Multiplicity in restricting small representations

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Abstract: We give a geometric criterion for the bounded multiplicity property of "small" infinite-dimensional representations of real reductive Lie groups in both induction and restrictions.

In particular, for a reductive symmetric pair (G, H), we determine the reductive subgroups G' having the property that any irreducible H-distinguished admissible representations of G are of bounded multiplicity when restricted to G'.

Key words: Branching law; multiplicity; reductive group; symmetric pair; visible action; spherical variety.

1. Introduction. By branching problems in representation theory, we mean the broad problem of understanding how irreducible representations of a group behave when restricted to a subgroup. As viewed in [12], we may divide the branching problems into the following three stages:

Stage A. Abstract features of the restriction;
Stage B. Branching law;
Stage C. Construction of symmetry breaking operators.

The role of Stage A is to develop a theory on the restriction of representations as generally as possible. In turn, we may expect a detailed study of the restriction in Stages B (decomposition of representations) and C (decomposition of vectors) in the "promising" settings that are suggested by the general theory in Stage A.

This article concerns a question in Stage A about "multiplicity" in branching problems.

Let G be a real reductive Lie group, $\mathcal{M}(G)$ the category of finitely generated, smooth admissible representations of G of moderate growth [31, Chap. 11], and $\mathrm{Irr}(G)$ the set of irreducible objects in $\mathcal{M}(G)$. We shall use the uppercase letter Π for representations of the group G, and the lowercase letter π for those of a reductive subgroup G'.

For Stage A, we may formulate an abstract feature of the restrictions as a property for

- the pair (G, G'),
- the triple (G, G', Π), or
- the quadruple (G, G', Π, π).

The formulation for *the triple* (G, G', Π) was adopted in the study of *G'-admissible restriction* of Π, namely, the restriction $\Pi|_{G'}$ of $\Pi \in \mathrm{Irr}(G)$ being discretely decomposable with finite multiplicity, see [5–7] for the general theory, and [17] for some classification theory of the triples (G, G', Π).

On the other hand, Fact 2.1 below is formulated as a property for *the pair* (G, G'). This is the study of "multiplicity" of the restrictions, see [11,16] for the general theory, and [15] for the classification of the pairs (G, G'). In this article, we discuss its refinement in a *formulation for the triple* (G, G', Ω) or for *the quadruple* (G, G', Ω, Ω') where $\Omega \subset \mathcal{M}(G)$ and $\Omega' \subset \mathcal{M}(G')$ are families of "small" infinite-dimensional representations, see Problems 2.3 and 4.1. This refinement reveals the underlying geometric structures of some concrete examples, e.g., [1,13,24], and yields much broader settings that seem to be promising for analysis of branching problems in Stage C.

Detail proofs of the theorems in this article will appear in [14].

2. Bounded multiplicity in restriction. Throughout this article, we shall assume that $G \supset G'$ are real forms of complex reductive algebraic Lie groups $G_{\mathbf{C}} \supset G'_{\mathbf{C}}$, respectively. Their compact real forms will be denoted by $G_U \supset G'_U$. The Lie algebras will be denoted by the corresponding lowercase German letters $\mathfrak{g}$, $\mathfrak{g}_{\mathbf{C}}$, $\mathfrak{g}_U$, $\mathfrak{g}'$, etc.

For $\Pi \in \mathcal{M}(G)$ and $\pi \in \mathcal{M}(G')$, we define the **multiplicity** of the restriction $\Pi|_{G'}$ in the category $\mathcal{M}$ by

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$$[\Pi|_{G'} : \pi] := \dim_{\mathbf{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbf{N} \cup \{\infty\},$$

where $\mathrm{Hom}_{G'}(\ ,\ )$ denotes the space of continuous G'-homomorphisms between the Fréchet representations.

In [16, Thms. C and D] we proved the following geometric criteria:

Fact 2.1. Let $G \supset G'$ be a pair of algebraic real reductive Lie groups.
(1) **Bounded multiplicity** for a pair (G, G'):

$$\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty \tag{2.1}$$

if and only if $(G_{\mathbf{C}} \times G'_{\mathbf{C}})/\mathrm{diag}\, G'_{\mathbf{C}}$ is spherical.
(2) **Finite multiplicity** for a pair (G, G'):

$$[\Pi|_{G'} : \pi] < \infty, \quad {}^{\forall}\Pi \in \mathrm{Irr}(G),\ {}^{\forall}\pi \in \mathrm{Irr}(G') \tag{2.2}$$

if and only if $(G \times G')/\mathrm{diag}\, G'$ is real spherical.

Here we recall that a complex $G_{\mathbf{C}}$-manifold X is called *spherical* if a Borel subgroup of $G_{\mathbf{C}}$ has an open orbit in X, and that a G-manifold Y is called *real spherical* if a minimal parabolic subgroup of G has an open orbit in Y.

