# Conjectures on reductive homogeneous spaces

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## Abstract

We address some conjectures and open problems in 'analysis of symmetries' which include the study of non-commutative harmonic analysis and discontinuous groups for reductive homogeneous spaces beyond the classical framework:

(1) discrete series for non-symmetric homogeneous spaces $G/H$;

(2) discontinuous group $\Gamma$ for $G/H$ beyond the Riemannian setting;

(3) analysis on pseudo-Riemannian locally homogeneous spaces.

## 1 Introduction

I have been working on various subjects of mathematics, and "symmetry" is a key word to create new interactions among different disciplines.

It was when I was an undergraduate student that Springer opened an office in Tokyo within walking distance of our mathematics building. In 1989 Catriona visited Japan for seven weeks to promote editorial activities in Japan including the forthcoming publication of the Proceedings of ICM 90 Kyoto, though I was not aware of it at that time. In 2006, the Mathematical Society of Japan relaunched the 3rd series of Japanese Journal of Mathematics (JJM) and started to collaborate with Springer as a global partner. Since then, as the founding Editor, I have been indebted by the high quality of Springer's staffs whom Catriona mentored. She visited frequently the Takagi Lectures, and we talked various topics over coffee. Each time, I was

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impressed by her care about what mathematicians think really important. Her work created new interactions among mathematicians from different disciplines. I admire her for all her invaluable contributions to the mathematics community over the years.

In this paper, I would like to address some conjectures and problems of the areas of "analysis of symmetries" in which I have been deeply involved, but to which I do not have a solution.

(1) Discrete series for non-symmetric homogeneous spaces $G/H$;

(2) Discontinuous group $\Gamma$ for $G/H$ beyond the Riemannian setting;

(3) Analysis on non-Riemannian locally homogeneous spaces $\Gamma \backslash G/H$.

These three topics are discussed in Sections 3–5, respectively, using a common setting which is explained in Section 2 with simple examples.

## 2  Basic settings

Throughout this paper, our basic geometric setting will be as follows.

**Setting 2.1.** $G$ is a real reductive linear Lie group, $H$ is a closed proper subgroup which is reductive in $G$, and $X := G/H$.

A distinguished feature of this setting is that the manifold $X$ carries a *pseudo-Riemannian structure* with a 'large' isometry group, namely, the reductive group $G$ acts transitively and isometrically on $X$. Such a pseudo-Riemannian structure is induced from the Killing form $B$ if $G$ is semisimple. For reductive $G$, one can take a maximal compact subgroup $K$ of $G$ such that $H \cap K$ is a maximal compact subgroup of $H$, and a $G$-invariant symmetric bilinear form $B$ on the Lie algebra $\mathfrak{g}$ of $G$ such that the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is an orthogonal decomposition with respect to $B$ and that $B$ is negative definite on $\mathfrak{k}$ and is positive definite on $\mathfrak{p}$. Then $B$ induces a $G$-invariant pseudo-Riemannian structure of signature $(p, q)$ on $X$, where $p + q = \dim X$ and $q = \dim K/H \cap K$.

Very special cases of homogeneous spaces in Setting 2.1 include:

**Example 2.2** (semisimple coadjoint orbits). For a reductive Lie group $G$, one can identify the Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ via $B$. The coadjoint orbit $\mathcal{O}_\lambda := \mathrm{Ad}^*(G)\lambda$ is called *semisimple*, *elliptic*, *hyperbolic*, or *regular* if the element in $\mathfrak{g}$ corresponding to $\lambda$ has that property. We write $\mathfrak{g}_{ss}^*$, $\mathfrak{g}_{ell}^*$, $\mathfrak{g}_{hyp}^*$, or $\mathfrak{g}_{reg}^*$ for the collection of such elements $\lambda$, respectively. By definition, $\mathfrak{g}_{ell}^*$, $\mathfrak{g}_{hyp}^* \subset \mathfrak{g}_{ss}^*$. The isotropy subgroup of $\lambda$ is reductive if $\lambda \in \mathfrak{g}_{ss}^*$, hence any semisimple coadjoint orbit $\mathcal{O}_\lambda$ gives an example of Setting 2.1.

For compact $G$, one has $\mathfrak{g}_{ell}^* = \mathfrak{g}_{ss}^* = \mathfrak{g}^*$ and $\mathcal{O}_\lambda$ is a generalized flag variety for any $\lambda \in \mathfrak{g}^*$; $\mathcal{O}_\lambda$ is a full flag variety iff $\lambda \in \mathfrak{g}_{reg}^*$.

The subclass $\{\mathcal{O}_\lambda : \lambda \in \mathfrak{g}_{ss}^*\}$ in Setting 2.1 plays a particular role in the unitary representation theory. The orbit philosophy due to Kirillov–Kostant–Duflo–Vogan suggests an intimate relationship between the set $\mathfrak{g}^*/\mathrm{Ad}^*(G)$ of coadjoint orbits and the set of equivalence classes of irreducible unitary representations of $G$ (the *unitary dual* $\widehat{G}$):

$$\mathfrak{g}^*/\mathrm{Ad}^*(G) \fallingdotseq \widehat{G}, \qquad \mathcal{O}_\lambda \leftrightarrow \pi_\lambda. \tag{1}$$

