Bounded multiplicity theorems for induction and restriction

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Abstract

We prove a geometric criterion for the bounded multiplicity property of "small" infinite-dimensional representations of real reductive Lie groups in both induction and restrictions.

Applying the criterion to symmetric pairs, we give a full description of the triples $H \subset G \supset G'$ such that any irreducible admissible representations of G with H-distinguished vectors have the bounded multiplicity property when restricted to the subgroup G'. This article also completes the proof of the general results announced in the previous paper [Adv. Math. 2021, Section 7].

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1 Introduction

In [K95, KO13] we initiated a new line of investigation on the finiteness or the boundedness of *multiplicities* in induction and restriction, and proposed a new avenue of research by clarifying a "nice framework" for both global analysis and branching problems with "firm grip" of group representations. This article gives its refinement by focusing on a family of "small" infinite-dimensional representations such as irreducible representations of G having H-distinguished vectors for reductive symmetric pairs (G, H).

Let G be a real reductive algebraic Lie group with Lie algebra $\mathfrak{g}$. We assume G is contained in a connected complex Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, though this assumption is easily relaxed. Let $\mathcal{M}(G)$ be the category of finitely generated, smooth admissible representations of G of moderate growth, sometimes referred to as the *Casselman–Wallach globalization* [Wal92, Chap. 11]. Denote by $\mathrm{Irr}(G)$ the set of irreducible objects in $\mathcal{M}(G)$, and by $\mathrm{Irr}(G)_f$ that of irreducible finite-dimensional ones.

We shall use the uppercase letter Π for representations of the group G, and the lowercase letter π for those of a subgroup.

Suppose G' is a reductive subgroup of G. For $\Pi \in \mathrm{Irr}(G)$ and $\pi \in \mathrm{Irr}(G')$, we define the *multiplicity* of the restriction $\Pi|_{G'}$ in the category $\mathcal{M}$ by

$$[\Pi|_{G'} : \pi] := \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\}, \tag{1.1}$$

where $\mathrm{Hom}_{G'}(\ ,\ )$ denotes the space of continuous G'-homomorphisms between the Fréchet representations.

In [KO13, Thms. C and D] we established the following geometric criteria:

Bounded multiplicity for a pair (G, G'):

$$\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty \tag{1.2}$$

if and only if $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag} G'_{\mathbb{C}}$ is spherical.

Finite multiplicity for a pair (G, G'):

$$[\Pi|_{G'} : \pi] < \infty \qquad \text{for all } \Pi \in \mathrm{Irr}(G) \text{ and } \pi \in \mathrm{Irr}(G') \tag{1.3}$$

if and only if $(G \times G')/\operatorname{diag} G'$ is real spherical.

Here we recall that a complex $G_{\mathbb{C}}$-manifold X is called *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in X, and that a G-manifold Y is called *real spherical* if a minimal parabolic subgroup of G has an open orbit in Y.

A remarkable feature of the above criterion is that the bounded multiplicity property (1.2) is determined only by the pair of the complexified Lie algebras $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$, hence the classification of the pairs (G, G') satisfying (1.2) is reduced to a classical one [Kr76]: the pair $(\mathfrak{g}, \mathfrak{g}')$ is any real form in the direct sum of the following pairs up to abelian ideals:

$$(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}) = (\mathfrak{sl}_n, \mathfrak{gl}_{n-1}), (\mathfrak{so}_n, \mathfrak{so}_{n-1}), \text{ or } (\mathfrak{so}_8, \mathfrak{spin}_7). \tag{1.4}$$

When (1.4) holds, the supremum in (1.2) equals one for many of the real forms such as $(SO(p,q), SO(p-1,q))$ or $(SU(p,q), U(p-1,q))$ [SZ12].

On the other hand, the finite multiplicity property (1.3) depends on real forms. It is fulfilled for a Riemannian symmetric pair by Harish-Chandra's admissibility theorem, whereas it is not the case for some reductive symmetric pairs such as $(G, G') = (SL(p+q, \mathbb{R}), SO(p,q))$. A complete classification of the irreducible symmetric pairs (G, G') satisfying the finite multiplicity property (1.3) was accomplished in [KM14] based on the above geometric criterion.

To go beyond these cases, we observe that even when the pair (G, G') does not satisfy the bounded multiplicity property (1.2) or more broadly, the finiteness property (1.3), there may still exist a specific $\Pi \in \mathrm{Irr}(G)$ for which a detailed study of the restriction $\Pi|_{G'}$ will be reasonable. Such Π should be a "small" representation relative to the subgroup G' in some sense. This observation suggests to look at the triple (Π, G, G') rather than a pair (G, G') of groups. This formulation has been successful in the study of G'-*admissible restriction* of Π, namely, the restriction $\Pi|_{G'}$ of $\Pi \in \mathrm{Irr}(G)$ being discretely decomposable with finite multiplicity, see [K94, K98a, K98b, K19b] for the general theory, and [KO12, KO15] for some classification theory of the triples (Π, G, G').

In this article, we allow the case where the restriction $\Pi|_{G'}$ is not "discretely decomposable", and highlight the *bounded multiplicity property*. For this purpose, we consider for $\Pi \in \mathrm{Irr}(G)$ the following quantity:

$$m(\Pi|_{G'}) := \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] \in \mathbb{N} \cup \{\infty\}. \tag{1.5}$$

We address the following:

Problem 1.1. Given a pair (G, G'), find a subset $\Omega \equiv \Omega(G')$ of $\mathrm{Irr}(G)$ (or of $\mathcal{M}(G)$) such that $\sup_{\Pi \in \Omega} m(\Pi|_{G'}) < \infty$.

We note that Problem 1.1 is nontrivial even when G is a compact Lie group where $m(\Pi|_{G'})$ is individually finite, see Example 9.5 for the case $(G, G') = (SU(3), SO(3))$. We begin with an observation of two opposite extremal choices of Ω: a singleton *vs* the whole set $\mathrm{Irr}(G)$. When Ω is a singleton, Problem 1.1 concerns the triple (Π, G, G') for which $\Pi \in \mathrm{Irr}(G)$ satisfies the bounded multiplicity property $m(\Pi|_{G'}) < \infty$, see [K15, Probl. 6.2 (2)]. When $\Omega = \mathrm{Irr}(G)$, Problem 1.1 is nothing but the bounded multiplicity property (1.2) for the pair (G, G'), and the aforementioned geometric criterion was proved in [KO13, Thm. D]. We are particularly interested in the intermediate case $\Omega = \mathrm{Irr}(G)_H$, the infinite set of H-distinguished irreducible representations of G. We also discuss Problem 1.1 when Ω is a subset of degenerate principal series representations, see Theorem 1.6 below.

Let us fix some notation. For $\Pi \in \mathrm{Irr}(G)$, we denote by $\Pi^{-\infty}$ the representation on the space of distribution vectors, that is, the topological dual of Π. For a closed subgroup H of G, we set

$$\mathrm{Irr}(G)_H := \{\Pi \in \mathrm{Irr}(G) : (\Pi^{-\infty})^H \neq \{0\}\}. \tag{1.6}$$

Let $\Pi^\vee$ be the contragredient representation in the category $\mathcal{M}(G)$. Then, one has $\Pi \in \mathrm{Irr}(G)_H$ if and only if $\mathrm{Hom}_G(\Pi^\vee, C^\infty(G/H)) \neq \{0\}$ by the Frobenius reciprocity. Elements Π in $\mathrm{Irr}(G)_H$ (or $\Pi^\vee$) are sometimes referred to as *H-distinguished*, or having *nonzero H-periods.* As a concrete setting of Problem 1.1, we study the following problem when (G, H) is a reductive symmetric pair. In this case, all elements in $\mathrm{Irr}(G)_H$ are quite "small" representations in general, see *e.g.*, Proposition 5.5 for an estimate of the Gelfand–Kirillov dimension.