A remarkable feature of Fact 2.1 (1) is that the bounded multiplicity property (2.1) is determined only by the complexifications of G and G', hence the classification of such pairs (G, G') is reduced to a classical result [20]: the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}'_{\mathbf{C}})$ is the direct sum of the following ones up to abelian ideals:

$$(\mathfrak{sl}_n, \mathfrak{gl}_{n-1}), (\mathfrak{so}_n, \mathfrak{so}_{n-1}), \text{ or } (\mathfrak{so}_8, \mathfrak{spin}_7). \tag{2.3}$$

On the other hand, the finite multiplicity property (2.2) depends on real forms. It is fulfilled for any Riemannian symmetric pair by Harish-Chandra's admissibility theorem, whereas it is not the case for some reductive symmetric pairs such as $(G, G') = (SL(p+q, \mathbf{R}), SO(p, q))$. A complete classification of the symmetric pairs (G, G') satisfying the finite multiplicity property (2.2) was accomplished in [15].

Example 2.2. Let $p_1 + p_2 = p$, $q_1 + q_2 = q$, and $(G, G') = (O(p, q), O(p_1, q_1) \times O(p_2, q_2))$. Suppose $p + q \geq 5$. The criteria in Fact 2.1 give the equivalences:

(2.1) $\Longleftrightarrow p_1 + q_1 = 1$ or $p_2 + q_2 = 1$.

(2.2) $\Longleftrightarrow p_1 + q_1 = 1,\ p_2 + q_2 = 1, p = 1,$ or $q = 1$.

This means that for general p_1, q_1, p_2, q_2, there exist $\Pi \in \mathrm{Irr}(G)$ and $\pi \in \mathrm{Irr}(G')$ such that $[\Pi|_{G'} : \pi] = \infty$. Nevertheless, a multiplicity-free theorem holds for the restriction $\Pi|_{G'}$ for any p_1, p_2, q_1, q_2, and for any discrete series representation Π for the symmetric space G/H with $H = O(p-1, q)$, see [13] for a precise statement.

This example suggests us to work with the *triple* (G, G', Π) rather than the *pair* (G, G') for the finer study of multiplicity estimates as mentioned in Introduction.

Take $\Pi \in \mathcal{M}(G)$. We say the restriction $\Pi|_{G'}$ has the *finite multiplicity property* if $[\Pi|_{G'} : \pi] < \infty$ for all $\pi \in \mathrm{Irr}(G')$, and has the *bounded multiplicity property* if $m(\Pi|_{G'}) < \infty$, where we set

$$m(\Pi|_{G'}) := \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] \in \mathbf{N} \cup \{\infty\}. \tag{2.4}$$

In search for broader settings in which we could expect a detailed study of the restriction $\Pi|_{G'}$ in Stages B and C, we address the following

Problem 2.3. Given a pair $G \supset G'$, find a subset Ω of $\mathcal{M}(G)$ such that $\sup_{\Pi \in \Omega} m(\Pi|_{G'}) < \infty$.

We bear in mind that branching problems often arise for a family of representations Π. For a better understanding of Problem 2.3, we first examine two opposite extremal choices of Ω. When Ω is a singleton, Problem 2.3 concerns the triple (G, G', Π) having the bounded multiplicity property. When Ω is the whole set $\mathrm{Irr}(G)$, Problem 2.3 asks the condition (2.1), and is solved by the geometric criterion for the pair (G, G'), as seen in Fact 2.1 (1). Second, we note that Problem 2.3 is nontrivial even when G is a compact Lie group where $m(\Pi|_{G'})$ is individually finite. In this article we discuss Problem 2.3 with focus on the following two cases:
(1) $\Omega = \mathrm{Irr}(G)_H$, the set of H-distinguished irreducible representations of G (Theorem 3.2);
(2) $\Omega = \Omega_P, \Omega_{P,\mathfrak{q}}$: families of degenerate principal series representations (Theorems 4.2 and 4.3).

Remark 2.4. One may wonder why we did not use $[\pi : \Pi|_{G'}] := \dim_{\mathbf{C}} \mathrm{Hom}_{G'}(\pi, \Pi|_{G'})$ instead of $[\Pi|_{G'} : \pi]$. The reason is that the space $\mathrm{Hom}_{G'}(\pi, \Pi|_{G'})$ may be too small to capture the whole picture of the restriction $\Pi|_{G'}$ in the category $\mathcal{M}$. This feature is akin to the fact in the category of Harish-Chandra modules that $\mathrm{Hom}_{\mathfrak{g}', K'}(\pi_{K'}, \Pi_K|_{\mathfrak{g}'})$ vanishes unless Π_K is "discretely decomposable" as a $(\mathfrak{g}', K')$-module [7].