We recall that any coadjoint orbit carries a canonical symplectic form called the *Kirillov–Kostant–Souriau form*. Then the correspondence $\mathcal{O}_\lambda \mapsto \pi_\lambda$ is supposed to be a "geometric quantization" of the Hamiltonian $G$-manifold $\mathcal{O}_\lambda$ if such $\pi_\lambda$ exists. This philosophy works fairly well for $\lambda \in \mathfrak{g}_{ss}^*$ satisfying an appropriate integral condition: loosely speaking, $\pi_\lambda$ is obtained by a unitary induction from a parabolic subgroup (real polarization of the para-Hermitian manifold $\mathcal{O}_\lambda$) for $\lambda \in \mathfrak{g}_{hyp}^*$, in a Dolbeault cohomology space on the pseudo-Kähler manifold $\mathcal{O}_\lambda$ as a generalization of the Borel–Weil–Bott theorem (complex polarization of $\mathcal{O}_\lambda$) or alternatively by a cohomological parabolic induction (*e.g.,* Zuckerman's derived functor module $A_{\mathfrak{q}}(\lambda)$) for $\lambda \in \mathfrak{g}_{ell}^*$, and by the combination of these two procedures for general $\lambda \in \mathfrak{g}_{ss}^*$, although there are some delicate issues about singular $\lambda$ and also about "$\rho$-shift" of the parameter, see [15, Chap. 2] for instance. The resulting "quantizations" $\pi_\lambda$ of semisimple coadjoint orbits $\mathcal{O}_\lambda$ give "large part" of the unitary dual $\widehat{G}$.

**Example 2.3** (symmetric spaces, real spherical spaces). Let $\sigma$ be an automorphism of a reductive Lie group $G$ of finite order, $G^\sigma$ the fixed point subgroup of $\sigma$, and $H$ an open subgroup of $G^\sigma$. Then $H$ is reductive and the homogeneous space $G/H$ provides another example of Setting 2.1. In particular, if the order of $\sigma$ is two, $G/H$ is called a (reductive) *symmetric space*. Geometrically, it is a symmetric space with respect to the Levi-Civita

connection of the pseudo-Riemannian structure in the sense that all geodesic symmetries are globally defined isometries. This is a subclass of Setting 2.1 for which the $L^2$-analysis has been extensively studied over 60 years. Group manifolds $(G \times G)/\operatorname{diag}(G)$, Riemannian symmetric spaces $G/K$ and irreducible affine symmetric spaces such as $SL(p+q, \mathbb{R})/SO(p,q)$ are examples of reductive symmetric spaces. More generally, in Setting 2.1, one has

$$\{\text{symmetric spaces}\} \subset \{\text{spherical spaces}\} \subset \{\text{real spherical spaces}\},$$

where we say $G/H$ is *spherical* if a Borel subgroup of the complexification $G_{\mathbb{C}}$ has an open orbit in $G_{\mathbb{C}}/H_{\mathbb{C}}$, and $G/H$ is *real spherical* if a minimal parabolic subgroup of $G$ has an open orbit in $G/H$. See Kobayashi–T. Oshima [21] for the roles that these geometric properties play in the global analysis on $G/H$. When $H$ is compact, $G/H$ is spherical if and only if it is a weakly symmetric space in the sense of Selberg.

The model space of non-zero constant sectional curvatures in pseudo-Riemannian geometry is a special case of reductive symmetric spaces:

**Example 2.4** (pseudo-Riemannian space form, see [33])**.** The hypersurface

$$X(p,q) := \{x \in \mathbb{R}^{p+q+1} : x_1^2 + \cdots + x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q+1}^2 = 1\}$$

in $\mathbb{R}^{p+1,q} := (\mathbb{R}^{p+q+1}, ds^2 = dx_1^2 + \cdots + dx_{p+1}^2 - dx_{p+2}^2 - \cdots - dx_{p+q}^2)$ carries a pseudo-Riemannian structure of signature $(p,q)$ with constant sectional curvature 1. Equivalently, we may regard $X(p,q)$ as a space of constant sectional curvature $-1$ with respect to the pseudo-Riemannian metric of signature $(q,p)$. If $q = 0$, $p = 0$, $q = 1$, or $p = 1$, then $X(p,q)$ is the sphere $\mathrm{S}^p$, the hyperbolic space $\mathrm{H}^q$, the de Sitter space $\mathrm{dS}^{p+1}$, or the anti-de Sitter space $\mathrm{AdS}^{q+1}$, respectively. For general $(p,q)$, the generalized Lorentz group $O(p+1,q)$ acts transitively and isometrically on $X(p,q)$, and one has a diffeomorphism $X(p,q) \simeq O(p+1,q)/O(p,q)$, giving an expression of $X(p,q)$ as a reductive symmetric space of rank one.

# 3 Problems on discrete series for $G/H$

The 'smallest units of symmetries' defined by group actions may be
        *irreducible representations* if the action is linear, and
        *homogeneous spaces* if the action is smooth on a manifold.

The objects of this section are irreducible subrepresentations in $L^2(X)$ for homogeneous spaces $X$, that is, *discrete series representations* for $X$ (Definition 3.2 below), which are supposed to be a building block in global analysis on $X$. For instance, when $X$ is a reductive symmetric space, parabolic inductions of discrete series representations for subsymmetric spaces yield the full spectrum in the Plancherel formula of $L^2(X)$, see [3] for instance.

This section elucidates the following problem in the generality of Setting 2.1 by using simple examples, and addresses some related conjectures.

**Problem 3.1.** Find all discrete series representations for $G/H$.

Let us fix some notation. Suppose a Lie group $G$ acts continuously on a manifold $X$. Then one has a natural unitary representation (*regular representation*) of $G$ on the Hilbert space $L^2(X)$ of $L^2$-sections for the half-density bundle $\mathcal{L} := (\wedge^{\dim X} T^* X \otimes or_X)^{\frac{1}{2}}$ of $X$ where $or_X$ stands for the orientation bundle.

**Definition 3.2.** An irreducible unitary representation $\pi$ of $G$ is said to be a *discrete series representation* for $X$ if there exists a non-zero continuous $G$-homomorphism from $\pi$ to the regular representation on $L^2(X)$. In other words, discrete series representations for $X$ are irreducible subrepresentations realized in closed subspaces of the Hilbert space $L^2(X)$.

We denote by $\mathrm{Disc}(X)$ the set of discrete series representations for $X$. It is a (possibly, empty) subset of the unitary dual $\widehat{G}$ of the group $G$.

