Multiplicity in restricting minimal representations

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Abstract

We discuss the action of a subgroup on small nilpotent orbits, and prove a bounded multiplicity property for the restriction of minimal representations of real reductive Lie groups with respect to arbitrary reductive symmetric pairs.

1 Statement of main results

This article is a continuation of our work [9, 13, 15, 17, 18, 21, 23] that concerns the restriction of irreducible representations Π of reductive Lie groups G to reductive subgroups G' with focus on the *bounded multiplicity property* of the restriction $\Pi|_{G'}$ (Definition 2). In this article we highlight the following specific setting:

· (G, G') is an arbitrary reductive symmetric pair;

· Π is of the smallest Gelfand–Kirillov dimension.

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We refer to [14] for some motivation and perspectives in the general branching problems, see also Section 2 for some aspects regarding finite/bounded multiplicity properties of the restriction.

To be rigorous about "multiplicities" for infinite-dimensional representations, we need to fix the topology of the representation spaces. For this, let G be a real reductive Lie group, $\mathcal{M}(G)$ the category of smooth admissible representations of G of finite length with moderate growth, which are defined on Fréchet topological vector spaces [33, Chap. 11]. We denote by $\mathrm{Irr}(G)$ the set of irreducible objects in $\mathcal{M}(G)$.

Suppose that G' is a reductive subgroup in G. For $\Pi \in \mathcal{M}(G)$, the *multiplicity* of $\pi \in \mathrm{Irr}(G')$ in the restriction $\Pi|_{G'}$ is defined by

$$[\Pi|_{G'} : \pi] := \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\},$$

where $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$ denotes the space of *symmetry breaking operators*, *i.e.*, continuous G'-homomorphisms between the Fréchet representations. For non-compact G', the multiplicity $[\Pi|_{G'} : \pi]$ may be infinite even when G' is a maximal subgroup of G, see Example 1 below.

By a *reductive symmetric pair* (G, G'), we mean that G is a real reductive Lie group and that G' is an open subgroup in the fixed point group G^σ of an involutive automorphism σ of G. The pairs $(SL(n, \mathbb{R}), SO(p, q))$ with $p + q = n$, $(O(p, q), O(p_1, q_1) \times O(p_2, q_2))$ with $p_1 + p_2 = p$, $q_1 + q_2 = q$, and the *group manifold case* $(`G \times `G, \mathrm{diag}(`G))$ are examples. For a reductive symmetric pair (G, G'), the subgroup G' is maximal amongst reductive subgroups of G.

One may ask for which pair (G, G') the finite multiplicity property

$$[\Pi|_{G'} : \pi] < \infty, \quad {}^\forall \Pi \in \mathrm{Irr}(G), {}^\forall \pi \in \mathrm{Irr}(G') \tag{1}$$

holds. Here are examples when (G, G') is a reductive symmetric pair:

Example 1 ([9, 19]). (1) For the symmetric pair $(SL(n, \mathbb{R}), SO(p, q))$ $(p + q = n)$, the finite multiplicity property (1) holds if and only if one of the following conditions holds: $p = 0$, $q = 0$, or $p = q = 1$.
(2) For the pair $(O(p, q), O(p_1, q_1) \times O(p_2, q_2))$ $(p_1 + p_2 = p$, $q_1 + q_2 = q)$, the finite multiplicity property (1) holds if and only if one of the following conditions holds: $p_1 + q_1 = 1$, $p_2 + q_2 = 1$, $p = 1$, or $q = 1$.
(3) For the group manifold case $(`G \times `G, \mathrm{diag}(`G))$ where $`G$ is a simple Lie group, the finite multiplicity property (1) holds if and only if $`G$ is compact or is locally isomorphic to $SO(n, 1)$.

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See Fact 15 (2) for a geometric criterion of the pair (G, G') to have the finite multiplicity property (1). A complete classification of such symmetric pairs (G, G') was accomplished in Kobayashi–Matsuki [19].

On the other hand, if we confine ourselves only to "small" representations Π of G, there will be a more chance that the multiplicity $[\Pi|_{G'} : \pi]$ becomes finite, or even stronger, the restriction $\Pi|_{G'}$ has the bounded multiplicity property in the following sense:

Definition 2. Let $\Pi \in \mathcal{M}(G)$. We say the restriction $\Pi|_{G'}$ has the *bounded multiplicity property* if $m(\Pi|_{G'}) < \infty$, where we set

$$m(\Pi|_{G'}) := \sup_{\pi \in \mathrm{Irr}(G')} \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\}. \tag{2}$$

In the series of the papers, we have explored the bounded multiplicity property of the restriction $\Pi|_{G'}$ not only uniformly with respect to $\pi \in \mathrm{Irr}(G')$ for the subgroup G' but also uniformly with respect to $\Pi \in \mathcal{M}(G)$, *e.g.*, either Π runs over the whole set $\mathrm{Irr}(G)$ [9, 13, 23] or Π belongs to certain family of "relatively small" representations of the group G [15, 17, 18, 21]. See Section 2 for some general results, which tell that the smaller Π is, the more subgroups G' tends to satisfy the bounded multiplicity property $\Pi|_{G'}$. In this article, we highlight the extremal case where Π is the "smallest", and give the bounded multiplicity theorems for *all* symmetric pairs (G, G').

What are "small representations" amongst infinite-dimensional representations? For this, the Gelfand–Kirillov dimension serves as a coarse measure of the "size" of representations. We recall that for $\Pi \in \mathcal{M}(G)$ the Gelfand–Kirillov dimension $\mathrm{DIM}(\Pi)$ is defined as half the dimension of the associated variety of $\mathcal{I}$ where $\mathcal{I}$ is the annihilator of Π in the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. The associated variety of $\mathcal{I}$ is a finite union of nilpotent coadjoint orbits in $\mathfrak{g}_{\mathbb{C}}^*$.

We recall for a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, there exists a unique nonzero minimal nilpotent $(\mathrm{Int}\, \mathfrak{g}_{\mathbb{C}})$-orbit in $\mathfrak{g}_{\mathbb{C}}^*$, which we denote by $\mathbb{O}_{\min, \mathbb{C}}$. The dimension of $\mathbb{O}_{\min, \mathbb{C}}$ is known as below, see [2] for example. We set $n(\mathfrak{g}_{\mathbb{C}})$ to be half the dimension of $\mathbb{O}_{\min, \mathbb{C}}$.

$\mathfrak{g}_{\mathbb{C}}$	A_n	$B_n\ (n \geq 2)$	C_n	D_n	$\mathfrak{g}_2^{\mathbb{C}}$	$\mathfrak{f}_4^{\mathbb{C}}$	$\mathfrak{e}_6^{\mathbb{C}}$	$\mathfrak{e}_7^{\mathbb{C}}$	$\mathfrak{e}_8^{\mathbb{C}}$
$n(\mathfrak{g}_{\mathbb{C}})$	n	$2n-2$	n	$2n-3$	3	8	11	17	29

For the rest of this section, let G be a non-compact connected simple Lie group without complex structure. This means that the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ is still a simple Lie algebra. By the definition, the Gelfand–Kirillov dimension has the following property: $\mathrm{DIM}(\Pi) = 0 \iff \Pi$ is finite-dimensional, and

$$n(\mathfrak{g}_\mathbb{C}) \le \mathrm{DIM}(\Pi) \le \frac{1}{2}(\dim \mathfrak{g} - \mathrm{rank}\,\mathfrak{g}), \qquad (3) \quad \boxed{\texttt{eqn:GKbdd}}$$

for any infinite-dimensional $\Pi \in \mathrm{Irr}(G)$. In this sense, if $\Pi \in \mathrm{Irr}(G)$ satisfies $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_\mathbb{C})$, then such Π is thought of as the "smallest" amongst infinite-dimensional irreducible representations of G.