3. *H*-distinguished representations of *G*. For $\Pi \in \mathrm{Irr}(G)$, we denote by $\Pi^{-\infty}$ the representation on the space of distribution vectors, that is, the topological dual of Π. For a closed subgroup H of G, we set

$$\text{(3.1)} \quad \operatorname{Irr}(G)_H := \{\Pi \in \operatorname{Irr}(G) : (\Pi^{-\infty})^H \neq \{0\}\}.$$

The Frobenius reciprocity tells $\Pi \in \operatorname{Irr}(G)_H$ if and only if $\operatorname{Hom}_G(\Pi^\vee, C^\infty(G/H)) \neq \{0\}$, where $\Pi^\vee$ is the contragredient representation in the category $\mathcal{M}(G)$. Elements Π in $\operatorname{Irr}(G)_H$ (or $\Pi^\vee$) are sometimes referred to as *H-distinguished*, or having *nonzero H-periods*.

For a reductive symmetric pair (G, H), the set $\operatorname{Irr}(G)_H$ is described by the Cartan–Helgason theorem when H is compact, whereas the full classification is far from being achieved in the general setting where H is not compact, although one has still some useful information about $\operatorname{Irr}(G)_H$, see e.g., Theorem 6.2 below.

The following notions are a key in answering Problem 2.3 for $\Omega = \operatorname{Irr}(G)_H$.

Definition 3.1. Let G/H be a reductive symmetric space defined by an involution σ of G. We take G_U ($\subset G_{\mathbf{C}}$) such that $G_U \cap H$ is a maximal compact subgroup of H.

(1) We say a complex parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbf{C}}$ is a *Borel subalgebra* for G/H if $\mathfrak{q}$ is defined by a generic element in $\sqrt{-1}\mathfrak{g}_U^{-\sigma}$.

(2) We say a real parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is a *minimal parabolic subalgebra* for G/H if $\mathfrak{p}$ is defined by a generic element in $\mathfrak{g} \cap \sqrt{-1}\mathfrak{g}_U^{-\sigma}$.

Borel subalgebras for the symmetric space G/H are unique up to inner automorphisms of $\mathfrak{g}_{\mathbf{C}}$. Likewise, minimal parabolic subalgebras for G/H are unique up to inner automorphisms of $\mathfrak{g}$. We shall write $B_{G/H}$ ($\subset G_{\mathbf{C}}$) and $P_{G/H}$ ($\subset G$) for the corresponding parabolic subgroups, referred to as a *Borel subgroup* and a *minimal parabolic subgroup* for the symmetric space G/H, respectively. We note that the Borel subalgebra $\mathfrak{b}_{G/H}$ for G/H is not necessarily solvable, and that it is determined only by the complexification $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$.

Here is an answer to Problem 2.3 for $\Omega = \operatorname{Irr}(G)_H$ when (G, H) is a reductive symmetric pair.

Theorem 3.2. *Let $B_{G/H}$ be a Borel subgroup for G/H. Suppose G' is an algebraic reductive subgroup of G. Then the following three conditions on the triple (G, H, G') are equivalent*:

(i) $\sup_{\Pi \in \operatorname{Irr}(G)_H} m(\Pi|_{G'}) < \infty$.

(ii) *$G_{\mathbf{C}}/B_{G/H}$ is G'_U-strongly visible.*

(iii) *$G_{\mathbf{C}}/B_{G/H}$ is $G'_{\mathbf{C}}$-spherical.*

See [8, Def. 3.3.1] for the definition of strongly visible actions on complex manifolds, and [29] for the equivalence (ii) $\Longleftrightarrow$ (iii).

The list of the triples (G, H, G') is given in Theorem 5.1 below in the setting that (G, G') is a symmetric pair and that $\mathfrak{g}_{\mathbf{C}}$ is simple.

We also discuss the following finite multiplicity property (**FM**) for the restriction $\Pi|_{G'}$, weaker than the bounded multiplicity property (i) in Theorem 3.2:

$$(\mathbf{FM}) \quad [\Pi|_{G'}; \pi] < \infty, \ {}^\forall \Pi \in \operatorname{Irr}(G)_H, \ {}^\forall \pi \in \operatorname{Irr}(G').$$

Proposition 3.3. *Let $P_{G/H}$ be a minimal parabolic subgroup for a reductive symmetric space G/H. Let G' be an algebraic reductive subgroup of G, and P' a minimal parabolic subgroup of G'.*

(1) *If $\#(P'_{\mathbf{C}} \backslash G_{\mathbf{C}}/(P_{G/H})_{\mathbf{C}}) < \infty$, then* (**FM**) *holds.*

(2) *If* (**FM**) *holds, $G/P_{G/H}$ is G'-real spherical.*

Proposition 3.3 (2) was proved in [11]. The converse statement of Proposition 3.3 (2) holds in the group manifold case, namely, if G/H is of the form $({}^\backprime G \times {}^\backprime G)/\operatorname{diag} {}^\backprime G$ and if G' is of the form $G'_1 \times G'_2$, see Fact 2.1 (2).