Hereafter, suppose we are in Setting 2.1. Then there is a $G$-invariant Radon measure $\mu$ on the homogeneous space $X = G/H$, hence $\mathcal{L}$ is trivial as a $G$-equivariant bundle and $L^2(X)$ may be identified with $L^2(X, d\mu)$.

If $G/H$ is spherical (see Example 2.3), then the ring

$$\mathbb{D}(G/H) := \{G\text{-invariant differential operators on } G/H\}$$

is commutative and the multiplicity of irreducible representations $\pi$ of $G$ in the regular representation on $C^\infty(G/H)$ is uniformly bounded, and vice versa [21]. In this case, the disintegration of the regular representation $L^2(X)$ into irreducibles (the Plancherel-type theorem) is essentially equivalent to the joint spectral decomposition for the commutative ring $\mathbb{D}(G/H)$, and Problem 3.1 highlights point spectra in $L^2(G/H)$.

Classical examples trace back to Gelfand–Graev (1962), Shintani (1967), Molchanov (1968), J. Faraut (1979), R. S. Strichartz (1983) and some others

on the analysis of the space form $X(p,q)$, which we review with some modern viewpoints, see [20, Thm. 2.1] and references therein.

**Example 3.3.** Let $(G,H) = (O(p+1,q), O(p,q))$ and $X = G/H$ as in Example 2.4. Then the ring $\mathbb{D}(G/H)$ is generated by the Laplacian $\Delta_X$, which is not an elliptic differential operator if $p, q > 0$. We set

$$L^2(X)_\lambda := \{f \in L^2(X) : \Delta_X f = \lambda f \text{ in the weak sense}\}.$$

Then $L^2(X)_\lambda$ is a closed subspace in $L^2(X)$, and the resulting unitary representation of $G$ on $L^2(X)_\lambda$ is irreducible whenever it is non-zero. Conversely, any discrete series representation for $X$ is realized on an $L^2$-eigenspace $L^2(X)_\lambda$ for some eigenvalue $\lambda$. In particular, one has the equivalence:

$$\text{Disc}(G/H) = \emptyset \iff \text{there is no point spectrum of } \Delta_X \text{ in } L^2(X)$$
$$\iff p = 0 \text{ and } q \geq 1.$$

Thus there exists point spectrum of the Laplacian $\Delta_X$ in $L^2(X)$ unless $X = X(p,q)$ is a hyperbolic space $H^q \equiv X(0,q)$. The description of the eigenspace $L^2(X)_\lambda$ for $q = 0$ is the classical theory of spherical harmonics on the sphere $S^p \equiv X(p,0)$. For $p, q \geq 1$, one has

$$L^2(X)_\lambda \neq 0 \text{ iff } \lambda = \lambda_k \text{ for some } k \in \mathbb{Z} \text{ with } -\frac{1}{2}(p+q-1) < k,$$

where $\lambda_k := -k(k+p+q-1)$. The resulting irreducible unitary representation on $L^2(X)_{\lambda_k}$ is isomorphic to a 'geometric quantization' of an elliptic coadjoint orbit of minimal dimension, or alternatively in an algebraic language, it is the unitarization of Zuckerman's derived functor module $A_{\mathfrak{q}}(k)$ with the normalization as in [31]. This algebraic description involves delicate questions for finitely many exceptional parameters, *i.e.,* those for $k < 0$, see Problem 3.10 below.

In the generality of Setting 2.1, we may divide Problem 3.1 into two subproblems:
(A) a characterization of the pairs $(G,H)$ for which $G/H$ admits at least one discrete series representation (Problem 3.4);
(B) a description of all discrete series representations for $X$.
We address Conjectures 3.6 and 3.7 as subproblems for (A), and formulate Conjecture 3.9 and Problem 3.10 for (B).

**Problem 3.4.** Find a characterization of the pairs $(G, H)$ such that $G/H$ admits a discrete series representation.

Similarly to the classical fact that there is no discrete spectrum of the Laplacian $\Delta_{\mathbb{R}^n}$ on $\mathbb{R}^n$ and that there is no continuous spectrum of the Laplacian $\Delta_{\mathbb{T}^n}$ on the $n$-torus $\mathbb{T}^n$, the Riemannian symmetric space $G/K$ does not admit any discrete series representation if it is of non-compact type and does not admit any continuous spectrum in the Plancherel formula if it is of compact type. The answer to Problem 3.4 is known for reductive symmetric spaces by the rank condition:

$$\operatorname{Disc}(G/H) \neq \emptyset \iff \operatorname{rank} G/H = \operatorname{rank} K/H \cap K. \tag{2}$$

The equivalence (2) was proved by Flensted-Jensen for $\Leftarrow$ and Matsuki–Oshima for $\Rightarrow$. It generalizes the Riemannian case $G/K$ as well as Harish-Chandra's rank condition for a group manifold, see [25] and references therein.

Beyond symmetric spaces, several approaches (*e.g.*, branching laws [12, 17], the wave front set [5], etc.) have been applied to find new families of (not necessarily, real spherical) homogeneous spaces $G/H$ that admit discrete series representations. It is more involved to prove the opposite direction, *i.e.*, to prove $\operatorname{Disc}(G/H) = \emptyset$ for non-symmetric spaces, and very little is known so far, except for certain families of spherical homogeneous spaces. For instance, one has:

**Example 3.5** (real forms of $SL(2n + 1, \mathbb{C})/Sp(n, \mathbb{C})$, [12]). $\operatorname{Disc}(G/H) = \emptyset$ if $G/H = SL(2n+1, \mathbb{R})/Sp(n, \mathbb{R})$, whereas $\#\operatorname{Disc}(G/H) = \infty$ for other real forms of $G_{\mathbb{C}}/H_{\mathbb{C}}$, *i.e.*, $SU(2p, 2q + 1)/Sp(p, q)$ or $SU(n, n + 1)/Sp(n, \mathbb{R})$.