In this article, we prove the following bounded multiplicity theorem of the restriction:

$\boxed{\texttt{thm:Joseph}}$ **Theorem 3.** *If the Gelfand–Kirillov dimension of $\Pi \in \mathrm{Irr}(G)$ is $n(\mathfrak{g}_\mathbb{C})$, then $m(\Pi|_{G'}) < \infty$ for any symmetric pair (G, G').*

For $\Pi_1, \Pi_2 \in \mathrm{Irr}(G)$, we consider the tensor product representation $\Pi_1 \otimes \Pi_2$, and define the upper bound of the multiplicity in $\Pi_1 \otimes \Pi_2$ by

$$m(\Pi_1 \otimes \Pi_2) := \sup_{\Pi \in \mathrm{Irr}(G)} \dim_\mathbb{C} \mathrm{Hom}_G(\Pi_1 \otimes \Pi_2, \Pi) \in \mathbb{N} \cup \{\infty\}.$$

The tensor product representation of two representations is a special case of the restriction with respect to symmetric pairs. We also prove the bounded multiplicity property of the tensor product:

$\boxed{\texttt{thm:tensormin}}$ **Theorem 4.** *If the Gelfand–Kirillov dimensions of $\Pi_1, \Pi_2 \in \mathrm{Irr}(G)$ are $n(\mathfrak{g}_\mathbb{C})$, then one has $m(\Pi_1 \otimes \Pi_2) < \infty$.*

Remark 5. Since the upper bound of the multiplicity $m(\Pi|_{G'})$ is defined in the category of admissible representations of moderate growth, $m(\Pi|_{G'})$ also gives an upper bound in the category of unitary representations where the multiplicity in the direct integral of irreducible unitary representations is defined as a measurable function on the unitary dual of the subgroup G'.

These results apply to "minimal representations" of G, which we recall now. For a complex simple Lie algebra $\mathfrak{g}_\mathbb{C}$ other than $\mathfrak{sl}(n, \mathbb{C})$, Joseph [6] constructed a completely prime two-sided primitive ideal $\mathcal{J}$ in $U(\mathfrak{g}_\mathbb{C})$, whose associated variety is the closure of the minimal nilpotent orbit $\mathbb{O}_{\min, \mathbb{C}}$. See also [3].

def:minrep **Definition 6** (minimal representation, see [4]). An irreducible admissible representation Π of G is called a *minimal representation* if the annihilator of the $U(\mathfrak{g}_\mathbb{C})$-module Π is the Joseph ideal $\mathcal{J}$ of $U(\mathfrak{g}_\mathbb{C})$.

The two irreducible components of the Segal–Shale–Weil representation are classical examples of a minimal representation of the metaplectic group $Mp(n, \mathbb{R})$, the connected double cover of the real symplectic group $Sp(n, \mathbb{R})$, which play a prominent role in number theory. The solution space of the Yamabe Laplacian on $S^p \times S^q$ gives the minimal representation of the conformal transformation group $O(p+1, q+1)$ when $p + q$ (≥ 6) is even ([20]). In general, there are at most four minimal representations for each connected simple Lie group G if exist, and they were classified [4, 30].

By the definition of the Joseph ideal, one has $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_\mathbb{C})$ if Π is a minimal representation. Thus Theorems 3 and 4 imply the following:

thm:minbdd **Theorem 7.** *Let Π be a minimal representation of G. Then the restriction $\Pi|_{G'}$ has the bounded multiplicity property $m(\Pi|_{G'}) < \infty$ for any symmetric pair (G, G').*

thm:tensor **Theorem 8.** *Let Π_1, Π_2 be minimal representations of G. Then the tensor product representation has the bounded multiplicity property $m(\Pi_1 \otimes \Pi_2) < \infty$.*

Example 9. The tensor product representation of the two copies of the Segal–Shale–Weil representations of the metaplectic group $Mp(n, \mathbb{R})$ is unitarily equivalent to the phase space representation of $Sp(n, \mathbb{R})$ on $L^2(\mathbb{R}^{2n})$ via the Wigner transform, see [22, Sect. 2] for instance.

In general, it is rare that the restriction $\Pi|_{G'}$ of $\Pi \in \mathcal{M}(G)$ is *almost irreducible* in the sense that the G'-module $\Pi|_{G'}$ remains irreducible or a direct sum of finitely many irreducible representations of G'. In [12, Sect. 5], we discussed such rare phenomena and gave a list of the triples (G, G', Π) where the restriction $\Pi|_{G'}$ is almost irreducible, in particular, in the following settings: $\Pi \in \mathcal{M}(G)$ is a degenerate principal series representation or Zuckerman's derived functor module $A_\mathfrak{q}(\lambda)$, which is supposed to be a "geometric quantization" of a hyperbolic coadjoint orbit or an elliptic coadjoint orbit, respectively, in the orbit philosophy, see [12, Thms. 3.8 and 3.5]. As a corollary of Theorem 7, we also prove the following theorem where Π is "attached to" the minimal nilpotent coadjoint orbit $\mathbb{O}_{\min, \mathbb{C}}$.

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Theorem 10. *Suppose that (G, G') is a symmetric pair such that the complexified Lie algebras $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ is in the list of Proposition 30 (vi). Then the restriction $\Pi|_{G'}$ is almost irreducible if Π is a minimal representation of G.*

Example 11. For the following symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$, there exists a minimal representation Π of some Lie group G with Lie algebra $\mathfrak{g}$ (*e.g.*, $G = Mp(n, \mathbb{R})$ for $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$), and Theorem 10 applies to (G, G', Π).

- $(\mathfrak{sp}(p + q, \mathbb{R}), \mathfrak{sp}(p, \mathbb{R}) \oplus \mathfrak{sp}(q, \mathbb{R}))$
- $(\mathfrak{so}(p, q), \mathfrak{so}(p - 1, q))$ or $(\mathfrak{so}(p, q), \mathfrak{so}(p, q - 1))$ for "$p \geq q \geq 4$ and $p \equiv q$ mod 2", "$p \geq 5$ and $q = 2$", or "$p \geq 4$ and $q = 3$".
- $(\mathfrak{f}_{4(4)}, \mathfrak{so}(5, 4))$,
- $(\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)})$, or $(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$.

We note that the upper bound $m(\Pi|_{G'})$ or $m(\Pi_1 \otimes \Pi_2)$ of the multiplicity can be larger than 1 in Theorems 3 and 4, see *e.g.* [21] for an explicit branching law of the restriction $\Pi|_{G'}$ when $(G, G') = (SL(n, \mathbb{R}), SO(p, q))$ with $p + q = n$. However, it is plausible that a multiplicity-free theorem holds in Theorems 7 and 8:

Conjecture 12. $m(\Pi|_{G'}) = 1$ in Theorem 7, and $m(\Pi_1 \otimes \Pi_2) = 1$ in Theorem 8.

Conjecture 12 holds when (G, G') is a Riemannian symmetric pair (G, K), see [4, Prop. 4.10].

Remark 13. (1) The Joseph ideal is not defined for $\mathfrak{sl}(n, \mathbb{C})$, hence there is no minimal representation in the sense of Definition 6 for $G = SL(n, \mathbb{R})$, for instance. However there exist continuously many $\Pi \in \mathrm{Irr}(G)$ (*e.g.*, degenerate principal series representations induced from a mirabolic subgroup) for $G = SL(n, \mathbb{R})$ such that $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$, and Theorems 3 and 4 apply to these representations. The Plancherel-type theorem for the restriction $\Pi|_{G'}$ is proved in [21] for all symmetric pairs (G, G') when Π is a *unitarily* induced representation. See also Example 16 below.
(2) The inequality (3) depends only on the complexification $\mathfrak{g}_{\mathbb{C}}$, and is not necessarily optimal for specific real forms $\mathfrak{g}$. In fact, one has a better inequality $n(\mathfrak{g}) \leq \mathrm{DIM}(\Pi)$ where $n(\mathfrak{g})$ depends on the real form $\mathfrak{g}$, see Section 3.2. For most of real Lie algebras one has $n(\mathfrak{g}) = n(\mathfrak{g}_{\mathbb{C}})$, but there are a few simple Lie algebras $\mathfrak{g}$ satisfying $n(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$. For example, if $G = Sp(p, q)$, $n(\mathfrak{g}) = 2(p + q) - 1 > n(\mathfrak{g}_{\mathbb{C}}) = p + q$, hence there is no $\Pi \in \mathrm{Irr}(G)$ with $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$, however, there exists a countable family of $\Pi \in \mathrm{Irr}(G)$

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with $\mathrm{DIM}(\Pi) = n(\mathfrak{g})$, to which another bounded multiplicity theorem (Theorem 34 in Section 3) applies.