Problem 1.2. Find a criterion for the triple $H \subset G \supset G'$ with bounded multiplicity property for the restriction:

$$\sup_{\Pi \in \mathrm{Irr}(G)_H} m(\Pi|_{G'}) < \infty. \tag{1.7}$$

In [K21, Thm. 7.6] we have given a geometric answer to Problem 1.2, see Theorem 1.4 below, together with some motivations, examples, and perspectives, but have postponed the detailed proof until this article. We also prove a full classification of the triples (G, H, G') satisfying the bounded multiplicity property (1.7) in the setting where (G, G') is a symmetric pair.

We recall that (G, H) is a *symmetric pair* defined by an involution σ of G, if H is an open subgroup of $G^\sigma = \{g \in G : \sigma g = g\}$. The same letter σ will be used to denote its differential. We take a maximal semisimple abelian subspace $\mathfrak{j}$ in $\mathfrak{g}^{-\sigma} = \{X \in \mathfrak{g} : \sigma X = -X\}$. The dimension of $\mathfrak{j}$ is independent of the choice of such a subspace, and is called the *rank* of the symmetric space G/H. We introduce the following terminology:

Definition 1.3 (Borel subalgebra for G/H)**.** A *Borel subalgebra* for the symmetric space G/H is the complex parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ associated to a positive system $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. We say the corresponding complex parabolic subgroup Q $(\subset G_{\mathbb{C}})$ is a *Borel subgroup* for the symmetric space G/H.

Borel subalgebras for the symmetric space G/H are unique up to inner automorphisms of $\mathfrak{g}_{\mathbb{C}}$. We sometimes write $\mathfrak{b}_{G/H}$ for $\mathfrak{q}$, and $B_{G/H}$ for Q.

We note that a Borel subalgebra $\mathfrak{b}_{G/H}$ for G/H is determined only by the complexification $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, and that it is not necessarily solvable. If $\mathfrak{b}_{G/H}$ is solvable then the regular representation on $L^2(G/H)$ is tempered [BK21, Thm. 1.1].

We prove the following.

Theorem 1.4. *Let $B_{G/H}$ be a Borel subgroup for a reductive symmetric space G/H. Suppose G' is an algebraic reductive subgroup of G, and G'_U is a compact real form of $G'_{\mathbb{C}}$. Then the following three conditions on the triple (G, H, G') are equivalent:*

(i) $\sup_{\Pi \in \mathrm{Irr}(G)_H} m(\Pi|_{G'}) < \infty$.

(ii) *$G_{\mathbb{C}}/B_{G/H}$ is $G'_{\mathbb{C}}$-spherical.*

(iii) *$G_{\mathbb{C}}/B_{G/H}$ is G'_U-strongly visible ([K05, Def. 3.3.1]).*

A special case of Theorem 1.4 includes the tensor product case. For $\Pi_1, \Pi_2 \in \mathrm{Irr}(G)$, we set

$$m(\Pi_1 \otimes \Pi_2) := \sup_{\Pi \in \mathrm{Irr}(G)} \dim_{\mathbb{C}} \mathrm{Hom}_G(\Pi_1 \otimes \Pi_2, \Pi) \in \mathbb{N} \cup \{\infty\}. \tag{1.8}$$

Theorem 1.5 (Tensor product)**.** *Suppose that (G, H_j) are reductive symmetric pairs, and that B_{G/H_j} are Borel subgroups for G/H_j for $j = 1, 2$. Then the following three conditions on the triple (G, H_1, H_2) are equivalent:*

(i)
$$\sup_{\Pi_1 \in \mathrm{Irr}(G)_{H_1}} \sup_{\Pi_2 \in \mathrm{Irr}(G)_{H_2}} m(\Pi_1 \otimes \Pi_2) < \infty. \tag{1.9}$$

(ii) *$(G_{\mathbb{C}} \times G_{\mathbb{C}})/(B_{G/H_1} \times B_{G/H_2})$ is $G_{\mathbb{C}}$-spherical via the diagonal action.*

(iii) *$(G_{\mathbb{C}} \times G_{\mathbb{C}})/(B_{G/H_1} \times B_{G/H_2})$ is G_U-strongly visible via the diagonal action.*

The classification theory for spherical varieties *e.g.*, [HNOO13], or alternatively that for strongly visible actions *e.g.*, [Tn12] leads us to the classification of the triples (G, H, G') for Theorem 1.4 and the triples (G, H_1, H_2) for Theorem 1.5. See Theorems 7.2, 7.6, 7.8, and 7.9 for a full description.

Although "smallness" of the representation $\Pi \in \mathrm{Irr}(G)$ should be necessary in some sense for the boundedness property $m(\Pi|_{G'}) < \infty$ of the restriction $\Pi|_{G'}$, invariants such as the associated variety are not informative enough for Problem 1.2, as one may notice that a delicate example already shows up in the compact setting, see Example 9.5. To overcome this difficulty, a key idea of our proof is to use "QP estimates" which implement a pair of parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p}_{\mathbb{C}}$ dealing with the induction from P to G, where $\mathfrak{q}$ is not necessarily defined over $\mathbb{R}$. For a finite-dimensional irreducible P-module η, we define $d_{\mathfrak{q}}(\eta)$ to be the minimum of the dimensions of non-zero $\mathfrak{q}$-submodules in η, and denote by $\mathrm{Irr}(P;\mathfrak{q})_f$ the subset of $\mathrm{Irr}(P)_f$ with $d_{\mathfrak{q}}(\eta) = 1$.

We deduce Theorem 1.4 from the following two results: Theorem 1.6 below gives "QP estimates for restriction" and Theorem 1.8 is a generalization of Harish-Chandra's subquotient theorem and Casselman's subrepresentation theorem for H-distinguished representations of G.

Let $\Omega_P := \{\mathrm{Ind}_P^G(\xi) : \xi \text{ is a character of } P\}$. We set

$$(\Omega_P \subset)\ \Omega_{P,\mathfrak{q}} := \{\mathrm{Ind}_P^G(\xi) : \xi \in \mathrm{Irr}(P;\mathfrak{q})_f\} \quad (\subset \mathcal{M}(G)). \tag{1.10}$$

Theorem 1.6 (see Thereom 4.2)**.** *Let $G \supset G'$ be a pair of real reductive algebraic Lie groups, P a parabolic subgroup of G, and Q a complex subgroup of $G_{\mathbb{C}}$ such that $\mathfrak{q} \subset \mathfrak{p}_{\mathbb{C}}$. Then one has the equivalence:*

$$G_{\mathbb{C}}/Q \text{ is } G'_{\mathbb{C}}\text{-spherical} \iff \sup_{\Pi \in \Omega_{P,\mathfrak{q}}} m(\Pi|_{G'}) < \infty. \tag{1.11}$$

A special case of Theorem 1.6 with $Q = P_{\mathbb{C}}$ shows:

Example 1.7 (see Example 4.5)**.** One has the equivalence from (1.11):

$$G_{\mathbb{C}}/P_{\mathbb{C}} \text{ is } G'_{\mathbb{C}}\text{-spherical} \iff \sup_{\Pi \in \Omega_P} m(\Pi|_{G'}) < \infty. \tag{1.12}$$

Let $P_{G/H}$ be a "minimal parabolic subgroup" for the symmetric space G/H (Definition 5.1), and $\mathfrak{b}_{G/H}$ a Borel subalgebra for G/H with $\mathfrak{b}_{G/H} \subset (\mathfrak{p}_{G/H})_{\mathbb{C}}$.