An optimistic solution to Problem 3.4 may be a combination of the following two conjectures:

**Conjecture 3.6** ([15, Conj. 6.9]). In Setting 2.1, one has the equivalence:

$$\operatorname{Disc}(G/H) \neq \emptyset \iff \#\operatorname{Disc}(G/H) = \infty.$$

**Conjecture 3.7.** In Setting 2.1, one has the following equivalence:

$$\#\operatorname{Disc}(G/H) = \infty \iff \mathfrak{h}^{\perp} \cap \mathfrak{g}_{\text{ell}}^* \text{ contains a non-empty open set of } \mathfrak{h}^{\perp}.$$

Both of the conjectures are true for reductive symmetric spaces $G/H$. In fact, Conjecture 3.7 is a reformulation of the rank condition (2) in the spirit of the orbit philosophy.

There are counterexamples for the implication $\Rightarrow$ of an analogous statement to Conjectures 3.6 and 3.7 if we drop the assumption that $H$ is reductive, for instance, they fail when $H$ is a parabolic sugroup and a cocompact discrete subgroup of $G$ with $\operatorname{rank} G > \operatorname{rank} K$, respectively. The implication $\Leftarrow$ in Conjecture 3.7 has been proved in Harris–Y. Oshima [5] recently without reductivity assumption on $H$.

**Remark 3.8.** Similarly to Conjecture 3.7, one might expect the equivalence:

$$L^2(G/H) \text{ is tempered } \iff \mathfrak{h}^\perp \cap \mathfrak{g}^*_{\mathrm{reg}} \text{ is dense in } \mathfrak{h}^\perp.$$

This is proved in Benoist–Kobayashi [1] for complex homogeneous spaces for any algebraic subgroup $H$ without reductivity assumption.

Once we know $\operatorname{Disc}(G/H) \neq \emptyset$, we may wish to capture *all* elements of $\operatorname{Disc}(G/H)$. We divide this exhaustion problem into two questions: one is geometric (Conjecture 3.9), and the other is algebraic (Problem 3.10).

**Conjecture 3.9.** Any $\pi \in \operatorname{Disc}(G/H)$ is obtained as a geometric quantization of some elliptic coadjoint orbit that meets $\mathfrak{h}^\perp$.

**Problem 3.10.** Find a necessary and sufficient condition for cohomologically parabolic induced modules $A_{\mathfrak{q}}(\lambda)$ not to vanish outside the good range of parameter $\lambda$.

Conjecture 3.9 strengthens Conjecture 3.7, and one can verify it for reductive symmetric spaces $X$, see [15, Ex. 2.9]. To be more precise, by using Matsuki–Oshima's theorem [25] and by using an algebraic characterization of Zuckeman's derived functor modules, one can identify any discrete series representation for $G/H$ as a "geometric quantization" $\pi_\lambda$ of an elliptic coadjoint orbit $\mathcal{O}_\lambda$ that meets $\mathfrak{h}^\perp$, with the normalization of "quantization" as in [15]. For "singular" $\lambda$, the above $\pi_\lambda$ may or may not vanish. A missing part of Problem 3.1 in the literature for symmetric spaces is the complete proof of the precise condition on $\lambda$ such that $\pi_\lambda \neq 0$, which is reduced to an algebraic question, that is, Problem 3.10. The algebraic results in [11, Chaps. 4,5] and [30] give an answer to Problem 3.10 for some classical symmetric spaces.

We examine Problem 3.10 by $X = X(p, q)$ with $p, q \geq 1$. As we saw in Example 3.3, the underlying $(\mathfrak{g}, K)$-modules (see [32, Chap. 3] for instance) of the $L^2$-eigenspace $L^2(X)_{\lambda_k}$ are expressed by $A_{\mathfrak{q}}(k)$. Then there are finitely many exceptional parameters $k \in \mathbb{Z}$ satisfying $-\frac{1}{2}(p + q - 1) < k < 0$, *i.e.,*

lying "outside the good range" for which the general algebraic representation theory does not guarantee the irreducibility/non-vanishing for the cohomological parabolic induction. This is the point that Problem 3.10 concerns with.

# 4    Problems on discontinuous groups for $G/H$

The local to global study of geometries was a major trend of 20th century geometry, with remarkable developments achieved particularly in Riemannian geometry. In contrast, in areas such as pseudo-Riemannian geometry, familiar to us as the space-time of relativity theory, and more generally in manifolds with indefinite metric tensor of arbitrary signature, surprisingly little is known about global properties of the geometry. For instance, the pseudo-Riemannian space form problem is unsolved, which asks the existence of a compact pseudo-Riemannian manifold $M$ with constant sectional curvature for a given signature $(p, q)$, see Conjecture 4.11 below.

When we highlight "homogeneous structure" as a local property, "discontinuous groups" are responsible for the global geometry. The theory of discontinuous groups beyond the Riemannian setting is a relatively "young area" in group theory that interacts with topology, differential geometry, representation theory, and number theory among others. See [13] for some background on this topic at an early stage of the developments. This theme was discussed also as a new topic of future research at the occasion of the World Mathematical Year 2000 by Kobayashi [18] and Margulis [24]. For over 30 years, there have been remarkable developments by various methods ranging from topology and differential geometry to representation theory and ergodic theory, however, some important problems are still unsolved, which we illustrate in this section by using simple examples.

Beyond the Riemannian setting, we highlight a substantial difference in "discrete subgroups" and "discontinuous groups", *e.g.,* [10].

**Definition 4.1.** Let $G$ be a Lie group acting on a manifold $X$. A discrete subgroup $\Gamma$ of $G$ is said to be a *discontinuous group* for $X$ if $\Gamma$ acts properly discontinuously and freely on $X$.