(3) Concerning Theorem 3, the bounded property of the multiplicity in the tensor product representations $\Pi_1 \otimes \Pi_2$ still holds for some other "small representations" Π_1 and Π_2 whose Gelfand–Kirillov dimensions are greater than $n(\mathfrak{g}_\mathbb{C})$. See [17, Thm. 1.5 and Cor. 4.10] for example.

This paper is organized as follows. In Section 2 we give a brief review of some background of the problem, examples, and known theorems. Section 3 is devoted to the proof of Theorems 3, 4 and 10.

2   Background and motivation

In this section, we explain some background, examples, and known theorems in relation to our main results.

If Π is an irreducible *unitary* representation of a group G, then one may consider the irreducible decomposition (*branching law*) of the restriction $\Pi|_{G'}$ to a subgroup G' by using the direct integral of Hilbert spaces. For non-unitary representations Π, such an irreducible decomposition does not make sense, but the computation of the multiplicity $[\Pi|_{G'} : \pi]$ for all $\pi \in \mathrm{Irr}(G')$ may be thought of as a variant of branching laws. Here we recall from Section 1 that for $\Pi \in \mathcal{M}(G)$ and $\pi \in \mathrm{Irr}(G')$ that the multiplicity $[\Pi|_{G'} : \pi]$ is the dimension of the space $\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$ of symmetry breaking operators.

By branching problems in representation theory, we mean the broad problem of understanding how irreducible (not necessarily, unitary) representations of a group behave when restricted to a subgroup. As viewed in [14], we may divide the branching problems into the following three stages:

Stage A. Abstract features of the restriction;

Stage B. Branching law;

Stage C. Construction of symmetry breaking operators.

The role of Stage A is to develop a theory on the restriction of representations as generally as possible. In turn, we may expect a detailed study

of the restriction in Stages B (decomposition of representations) and C (decomposition of vectors) in the "promising" settings that are suggested by the general theory in Stage A.

The study of the upper estimate of the multiplicity in this article is considered as a question in Stage A of branching problems.

For a detailed analysis on the restriction $\Pi|_{G'}$ in Stages B and C, it is desirable to have the bounded multiplicity property $m(\Pi|_{G'}) < \infty$ (see Definition 2), or at least to have the finite multiplicity property

$$[\Pi|_{G'} : \pi] < \infty \quad \text{for } \pi \in \mathrm{Irr}(G'). \tag{4}$$

In the previous papers [10, 13, 16, 17, 18, 23] we proved some general theorems for bounded/finite multiplicities of the restriction $\Pi|_{G'}$, which we review briefly now.

2.1 Bounded multiplicity pairs (G, K') with K' compact

Harish-Chandra's admissibility theorem tells the finiteness property (4) holds for any $\Pi \in \mathcal{M}(G)$ if G' is a maximal compact subgroup K of G. More generally, the finiteness property (4) for a compact subgroup plays a crucial role in the study of discretely decomposable restriction with respect to reductive subgroups [8, 10, 11, 16]. We review briefly the necessary and sufficient condition for (4) when G' is compact. In this subsection, we use the letter K' instead of G' to emphasize that G' is compact. Without loss of generality, we may and do assume that K' is contained in K.

Fact 14 ([10, 16]). Suppose that K' is a compact subgroup of a real reductive group G. Let $\Pi \in \mathcal{M}(G)$. Then the following two conditions on the triple (G, K', Π) are equivalent:
(i) The finite multiplicity property (4) holds.
(ii) $\mathrm{AS}_K(\Pi) \cap C_K(K') = \{0\}$.

Here $\mathrm{AS}_K(\Pi)$ is the asymptotic K-support of Π, and $C_K(K')$ is the momentum set for the natural action on the cotangent bundle $T^*(K/K')$. There are two proofs for the implication (ii) $\Rightarrow$ (i): by using the singularity spectrum (or the wave front set) [10] and by using symplectic geometry [16]. The proof for the implication (i) $\Rightarrow$ (ii) is given in [16]. See [24] for some classification theory.

2.2 Bounded/finite multiplicity pairs (G, G')

We now consider the general case where G' is not necessarily compact. In [13] and [23, Thms. C and D] we proved the following geometric criteria that concern *all* $\Pi \in \mathrm{Irr}(G)$ and *all* $\pi \in \mathrm{Irr}(G')$:

Fact 15. Let $G \supset G'$ be a pair of real reductive algebraic Lie groups.
(1) **Bounded multiplicity** for a pair (G, G'):

$$\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty \qquad (5)$$

if and only if $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag} G'_{\mathbb{C}}$ is spherical.
(2) **Finite multiplicity** for a pair (G, G'):

$$[\Pi|_{G'} : \pi] < \infty, \qquad {}^{\forall}\Pi \in \mathrm{Irr}(G), {}^{\forall}\pi \in \mathrm{Irr}(G')$$

if and only if $(G \times G')/\operatorname{diag} G'$ is real spherical.

Here we recall that a complex $G_{\mathbb{C}}$-manifold X is called *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in X, and that a G-manifold Y is called *real spherical* if a minimal parabolic subgroup of G has an open orbit in Y.

A remarkable discovery in Fact 15 (1) was that the bounded multiplicity property (5) is determined only by the complexified Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}'_{\mathbb{C}}$. In particular, the classification of such pairs (G, G') is very simple, because it is reduced to a classical result when G is compact [27]: the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ is the direct sum of the following ones up to abelian ideals:

$$(\mathfrak{sl}_n, \mathfrak{gl}_{n-1}), (\mathfrak{so}_n, \mathfrak{so}_{n-1}), \text{ or } (\mathfrak{so}_8, \mathfrak{spin}_7). \qquad (6)$$

See [25, 26] *e.g.*, for some recent developments in Stage C such as detailed analysis on symmetry breaking operators for some non-compact real forms of the pairs (6).

On the other hand, the finite multiplicity property in Fact 15 (2) depends on real forms G and G'. It is fulfilled for any Riemannian symmetric pair, which is Harish-Chandra's admissibility theorem. More generally for non-compact G', the finite-multiplicity property (4) often holds when the restriction $\Pi|_{G'}$ decomposes discretely, see [8, 10, 11] for the general theory of "G'-admissible restriction". However, for some reductive symmetric pairs such as $(G, G') = (SL(p+q, \mathbb{R}), SO(p, q))$, there exists $\Pi \in \mathrm{Irr}(G)$ for which the finite multiplicity property (4) of the restriction $\Pi|_{G'}$ fails, as we have seen in Example 1. Such Π is fairly "large".

2.3 Uniform estimates for a family of small representations

The classification in [19] tells that the class of the reductive symmetric pairs (G, G') satisfying the finite multiplicity property (1) is much broader than that of real forms (G, G') corresponding to those complex pairs in (5). However, there also exist pairs (G, G') beyond the list of [19] for which we can still expect fruitful branching laws of the restriction $\Pi|_{G'}$ in Stages B and C for some $\Pi \in \mathrm{Irr}(G)$. Such Π must be a "small representation". Here are some known examples:

Example 16. (1) (**Stage B**) See-saw dual pairs ([28]) yield explicit formulæ of the multiplicity for the restriction of small representations, with respect to some classical symmetric pairs (G, G').