4. Degenerate principal series representations. Let P be a parabolic subgroup of G. We write $\operatorname{Irr}(P)_f$ for the set of equivalence classes of irreducible finite-dimensional representations of P. Let $\operatorname{Ind}_P^G(\xi)$ be the degenerate principal series representation of G obtained as a smooth induction from $\xi \in \operatorname{Irr}(P)_f$. Then $\operatorname{Ind}_P^G(\xi) \in \mathcal{M}(G)$.

Suppose that P' is a parabolic subgroup of a real reductive algebraic subgroup G' of G. Degenerate principal series representations $\operatorname{Ind}_{P'}^{G'}(\eta)$ of G' are defined similarly for $\eta \in \operatorname{Irr}(P')_f$. This section studies the multiplicity $[\operatorname{Ind}_P^G(\xi)|_{G'} : \operatorname{Ind}_{P'}^{G'}(\eta)]$, namely, the dimension of the space $\operatorname{Hom}_{G'}(\operatorname{Ind}_P^G(\xi)|_{G'}, \operatorname{Ind}_{P'}^{G'}(\eta))$ of "symmetry breaking operators".

In the case $(G, G') = (O(n+1, 1), O(n, 1))$, this is the space of conformally covariant symmetry breaking operators for the totally geodesic embedding $S^{n-1} \hookrightarrow S^n$. All such operators have been constructed and classified recently, see [18] for the scalar case, and [19] for differential forms. In this case, the multiplicity takes the values in $\{0, 1, 2\}$.

For a finer estimate of the multiplicity $[\operatorname{Ind}_P^G(\xi)|_{G'} : \operatorname{Ind}_{P'}^{G'}(\eta)]$ in the general setting, we implement yet other parabolic subgroups $Q \subset P_{\mathbf{C}}$ and $Q' \subset P'_{\mathbf{C}}$. What we call a "$QP$ estimate" of the multiplicity will play a key role in the proof of Theorem 3.2 for H-distinguished representations.

Let Q be a complex parabolic subgroup of $G_{\mathbf{C}}$ with $\mathfrak{q} \subset \mathfrak{p}_{\mathbf{C}}$. We do not require $\mathfrak{q}$ to be defined over $\mathbf{R}$. For $\xi \in \mathrm{Irr}(P)_f$, we define $d_{\mathfrak{q}}(\xi)$ to be the minimum of the dimensions of non-zero $\mathfrak{q}$-submodules in η, and denote by $\mathrm{Irr}(P;\mathfrak{q})_f$ the subset of $\mathrm{Irr}(P)_f$ with $d_{\mathfrak{q}}(\xi) = 1$.

We define subsets of $\mathcal{M}(G)$ by

$$\Omega_P := \{\mathrm{Ind}_P^G(\xi) : \xi \text{ is a character of } P\}, \tag{4.1}$$

$$\Omega_{P,\mathfrak{q}} := \{\mathrm{Ind}_P^G(\xi) : \xi \in \mathrm{Irr}(P;\mathfrak{q})_f\}. \tag{4.2}$$

Obviously, one has $\Omega_P \subset \Omega_{P,\mathfrak{q}}$. Moreover, $\Omega_{P,\mathfrak{q}}$ is the whole set $\{\mathrm{Ind}_P^G(\xi) : \xi \in \mathrm{Irr}(P)_f\}$ if $\mathfrak{q}$ is a Borel subalgebra of $\mathfrak{g}_{\mathbf{C}}$.

We consider the following refinement of Problem 2.3:

Problem 4.1. Given a pair $G \supset G'$, find subsets $\Omega \subset \mathcal{M}(G)$ and $\Omega' \subset \mathcal{M}(G')$ such that

$$\sup_{\Pi \in \Omega} \sup_{\pi \in \Omega'} [\Pi|_{G'} : \pi] < \infty.$$

One observes that Problem 2.3 corresponds to the case where $\Omega' = \mathrm{Irr}(G')$.