Theorem 1.8 (see Theorem 5.4)**.** *Any $\Pi \in \mathrm{Irr}(G)_H$ is obtained as the quotient of the induced representation $\mathrm{Ind}_{P_{G/H}}^G(\xi)$ for some $\xi \in \mathrm{Irr}(P_{G/H};\mathfrak{b}_{G/H})$.*

Along the same line as in [KO13, K14], the "QP estimate" for *restriction* (*e.g.,* the implication (ii) $\Rightarrow$ (i) in Theorem 1.4) is derived from the following "QP estimates for *induction*" applied to $(G \times G')/\operatorname{diag} G'$. Theorem 1.9 is a generalization of some results in [KO13] relying on the theory of "boundary valued maps" and in Tauchi [Tu19] relying on the theory of holonomic $\mathcal{D}$-modules [Ka83, KK81].

Theorem 1.9 (see Theorem 3.1 (1))**.** *Let P be a parabolic subgroup of a real reductive algebraic Lie group G, and H an algebraic subgroup. Suppose that Q is a parabolic subgroup of $P_{\mathbb{C}}$ such that $\#(Q\backslash G_{\mathbb{C}}/H_{\mathbb{C}}) < \infty$. Then one has*

$$\sup_{\eta\in\mathrm{Irr}(P)_f}\ \sup_{\tau\in\mathrm{Irr}(H)_f} \frac{1}{d_{\mathfrak{q}}(\eta)\dim\tau}\dim_{\mathbb{C}}\mathrm{Hom}_G(\mathrm{Ind}_P^G(\eta),\mathrm{Ind}_H^G(\tau)) < \infty.$$

Individual finite multiplicity results for induction and restriction can be read from the "QP estimates" for induction and restriction, respectively, by putting $Q := P_{\mathbb{C}}$, see Remark 3.3 and Example 4.6 for instance.

Organization of the paper
In Section 2 we introduce the set $\mathrm{Irr}(P;\mathfrak{q})_f$ and discuss some basic properties of finite-dimensional representations. Bounded multiplicity theorems for induction and restriction of degenerate principal series representations with "QP estimates" are given in Sections 3 and 4, respectively. Section 5 is devoted to a refinement of Casselman's subrepresentation theorem for H-distinguished representations (Theorem 1.8). With these preparations, our main results (Theorems 1.4 and 1.5) will be proved in Section 6. The classification of the triples (G, H, G') with the bounded multiplicity property (1.7) is given in Section 7, and is proved in Section 8 based on the geometric criteria in Theorems 1.4 and 1.5.

2 Preliminaries on $\mathrm{Irr}(P;\mathfrak{q})_f$

We prepare some finer properties of finite-dimensional representations that we shall need in the "QP estimate", the uniform estimate of multiplicities for a family of representations in induction and restriction.

2.1 Definition of $\mathrm{Irr}(P;\mathfrak{q})_f$

In this subsection we examine finite-dimensional irreducible representations of a Lie group P with respect to a parabolic subalgebra $\mathfrak{q}$.

Let P be a real algebraic group or its open subgroup in the classical topology. We write $P = LN$ for its Levi decomposition, and $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ for the corresponding decomposition of the Lie algebras. We denote by $\mathrm{Irr}(P)_f$ the set of equivalence classes of irreducible finite-dimensional representations of P, and by $\mathrm{Irr}(\mathfrak{p})_f$ that of the Lie algebra $\mathfrak{p}$. If P is connected, one may regard $\mathrm{Irr}(P)_f$ as a subset of $\mathrm{Irr}(\mathfrak{p})_f$. Since the unipotent subgroup N acts trivially on any irreducible finite-dimensional representation of P, one has a natural bijection $\mathrm{Irr}(P)_f \simeq \mathrm{Ind}(L)_f$ via the quotient map $P \to L \simeq P/N$.

Definition 2.1. Let $\mathfrak{p}_{\mathbb{C}} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}$ be the complexified Lie algebra of $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$, and $\mathfrak{q}$ a parabolic subalgebra in $\mathfrak{p}_{\mathbb{C}}$, namely, $\mathfrak{q}$ is the full inverse of a parabolic subalgebra of $\mathfrak{l}_{\mathbb{C}}$ via the quotient map $\varpi\colon \mathfrak{p}_{\mathbb{C}} \to \mathfrak{p}_{\mathbb{C}}/\mathfrak{n}_{\mathbb{C}} \simeq \mathfrak{l}_{\mathbb{C}}$. For $(\xi, V) \in \mathrm{Irr}(P)_f$, we define $d_{\mathfrak{q}}(\xi)$ to be the minimal dimension of an irreducible $\mathfrak{q}$-submodule in V. We set

$$\mathrm{Irr}(P;\mathfrak{q})_f := \{\xi \in \mathrm{Irr}(P)_f : d_{\mathfrak{q}}(\xi) = 1\}, \tag{2.1}$$

$$\mathrm{Irr}(\mathfrak{p};\mathfrak{q})_f := \{\xi \in \mathrm{Irr}(\mathfrak{p})_f : d_{\mathfrak{q}}(\xi) = 1\}. \tag{2.2}$$

We say v is a *relatively* $\mathfrak{q}$*-invariant* vector, if there is a complex linear form λ on $\mathfrak{q}$ such that $\xi(X)v = \lambda(X)v$ for all $X \in \mathfrak{q}$. By definition $d_{\mathfrak{q}}(\xi) = 1$ if and only if there is a non-zero relatively $\mathfrak{q}$-invariant vector. Let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{q}$. If $(\xi, V) \in \mathrm{Irr}(\mathfrak{p})$, then

$$V^{\mathfrak{u}} := \{v \in V : \xi(X)v = 0 \quad {}^{\forall}X \in \mathfrak{u}\}$$

is the unique irreducible $\mathfrak{q}$-submodule of V, and $d_{\mathfrak{q}}(\xi) = \dim V^{\mathfrak{u}}$.

Unlike the notation $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$, we do not use the letter $\mathfrak{q}_{\mathbb{C}}$ in Definition 2.1 to denote the (complex) parabolic subalgebra $\mathfrak{q}$, because the subalgebra $\mathfrak{q}$ is not necessarily defined over $\mathbb{R}$.

We shall formulate Theorems 3.1 and 4.1 ("*QP* estimate") and Theorem 5.4 (quotient representation theorem for H-distinguished representations) by using the pair P and $\mathfrak{q}$, which is a key in proving Theorem 1.4 through a unified treatment both for the real polarization (the usual parabolic induction) when $\mathfrak{p} \cap \mathfrak{q}$ is a real form of $\mathfrak{q}$ and for the complex polarization (the Borel–Weil type induction) when $(\mathfrak{p} \cap \mathfrak{q})/\mathfrak{n}$ is reductive.

We collect some basic properties on $\mathrm{Irr}(P;\mathfrak{q})_f$. The proof is straightforward from the definition.