The quotient space $X_\Gamma := \Gamma \backslash X$ by a discontinuous group $\Gamma$ is a (Hausdorff) $C^\infty$-manifold, and any $G$-invariant local geometric structure on $X$ can

be pushed forward to $X_\Gamma$ via the covering map $X \to X_\Gamma$. Such quotients $X_\Gamma$ are complete $(G, X)$-manifolds in the sense of Ehresmann and Thurston.

A classical example is a compact Riemann surface $\Sigma_g$ with genus $g \geq 2$, which can be expressed as $X_\Gamma$ where $\Gamma \simeq \pi_1(\Sigma_g)$ (surface group) and $X \simeq PSL(2, \mathbb{R})/PSO(2)$ by the uniformization theory. More generally, any complete affine locally symmetric space is given as the form $\Gamma \backslash G/H$ where $\Gamma$ is a discontinuous group for a symmetric space $G/H$.

**Remark 4.2.** The crucial assumption in Definition 4.1 is proper discontinuity of the action, and freeness is less important. In [13, Def. 2.5], we did not include the freeness assumption in the definition of discontinuous groups, allowing $X_\Gamma = \Gamma \backslash X$ to be an orbifold.

We discuss the following problems in the generality of Setting 2.1, *cf.* [18, Problems B and C].

**Problem 4.3.** Determine all pairs $(G, H)$ such that $G/H$ admits cocompact discontinuous groups.

**Problem 4.4** (higher Teichmüller theory for $G/H$)**.** Describe the moduli of all deformations of a discontinuous group $\Gamma$ for $G/H$.

In the classical case where $H$ is compact, a theorem of Borel answers Problem 4.3 in the affirmative by the existence of cocompact arithmetic discrete subgroups in $G$, whereas the Selberg–Weil local rigidity theorem tells that Problem 4.4 makes sense for a cocompact $\Gamma$ in a simple Lie group $G$ only if $\mathfrak{g} \simeq \mathfrak{sl}(2, \mathbb{R})$, and in this case the deformation of discontinuous groups gives rise to that of complex structures on the Riemann surface.

Such features change dramatically when $H$ is non-compact: some homogeneous spaces may not admit any discontinuous group of infinite order (the Calabi–Markus phenomenon [2]), showing an obstruction to the existence of cocompact discontinuous groups for $G/H$. On the other hand, discontinuous groups for pseudo-Riemannian manifolds $G/H$ tend to be "more flexible" in contrast to the classical rigidity theorems in the Riemannian case. For instance, some irreducible symmetric spaces of arbitrarily higher dimension admit cocompact discontinuous groups which are not locally rigid ([7, 16]), providing wide open settings for Problem 4.4.

As we mentioned, the notion "discontinuous group for $G/H$" is much stronger than "discreteness in $G$" when $H$ is non-compact. For instance, a cocompact discrete subgroup $\Gamma$ of $G$ never acts properly discontinuously on

$G/H$ unless $H$ is compact. Thus the existence of a lattice in $G$ does not imply that $G/H$ admits a cocompact discontinuous group.

We examine some related questions and conjectures to Problem 4.3. First, by relaxing the "cocompactness" assumption of $\Gamma$ in Problem 4.3, one may ask the following:

**Problem 4.5.** Find a necessary and sufficient condition for $G/H$ in Setting 2.1 to admit a discontinuous group $\Gamma$ for $G/H$ such that

(1) $\Gamma \simeq \mathbb{Z}$;

(2) $\Gamma \simeq$ a surface group $\pi_1(\Sigma_g)$ with $g \geq 2$.

Problem 4.5 (1) was solved in [10] in terms of the real rank condition $\mathrm{rank}_{\mathbb{R}} G > \mathrm{rank}_{\mathbb{R}} H$, which revealed the Calabi–Markus phenomenon [2] in the generality of Setting 2.1. Problem 4.5 (2) was solved by Okuda [27] for irreducible symmetric spaces, but is unsolved in the generality of Setting 2.1.

Cocompact discontinuous groups for $G/H$ are much smaller than cocompact lattices of $G$, for instance, their cohomological dimensions are strictly smaller [10]. A simple approach to Problem 4.3 is to utilize a 'continuous analog' of discontinuous groups $\Gamma$:

**Definition 4.6** (standard quotients $\Gamma\backslash G/H$ [8, Def. 1.4]). Suppose $L$ is a reductive subgroup of $G$ such that $L$ acts properly on $G/H$. Then any torsion-free discrete subgroup $\Gamma$ of $L$ is a discontinuous group for $G/H$. The quotient space $\Gamma\backslash G/H$ is called a *standard quotient* of $G/H$.

If such an $L$ acts cocompactly on $G/H$, then $G/H$ admits a cocompact discontinuous group $\Gamma$ by taking $\Gamma$ to be a torsion-free cocompact discrete subgroup in $L$, which always exists by Borel's theorem. We address the following conjecture and a subproblem to Problem 4.3.

**Conjecture 4.7** ([18, Conj. 4.3]). In Setting 2.1, $G/H$ admits a cocompact discontinuous group, only if $G/H$ admits a compact standard quotient.

If Conjecture 4.7 were proved to be true, then Problem 4.3 would be reduced to the following one:

**Problem 4.8.** Classify the pairs $(G, H)$ such that $G/H$ admits a compact standard quotient.

This problem should be manageable because one could use the general theory of real finite-dimensional representations of semisimple Lie algebras and apply the properness criterion and the cocompactness criterion in [10, Thms 4.1 and 4.7]. See [29] for some developments.

**Remark 4.9.** (1) Conjecture 4.7 does not assert that any cocompact discontinuous group is a standard one. In fact, there exist triples $(G, H, \Gamma)$ such that $\Gamma$ is a cocompact discontinuous group for $G/H$ and that the Zariski closure of $\Gamma$ does not act properly on $G/H$, see [7, 16].
(2) An analogous statement to Conjecture 4.7 fails if we drop the reductivity assumption on the groups $G$, $H$ and $L$.
(3) An analogous statement to Conjecture 4.7 is proved in Okuda [27] for semisimple symmetric spaces $G/H$ if we replace the "cocompactness" assumption with the condition that $\Gamma$ is a surface group $\pi_1(\Sigma_g)$.