Theorem 4.2 ("QP estimate" for restriction)**.** *Suppose that Q and Q' are complex parabolic subgroups of $G_{\mathbf{C}}$ and $G'_{\mathbf{C}}$, respectively, such that $\mathfrak{q} \subset \mathfrak{p}_{\mathbf{C}}$, $\mathfrak{q}' \subset \mathfrak{p}'_{\mathbf{C}}$, and $\#(Q'_{\mathrm{opp}} \backslash G_{\mathbf{C}}/Q) < \infty$. Here Q'_{opp} stands for the opposite parabolic subgroup of Q' in $P'_{\mathbf{C}}$. Then there exists $C > 0$ such that*

$$[\mathrm{Ind}_P^G(\xi)|_{G'} : \mathrm{Ind}_{P'}^{G'}(\eta)] \le C d_{\mathfrak{q}}(\xi) d_{\mathfrak{q}'}(\eta) \tag{4.3}$$

for any $\xi \in \mathrm{Irr}(P)_f$ and any $\eta \in \mathrm{Irr}(P')_f$. In particular, one has

$$\sup_{\xi \in \mathrm{Irr}(P;\mathfrak{q})_f} \sup_{\eta \in \mathrm{Irr}(P';\mathfrak{q}')_f} [\mathrm{Ind}_P^G(\xi)|_{G'} : \mathrm{Ind}_{P'}^{G'}(\eta)] \le C.$$

When Q' is a Borel subgroup of $G'_{\mathbf{C}}$, one obtains the converse statement of Theorem 4.2 as follows:

Theorem 4.3. *Let $G \supset G'$ be a pair of real reductive algebraic Lie groups, P a parabolic subgroup of G, and Q a complex parabolic subgroup of $G_{\mathbf{C}}$ such that $\mathfrak{q} \subset \mathfrak{p}_{\mathbf{C}}$. Then the following four conditions on $(G, G'; P, Q)$ are equivalent:*

(i) $\sup_{\Pi \in \Omega_{P,\mathfrak{q}}} m(\Pi|_{G'}) < \infty$.

(ii) *There exists $C > 0$ such that*

$$m(\mathrm{Ind}_P^G(\xi)|_{G'}) \le C d_{\mathfrak{q}}(\xi) \text{ for all } \xi \in \mathrm{Irr}(P)_f.$$

(iii) *$G_{\mathbf{C}}/Q$ is G'_U-strongly visible.*

(iv) *$G_{\mathbf{C}}/Q$ is $G'_{\mathbf{C}}$-spherical.*

The parabolic subgroups Q in (iv) are classified in [2] in the setting where $(G_{\mathbf{C}}, G'_{\mathbf{C}})$ is a symmetric pair. Theorem 4.3 with $Q = P_{\mathbf{C}}$ shows:

Corollary 4.4. *Let P be a parabolic subgroup of G, and G' an algebraic subgroup of G. Then one has the equivalence on the triple $(G, G'; P)$:*

$$G_{\mathbf{C}}/P_{\mathbf{C}} \text{ is } G'_{\mathbf{C}}\text{-spherical} \iff \sup_{\Pi \in \Omega_P} m(\Pi|_{G'}) < \infty.$$

Example 4.5. If the unipotent radical of P is abelian, then Corollary 4.4 applies for any symmetric pair (G, G') by [8, Cor. 15].

Theorem 4.2 also implies the following

Theorem 4.6 (Invariant trilinear forms)**.** *Let G be a real reductive algebraic Lie group, and P_j $(j = 1,2,3)$ parabolic subgroups of G. Suppose that Q_j $(j = 1,2,3)$ are complex parabolic subgroups of $G_{\mathbf{C}}$ such that $Q_j \subset (P_j)_{\mathbf{C}}$ $(1 \le j \le 3)$ and $\#(\mathrm{diag}(G_{\mathbf{C}}) \backslash (G_{\mathbf{C}} \times G_{\mathbf{C}} \times G_{\mathbf{C}}) / (Q_1 \times Q_2 \times Q_3)) < \infty$. Then there exists $C > 0$ such that*

$$\dim_{\mathbf{C}} \mathrm{Hom}_G\left(\bigotimes_{j=1}^{3} \mathrm{Ind}_{P_j}^G(\xi_j), \mathbf{C}\right) \le C \prod_{j=1}^{3} d_{\mathfrak{q}_j}(\xi_j)$$

for all $\xi_j \in \mathrm{Irr}(P_j)_f$ $(j = 1,2,3)$.

See [22,23] for a classification of (Q_1, Q_2, Q_3) with the above geometric property for some classical groups $G_{\mathbf{C}}$.

For $\Pi_1, \Pi_2 \in \mathcal{M}(G)$, we consider the tensor product representation $\Pi_1 \otimes \Pi_2$, and set

$$m(\Pi_1 \otimes \Pi_2) := \sup_{\Pi \in \mathrm{Irr}(G)} \dim_{\mathbf{C}} \mathrm{Hom}_G(\Pi_1 \otimes \Pi_2, \Pi).$$

A special case of Theorem 4.6 implies (v) $\Rightarrow$ (i) of the theorem below.

Theorem 4.7. *Let G be a real reductive algebraic Lie group, and P_j $(j = 1,2)$ parabolic subgroups. Then the following five conditions on the triple (G, P_1, P_2) are equivalent:*

(i) *There exists $C > 0$ such that*

$$m(\mathrm{Ind}_{P_1}^G(\xi_1) \otimes \mathrm{Ind}_{P_2}^G(\xi_2)) \le C \dim \xi_1 \ \dim \xi_2$$

for all $\xi_j \in \mathrm{Irr}(P_j)_f$ $(j = 1,2)$.