Lemma 2.2. *Let* $\mathfrak{q}$ *and* $\mathfrak{q}'$ *be parabolic subalgebras of* $\mathfrak{p}_{\mathbb{C}}$. *One has*

$$\begin{aligned}
&\mathrm{Irr}(\mathfrak{p};\mathfrak{p}_{\mathbb{C}})_f = \{\text{characters of } \mathfrak{p}\}, \\
&\mathrm{Irr}(P;\mathfrak{q})_f \subset \mathrm{Irr}(\mathfrak{p};\mathfrak{q})_f && \text{if } P \text{ is connected}; \\
&\mathrm{Irr}(P;\mathfrak{q})_f \simeq \mathrm{Irr}(L;\mathfrak{q}\cap\mathfrak{l}_{\mathbb{C}})_f, && (2.3)\\
&\mathrm{Irr}(P;\mathfrak{q})_f = \mathrm{Irr}(P;\mathrm{Ad}(g)\mathfrak{q})_f && \text{for any } g \in P_{\mathbb{C}}; \quad (2.4)\\
&\mathrm{Irr}(P;\mathfrak{q})_f \supset \mathrm{Irr}(P;\mathfrak{q}')_f && \text{if } \mathfrak{q} \subset \mathfrak{q}'; \\
&\mathrm{Irr}(P;\mathfrak{b})_f = \mathrm{Irr}(P)_f && \text{if } \mathfrak{b} \text{ is a Borel subalgebra of } \mathfrak{p}_{\mathbb{C}}.
\end{aligned}$$

It is convenient to prepare notation for (finite-dimensional) holomorphic representations of a complex Lie group.

Definition 2.3. For a connected complex Lie group $P_{\mathbb{C}}$, we denote by $\mathrm{Irr}(P_{\mathbb{C}})_{\mathrm{hol}}$ the set of equivalence classes of finite-dimensional irreducible holomorphic representations of $P_{\mathbb{C}}$. If P is a real form of $P_{\mathbb{C}}$, we have a natural inclusion $\mathrm{Irr}(P_{\mathbb{C}})_{\mathrm{hol}} \hookrightarrow \mathrm{Irr}(P)_f$ by restriction. Accordingly, we set $\mathrm{Irr}(P_{\mathbb{C}}; \mathfrak{q})_{\mathrm{hol}} := \mathrm{Irr}(P; \mathfrak{q})_f \cap \mathrm{Irr}(P_{\mathbb{C}})_{\mathrm{hol}}$.

The concept of opposite parabolic subalgebras in reductive Lie algebras is naturally extended to the non-reductive case:

Definition 2.4. Let $\varpi\colon \mathfrak{p}_{\mathbb{C}} \to \mathfrak{p}_{\mathbb{C}}/\mathfrak{n}_{\mathbb{C}} \simeq \mathfrak{l}_{\mathbb{C}}$ be the projection as before, and $\mathfrak{q}$ a parabolic subalgebra of $\mathfrak{p}_{\mathbb{C}}$. We say $\mathfrak{q}_{\mathrm{opp}}$ is the opposite parabolic subalgebra of $\mathfrak{q}$ in $\mathfrak{p}_{\mathbb{C}}$ if $\mathfrak{q}_{\mathrm{opp}}$ is the full inverse of the opposite parabolic subalgebra of $\varpi(\mathfrak{q})$ in $\mathfrak{l}_{\mathbb{C}}$ (with respect to a fixed Cartan subalgebra).

We denote by $\xi^\vee$ the contragredient representation of ξ. Then one has the following.

Lemma 2.5. (1) $\xi \in \mathrm{Irr}(P; \mathfrak{q})_f$ *if and only if* $\xi^\vee \in \mathrm{Irr}(P; \mathfrak{q}_{\mathrm{opp}})_f$.

(2) $\xi \in \mathrm{Irr}(\mathfrak{p}; \mathfrak{q})_f$ *if and only if* $\xi^\vee \in \mathrm{Irr}(\mathfrak{p}; \mathfrak{q}_{\mathrm{opp}})_f$.

By (2.3), it suffices to prove Lemma 2.5 in the reductive case, which is shown in Lemma 2.6 (2) below.

2.2 Description of $\mathrm{Irr}(P; \mathfrak{q})_f$

As we saw in (2.3), the description of $\mathrm{Irr}(P; \mathfrak{q})_f$ reduces to the case where P is a reductive group, for which we use the letter G in this subsection for later purpose. Let $\widetilde{\mathfrak{j}}$ be a Cartan subalgebra of $\mathfrak{g}$, W the Weyl group of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$, and w_0 the longest element in W. We fix a W-invariant non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ on the dual space $\widetilde{\mathfrak{j}}_{\mathbb{C}}^{\,*}$. We take a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$, denote by Ψ the set of simple roots, and write $\Lambda_+ \equiv \Lambda_+(\mathfrak{g}_{\mathbb{C}})$ for the set of dominant integral weights of $\widetilde{\mathfrak{j}}_{\mathbb{C}}$. Then the Cartan–Weyl highest weight theory establishes the bijection:

$$\mathrm{Irr}(\mathfrak{g})_f \simeq \Lambda_+, \qquad \Pi_\lambda \leftrightarrow \lambda. \tag{2.5}$$

If Π_λ lifts to a holomorphic representation of a connected complex reductive Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, we use the same letter Π_λ to denote the lift.

Given a subset Θ in Ψ, we write $\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}_-^\Theta \oplus \mathfrak{l}_{\mathbb{C}}^\Theta \oplus \mathfrak{n}_+^\Theta$ for the Gelfand–Naimark decomposition, where $\mathfrak{l}_{\mathbb{C}}^\Theta$ is a reductive subalgebra containing $\widetilde{\mathfrak{j}_{\mathbb{C}}}$ with $\Delta(\mathfrak{l}_{\mathbb{C}}^\Theta, \widetilde{\mathfrak{j}_{\mathbb{C}}}) = \Delta(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}_{\mathbb{C}}}) \cap \mathbb{Z}\text{-span}\Theta$, $\mathfrak{p}^\Theta \equiv \mathfrak{p}_+^\Theta := \mathfrak{l}_{\mathbb{C}}^\Theta \oplus \mathfrak{n}_+^\Theta$ is a parabolic subalgebra with $\Delta(\mathfrak{n}_+^\Theta, \widetilde{\mathfrak{j}_{\mathbb{C}}}) \subset \Delta^+(\mathfrak{g}, \widetilde{\mathfrak{j}_{\mathbb{C}}})$ and $\mathfrak{p}_-^\Theta := \mathfrak{l}_{\mathbb{C}}^\Theta \oplus \mathfrak{n}_-^\Theta$ is the opposite parabolic subalgebra.

Lemma 2.6. *Suppose Θ is a subset of Ψ.*

(1) *The map* (2.5) *of taking highest weights induces the following bijection:*

$$\operatorname{Irr}(\mathfrak{g}; \mathfrak{p}^\Theta)_f \simeq \{\lambda \in \Lambda_+ : \langle \lambda, \alpha \rangle = 0 \quad {}^\forall \alpha \in \Theta\}. \tag{2.6}$$

(2) $\xi \in \operatorname{Irr}(\mathfrak{g}; \mathfrak{p}^\Theta)_f$ *if and only if* $\xi^\vee \in \operatorname{Irr}(\mathfrak{g}; \mathfrak{p}_-^\Theta)_f$.