(2) (**Stage C**) For $G = SL(n, \mathbb{R})$, any degenerate representation $\Pi = \mathrm{Ind}_P^G(\mathbb{C}_\lambda)$ induced from a "mirabolic subgroup" P of G has the smallest Gelfand–Kirillov dimension $n(\mathfrak{g}_\mathbb{C})$. For a unitary character $\mathbb{C}_\lambda$, the Plancherel-type formula of the restriction $\Pi|_{G'}$ is determined in [21] for *all* symmetric pairs (G, G'). The feature of the restriction $\Pi|_{G'}$ is summarized as follows: let $p + q = n$, and when n is even we write $n = 2m$.

$\cdot\ G' = S(GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$.

$\quad \cdots$ Only continuous spectrum appears with multiplicity one.

$\cdot\ G' = SL(m, \mathbb{C}) \cdot \mathbb{T}$.

$\quad \cdots$ Only discrete spectrum appears with multiplicity one.

$\cdot\ G' = SO(p, q)$.

$\quad \cdots$ Discrete spectrum appears with multiplicity one,

$\qquad$ and continuous spectrum appears with multiplicity two.

$\cdot\ G' = Sp(m, \mathbb{R})$

$\quad \cdots$ Almost irreducible (See also Theorem 10).

$\quad$ The uniform bounded multiplicity property in all these cases (**Stage A**) is guaranteed by Theorem 3 in this article because $\mathrm{DIM}(\Pi)$ attains $n(\mathfrak{g}_\mathbb{C})$, and alternatively, by another general result [17, Thm. 4.2].

(3) (**Stage C**) For the symmetric pair $(G, G') = (O(p, q), O(p_1, q_1) \times O(p_2, q_2))$ with $p_1 + p_2 = p$ and $q_1 + q_2 = q$, by using the Yamabe operator in conformal geometry, discrete spectrum in the restriction $\Pi|_{G'}$ of the minimal representation Π was obtained geometrically in [20]. Moreover, for the same pair (G, G'), discrete spectrum in the restriction $\Pi|_{G'}$ was explicitly constructed and classified when Π belongs to cohomologically parabolic induced repre-

sentation $A_{\mathfrak{q}}(\lambda)$ from a maximal θ-stable parabolic subalgebra $\mathfrak{q}$ in [15]. In contrast to Example 1 (2), the multiplicity is one for any p_1, q_1, p_2, and q_2.

In view of these nice cases, and also in search for further broader settings in which we could expect a detailed study of the restriction $\Pi|_{G'}$ in Stages B and C, we addressed the following:

Problem 17 ([14, Prob. 6.2], [17, Prob. 1.1]). Given a pair $G \supset G'$, find a subset Ω of $\mathcal{M}(G)$ such that $\sup_{\Pi \in \Omega} m(\Pi|_{G'}) < \infty$.

Since branching problems often arise for a family of representations Π, the formulation of Problem 17 is to work with the *triple* (G, G', Ω) rather than the pair (G, G') for the finer study of multiplicity estimates of the restriction $\Pi|_{G'}$. Fact 15 (1) deals with the case $\Omega = \mathrm{Irr}(G)$. In [15, 17], we have considered Problem 17 including the following cases:

(1) $\Omega = \mathrm{Irr}(G)_H$, the set of H-distinguished irreducible representations of G where (G, H) is a reductive symmetric pair;

(2) $\Omega = \Omega_P$, the set of induced representations from characters of a parabolic subgroup P of G;

(3) $\Omega = \Omega_{P,\mathfrak{q}}$, certain families of (vector-bundle valued) degenerate principal series representations.

For the readers' convenience, we give a flavor of the solutions to Problem 17 in the above cases by quoting the criteria from [17]. See [18] for a brief survey.

We write $G_{\mathbb{C}}$ for the complexified Lie group G, and G_U for the compact real form of $G_{\mathbb{C}}$. For a reductive symmetric pair (G, H), one can define a Borel subgroup $B_{G/H}$ which is a parabolic subgroup in $G_{\mathbb{C}}$, see [18, Def. 3.1]. Note that $B_{G/H}$ is not necessarily solvable. For $\Omega = \mathrm{Irr}(G)_H$ when (G, H) is a reductive symmetric pair, one has the following answer to Problem 17:

Fact 18 ([17, Thm. 1.4]). Let $B_{G/H}$ be a Borel subgroup for G/H. Suppose G' is an algebraic reductive subgroup of G. Then the following three conditions on the triple (G, H, G') are equivalent:

(i) $\sup_{\Pi \in \mathrm{Irr}(G)_H} m(\Pi|_{G'}) < \infty$.

(ii) $G_{\mathbb{C}}/B_{G/H}$ is G'_U-strongly visible.

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(iii) $G_{\mathbb{C}}/B_{G/H}$ is $G'_{\mathbb{C}}$-spherical.

For $\Omega = \Omega_P$, one has the following answer to Problem 17:

fact:introQsph2 | **Fact 19** ([17, Ex. 4.5], [31]). Let $G \supset G'$ be a pair of real reductive algebraic Lie groups, and P a parabolic subgroup of G. Then one has the equivalence on the triple $(G, G'; P)$:

(i) $\sup\limits_{\Pi \in \Omega_P} m(\Pi|_{G'}) < \infty$.

(ii) $G_{\mathbb{C}}/P_{\mathbb{C}}$ is strongly G'_U-visible.

(iii) $G_{\mathbb{C}}/P_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.

The following is a useful extension of Fact 19.

fact:Qsph | **Fact 20** ([17, Thm. 4.2]). Let $G \supset G'$ be a pair of real reductive algebraic Lie groups, P a parabolic subgroup of G, and Q a complex parabolic subgroup of $G_{\mathbb{C}}$ such that $\mathfrak{q} \subset \mathfrak{p}_{\mathbb{C}}$. One defines a subset $\Omega_{P,\mathfrak{q}}$ in $\mathcal{M}(G)$ that contains Ω_P (see [17] for details). Then the following three conditions on $(G, G'; P, Q)$ are equivalent:

(i) $\sup\limits_{\Pi \in \Omega_{P,\mathfrak{q}}} m(\Pi|_{G'}) < \infty$.

(ii) $G_{\mathbb{C}}/Q$ is G'_U-strongly visible.

(iii) $G_{\mathbb{C}}/Q$ is $G'_{\mathbb{C}}$-spherical.

These criteria lead us to classification results for the triples (G, G', Ω), see [15, 17, 18] and references therein.

The representations Π in $\Omega = \mathrm{Irr}(G)_H$ or Ω_P, $\Omega_{P,\mathfrak{q}}$ are fairly small, however, the classification results in [17] indicate that some symmetric pairs (G, G') still do not appear for such a family Ω. A clear distinction from these previous results is that Theorem 3 allows *all* symmetric pairs (G, G') for an affirmative answer to Problem 17 in the extremal case where $\Omega = \{\Pi\}$ with $\mathrm{DIM}(\Pi) = n(\mathfrak{g}_{\mathbb{C}})$.

Concerning the method of the proof, we utilized in [23] hyperfunction boundary maps for the "if" part (*i.e.*, the sufficiency of the bounded multiplicity property) and a generalized Poisson transform [13] for the "only if" part in the proof of Fact 15. The proof in [17, 31] used a theory of holonomic $\mathcal{D}$-modules for the "if" part. Our proof in this article still uses a theory of $\mathcal{D}$-modules, and more precisely, the following:

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Fact 21 ([7]). Let $\mathcal{I}$ be the annihilator of $\Pi \in \mathcal{M}(G)$ in the enveloping algebra $U(\mathfrak{g}_\mathbb{C})$. Assume that the $G'_\mathbb{C}$-action on the associated variety of $\mathcal{I}$ is coisotropic (Definition 22). Then the restriction $\Pi|_{G'}$ has the bounded multiplicity property (Definition 2).