Special cases of Conjecture 4.7 include:

**Conjecture 4.10.** $SL(n, \mathbb{R})/SL(m, \mathbb{R})$ does not admit a cocompact discontinuous group for any $n > m$.

**Conjecture 4.11** (Space form conjecture [22])**.** There exists a compact, complete, pseudo-Riemannian manifold of signature $(p, q)$ with constant sectional curvature 1 if and only if $(p, q)$ is in the list of Example 4.14 (4) below.

A criterion on triples $(G, H, L)$ of reductive Lie groups for $L$ to act properly on $X = G/H$ was established in [10], and a list of irreducible symmetric spaces $G/H$ admitting proper and cocompact actions of reductive subgroups $L$ was given in [22]. Tojo [29] worked with simple Lie groups $G$ and announced that the list in [22] exhausts all such triples $(G, H, L)$ with $L$ maximal, giving a solution to Problem 4.8 for symmetric spaces $G/H$ with $G$ simple.

A number of obstructions to the existence of cocompact discontinuous groups for $G/H$ with $H$ non-compact have been found for over 30 years. One of the recent developments includes the affirmative solution to the "rank conjecture" raised by the author in 1989: it was proved in the case $\operatorname{rank} G = \operatorname{rank} H$ by Kobayashi–Ono (1990), and has been proved recently in the general case by Morita [26] and Tholozan [28], independently.

**Conjecture 4.12** ([13, Conj. 4.15])**.** If $G/H$ admits a cocompact discontinuous group, then $\operatorname{rank} G + \operatorname{rank}(H \cap K) \geq \operatorname{rank} H + \operatorname{rank} K$.

Whereas the idea of standard quotients $\Gamma \backslash G/H$ is to replace a discrete subgroup $\Gamma$ with a connected subgroup $L$ (Definition 4.6), one may consider an "approximation" of Problem 4.3, by taking the *tangential homogeneous space* $X_\theta = G_\theta/H_\theta$ in replacement of $X = G/H$, where $G_\theta := K \ltimes \mathfrak{p}$ is the Cartan motion group of the real reductive group $G = K \exp \mathfrak{p}$ and similarly

for $H_\theta$. If $G/H$ admits a compact standard quotient, then the tangential homogeneous space $G_\theta/H_\theta$ admits a cocompact discontinuous group. The group $G_\theta$ is a compact extension of the abelian group $\mathfrak{p}$, and is of much simpler structure. We ask the following digression of Problem 4.3:

**Problem 4.13** ([22]). For which pairs $(G, H)$ in Setting 2.1, does $G_\theta/H_\theta$ admit a cocompact discontinuous group?

This problem is unsolved even for symmetric spaces in general, but has a complete answer in some special settings, *e.g.*, Example 4.14 (6) below.

We end this section with a brief summary of the state-of-art for these problems and conjectures by taking the space form $X(p, q)$ as an example.

**Example 4.14** ([2, 6, 7, 13, 16, 22, 23, 26, 27, 28]). Let $(G, H) = (O(p + 1, q), O(p, q))$, and $X = X(p, q) = G/H$ the pseudo-Riemannian space form of signature $(p, q)$ as in Example 2.4.
(1) $X(p, q)$ admits a discontinuous group of infinite order iff $p < q$.
(2) $X(p, q)$ admits a discontinuous group which is isomorphic to a surface group iff $p + 1 < q$ or $p + 1 = q \in 2\mathbb{N}$.
(3) If $X(p, q)$ admits a cocompact discontinuous group, then $p = 0$ or "$p < q$ and $q \in 2\mathbb{N}$".
(4) $X(p, q)$ admits a cocompact discontinuous group if $(p, q)$ is in the list below. (The converse assertion was stated as Conjecture 4.11.)

| $p$ | $\mathbb{N}$ | 0 | 1 | 3 | 7 |
|---|---|---|---|---|---|
| $q$ | 0 | $\mathbb{N}$ | $2\mathbb{N}$ | $4\mathbb{N}$ | 8 |

(5) If $(p, q) = (0, 2)$, $(1, 2)$, or $(3, 4)$, then $X(p, q)$ admits a cocompact discontinuous group which can be deformed continuously into a Zariski dense subgroup of $G$ by keeping proper discontinuity of the action. For $(p, q) = (1, 2n)$ $(n \geq 2)$, the anti-de Sitter space $X(1, 2n)$ admits a compact quotient which has a non-trivial continuous deformation as standard quotients.
(6) The tangential homogeneous space $G_\theta/H_\theta$ admits a cocompact discontinuous group if and only if $p < \rho(q)$ where $\rho(q)$ is the Radon–Hurwitz number, or equivalently, if and only if $(p, q)$ is in the following list:

| $p$ | $\mathbb{N}$ | 0 | 1 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $q$ | 0 | $\mathbb{N}$ | $2\mathbb{N}$ | $4\mathbb{N}$ | $8\mathbb{N}$ | $8\mathbb{N}$ | $8\mathbb{N}$ | $8\mathbb{N}$ | $16\mathbb{N}$ | $32\mathbb{N}$ | $64\mathbb{N}$ | $64\mathbb{N}$ | $\cdots$ |

# 5 Spectral analysis for pseudo-Riemannian locally homogeneous spaces $\Gamma\backslash G/H$

This section discusses briefly a new direction of analysis on pseudo-Riemannian locally homogeneous spaces $\Gamma\backslash G/H$.