(ii) *There exists $C > 0$ such that*

$$m(\mathrm{Ind}_{P_1}^G(\xi_1) \otimes \mathrm{Ind}_{P_2}^G(\xi_2)) \le C$$

for all characters ξ_j of P_j $(j = 1,2)$.

(iii) *$\mathcal{O}(G_{\mathbf{C}}/P_{1\mathbf{C}}, \mathcal{L}_1) \otimes \mathcal{O}(G_{\mathbf{C}}/P_{2\mathbf{C}}, \mathcal{L}_2)$ is a multiplicity free $G_{\mathbf{C}}$-module for any $G_{\mathbf{C}}$-equivariant holomorphic line bundles $\mathcal{L}_j$ on $G_{\mathbf{C}}/P_{j\mathbf{C}}$ $(j = 1,2)$.*

(iv) *$G_{\mathbf{C}}/P_{1\mathbf{C}} \times G_{\mathbf{C}}/P_{2\mathbf{C}}$ is $\mathrm{diag}(G_U)$-strongly visible.*

(v) $G_{\mathbf{C}}/P_{1\mathbf{C}} \times G_{\mathbf{C}}/P_{2\mathbf{C}}$ *is* diag$(G_{\mathbf{C}})$*-spherical.*

The classification of such pairs $(P_{1\mathbf{C}}, P_{2\mathbf{C}})$ appeared in different contexts. For instance, one may read from [26] for the multiplicity-free results on finite-dimensional representations (iii). The classification theory of visible actions also gives a complete list of the pairs $(P_{1\mathbf{C}}, P_{2\mathbf{C}})$ satisfying (iv), see [10] for type A, and [28] for the other cases. See also [21] for the list satisfying (v) when $P_{j\mathbf{C}}$ are maximal.

Example 4.8. Let G be a real reductive Lie group, and P_1, P_2 parabolic subgroups with abelian unipotent radical. The double flag variety $G_{\mathbf{C}}/P_{1\mathbf{C}} \times G_{\mathbf{C}}/P_{2\mathbf{C}}$ is strongly visible via the diagonal G_U-action [9, Thm. 1.7], hence Theorem 4.7 applies. In particular, by taking P_2 to be the opposite parabolic subgroup of P_1, one sees from Theorem 4.7 the uniform bounded multiplicity property in the Plancherel formula for any para-Hermitian symmetric space.

5. Classification of triples (G, H, G'). In this section, we present the classification of the triples (G, H, G') satisfying

$$\sup_{\Pi \in \mathrm{Irr}(G)_H} m(\Pi|_{G'}) < \infty \tag{5.1}$$

on the level of Lie algebras up to outer automorphisms in the following setting:

- both (G, H) and (G, G') are symmetric pairs,
- $\mathfrak{g}_{\mathbf{C}}$ is simple.

Theorem 5.1. *Suppose that* $\mathfrak{g}_{\mathbf{C}}$ *is simple and that* (G, H) *and* (G, G') *are symmetric pairs. Then the triple* (G, H, G') *satisfies the bounded multiplicity property* (5.1) *if and only if the triple* $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}, \mathfrak{g}'_{\mathbf{C}})$ *of the complexified Lie algebras is in Table* 5.1 *or the pair* $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}'_{\mathbf{C}})$ *is in* (2.3)*. In the table, p, q are arbitrary subject to* $n = p + q$.

Example 5.2. The triple (G, H, G') in Example 2.2 is a real form of the triple $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}, \mathfrak{g}'_{\mathbf{C}})$ in the fourth row of Table 5.1, hence Theorem 5.1 guarantees the bounded multiplicity property of the restriction $\Pi|_{G'}$ for all $\Pi \in \mathrm{Irr}(G)_H$, see [13,24].

Remark 5.3. When the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}'_{\mathbf{C}})$ is in the list (2.3), the supremum of the multiplicity (2.1) is equal to one for many of the real forms such as $(SO(p, q), SO(p-1, q))$, see [27].