Proof. (1) We set $\widetilde{\mathfrak{j}}_{\mathbb{C}}^\Theta := [\mathfrak{l}_{\mathbb{C}}^\Theta, \mathfrak{l}_{\mathbb{C}}^\Theta] \cap \widetilde{\mathfrak{j}_{\mathbb{C}}}$, which is a Cartan subalgebra of the semisimple part $[\mathfrak{l}_{\mathbb{C}}^\Theta, \mathfrak{l}_{\mathbb{C}}^\Theta]$ of $\mathfrak{l}_{\mathbb{C}}^\Theta$. Then the right-hand side of (2.6) equals $\{\lambda \in \Lambda_+ : \lambda \text{ vanishes on } \widetilde{\mathfrak{j}}_{\mathbb{C}}^\Theta\}$.

Suppose v is a relatively $\mathfrak{p}^\Theta$-invariant vector of $\xi \in \operatorname{Irr}(\mathfrak{g}; \mathfrak{p}^\Theta)_f$, namely, v satisfies $\xi(X)v = \lambda(X)v$ (${}^\forall X \in \mathfrak{p}^\Theta$) for some linear form λ on $\mathfrak{p}^\Theta$. Then λ vanishes on $[\mathfrak{p}^\Theta, \mathfrak{p}^\Theta]$, hence, $\lambda|_{\widetilde{\mathfrak{j}_{\mathbb{C}}}}$ is the highest weight of ξ and λ vanishes on $\widetilde{\mathfrak{j}}_{\mathbb{C}}^\Theta$. Conversely, if $\lambda \in \Lambda_+$ vanishes on $\widetilde{\mathfrak{j}}_{\mathbb{C}}^\Theta$, then λ extends to a character of $\mathfrak{p}^\Theta$ via the quotient map $\mathfrak{p}^\Theta \to \mathfrak{p}^\Theta/[\mathfrak{p}^\Theta, \mathfrak{p}^\Theta] \simeq \widetilde{\mathfrak{j}_{\mathbb{C}}}/\widetilde{\mathfrak{j}}_{\mathbb{C}}^\Theta$, and the highest vector v of Π_λ satisfies $\xi(X)v = \lambda(X)v$ ${}^\forall X \in \mathfrak{p}^\Theta$. Hence $\lambda \in \operatorname{Irr}(\mathfrak{g}; \mathfrak{p}^\Theta)_f$.
(2) Since $-w_0\lambda$ is the highest weight of the contragredient representation $\xi^\vee$, one has $\xi^\vee \in \operatorname{Irr}(\mathfrak{g}; \mathfrak{p}^{-w_0\Theta})_f$ if $\xi \in \operatorname{Irr}(\mathfrak{g}; \mathfrak{p}^\Theta)_f$, and vice versa. Since $\mathfrak{p}^{-w_0\Theta}$ is conjugate to $\mathfrak{p}_-^\Theta$ by an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$, one has $\operatorname{Irr}(\mathfrak{g}; \mathfrak{p}^{-w_0\Theta})_f = \operatorname{Irr}(\mathfrak{g}; \mathfrak{p}_-^\Theta)_f$, whence the assertion follows. □

2.3 Geometric realization for $\operatorname{Irr}(P; \mathfrak{q})_f$

Suppose we are in the setting of Definition 2.1. In this subsection, we provide a geometric interpretation of $\operatorname{Irr}(P; \mathfrak{q})_f$.

We let the Lie algebra $\mathfrak{p}$ of P act on $C^\infty(P)$ as left invariant vector fields by $(dR(X)f)(g) := \frac{d}{dt}|_{t=0} f(g \exp tX)$, and the same letter dR is used to denote its complex linear extension to $\mathfrak{p}_{\mathbb{C}}$. For a $\mathfrak{q}$-module (τ, W), we let

$\mathfrak{q}$ act on $C^\infty(P) \otimes W$ by $dR \otimes \mathrm{id} + \mathrm{id} \otimes \tau$, which may be written simply as $dR \otimes \tau$.

Lemma 2.7. *Suppose that (τ, W) is a quotient of $(\xi, V) \in \mathrm{Irr}(P)_f$ as a $\mathfrak{q}$-module. Then the left translation of P leaves $(C^\infty(P) \otimes W)^{\mathfrak{q}}$ invariant, and there is a natural injective P-homomorphism $\widetilde{T}\colon V \to (C^\infty(P) \otimes W)^{\mathfrak{q}}$.*

Proof. Denote by $(\xi^\vee, V^\vee)$ the contragredient representation of ξ, and consider a bilinear map

$$T\colon V \times V^\vee \to C^\infty(P), \quad (v, u) \mapsto (g \mapsto \langle \xi(g^{-1})v, u\rangle).$$

Taking the dual of the quotient map $V \to W$, one has an injective $\mathfrak{q}$-homomorphism $W^\vee \hookrightarrow V^\vee$, hence the restriction $T|_{V \times W^\vee}$ induces a P-homomorphism $\widetilde{T}\colon V \to \mathrm{Hom}_{\mathbb{C}}(W^\vee, C^\infty(P)) \simeq C^\infty(P) \otimes W$ by $\widetilde{T}(v)(g) := T(v, \cdot)(g)$ for $v \in V$ and $g \in P$. Then $\widetilde{T}(v) \in \mathrm{Hom}_{\mathfrak{q}}(W^\vee, C^\infty(P)) \simeq (C^\infty(P) \otimes W)^{\mathfrak{q}}$ because $dR(X)T(v, u) = T(v, d\xi^\vee(X)u) = T(v, \tau^\vee(X)u)$ for any $u \in W^\vee$ and any $X \in \mathfrak{p}_{\mathbb{C}}$. The resulting P-homomorphism $\widetilde{T}\colon V \to (C^\infty(P) \otimes W)^{\mathfrak{q}}$ is injective because (ξ, V) is irreducible. □

Remark 2.8. In general, the P-homomorphism $\widetilde{T}\colon V \to (C^\infty(P) \otimes W)^{\mathfrak{q}}$ is not surjective. We note that $\widetilde{T}$ is bijective if P is a connected compact Lie group by the Borel–Weil theorem.

Suppose $P_{\mathbb{C}}$ is a connected complex Lie group, and Q a parabolic subgroup of $P_{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$. For a holomorphic character $\mathbb{C}_\lambda$ of Q, we denote by $\mathcal{L}_\lambda$ the $P_{\mathbb{C}}$-equivariant holomorphic line bundle $P_{\mathbb{C}} \times_Q \mathbb{C}_\lambda$ over the flag variety $P_{\mathbb{C}}/Q$, and by $\mathcal{O}(P_{\mathbb{C}}/Q, \mathcal{L}_\lambda)$ the space of holomorphic sections for $\mathcal{L}_\lambda$.

Lemma 2.9 (Geometric realization for $\mathrm{Irr}(P_{\mathbb{C}}; \mathfrak{q})_{\mathrm{hol}}$)**.** (1) *The regular representation of $P_{\mathbb{C}}$ on $\mathcal{O}(P_{\mathbb{C}}/Q, \mathcal{L}_\lambda)$ belongs to* $\mathrm{Irr}(P_{\mathbb{C}}; \mathfrak{q}_{\mathrm{opp}})_{\mathrm{hol}}$ *if it is non-zero, and its contragredient representation belongs to* $\mathrm{Irr}(P_{\mathbb{C}}; \mathfrak{q})_{\mathrm{hol}}$.
(2) *Assume that (ξ, V) is an irreducible holomorphic representation of $P_{\mathbb{C}}$ such that its contragredient representation $\xi^\vee$ belongs to* $\mathrm{Irr}(P_{\mathbb{C}}; \mathfrak{q})_{\mathrm{hol}}$. *Then there exists a holomorphic character λ of Q such that ξ is isomorphic to the regular representation of $P_{\mathbb{C}}$ on $\mathcal{O}(P_{\mathbb{C}}/Q, \mathcal{L}_\lambda)$.*

We shall apply the above lemma also to real forms P of $P_{\mathbb{C}}$, where we do not assume P to be connected.