Suppose we are in Setting 2.1. Let $\Gamma$ be a discontinuous group for $X = G/H$. Then any $G$-invariant differential operator $D \in \mathbb{D}(G/H)$ induces a differential operator $D_\Gamma$ on the quotient $X_\Gamma := \Gamma\backslash G/H$ via the covering $X \to X_\Gamma$. We think of the set $\mathbb{D}(X_\Gamma) := \{D_\Gamma : D \in \mathbb{D}(G/H)\}$ as the algebra of *intrinsic differential operators* on the locally homogeneous space $X_\Gamma$.

**Example 5.1.** (1) In Setting 2.1, $X_\Gamma$ inherits a pseudo-Riemannian structure from $X$, and the Laplacian $\Delta_{X_\Gamma}$ belongs to $\mathbb{D}(X_\Gamma)$.
(2) For $X = X(p,q)$, $\mathbb{D}(X_\Gamma)$ is a polynomial ring in the Laplacian $\Delta_{X_\Gamma}$ for any discontinuous group $\Gamma$.

We address the following problem:

**Problem 5.2** (see [8, 9])**.** For intrinsic differential operators on $X_\Gamma = \Gamma\backslash G/H$,
(1) construct joint eigenfunctions on $X_\Gamma$;
(2) find a spectral theory on $L^2(X_\Gamma)$.

With the same spirit as in Section 3, we highlight "discrete spectrum".

**Definition 5.3.** We say $\lambda \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}(X_\Gamma), \mathbb{C})$ is a *discrete spectrum* for intrinsic differential operators on $X_\Gamma$ if $L^2(X_\Gamma)_\lambda \neq \{0\}$, where we set

$$L^2(X_\Gamma)_\lambda := \{f \in L^2(X_\Gamma) : Df = \lambda(D)f \quad {}^\forall D \in \mathbb{D}(X_\Gamma)\}.$$

We write $\mathrm{Spec}_d(X_\Gamma)$ for the set of discrete spectra.

A subproblem to Problem 5.2 (1) includes:

**Problem 5.4.** Construct joint $L^2$-eigenfunctions on $X_\Gamma$.

In relation to Problem 4.4 about the deformations of a discontinuous group $\Gamma$ for $G/H$, one may also ask the following:

**Problem 5.5.** Understand the behavior of $\mathrm{Spec}_d(X_\Gamma)$ under small deformations of $\Gamma$ inside $G$.

These problems have been studied extensively in the following special settings for $X_\Gamma = \Gamma \backslash G/H$:

(1) $(H = K)$. When $H$ is a maximal compact subgroup $K$ of $G$, *i.e.,* $X_\Gamma$ is a *Riemannian* locally symmetric space, a vast theory has been developed over several decades, in particular, in connection with the theory of automorphic forms when $\Gamma$ is arithmetic.

(2) $(\Gamma = \{e\})$. This case is related to the topic in Section 3. In particular, Problem 5.2 has been extensively studied in the case where $G/H$ is a reductive symmetric space and $\Gamma = \{e\}$.

(3) $G = \mathbb{R}^{p,q}$, $\Gamma = \mathbb{Z}^{p+q}$, and $H = \{0\}$. In this case, $\operatorname{Spec}_d(X_\Gamma)$ is the set of values of indefinite quadratic forms at integral points, see [19] for a discussion on Problem 5.5 in relation to the Oppenheim conjecture (proved by Margulis) in Diophantine approximation.

The situation changes drastically beyond the classical setting, namely, when $H$ is not compact any more and $\Gamma \neq \{e\}$. New difficulties include:

(1) (Representation theory) Even when $\Gamma \backslash G/H$ is compact, the regular representation of $G$ on $L^2(\Gamma \backslash G)$ has infinite multiplicities, as opposed to a classical theorem of Gelfand–Piatetski–Shapiro.

(2) (Analysis) In contrast to the Riemannian case where $H = K$, the Laplacian $\Delta_{X_\Gamma}$ is not an elliptic differential operator any more.

As we saw in Section 4, if $H$ is not compact, then not all homogeneous spaces $G/H$ admit discontinuous groups of infinite order, but fortunately, there exist a family of reductive symmetric spaces $G/H$ that admit "large" discontinuous groups $\Gamma$, *e.g.,* such that $X_\Gamma = \Gamma \backslash G/H$ is compact or of finite volume. Moreover, there also exist triples $(G, H, \Gamma)$ such that discontinuous groups $\Gamma$ for $G/H$ can be deformed continuously. These examples offer broad settings for Problems 5.2 and its subproblems.

For Problem 5.5, we consider two notions for stability:

**Definition 5.6.** (1) (stability for proper discontinuity) A discontinuous group $\Gamma$ is *stable under small deformations* if the group $\varphi(\Gamma)$ acts properly discontinuously and freely on $X$ for all $\varphi \in \operatorname{Hom}(\Gamma, G)$ in some neighbourhood $\mathcal{U}$ of the natural inclusion $\Gamma$ in $G$.

(2) (stability for $L^2$-spectrum) We say $\lambda \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}(X_\Gamma), \mathbb{C})$ is a *stable spectrum* if $L^2(X_{\varphi(\Gamma)})_\lambda \neq \{0\}$ for any $\varphi \in \mathrm{Hom}(\Gamma, G)$ in some neighbourhood $\mathcal{U}$ of the natural inclusion $\Gamma$ in $G$.

**Conjecture 5.7.** Suppose that $\Gamma$ is a finitely generated discontinuous group for $G/H$ having non-trivial continuous deformations (up to inner automorphisms) with stability of proper discontinuity. Then the following conditions on the pair $(G, H)$ are equivalent.
(i) There exist infinitely many stable spectra on $L^2(\Gamma \backslash G/H)$.
(ii) $\mathrm{Disc}(G/H) \neq \emptyset$.

See [8] for some results in the direction (ii) $\Rightarrow$ (i), which treat also the case $\mathrm{vol}(\Gamma \backslash G/H) = \infty$.

