $\mathbf{H}(\Pi)$ is given by a diagonal embedding of $Y = \Pi$ into $X := Y \times Y$. Instead of using $X = \Pi \times \Pi$, we consider $\widetilde{X} := U_\Pi$ as in Example 2.2. In this case the "normal derivative" of ℓ-th order with respect to $Y \hookrightarrow \widetilde{X}$ is given simply by

$$N_\ell := \operatorname{Rest}_{t=0} \circ (\frac{\partial}{\partial t})^\ell.$$

A distinguishing feature of the generating operator T is that all the normal derivatives N_ℓ give rise to symmetry breaking operators after the transformation by T, symbolically written in the following diagram (see (6.2) for the notation π_λ):

$$\begin{array}{ccc} \mathcal{O}(X) & \overset{T}{\hookrightarrow} & \mathcal{O}(\widetilde{X}) \\ \text{SBO} \searrow & \circlearrowright & \swarrow \ell\text{-th normal derivative } N_\ell \\ & (\pi_{2+2\ell}, \mathcal{O}(Y)) & \end{array}$$

6.2. Modular forms and the generating operator T. The Rankin–Cohen brackets were introduced in [2, 9] to construct holomorphic modular forms of higher weight from those of lower weight. This section highlights the relationship of our generating operator T in (2.1) and modular forms.

By Theorem 2.3, one has

$$N_\ell \circ T = R_\ell, \tag{6.1}$$

where R_ℓ are the Rankin–Cohen brackets (2.4). Then by a direct computation [2] or by the F-method [6], one sees the following covariance property:

Proposition 6.1. *For all* $\ell \in \mathbb{N}$*, for any* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ *and for any* $f \in \mathcal{O}(\Pi \times \Pi)$*, one has*

$$N_\ell \circ (Tf^g)(z) = (cz+d)^{-2\ell-2}((N_\ell \circ T)f)(\frac{az+b}{cz+d})$$

where $f^g(\zeta_1, \zeta_2) := (c\zeta_1 + d)^{-1}(c\zeta_2 + d)^{-1} f(\frac{a\zeta_1+b}{c\zeta_1+d}, \frac{a\zeta_2+b}{c\zeta_2+d})$.

To clarify its representation-theoretic meaning, we write π_λ ($\lambda \in \mathbb{Z}$) for a representation of $SL(2,\mathbb{R})$ on $\mathcal{O}(\Pi)$ given by

$$\pi_\lambda(g)h(z) = (cz+d)^{-\lambda} h(\frac{az+b}{cz+d}) \qquad \text{for } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.2}$$

Then Proposition 6.1 tells us that

$$(N_\ell \circ T) \circ (\pi_1(g) \boxtimes \pi_1(g)) = \pi_{2\ell+2}(g) \circ (N_\ell \circ T) \tag{6.3}$$

for any $g \in SL(2,\mathbb{R})$. Therefore, for a subgroup Γ, $N \circ T(f)$ is Γ-invariant whenever f is $(\Gamma \times \Gamma)$-invariant.

Suppose that Γ is a congruence subgroup of $SL(2,\mathbb{Z})$. For any modular form h of level Γ and weight 1, we set

$$H(z,t) := \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_1} \oint_{C_2} \frac{h(\zeta_1)h(\zeta_2)}{(\zeta_1 - z)(\zeta_2 - z) + t(\zeta_1 - \zeta_2)} d\zeta_1 d\zeta_2.$$

It follows from (6.1) that $(N_\ell H)(z) = (\frac{\partial}{\partial t})^\ell\big|_{t=0} H(z,t) = R_\ell(h(\zeta_1)h(\zeta_2))(z)$ is a modular form of level Γ and weight $2\ell + 2$ for all $\ell \in \mathbb{N}$.

6.3. Unitary representation and the generating operator T.

Viewed as a representation of the universal covering group $SL(2,\mathbb{R})\tilde{}$, the representation π_λ is well-defined for all $\lambda \in \mathbb{C}$. For $\lambda > 1$, π_λ leaves the weighted Bergman space $\mathbf{H}^2(\Pi)_\lambda = \mathcal{O}(\Pi) \cap L^2(\Pi, y^{\lambda-2}dxdy)$ invariant, and $SL(2,\mathbb{R})\tilde{}$ acts as an irreducible unitary representation on the Hilbert space $\mathbf{H}^2(\Pi)_\lambda$. These unitary representations $(\pi_\lambda, \mathbf{H}^2(\Pi)_\lambda)$ are referred to as (relative) *holomorphic discrete series representations* of $SL(2,\mathbb{R})\tilde{}$. In particular, the set of holomorphic discrete series representations of the group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I_2\} \simeq \mathrm{Aut}(\Pi)$ is given by $\{\pi_\lambda : \lambda = 2, 4, 6, \dots\}$.