Proof. Let $L_{\mathbb{C}}$ be a Levi subgroup of $P_{\mathbb{C}}$. We note that $L_{\mathbb{C}}$ is connected.
(1) By (2.3), it suffices to prove the assertion when $P_{\mathbb{C}} = L_{\mathbb{C}}$. We take a Cartan subalgebra $\widetilde{\mathfrak{j}}_{\mathbb{C}}$ of $\mathfrak{l}_{\mathbb{C}}$, and fix a positive system $\Delta^+(\mathfrak{l}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$ such that $\Delta^+(\mathfrak{l}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}}) \subset \Delta(\mathfrak{q}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$. We use the same letter λ to denote the differential, and also its restriction to the Cartan subalgebra $\widetilde{\mathfrak{j}}_{\mathbb{C}}$. Then the Borel–Weil theorem for the connected complex reductive Lie group $L_{\mathbb{C}}$ tells that $\mathcal{O}(P_{\mathbb{C}}/Q, \mathcal{L}_\lambda)$ is non-zero if and only if $-\lambda$ is dominant with respect to $\Delta^+(\mathfrak{l}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$. In this case, its contragredient representation contains $\mathbb{C}_\lambda$ as a $\mathfrak{q}$-submodule, namely, has the highest weight λ, as seen in Lemma 2.6.
(2) Retain the notation as in the proof of Lemma 2.7. Then the matrix coefficient $T(v, u)$ is a holomorphic function on $P_{\mathbb{C}}$, because ξ is a holomorphic representation of $P_{\mathbb{C}}$. Since $(\xi^\vee, V^\vee) \in \mathrm{Irr}(P_{\mathbb{C}}; \mathfrak{q})_{\mathrm{hol}}$, there exists a one-dimensional $\mathfrak{q}$-submodule $\mathbb{C}u$ in $V^\vee$, on which the connected group Q acts as a holomorphic character, to be denoted by λ. Then $T(\cdot, u)$ induces a $P_{\mathbb{C}}$-homomorphism from V to $(\mathcal{O}(P_{\mathbb{C}}) \otimes \mathbb{C}_\lambda)^{\mathfrak{q}} \simeq \mathcal{O}(P_{\mathbb{C}}/Q, \mathcal{L}_\lambda)$, which is bijective by the irreducibility. Thus the lemma is shown. □

2.4 Multiplicities in finite-dimensional representations

For finite-dimensional representations, the boundedness of multiplicity is equivalent to multiplicity-freeness in many settings. In this subsection we prepare two lemmas in a way that we need later. For the sake of completeness, we give a proof of the first one.

Let $G_{\mathbb{C}}$ be a connected complex reductive Lie group. We take a Cartan subalgebra $\widetilde{\mathfrak{j}}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ and fix a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$. We denote by $\mathfrak{b}$ the corresponding Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$, by Λ_+ the set of dominant integral weights, and by Π_λ the irreducible holomorphic representation of $G_{\mathbb{C}}$ if $\lambda \in \Lambda_+$ lifts to a character of the Cartan subgroup as before.

Suppose that $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Without loss of generality, we may assume that $\mathfrak{q}$ contains $\mathfrak{b}$. As in Lemma 2.6 (1), we regard $\mathrm{Irr}(G_{\mathbb{C}}; \mathfrak{q})_{\mathrm{hol}}$ as a subset of Λ_+ via the bijection (2.5).

First, suppose that $G'_{\mathbb{C}}$ is a connected complex reductive subgroup. For $\Pi \in \mathrm{Irr}(G_{\mathbb{C}})_{\mathrm{hol}}$, we set

$$m(\Pi|_{G'_{\mathbb{C}}}) := \max_{\pi \in \mathrm{Irr}(G'_{\mathbb{C}})_{\mathrm{hol}}} [\Pi|_{G'_{\mathbb{C}}} : \pi], \tag{2.7}$$

as an analog of $m(\Pi|_{G'})$ in (1.5).

Lemma 2.10. *If $G_{\mathbb{C}}/Q$ is not $G'_{\mathbb{C}}$-spherical, then there exists $\lambda \in \Lambda_+$ satisfying $\Pi_{N\lambda} \in \mathrm{Irr}(G_{\mathbb{C}}; \mathfrak{q})_{\mathrm{hol}}$ and $m(\Pi_{N\lambda}|_{G'_{\mathbb{C}}}) \geq N+1$ for all $N \in \mathbb{N}$.*

Proof. Suppose $G_{\mathbb{C}}/Q$ is not $G'_{\mathbb{C}}$-spherical. By a result of Vinberg–Kimelfeld [VK78, Cor. 1], there exists a $G_{\mathbb{C}}$-homogeneous holomorphic line bundle $\mathcal{L}$ over $G_{\mathbb{C}}/Q$ such that the irreducible $G_{\mathbb{C}}$-module $\mathcal{O}(G_{\mathbb{C}}/Q, \mathcal{L})$ contains an irreducible representation of $G'_{\mathbb{C}}$ with multiplicity. This means that there exist linearly independent sections f_1, $f_2 \in \mathcal{O}(G_{\mathbb{C}}/Q, \mathcal{L})$ and a dominant character μ of a Borel subgroup B' of $G'_{\mathbb{C}}$ satisfying $f_j(b^{-1}g) = \mu(b)f_j(g)$ $(j = 1, 2)$ for any $b \in B'$, $g \in G_{\mathbb{C}}$.

We claim that the holomorphic sections $f_1^i f_2^{N-i} \in \mathcal{O}(G_{\mathbb{C}}/Q, \mathcal{L}^{\otimes N})$ $(0 \leq i \leq N)$ are linearly independent. Indeed, if $a_0 f_1^N + a_1 f_1^{N-1} f_2 + \cdots + a_N f_2^N = 0$ were a linear dependence, then one would have $f_1 - t f_2 = 0$ where t is a zero of the equation $a_0 t^N + a_1 t^{N-1} + \cdots + a_N = 0$ because the ring $\mathcal{O}(G_{\mathbb{C}})$ has no divisor. This means $\dim_{\mathbb{C}} \mathrm{Hom}_{G'_{\mathbb{C}}}(\pi_{N\mu}, \mathcal{O}(G_{\mathbb{C}}/Q, \mathcal{L}^{\otimes N})|_{G'_{\mathbb{C}}}) \geq N+1$ because B' acts on $f_1^i f_2^{N-i}$ as the character $\mathbb{C}_{N\mu}$. Let λ be the character of Q acting on the fiber of $\mathcal{L}^{-1}$ at the origin $o = eQ \in G_{\mathbb{C}}/Q$. Then $\Pi_{N\lambda}$ is the contragredient representation on $\mathcal{O}(G_{\mathbb{C}}/Q, \mathcal{L}^{\otimes N})$ and belongs to $\mathrm{Irr}(G_{\mathbb{C}}; \mathfrak{q})_{\mathrm{hol}}$ by Lemma 2.9 (the Borel–Weil theorem). Hence $\dim_{\mathbb{C}} \mathrm{Hom}_{G'_{\mathbb{C}}}(\Pi_{N\lambda}|_{G'_{\mathbb{C}}}, \pi_{N\mu}^{\vee}) \geq N+1$, showing the lemma. □

Second, we drop the reductive assumption of a subgroup. By a similar argument as in Lemma 2.10, one obtains the following:

Lemma 2.11. *Let $H_{\mathbb{C}}$ be a complex algebraic subgroup of $G_{\mathbb{C}}$ (not necessarily reductive). If $H_{\mathbb{C}}$ does not have an open orbit in $G_{\mathbb{C}}/Q$, then there exist a holomorphic character χ of $H_{\mathbb{C}}$ and $\lambda \in \Lambda_+$ satisfying $\Pi_{N\lambda} \in \mathrm{Irr}(G_{\mathbb{C}}; \mathfrak{q})_{\mathrm{hol}}$ and $\dim_{\mathbb{C}} \mathrm{Hom}_{H_{\mathbb{C}}}(\Pi_{N\lambda}|_{H_{\mathbb{C}}}, \chi^N) \geq N+1$ for all $N \in \mathbb{N}$.*

2.5 Complex symmetric pair and the Satake diagram

This subsection and the next one will not be used until Section 6.