We note that the assumption in Fact 21 depends only on the complexification of the pair $(\mathfrak{g}, \mathfrak{g}')$ of the Lie algebras. Thus the proof of Theorems 3 and 4 is reduced to a geometric question on *holomorphic* coisotropic actions on *complex* nilpotent coadjoint orbits, which will be proved in Theorem 23.

3 Coisotropic action on coadjoint orbits

Let V be a vector space endowed with a symplectic form ω. A subspace W is called *coisotropic* if $W^\perp \subset W$, where

$$W^\perp := \{v \in V : \omega(v, \cdot) \text{ vanishes on } W\}.$$

The concept of coisotropic actions is defined infinitesimally as follows.

Definition 22 (Huckleberry–Wurzbacher [5]). Let H be a connected Lie group, and X a Hamiltonian H-manifold. The H-action is called *coisotropic* if there is an H-stable open dense subset U of X such that $T_x(H \cdot x)$ is a coisotropic subspace in the tangent space $T_x X$ for all $x \in U$.

Any coadjoint orbit of a Lie group G is a Hamiltonian G-manifold with the Kirillov–Kostant–Souriau symplectic form. The main result of this section is the following:

Theorem 23. *Let $\mathbb{O}_{\min, \mathbb{C}}$ be the minimal nilpotent coadjoint orbit of a connected complex simple Lie group $G_\mathbb{C}$.*

1) *The diagonal action of $G_\mathbb{C}$ on $\mathbb{O}_{\min, \mathbb{C}} \times \mathbb{O}_{\min, \mathbb{C}}$ is coisotropic.*

2) *For any symmetric pair $(G_\mathbb{C}, K_\mathbb{C})$, the $K_\mathbb{C}$-action on $\mathbb{O}_{\min, \mathbb{C}}$ is coisotropic.*

3.1 Generalities: coisotropic actions on coadjoint orbits

We begin with a general setting for a *real* Lie group. Suppose that $\mathbb{O}$ is a coadjoint orbit of a connected Lie group G through $\lambda \in \mathfrak{g}^*$. Denote by

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G_λ the stabilizer subgroup of λ in G, and by $Z_{\mathfrak{g}}(\lambda)$ its Lie algebra. Then the Kirillov–Kostant–Souriau symplectic form ω on the coadjoint orbit $\mathbb{O} = \mathrm{Ad}^*(G) \simeq G/G_\lambda$ is given at the tangent space $T_\lambda \mathbb{O} \simeq \mathfrak{g}/Z_{\mathfrak{g}}(\lambda)$ by

$$\omega\colon \mathfrak{g}/Z_{\mathfrak{g}}(\lambda) \times \mathfrak{g}/Z_{\mathfrak{g}}(\lambda) \to \mathbb{R}, \quad (X,Y) \mapsto \lambda([X,Y]). \tag{7}$$ `eqn:KKS`

Suppose that H is a connected subgroup with Lie algebra $\mathfrak{h}$. For $\lambda \in \mathfrak{g}^*$, we define a subspace of the Lie algebra $\mathfrak{g}$ by

$$Z_{\mathfrak{g}}(\mathfrak{h}; \lambda) := \{Y \in \mathfrak{g} : \lambda([X,Y]) = 0 \text{ for all } X \in \mathfrak{h}\}. \tag{8}$$ `eqn:zhlmd`

Clearly, $Z_{\mathfrak{g}}(\mathfrak{h}; \lambda)$ contains the Lie algebra $Z_{\mathfrak{g}}(\lambda) \equiv Z_{\mathfrak{g}}(\mathfrak{g}; \lambda)$ of G_λ.

We shall use the following:

`lem:1.7` **Lemma 24.** *The H-action on a coadjoint orbit $\mathbb{O}$ in $\mathfrak{g}^*$ is coisotropic if there exists a subset S (slice) in $\mathbb{O}$ with the following two properties:*

$$\mathrm{Ad}^*(H)S \text{ is open dense in } \mathbb{O},$$
$$Z_{\mathfrak{g}}(\mathfrak{h}; \lambda) \subset \mathfrak{h} + Z_{\mathfrak{g}}(\lambda) \text{ for any } \lambda \in S. \tag{9}$$ `eqn:coiso`

Proof. It suffices to verify that $T_\lambda(\mathrm{Ad}^*(H)\lambda)$ is a coisotropic subspace in $T_\lambda \mathbb{O}$ for any $\lambda \in S$ because the condition (9) `eqn:coiso` is H-invariant. Via the identification $T_\lambda \mathbb{O} \simeq \mathfrak{g}/Z_{\mathfrak{g}}(\lambda)$, one has $T_\lambda(\mathrm{Ad}^*(H)\lambda) \simeq (\mathfrak{h} + Z_{\mathfrak{g}}(\lambda))/Z_{\mathfrak{g}}(\lambda)$. By the formula (7) `eqn:KKS` of the symplectic form ω on $\mathbb{O}$, one has $T_\lambda(\mathrm{Ad}^*(H)\lambda)^\perp \simeq Z_{\mathfrak{g}}(\mathfrak{h}; \lambda)/Z_{\mathfrak{g}}(\lambda)$. Hence $T_\lambda(\mathrm{Ad}^*(H)\lambda)$ is a coisotropic subspace in $T_\lambda \mathbb{O}$ if and only if $Z_{\mathfrak{g}}(\mathfrak{h}; \lambda) \subset \mathfrak{h} + Z_{\mathfrak{g}}(\lambda)$, whence the lemma. $\square$

For semisimple $\mathfrak{g}$, the Killing form B induces the following G-isomorphism

$$\mathfrak{g}^* \simeq \mathfrak{g}, \quad \lambda \mapsto X_\lambda. \tag{10}$$ `eqn:Xlmd`

By definition, one has $\lambda([X,Y]) = B(X_\lambda, [X,Y]) = B([X_\lambda, X], Y)$, and thus

$$Z_{\mathfrak{g}}(\mathfrak{h}; \lambda) = [X_\lambda, \mathfrak{h}]^{\perp B},$$

where the right-hand side stands for the orthogonal complement subspace of $[X_\lambda, \mathfrak{h}] := \{[X_\lambda, X] : X \in \mathfrak{h}\}$ in $\mathfrak{g}$ with respect to the Killing form B. Hence we have the following.

`lem:Hcoiso` **Lemma 25.** *For semisimple $\mathfrak{g}$, one may replace the condition (9)* `eqn:coiso` *in Lemma 24* `lem:1.7` *by*

$$(\mathfrak{h} + Z_{\mathfrak{g}}(\lambda))^{\perp B} \subset [X_\lambda, \mathfrak{h}] \quad \text{for any } \lambda. \tag{11}$$ `eqn:coisoB`

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3.2 Real minimal nilpotent orbits

Let G be a connected non-compact simple Lie group without complex structure. Denote by $\mathcal{N}$ the nilpotent cone in $\mathfrak{g}$, and $\mathcal{N}/G$ the set of nilpotent orbits, which may be identified with nilpotent coadjoint orbits in $\mathfrak{g}^*$ via (10). The finite set $\mathcal{N}/G$ is a poset with respect to the closure ordering, and there are at most two minimal elements in $(\mathcal{N} \setminus \{0\})/G$, which we refer to as *real minimal nilpotent (coadjoint) orbits*. See [1, 24, 29] for details. The relationship with the complex minimal nilpotent orbits $\mathbb{O}_{\min,\mathbb{C}}$ in the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is given as below. Let K be a maximal compact subgroup of G modulo center.