6. Sketch of the proof for our main results. We give two ingredients that are used in the proof of out main results.

In the classical harmonic analysis on the Riemannian symmetric space G/K, building blocks of representations in $C^\infty(G/K)$ are constructed by the twisted Poisson transform, an integral G-intertwining operator from the spherical principal series representation to $C^\infty(G/K)$. More generally, for a closed subgroup H in G, we consider the space $\mathrm{Hom}_G(\mathrm{Ind}_P^G(\xi), \mathrm{Ind}_H^G(\tau))$ of generalized Poisson transforms, where P is a parabolic subgroup of G, $\xi \in \mathrm{Irr}(P)_f$, and $\tau \in \mathrm{Irr}(H)_f$. We give a "$QP$ estimate" of the dimension of this space. Along the same line as in [11,16], the "QP estimate" for *restriction* (e.g., the implication (iv) $\Rightarrow$ (i) in Theorem 4.3) is deduced from the following "QP estimates for *induction*" applied to $(G \times G')/\mathrm{diag}\, G'$. Theorem 6.1 (1) below is a generalization of some results in [16] relying on the "boundary valued maps" and in Tauchi [30] relying on the theory of holonomic $\mathcal{D}$-modules [3,4].

Table 5.1. Triples $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}, \mathfrak{g}'_{\mathbf{C}})$ with $\mathfrak{g}_{\mathbf{C}}$ simple in Theorem 5.1

$\mathfrak{g}_{\mathbf{C}}$	$\mathfrak{h}_{\mathbf{C}}$	$\mathfrak{g}'_{\mathbf{C}}$
$\mathfrak{sl}_n$	$\mathfrak{gl}_{n-1}$	$\mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus \mathbf{C}$
$\mathfrak{sl}_{2m}$	$\mathfrak{gl}_{2m-1}$	$\mathfrak{sp}_m$
$\mathfrak{sl}_6$	$\mathfrak{sp}_3$	$\mathfrak{sl}_4 \oplus \mathfrak{sl}_2 \oplus \mathbf{C}$
$\mathfrak{so}_n$	$\mathfrak{so}_{n-1}$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$
$\mathfrak{so}_{2m}$	$\mathfrak{so}_{2m-1}$	$\mathfrak{gl}_m$
$\mathfrak{so}_{2m}$	$\mathfrak{so}_{2m-2} \oplus \mathbf{C}$	$\mathfrak{gl}_m$
$\mathfrak{sp}_n$	$\mathfrak{sp}_{n-1} \oplus \mathfrak{sp}_1$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_q$
$\mathfrak{sp}_n$	$\mathfrak{sp}_{n-2} \oplus \mathfrak{sp}_2$	$\mathfrak{sp}_{n-1} \oplus \mathfrak{sp}_1$
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathfrak{so}_{10} \oplus \mathbf{C}$
$\mathfrak{f}_4$	$\mathfrak{so}_9$	$\mathfrak{so}_9$

Theorem 6.1 ("QP estimate" for induction)**.** *Let G be a real reductive algebraic Lie group, H an algebraic subgroup, P a parabolic subgroup of G, and Q a complex parabolic subgroup of $G_{\mathbf{C}}$ with $Q \subset P_{\mathbf{C}}$.*

(1) *If* $\#(Q \backslash G_{\mathbf{C}}/H_{\mathbf{C}}) < \infty$*, then there exists* $C > 0$ *such that for all* $\eta \in \mathrm{Irr}(P)_f$ *and all* $\tau \in \mathrm{Irr}(H)_f$

$$\dim_{\mathbf{C}} \mathrm{Hom}_G(\mathrm{Ind}_P^G(\eta), \mathrm{Ind}_H^G(\tau)) \le C d_{\mathfrak{q}}(\eta) \dim \tau.$$

(2) *Conversely, if the conclusion in* (1) *holds, then* Q *has an open orbit in* $G_{\mathbf{C}}/H_{\mathbf{C}}$.

For the proof of Theorem 3.2, we also use the following reformulation [14] of Casselman–Oshima's subrepresentation theorem [25,31].

Theorem 6.2 (Quotient representation theorem)**.** *Let G/H be a reductive symmetric space, and $P_{G/H}$ and $\mathfrak{b}_{G/H}$ a minimal parabolic subgroup and a Borel subalgebra for G/H, respectively, with $\mathfrak{b}_{G/H} \subset (\mathfrak{p}_{G/H})_{\mathbf{C}}$. Then for any $\Pi \in \mathrm{Irr}(G)_H$, there* *exists* $\xi \in \mathrm{Irr}(P_{G/H}; \mathfrak{b}_{G/H})_f$ *such that* Π *is a quotient of the degenerate principal series representation* $\mathrm{Ind}_{P_{G/H}}^{G}(\xi)$.

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References

[1] J.-L. Clerc, T. Kobayashi, B. Ørsted and M. Pevzner, Generalized Bernstein-Reznikov integrals, Math. Ann. **349** (2011), no. 2, 395–431.

[2] X. He, K. Nishiyama, H. Ochiai and Y. Oshima, On orbits in double flag varieties for symmetric pairs, Transform. Groups **18** (2013), no. 4, 1091–1136.

[3] M. Kashiwara, *Systems of microdifferential equations*, based on lecture notes by Teresa Monteiro Fernandes translated from the French, Progress in Mathematics, 34, Birkhäuser Boston, Inc., Boston, MA, 1983.