Let G be a connected semisimple Lie group, σ an involutive automorphism of G, and H an open subgroup of the fixed point group G^σ. We use the same letter σ to denote the complex linear extension of its differential on $\mathfrak{g}_{\mathbb{C}}$. Then the Lie algebra $\mathfrak{g}$ of G has a decomposition $\mathfrak{g} = \mathfrak{g}^\sigma \oplus \mathfrak{g}^{-\sigma}$ into the eigenspaces of σ with eigenvalues 1 and -1. We note that the Lie algebra $\mathfrak{h}$ of H equals $\mathfrak{g}^\sigma$. We take a maximal semisimple abelian subspace $\mathfrak{j}$ in $\mathfrak{g}^{-\sigma}$, and fix $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ as in Definition 1.3. We extend $\mathfrak{j}$ to a Cartan subalgebra $\widetilde{\mathfrak{j}}$ of $\mathfrak{g}$. We refer to

$\widetilde{\mathfrak{j}}$ as a *σ-split Cartan subalgebra*. Via the direct sum decomposition $\widetilde{\mathfrak{j}} = \mathfrak{t} \oplus \mathfrak{j}$ with $\mathfrak{t} := \widetilde{\mathfrak{j}} \cap \mathfrak{h}$, we may regard $\mathfrak{j}_{\mathbb{C}}^*$ as a subspace of $\widetilde{\mathfrak{j}}_{\mathbb{C}}^*$. We choose a compatible positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$ such that the restriction map $\alpha \mapsto \alpha|_{\mathfrak{j}_{\mathbb{C}}}$ sends $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$ to $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}}) \cup \{0\}$, and denote by Ψ the set of simple roots in $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$ as in Section 2.2. We set

$$\Theta := \{\alpha \in \Psi : \alpha|_{\mathfrak{j}_{\mathbb{C}}} \equiv 0\}. \tag{2.8}$$

By Definition 1.3, one has the following:

Lemma 2.12. *The parabolic subalgebra $\mathfrak{p}^{\Theta}$ of $\mathfrak{g}_{\mathbb{C}}$ is a Borel subalgebra $\mathfrak{q}$ for the symmetric space G/H.*

One can read Θ from the Satake diagram (*e.g.*, [He78, p. 531]) of another real form $\mathfrak{g}_{\mathbb{R}}$ of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$, which we explain below. We take a Cartan involution θ of $\mathfrak{g}_{\mathbb{C}}$ commuting with σ. Since σ is complex linear and θ is antilinear on $\mathfrak{g}_{\mathbb{C}}$, $\sigma\theta$ is a complex conjugation of $\mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{R}} := \mathfrak{g}_{\mathbb{C}}^{\sigma\theta}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$. This yields a one-to-one correspondence:

$$\text{a real form } \mathfrak{g}_{\mathbb{R}} \text{ of } \mathfrak{g}_{\mathbb{C}} \longleftrightarrow \text{a complex symmetric pair } (\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}). \tag{2.9}$$

We set $\mathfrak{k}_{\mathbb{R}} := \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{R}}$. Note that σ leaves $\mathfrak{g}_{\mathbb{R}}$ invariant, and the restriction $\sigma|_{\mathfrak{g}_{\mathbb{R}}}$ is a Cartan involution of $\mathfrak{g}_{\mathbb{R}}$. In particular, $\widetilde{\mathfrak{j}}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{R}}$ is a maximally split Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Since a Borel subgroup for the symmetric space G/H is determined only by the complexified Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$, and since $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \simeq ((\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}}, (\mathfrak{k}_{\mathbb{R}})_{\mathbb{C}})$, one obtains from Lemma 2.12 the following:

Lemma 2.13. *The complexification of a minimal parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$ is a Borel subalgebra for the symmetric space G/H. In particular, if we take Θ to be the set of black circles in the Satake diagram, then $\mathfrak{p}^{\Theta}$ is a Borel subalgebra of G/H.*

We shall use Lemma 2.13 in Section 8 in the proof of the classification results in Section 7.

2.6 The Cartan–Helgason theorem vs $\mathrm{Irr}(\mathfrak{g}; \mathfrak{p}^{\Theta})_f$

In this subsection, we examine $\mathrm{Irr}(G)_{H,f}$ for a symmetric space G/H and compare it with $\mathrm{Irr}(\mathfrak{g}; \mathfrak{p}^{\Theta})_f$, see (2.2), where $\mathfrak{p}^{\Theta}$ is the Borel subalgebra of the symmetric space G/H.

We retain the notation as in Section 2.5, and set

$$\Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}}) := \{\lambda \in \mathfrak{j}_{\mathbb{C}}^* : \frac{\langle \lambda, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{N} \quad \text{for all } \beta \in \Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})\}. \tag{2.10}$$

We regard $\Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}})$ $(\subset \mathfrak{j}_{\mathbb{C}}^*)$ as a subset of Λ_+ $(\subset \widetilde{\mathfrak{j}_{\mathbb{C}}}^*)$ via the decomposition $\widetilde{\mathfrak{j}} = \mathfrak{t} \oplus \mathfrak{j}$. Since $\langle \lambda, \alpha \rangle = 0$ for any $\lambda \in \mathfrak{j}_{\mathbb{C}}^*$ and any $\alpha \in \Theta$, Lemma 2.6 (1) implies the following:

Lemma 2.14. *Via the Cartan–Weyl bijection* (2.5)*, one has*

$$\Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}}) \subset \operatorname{Irr}(\mathfrak{g};\mathfrak{p}^{\Theta})_f. \tag{2.11}$$

Remark 2.15. Both $\Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}})$ and $\operatorname{Irr}(\mathfrak{g};\mathfrak{p}^{\Theta})_f$ are free semigroups, but the former may be much smaller than the latter. For example, if $(G,H) = ({}'G \times {}'G, \operatorname{diag} {}'G)$, then the rank of the semigroup $\operatorname{Irr}(\mathfrak{g};\mathfrak{p}^{\Theta})_f$ is twice the rank of the semigroup $\Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}})$.

For the simplicity of the proof, we adopt the definition of $\operatorname{Irr}(G)_{H,f}$ as the set $\{\Pi \in \operatorname{Irr}(G)_f : \Pi^H \neq \{0\}\}$ rather than $\{\Pi \in \operatorname{Irr}(G)_f : (\Pi^\vee)^H \neq \{0\}\}$ in the next lemma by an abuse of notation, however, this definition coincides with the previous one as we shall prove in Lemma 2.17 below.