The last section has been devoted to a "very young" topic, though special cases trace back to rich and deep classical theories. I expect this topic will create new interactions with different subjects of mathematics, and this is why I include it as a part of my article for "Mathematics Going Forward".

# References

[1] Y. Benoist and T. Kobayashi, *Tempered homogeneous spaces* IV, to appear in J. Inst. Math. Jussieu. Available also at arXiv: 2009.10391.

[2] E. Calabi, L. Markus, *Relativistic space forms*, Ann. of Math. **75** (1962), 63–76.

[3] P. Delorme, *Formule de Plancherel pour les espaces symétriques réductifs*, Ann. of Math. (2), **147** (1998), 417–452.

[4] W. M. Goldman, *Nonstandard Lorentz space forms*, J. Differential Geom. **21** (1985), 301–308.

[5] B. Harris, Y. Oshima, *On the asymptotic support of Plancherel measures for homogeneous spaces*, arXiv:2201.11293.

[6] K. Kannaka, *Deformation of standard pseudo-Riemannian locally symmetric spaces*, in preparation.

[7] F. Kassel, *Deformation of proper actions on reductive homogeneous spaces*, Math. Ann. **353** (2012), 599–632.

[8] F. Kassel, T. Kobayashi, *Poincaré series for non-Riemannian locally symmetric spaces*, Adv. Math. **287** (2016), 123–236.

[9] F. Kassel and T. Kobayashi, *Spectral analysis on standard locally homogeneous spaces*, arXiv: 1912.12601.

[10] T. Kobayashi, *Proper action on a homogeneous space of reductive type*, Math. Ann. **285** (1989), 249–263.

[11] T. Kobayashi, Singular unitary representations and discrete series for indefinite Stiefel manifolds $U(p, q; F)/U(p-m, q; F)$. Mem. Amer. Math. Soc. **95** (1992), no. 462, vi+106 pp.

[12] T. Kobayashi, *Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications*, Invent. Math., **117** (1994), 181–205.

[13] T. Kobayashi, *Discontinuous groups and Clifford-Klein forms of pseudo-Riemannian homogeneous manifolds.* Algebraic and analytic methods in representation theory (Sønderborg, 1994), 99–165, Perspect. Math., **17**. Academic Press, San Diego, CA, 1997.

[14] T. Kobayashi, *Discrete series representations for the orbit spaces arising from two involutions of real reductive Lie groups*, J. Funct. Anal., **152** (1998), pp. 100–135.

[15] T. Kobayashi, *Harmonic analysis on homogeneous manifolds of reductive type and unitary representation theory, Translations, Series* II, Selected Papers on Harmonic Analysis, Groups, and Invariants, **183** (1998), Amer. Math. Soc., 1–31.

[16] T. Kobayashi, *Deformation of compact Clifford–Klein forms of indefinite-Riemannian homogeneous manifolds*, Math. Ann. **310** (1998), 395–409.

[17] T. Kobayashi, *Discrete series representations for the orbit spaces arising from two involutions of real reductive Lie groups*, J. Funct. Anal., **152** (1998), pp. 100–135.

[18] T. Kobayashi, *Discontinuous groups for non-Riemannian homogeneous spaces.* Mathematics Unlimited–2001 and Beyond, 723–747, Springer, Berlin, 2001.

[19] T. Kobayashi, *Intrinsic sound of anti-de Sitter manifolds.* Springer Proc. Math. Stat. **191** (2016), 83–99.

[20] T. Kobayashi, *Branching laws of unitary representations associated to minimal elliptic orbits for indefinite orthogonal group $O(p,q)$*, Adv. Math., **388** (2021), Paper No. 107862, 38pp.

[21] T. Kobayashi, T. Oshima, *Finite multiplicity theorems for induction and restriction*, Adv. Math., **248** (2013), 921–944.

[22] T. Kobayashi, T. Yoshino, *Compact Clifford–Klein forms of symmetric spaces—revisited*, Pure and Appl. Math. Quarterly, **1**, (2005), 603–684. Special Issue: In Memory of Armand Borel.

[23] R. S. Kulkarni, *Proper actions and pseudo-Riemannian space forms*, Adv. Math. **40** (1981), 10–51.

[24] G. Margulis, *Problems and conjectures in rigidity theory.* Mathematics: frontiers and perspectives, 161–174, Amer. Math. Soc., 2000.

[25] T. Matsuki, T. Oshima, *A description of discrete series for semisimple symmetric spaces.* Adv. Stud. Pure Math. **4** (1984), 331–390.

[26] Y. Morita, *Proof of Kobayashi's rank conjecture on Clifford–Klein forms.* J. Math. Soc. Japan **71** (2019), no. 4, 1153–1171.

[27] T. Okuda, *Classification of semisimple symmetric spaces with proper $SL(2,\mathbb{R})$-actions*, J. Differential Geom. **94** (2013), 301–342.

[28] N. Tholozan, *Volume and non-existence of compact Clifford–Klein forms*, arXiv:1511.09448v2, preprint.

[29] K. Tojo, *Classification of irreducible symmetric spaces which admit standard compact Clifford–Klein forms.* Proc. Japan Acad. Ser. A Math. Sci. 95 (2019), 11–15.

[30] P. E. Trapa, *Annihilators and associated varieties of $A_{\mathfrak{q}}(\lambda)$ modules for $U(p,q)$.* Compositio Math. **129** (2001), 1–45.

[31] D. A. Vogan, G. J. Zuckerman, *Unitary representations with nonzero cohomology.* Compositio Math. **53** (1984), 51–90.

[32] N. R. Wallach, Real Reductive Groups. I, Pure Appl. Math. **132** Academic Press, Inc., Boston, MA, 1988

[33] J. A. Wolf, Spaces of Constant Curvature. Sixth edition. AMS Chelsea Publishing, Providence, RI, 2011. xviii+424 pages.