Lemma 26. *In the setting above, exactly one of the following cases occurs.*

(1) $(\mathfrak{g}, \mathfrak{k})$ *is not of Hermitian type, and* $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g} = \emptyset$.

(2) $(\mathfrak{g}, \mathfrak{k})$ *is not of Hermitian type, and* $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g}$ *is a single orbit of* G.

(3) $(\mathfrak{g}, \mathfrak{k})$ *is of Hermitian type, and* $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g}$ *consists of two orbits of* G.

As the G-orbit decomposition of $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g}$, we write $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g} = \{\mathbb{O}_{\min,\mathbb{R}}\}$ in Case (2), $\mathbb{O}_{\min,\mathbb{C}} \cap \mathfrak{g} = \{\mathbb{O}_{\min,\mathbb{R}}^+, \mathbb{O}_{\min,\mathbb{R}}^-\}$ in Case (3). Then they exhaust all real minimal nilpotent orbits in Cases (2) and (3). Real minimal nilpotent orbits are unique in Case (1), to be denoted by $\mathbb{O}_{\min,\mathbb{R}}$. We set

$$n(\mathfrak{g}) := \begin{cases} \frac{1}{2} \dim \mathbb{O}_{\min,\mathbb{R}} & \text{in Cases (1) and (2),} \\ \frac{1}{2} \dim \mathbb{O}_{\min,\mathbb{R}}^+ = \frac{1}{2} \dim \mathbb{O}_{\min,\mathbb{R}}^- & \text{in Case (3).} \end{cases} \quad (12)$$

Then $n(\mathfrak{g}) = n(\mathfrak{g}_{\mathbb{C}})$ in Cases (2) and (3), and $n(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$ in Case (1). The formula of $n(\mathfrak{g})$ in Case (1) is given in [29] as follows.

$\mathfrak{g}$	$\mathfrak{su}^*(2n)$	$\mathfrak{so}(n-1,1)$	$\mathfrak{sp}(m,n)$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{e}_{6(-26)}$
$n(\mathfrak{g})$	$4n-4$	$n-2$	$2(m+n)-1$	11	16

For any $\Pi \in \mathrm{Irr}(G)$, the Gelfand–Kirillov dimension $\mathrm{DIM}(\Pi)$ satisfies $n(\mathfrak{g}) \leq \mathrm{DIM}(\Pi)$, which is equivalent to $n(\mathfrak{g}_{\mathbb{C}}) \leq \mathrm{DIM}(\Pi)$ in Cases (2) and (3). We shall give a brief review of several conditions that are equivalent to $n(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$ in Proposition 30.

We prove the following.

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Theorem 27. *Let $\mathbb{O}$ be a real minimal nilpotent coadjoint orbit in $\mathfrak{g}^*$. Then the K-action on $\mathbb{O}$ is coisotropic.*

For the proof, we recall some basic facts on real minimal nilpotent orbits.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition, and θ the corresponding Cartan involution. We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and fix a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. We denote by μ the highest element in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, and $A_\mu \in \mathfrak{a}$ the coroot of μ. It is known (*e.g.*, [29]) that any minimal nilpotent coadjoint orbit $\mathbb{O}$ is of the form $\mathbb{O} = \mathrm{Ad}(G)X$ via the identification $\mathfrak{g}^* \simeq \mathfrak{g}$ for some non-zero element $X \in \mathfrak{g}(\mathfrak{a}; \mu) := \{X \in \mathfrak{g} : [H, X] = \mu(H)X \text{ for all } H \in \mathfrak{a}\}$. Let G_X be the stabilizer subgroup of X in G. Then one has the decomposition:

Lemma 28. $G = K \exp(\mathbb{R}A_\mu)G_X$.

Proof. We set $\mathfrak{a}^{\perp\mu} := \{H \in \mathfrak{a} : \mu(H) = 0\}$, $\mathfrak{n} = \bigoplus_{\nu \in \Sigma^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}(\mathfrak{a}; \nu)$, and $\mathfrak{m} := Z_\mathfrak{k}(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. We note that $\mathfrak{a} = \mathbb{R}A_\mu \oplus \mathfrak{a}^{\perp\mu}$ is the orthogonal direct sum decomposition with respect to the Killing form.

Since μ is the highest element in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, the Lie algebra $Z_\mathfrak{g}(X)$ of G_X contains $\mathfrak{a}^{\perp\mu} \oplus \mathfrak{n}$. In particular, G_X contains the subgroup $\exp(\mathfrak{a}^{\perp\mu})N$. Since $A = \exp(\mathbb{R}A_\mu)\exp(\mathfrak{a}^{\perp\mu})$, the Iwasawa decomposition $G = KAN$ implies $G = K \exp(\mathbb{R}A_\mu)G_X$. $\square$

Proof of Theorem 27. Retain the above notation and convention. In particular, we write as $\mathbb{O} = \mathrm{Ad}^*(G)X$. By Lemmas 25 and 28, it suffices to verify

$$(\mathfrak{k} + Z_\mathfrak{g}(X'))^\perp \subset [X', \mathfrak{k}] \quad \text{for any } X' \in \mathrm{Ad}(\exp \mathbb{R}A_\mu)X. \qquad (13)$$

Since $X \in \mathfrak{g}(\mathfrak{a}; \mu)$, any $X' \in \mathrm{Ad}(\exp \mathbb{R}A_\beta)X$ is of the form $X' = cX$ for some $c > 0$. Thus it is enough to show (13) when $X' = X$. Since $Z_\mathfrak{g}(X) \supset \mathfrak{a}^{\perp\mu} \oplus \mathfrak{n}$, one has $\mathfrak{k} + Z_\mathfrak{g}(X) \supset \theta\mathfrak{n} \oplus \mathfrak{a}^{\perp\mu} \oplus \mathfrak{m} \oplus \mathfrak{n}$, hence $(\mathfrak{k} + Z_\mathfrak{g}(X))^{\perp\mu} \subset \mathbb{R}A_\mu$. In view that $(A_\mu, X, c'\theta X)$ forms an $\mathfrak{sl}_2(\mathbb{R})$-triple for some $c' \in \mathbb{R}$, one has $A_\mu \in [X, \mathfrak{k}]$. Thus (13) is verified for $X' = X$. Hence the K-action on $\mathbb{O}$ is coisotropic by Lemma 24. $\square$

3.3 Complex minimal nilpotent orbit

In this section we give a proof of Theorem 23.

Suppose that $G_\mathbb{C}$ is a connected complex simple Lie group. We take a Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ of the Lie algebra $\mathfrak{g}_\mathbb{C}$ of $G_\mathbb{C}$, choose a positive system

$\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, and set $\mathfrak{n}_\mathbb{C}^+ := \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})} \mathfrak{g}_\mathbb{C}(\mathfrak{h}_\mathbb{C}; \alpha)$, $\mathfrak{n}_\mathbb{C}^- := \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})} \mathfrak{g}_\mathbb{C}(\mathfrak{h}_\mathbb{C}; -\alpha)$. Let β be the highest root in $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, and $H_\beta \in \mathfrak{h}_\mathbb{C}$ the coroot of β. Then one has the direct sum decomposition $\mathfrak{h}_\mathbb{C} = \mathbb{C}H_\beta \oplus \mathfrak{h}_\mathbb{C}^{\perp\beta}$ where $\mathfrak{h}_\mathbb{C}^{\perp\beta} := \{H \in \mathfrak{h}_\mathbb{C} : \beta(H) = 0\}$. The minimal nilpotent coadjoint orbit $\mathbb{O}_{\min, \mathbb{C}}$ is of the form $\mathbb{O}_{\min, \mathbb{C}} = \mathrm{Ad}(G_\mathbb{C})X \simeq G_\mathbb{C}/(G_\mathbb{C})_X$ for any non-zero $X \in \mathfrak{g}(\mathfrak{h}_\mathbb{C}; \beta)$ via the identification $\mathfrak{g}_\mathbb{C}^* \simeq \mathfrak{g}_\mathbb{C}$. One can also write as $\mathbb{O}_{\min, \mathbb{C}} = \mathrm{Ad}(G_\mathbb{C})Y \simeq G_\mathbb{C}/(G_\mathbb{C})_Y$ for any non-zero $Y \in \mathfrak{g}(\mathfrak{h}_\mathbb{C}; -\beta)$.