[4] M. Kashiwara and T. Kawai, On holonomic systems of microdifferential equations. III. Systems with regular singularities, Publ. Res. Inst. Math. Sci. **17** (1981), no. 3, 813–979.

[5] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications, Invent. Math. **117** (1994), no. 2, 181–205.

[6] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups. II. Micro-local analysis and asymptotic K-support, Ann. of Math. (2) **147** (1998), no. 3, 709–729.

[7] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups. III. Restriction of Harish-Chandra modules and associated varieties, Invent. Math. **131** (1998), no. 2, 229–256.

[8] T. Kobayashi, Multiplicity-free representations and visible actions on complex manifolds, Publ. Res. Inst. Math. Sci. **41** (2005), no. 3, 497–549.

[9] T. Kobayashi, Visible actions on symmetric spaces, Transform. Groups **12** (2007), no. 4, 671–694.

[10] T. Kobayashi, A generalized Cartan decomposition for the double coset space $(U(n_1) \times U(n_2) \times U(n_3)) \backslash U(n)/(U(p) \times U(q))$, J. Math. Soc. Japan **59** (2007), no. 3, 669–691.

[11] T. Kobayashi, Shintani functions, real spherical manifolds, and symmetry breaking operators, in *Developments and retrospectives in Lie theory*, Dev. Math., 37, Springer, Cham, 2014, pp. 127–159.

[12] T. Kobayashi, A program for branching problems in the representation theory of real reductive groups, in *Representations of reductive groups*, Progr. Math., 312, Birkhäuser/Springer, Cham, 2015, pp. 277–322.

[13] T. Kobayashi, Branching laws of unitary representations associated to minimal elliptic orbits for indefinite orthogonal group $O(p, q)$, Adv. Math., 388, (2021), Paper No. 107862.

[14] T. Kobayashi, Bounded multiplicity theorems for induction and restriction, J. Lie Theory **32** (2022), no. 1, 197–238.

[15] T. Kobayashi and T. Matsuki, Classification of finite-multiplicity symmetric pairs, Transform. Groups **19** (2014), no. 2, 457–493.

[16] T. Kobayashi and T. Oshima, Finite multiplicity theorems for induction and restriction, Adv. Math. **248** (2013), 921–944.

[17] T. Kobayashi and Y. Oshima, Classification of symmetric pairs with discretely decomposable restrictions of $(\mathfrak{g}, K)$-modules, J. Reine Angew. Math. **703** (2015), 201–223.

[18] T. Kobayashi and B. Speh, *Symmetry breaking for representations of rank one orthogonal groups*, Mem. Amer. Math. Soc. **238** (2015), no. 1126, v+110 pp.

[19] T. Kobayashi and B. Speh, *Symmetry breaking for representations of rank one orthogonal groups II*, Lecture Notes in Mathematics, 2234, Springer, Singapore, 2018.

[20] M. Krämer, Multiplicity free subgroups of compact connected Lie groups, Arch. Math. (Basel) **27** (1976), no. 1, 28–36.

[21] P. Littelmann, On spherical double cones, J. Algebra **166** (1994), no. 1, 142–157.

[22] P. Magyar, J. Weyman and A. Zelevinsky, Multiple flag varieties of finite type, Adv. Math. **141** (1999), no. 1, 97–118.

[23] T. Matsuki, Orthogonal multiple flag varieties of finite type I: Odd degree case, J. Algebra **425** (2015), 450–523.

[24] B. Ørsted and B. Speh, Branching laws for discrete series of some affine symmetric spaces, preprint, arXiv:1907.07544.

[25] T. Oshima, A realization of semisimple symmetric spaces and construction of boundary value maps, in *Representations of Lie groups, Kyoto, Hiroshima, 1986*, Adv. Stud. Pure Math., 14, Academic Press, Boston, MA, 1988, pp. 603–650.

[26] J. R. Stembridge, Multiplicity-free products and restrictions of Weyl characters, Represent. Theory **7** (2003), 404–439.

[27] B. Sun and C.-B. Zhu, Multiplicity one theorems: the Archimedean case, Ann. of Math. (2) **175** (2012), no. 1, 23–44.

[28] Y. Tanaka, Classification of visible actions on flag varieties, Proc. Japan Acad. Ser. A Math. Sci. **88** (2012), no. 6, 91–96.

[29] Y. Tanaka, Visible actions of compact Lie groups on complex spherical varieties. (to appear in J. Diff. Geom).

[30] T. Tauchi, A generalization of the Kobayashi-Oshima uniformly bounded multiplicity theorem, Internat. J. Math. **32** (2021), no. 14, 2150107.

[31] N. R. Wallach, *Real reductive groups. II*, Pure and Applied Mathematics, 132-II, Academic Press, Inc., Boston, MA, 1992.