If $\lambda = 1$ then $\mathbf{H}^2(\Pi)_\lambda = \{0\}$, however, the Hardy space $\mathbf{H}(\Pi)$ is an invariant subspace of $(\pi_\lambda, \mathcal{O}(\Pi))$ with $\lambda = 1$, and $SL(2, \mathbb{R})$ acts on $\mathbf{H}(\Pi)$ as an irreducible unitary representation, too.

With these notations, one may interpret Theorem 5.1 as a decomposition of the completed tensor product of two copies of the unitary representation $(\pi_1, \mathbf{H}(\Pi))$ on the Hardy space into a multiplicity-free discrete sum of irreducible unitary representations:

$$\mathbf{H}(\Pi)\widehat{\otimes}\mathbf{H}(\Pi) \simeq {\sum_{\ell=0}^{\infty}}^{\oplus} \mathbf{H}^2(\Pi)_{2+2\ell} \qquad \text{(Hilbert direct sum)}.$$

The right-hand side may be thought of as a "model" of holomorphic discrete series representations of $PSL(2, \mathbb{R})$ in the sense that all such representations occur exactly once.

6.4. Limit of the weighted Bergman spaces.

The Hardy norm $\|\cdot\|_{\mathbf{H}(\Pi)}$ may be regarded as the residue of the analytic continuation of the norm of the weighted Bergman space $\mathbf{H}^2(\Pi)_\lambda$ which is originally defined for real $\lambda > 1$:

$$\|\cdot\|^2_{\mathbf{H}(\Pi)} = \lim_{\lambda\downarrow 1}(\lambda - 1)\|\cdot\|^2_{\mathbf{H}^2(\Pi)_\lambda}.$$

Then the exact formula (5.1) in Theorem 5.1 may be thought of as the limit of [7, Thm. 2.7] which dealt with the weighted Bergman spaces, namely, our b_ℓ in Theorem 5.1 may be rediscovered by the following limit procedure with the notation as in [7, (2.3) and (2.4)]:

$$\begin{aligned}
&\frac{1}{(\ell!)^2}\lim_{\lambda'\downarrow 1}\lim_{\lambda''\downarrow 1}\frac{c_\ell(\lambda',\lambda'')r_\ell(\lambda',\lambda'')}{(\lambda'-1)(\lambda''-1)} \\
=&\frac{1}{(\ell!)^2}\lim_{\lambda'\downarrow 1}\lim_{\lambda''\downarrow 1}\frac{\Gamma(\lambda'+\ell)\Gamma(\lambda''+\ell)}{(\lambda'+\lambda''+2\ell-1)\Gamma(\lambda'+\lambda''+\ell-1)\ell!}\cdot\frac{\Gamma(\lambda'+\lambda''+2\ell-1)}{2^{2\ell+2}\pi\Gamma(\lambda')\Gamma(\lambda'')} \\
=&\frac{(2\ell)!}{(2\ell+1)\pi(\ell!)^2 2^{2\ell+2}} = \frac{(2\ell-1)!!}{4\pi(2\ell+1)(2\ell)!!} = b_\ell.
\end{aligned}$$

7 Appendix: The Legendre polynomials

Suppose $\ell \in \mathbb{N}$. The Legendre polynomial $P_\ell(v)$ is a polynomial solution to the Legendre differential equation:

$$((1-v^2)\frac{d^2}{dv^2} - 2v\frac{d}{dv} + \ell(\ell+1))f = 0$$

which is normalized by $P_\ell(1) = 1$. Then it satisfies the Rodrigues formula

$$P_\ell(v) := \frac{1}{2^\ell}\sum_{j=0}^{\ell}\binom{\ell}{j}^2 (v-1)^{\ell-j}(v+1)^j. \tag{7.1}$$

Fact 7.1 ([8, 10]). (1) *The Legendre polynomials $\{P_\ell(v)\}_{\ell\in\mathbb{N}}$ form an orthogonal basis in the Hilbert space $L^2((-1,1),dv)$ with the following norm:*

$$(P_\ell, P_{\ell'})_{L^2(-1,1)} = \frac{2}{2\ell+1}\delta_{\ell\ell'}.$$

(2) *The differential operator $(1-v^2)\frac{d^2}{dv^2} - 2v\frac{d}{dv}$ is essentially self-adjoint on the Hilbert space $L^2((-1,1),dv)$.*

The Legendre polynomials $P_\ell(x)$ are particular cases of the Jacobi polynomials $P_\ell^{\alpha,\beta}(x)$ with $\alpha = \beta = 0$.

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