By an elementary representation theory of $\mathfrak{sl}_2$, one sees (*e.g.*, [2]) that the Lie algebras $Z_{\mathfrak{g}_\mathbb{C}}(X)$ and $Z_{\mathfrak{g}_\mathbb{C}}(Y)$ of the isotropy subgroups $(G_\mathbb{C})_X$ and $(G_\mathbb{C})_Y$ are given respectively by

$$Z_{\mathfrak{g}_\mathbb{C}}(X) = \bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \\ \alpha \perp \beta}} \mathfrak{g}_\mathbb{C}(\mathfrak{h}_\mathbb{C}; -\alpha) \oplus \mathfrak{h}_\mathbb{C}^{\perp\beta} \oplus \mathfrak{n}_\mathbb{C}^+, \tag{14} \quad \boxed{\texttt{eqn:Zglmd}}$$

$$Z_{\mathfrak{g}_\mathbb{C}}(Y) = \mathfrak{n}_\mathbb{C}^- \oplus \mathfrak{h}_\mathbb{C}^{\perp\beta} \oplus \bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) \\ \alpha \perp \beta}} \mathfrak{g}_\mathbb{C}(\mathfrak{h}_\mathbb{C}; \alpha).$$

Proof of Theorem 23 (1). We set $S := \exp \mathbb{C}(H_\beta, -H_\beta) \cdot (X, Y)$ in $\mathbb{O}_{\min, \mathbb{C}} \times \mathbb{O}_{\min, \mathbb{C}}$. We claim that $\mathrm{diag}(G_\mathbb{C})S$ is open dense in $\mathbb{O}_{\min, \mathbb{C}} \times \mathbb{O}_{\min, \mathbb{C}}$. To see this, we observe that $(G_\mathbb{C})_X \exp(\mathbb{C}H_\beta)(G_\mathbb{C})_Y$ contains the open Bruhat cell $N_\mathbb{C}^+ H_\mathbb{C} N_\mathbb{C}^- = N_\mathbb{C}^+ \exp(\mathfrak{h}_\mathbb{C}^{\perp\beta}) \exp(\mathbb{C}H_\beta) N_\mathbb{C}^-$ in $G_\mathbb{C}$ as is seen from (14), and thus $\mathrm{diag}(G_\mathbb{C}) \exp \mathbb{C}(H_\beta, 0)((G_\mathbb{C})_X \times (G_\mathbb{C})_Y)$ is open dense in the direct product group $G_\mathbb{C} \times G_\mathbb{C}$ via the identification $\mathrm{diag}(G_\mathbb{C}) \backslash (G_\mathbb{C} \times G_\mathbb{C}) \simeq G_\mathbb{C}$, $(x, y) \mapsto x^{-1}y$.

By Lemma 25, Theorem 23 (1) will follow if we show

$$(\mathrm{diag}(\mathfrak{g}_\mathbb{C}) + Z_{\mathfrak{g}_\mathbb{C} \oplus \mathfrak{g}_\mathbb{C}}(\mathrm{Ad}(a)X, \mathrm{Ad}(a)^{-1}Y))^{\perp B} \subset [(\mathrm{Ad}(a)X, \mathrm{Ad}(a)^{-1}Y), \mathrm{diag}(\mathfrak{g}_\mathbb{C})] \tag{15} \quad \boxed{\texttt{eqn:ZZdiag}}$$

for any $a \in \exp(\mathbb{C}H_\beta)$. Since $\mathrm{Ad}(a)X = cX$ and $\mathrm{Ad}(a)^{-1}Y = c^{-1}Y$ for some $c \in \mathbb{C}^\times$, and since X and Y are arbitrary non-zero elements in $\mathfrak{g}_\mathbb{C}(\mathfrak{h}_\mathbb{C}; \beta)$ and $\mathfrak{g}_\mathbb{C}(\mathfrak{h}_\mathbb{C}; -\beta)$, respectively, it suffices to verify (15) for $a = e$. By (14), one has

$$(\mathrm{diag}(\mathfrak{g}_\mathbb{C}) + (Z_{\mathfrak{g}_\mathbb{C}}(X) \oplus Z_{\mathfrak{g}_\mathbb{C}}(Y)))^{\perp B} = \mathbb{C}(H_\beta, -H_\beta).$$

Since $[X, Y] = c'H_\beta$ for some $c' \in \mathbb{C}^\times$, one has $[(X, Y), (X + Y, X + Y)] = c'(H_\beta, -H_\beta)$, showing $(H_\beta, -H_\beta) \in [(X, Y), \mathrm{diag}(\mathfrak{g}_\mathbb{C})]$. Thus Theorem 23 (1) is proved. $\qquad \square$

Next, we consider the setting in Theorem 23 (2). Let $(G_{\mathbb{C}}, K_{\mathbb{C}})$ be a symmetric pair defined by a holomorphic involutive automorphism θ of $G_{\mathbb{C}}$. Then there is a real form $\mathfrak{g}_{\mathbb{R}}$ of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ such that $\theta|_{\mathfrak{g}_{\mathbb{R}}}$ defines the Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ of the real simple Lie algebra $\mathfrak{g}_{\mathbb{R}}$ with $\mathfrak{k}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ being the Lie algebra $\mathfrak{k}_{\mathbb{C}}$ of $K_{\mathbb{C}}$. We denote by $G_{\mathbb{R}}$ the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{R}}$.

We take a maximal abelian subspace $\mathfrak{a}_{\mathbb{R}}$ in $\mathfrak{p}_{\mathbb{R}}$, and apply the results of Section 3.2 by replacing the notation $\mathfrak{g}, \mathfrak{k}, \mathfrak{p}, \mathfrak{a}, \cdots$ with $\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}, \mathfrak{a}_{\mathbb{R}}$, etc.

Let $\mathcal{N}_{\mathbb{C}}$ be the nilpotent cone in $\mathfrak{g}_{\mathbb{C}}$, and $\mathcal{N}_{\mathbb{R},\mathbb{C}} := \{X \in \mathcal{N}_{\mathbb{C}} : \mathrm{Ad}(G_{\mathbb{C}})X \cap \mathfrak{g}_{\mathbb{R}} \neq \emptyset\}$. Then there exists a unique $G_{\mathbb{C}}$-orbit, to be denoted by $\mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$, which is minimal in $(\mathcal{N}_{\mathbb{R},\mathbb{C}} \setminus \{0\})/G_{\mathbb{C}}$ with respect to the closure relation, and $\mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}} = \mathrm{Ad}(G_{\mathbb{C}})X$ for any non-zero $X \in \mathfrak{g}_{\mathbb{R}}(\mathfrak{a}_{\mathbb{R}}; \beta)$ ([29]).

We extend $\mathfrak{a}_{\mathbb{R}}$ to a maximally split Cartan subalgebra $\mathfrak{h}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$ where $\mathfrak{t}_{\mathbb{R}} := \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{k}_{\mathbb{R}}$, write $\mathfrak{h}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}}$ for the complexification, and take a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ which is compatible with $\Sigma^+(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_{\mathbb{R}})$.

The proof of Theorem 27 shows its complexified version as follows.

Theorem 29. *The action of $K_{\mathbb{C}}$ on $\mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$ is coisotropic.*

This confirms Theorem 23 (2) when $\mathbb{O}_{\min,\mathbb{C}} = \mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$, or equivalently, in Cases (2) and (3) of Lemma 26.

Let us verify Theorem 23 (2) in the case $\mathbb{O}_{\min,\mathbb{C}} \neq \mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$.

We need the following:

Proposition 30 ([24, Cor. 5.9], [29, Prop. 4.1]). *Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{k}_{\mathbb{C}}$ the complexified Lie algebra of $\mathfrak{k}_{\mathbb{R}}$, the Lie algebra $\mathfrak{k}_{\mathbb{R}}$ of a maximal compact subgroup $K_{\mathbb{R}}$ of the analytic subgroup $G_{\mathbb{R}}$ in $\mathrm{Int}\,\mathfrak{g}_{\mathbb{C}}$. Then the following six conditions on $\mathfrak{g}_{\mathbb{R}}$ are equivalent:*

(i) $\mathbb{O}_{\min} \cap \mathfrak{g}_{\mathbb{R}} = \emptyset$.

(ii) $\mathbb{O}_{\min,\mathbb{C}} \neq \mathbb{O}^{\mathbb{C}}_{\min,\mathbb{R}}$.

(iii) $\theta\beta \neq -\beta$.

(iv) $n(\mathfrak{g}) > n(\mathfrak{g}_{\mathbb{C}})$.

(v) $\mathfrak{g}_{\mathbb{R}}$ *is compact or is isomorphic to* $\mathfrak{su}^*(2n)$, $\mathfrak{so}(n-1, 1)$ $(n \geq 5)$, $\mathfrak{sp}(m, n)$, $\mathfrak{f}_{4(-20)}$, *or* $\mathfrak{e}_{6(-26)}$.

(vi) $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C}$ *or the pair* $(\mathfrak{g}_\mathbb{C}, \mathfrak{k}_\mathbb{C})$ *is isomorphic to* $(\mathfrak{sl}(2n, \mathbb{C}), \mathfrak{sp}(n, \mathbb{C}))$, $(\mathfrak{so}(n, \mathbb{C}), \mathfrak{so}(n-1, \mathbb{C}))$ $(n \geq 5)$, $(\mathfrak{sp}(m+n, \mathbb{C}), \mathfrak{sp}(m, \mathbb{C}) \oplus \mathfrak{sp}(n, \mathbb{C}))$, $(\mathfrak{f}_4^\mathbb{C}, \mathfrak{so}(9, \mathbb{C}))$, *or* $(\mathfrak{e}_6^\mathbb{C}, \mathfrak{f}_4^\mathbb{C})$.

Remark 31. The equivalence (i) $\iff$ (v) was stated in [1, Prop. 4.1] without proof, and Okuda [29] supplied a complete proof.

Lemma 32. *Suppose X is a highest root vector, namely, $0 \neq X \in \mathfrak{g}_\mathbb{C}(\mathfrak{h}_\mathbb{C}; \beta)$. If $\theta\beta \neq -\beta$, then $H_\beta \in \mathfrak{k}_\mathbb{C} + Z_{\mathfrak{g}_\mathbb{C}}(X)$.*

Proof. Since $\theta\beta \neq -\beta$, one has $\beta|_{\mathfrak{t}_\mathbb{C}} \not\equiv 0$, namely, $\mathfrak{t}_\mathbb{C} \not\subset \mathfrak{h}_\mathbb{C}^{\perp\beta}$. Since $\mathfrak{h}_\mathbb{C}^{\perp\beta}$ is of codimension one in $\mathfrak{h}_\mathbb{C}$, we get $\mathfrak{t}_\mathbb{C} + \mathfrak{h}_\mathbb{C}^{\perp\beta} = \mathfrak{h}_\mathbb{C}$. Thus $H_\beta \in \mathfrak{h}_\mathbb{C} \subset \mathfrak{k}_\mathbb{C} + Z_{\mathfrak{g}_\mathbb{C}}(X)$. $\square$

Proposition 33. *If one of (and therefore any of) the equivalent conditions in Proposition 30 holds, then $K_\mathbb{C}$ has a Zariski open orbit in $\mathbb{O}_{\min, \mathbb{C}}$. In particular, the $K_\mathbb{C}$-action on $\mathbb{O}_{\min, \mathbb{C}}$ is coisotropic.*

Proof. Since $\mathbb{O}_{\min, \mathbb{C}} = \mathrm{Ad}(G_\mathbb{C})X$ for a non-zero $X \in \mathfrak{g}_\mathbb{C}(\mathfrak{h}_\mathbb{C}; \beta)$, the proposition is clear. $\square$

Proof of Theorem 23 (2). The Case (1) in Lemma 26 is proved in Proposition 33, and the Cases (2) and (3) are proved in Theorem 29. $\square$

3.4 Proof of Theorems in Section 1

As we saw at the end of Section 2, Theorems 3 and 4 are derived from the geometric result, namely, from Theorem 23, and thus the proof of these theorems has been completed.

In the same manner, one can deduce readily from Theorem 29 the following bounded multiplicity property which is not covered by Theorem 3 for the five cases in Proposition 30 where $n(\mathfrak{g}) > n(\mathfrak{g}_\mathbb{C})$.

Theorem 34. *Suppose that the Gelfand–Kirillov dimension of $\Pi \in \mathrm{Irr}(G)$ is $n(\mathfrak{g})$. If (G, G') is a symmetric pair such that $\mathfrak{g}'_\mathbb{C}$ is conjugate to $\mathfrak{k}_\mathbb{C}$ by $\mathrm{Int}\,\mathfrak{g}_\mathbb{C}$, then $m(\Pi|_{G'}) < \infty$.*

Proof. We write $G'_\mathbb{C}$ and $K_\mathbb{C}$ for the analytic subgroups of $G_\mathbb{C} = \mathrm{Int}\,\mathfrak{g}_\mathbb{C}$ with Lie algebras $\mathfrak{g}'_\mathbb{C}$ and $\mathfrak{k}_\mathbb{C}$, respectively. Then the $K_\mathbb{C}$-action on $\mathbb{O}_{\min, \mathbb{R}}^\mathbb{C}$ is coisotropic by Theorem 29, and so is the $G'_\mathbb{C}$-action on $\mathbb{O}_{\min, \mathbb{R}}^\mathbb{C}$ because $G'_\mathbb{C}$ and $K_\mathbb{C}$ are conjugate by an element of $G_\mathbb{C}$. Hence the theorem follows from Fact 21. $\square$

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Finally, we give a proof of Theorem 10.

Proof of Theorem 10. Let $\mathcal{J}$ be the Joseph ideal. Let $(U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J})^{\mathfrak{g}'_{\mathbb{C}}}$ be the algebra of $\mathfrak{g}'_{\mathbb{C}}$-invariant elements in $U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}$ via the adjoint action. Then one has

$$(U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J})^{\mathfrak{g}'_{\mathbb{C}}} = \mathbb{C}$$

if one of (therefore, all of) the equivalent conditions in Proposition 30 is satisfied, see [30, Lem. 3.4]. In particular, the center $Z(\mathfrak{g}'_{\mathbb{C}})$ of the enveloping algebra $U(\mathfrak{g}'_{\mathbb{C}})$ of the subalgebra $\mathfrak{g}'_{\mathbb{C}}$ acts as scalars on the minimal representation Π because the action factors through the following composition of homomorphisms:

$$Z(\mathfrak{g}'_{\mathbb{C}}) \to U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J} \to \mathrm{End}_{\mathbb{C}}(\Pi).$$

Since any minimal representation is unitarizable by the classification [30], and since there are at most finitely many elements in $\mathrm{Irr}(G')$ having a fixed $Z(\mathfrak{g}'_{\mathbb{C}})$-infinitesimal character, the restriction $\Pi|_{G'}$ splits into a direct sum of at most finitely many irreducible representations of G' with multiplicity being finite by Theorem 7. Thus the proof of Theorem 10 is completed. $\square$

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