Lemma 2.16 (Cartan–Helgason theorem)**.** *Let (G,H) be a symmetric pair defined by an involution σ of a connected semisimple Lie group G. We regard both* $\operatorname{Irr}(G)_f$ *and* $\Lambda_+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$ *as a subset of* Λ_+.

(1) $\Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}}) = \operatorname{Irr}(G)_{H,f}$ *if σ is a Cartan involution of G.*

(2) *If G is a real form of the simply connected complex Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, then* $\Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}}) = \operatorname{Irr}(G)_{H,f}$.

(3) *For a general semisimple symmetric pair (G,H), there exists a positive integer k such that*

$$k\Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}}) \subset \operatorname{Irr}(G)_{H,f} \subset \Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}}). \tag{2.12}$$

Proof of Lemma 2.16. (1) This is the (usual) Cartan–Helgason theorem. See [War72, Thm. 3.3.1.1] or [He94, p. 139] for example.
(2) The involution σ of G lifts to a holomorphic involution of the simply connected complex group $G_{\mathbb{C}}$, for which we shall use the same letter σ. We

take a Cartan involution θ of $G_{\mathbb{C}}$ commuting with σ. Then $\sigma\theta$ is an anti-holomorphic involution of $G_{\mathbb{C}}$. We set $H_{\mathbb{C}} := G_{\mathbb{C}}^{\sigma}$, $G_{\mathbb{R}} := G_{\mathbb{C}}^{\theta\sigma}$. Since $G_{\mathbb{C}}$ is simply connected, both $H_{\mathbb{C}}$ and $G_{\mathbb{R}}$ are connected by a result of Borel [Bo61], and $K_{\mathbb{R}} := H_{\mathbb{C}} \cap G_{\mathbb{R}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$. Therefore, one has a natural bijection $\mathrm{Irr}(G_{\mathbb{R}})_f \simeq \mathrm{Irr}(G)_f \simeq \Lambda_+$ via the holomorphic continuation because $G_{\mathbb{C}}$ is simply connected. Since both H and $K_{\mathbb{R}}$ are real forms of the connected complex Lie group $H_{\mathbb{C}}$, there is a one-to-one correspondence between $\mathrm{Irr}(G_{\mathbb{R}})_{K_{\mathbb{R}},f}$ and $\mathrm{Irr}(G)_{H,f}$, and the former identifies with $\Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}})$ by (1).
(3) We now consider the general case where G is not necessarily a subgroup of the simply connected group $G_{\mathbb{C}}$, and use the letter $\widetilde{G}$ to denote the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$. (Note that G in the proof of (2) played the role of $\widetilde{G}$ here.) The holomorphic involution σ of $G_{\mathbb{C}}$ leaves $\widetilde{G}$ invariant. We set $\widetilde{H} := \widetilde{G}^{\sigma}$. Then $\mathrm{Irr}(\widetilde{G})_{\widetilde{H},f} \simeq \Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}})$ by (2).

For any $(\Pi, V) \in \mathrm{Irr}(G)_f$, the simply connected group $G_{\mathbb{C}}$ acts holomorphically on V, and thus one has a natural quotient map $\widetilde{G} \to G/\operatorname{Ker}\Pi$. In turn, one has an injection $\mathrm{Irr}(G)_f \hookrightarrow \mathrm{Irr}(\widetilde{G})_f \simeq \Lambda_+$, which induces

$$\mathrm{Irr}(G)_{H,f} \subset \mathrm{Irr}(G)_{H_0,f} \hookrightarrow \mathrm{Irr}(\widetilde{G})_{\widetilde{H},f} = \Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}}),$$

where H_0 denotes the identity component of H. For the middle inclusion, we have used that $\widetilde{H} = \widetilde{G}^{\sigma}$ is contained in the connected subgroup $G_{\mathbb{C}}^{\sigma}$. Hence we have shown the right inclusion in (2.12).

To see the left inclusion in (2.12), we set $b := [H : H_0]$, the number of connected components in H. We claim that $\Pi_{b\lambda} \in \mathrm{Irr}(G)_{H,f}$ for any $(\Pi_\lambda, V) \in \mathrm{Irr}(G)_{H_0,f}$. In fact, we take a generator v in the space V^{H_0} of H_0-fixed vectors in V, which is one-dimensional. Then the quotient group H/H_0 leaves $V^{H_0} = \mathbb{C}v$ invariant. On the other hand, the b-th tensor product representation $V \otimes \cdots \otimes V$ contains uniquely an irreducible subrepresentation $\Pi_{b\lambda}$. Let $S\colon V \otimes \cdots \otimes V \to \Pi_{b\lambda}$ be the projection. Then the H_0-fixed indecomposable vector $v \otimes \cdots \otimes v \in V \otimes \cdots \otimes V$ has a non-zero image, say v_b, in $\Pi_{b\lambda}$. Moreover, since H/H_0 acts on $\mathbb{C}v$ as a scalar, its diagonal action on $\mathbb{C}v \otimes \cdots \otimes v$ is trivial because b is the order of the finite group H/H_0. Thus $\Pi_{b\lambda} \in \mathrm{Irr}(G)_{H,f}$.

We take a positive integer a such that $a\Lambda_+ = a\,\mathrm{Irr}(\widetilde{G})_f \subset \mathrm{Irr}(G)_f$. Then one has $a\Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}}) = a\,\mathrm{Irr}(\widetilde{G})_{\widetilde{H},f} \subset \mathrm{Irr}(G)_{H_0,f}$. Hence we have shown

$$ab\Lambda_+(\mathfrak{g}_{\mathbb{C}};\mathfrak{h}_{\mathbb{C}}) \subset b\,\mathrm{Irr}(G)_{H_0,f} \subset \mathrm{Irr}(G)_{H,f}.$$

This proves the left inclusion of (2.12) with $k = ab$. $\square$

Lemma 2.17. *We consider two involutions of* $\mathrm{Irr}(G)_f$ *given by*

$$\Pi \mapsto \Pi^\sigma := \Pi \circ \sigma,$$
$$\Pi \mapsto \Pi^\vee \quad \textit{(contragredient representation).}$$

Then $\Pi^\sigma \simeq \Pi^\vee$ *for all* $\Pi \in \mathrm{Irr}(G)_{H,f}$*. In particular,* $\Pi^\vee \in \mathrm{Irr}(G)_{H,f}$ *if and only if* $\Pi \in \mathrm{Irr}(G)_{H,f}$*.*

Proof. Suppose λ is the highest weight of Π. Then $\Pi^\vee$ has an extremal weight $-\lambda$, whereas Π^σ has an extremal weight $\sigma\lambda$ which equals $-\lambda$ by Lemma 2.16. Hence $\Pi^\sigma \simeq \Pi^\vee$.

Suppose $\Pi \in \mathrm{Irr}(G)_{H,f}$. Then obviously $\Pi^\sigma \in \mathrm{Irr}(G)_{H,f}$, hence $\Pi^\vee \in \mathrm{Irr}(G)_{H,f}$. $\square$

3 Bounded multiplicity results for induction

In the classical harmonic analysis on the Riemannian symmetric space G/K, building blocks of representations in $C^\infty(G/K)$ are constructed by the twisted Poisson transform, an integral G-intertwining operator from the spherical principal series representation to $C^\infty(G/K)$, see [He94] for instance. More generally, for a closed subgroup H in G, we consider the space $\mathrm{Hom}_G(\mathrm{Ind}_P^G(\xi), \mathrm{Ind}_H^G(\tau))$ of generalized Poisson transforms, where P is a parabolic subgroup of G, $\xi \in \mathrm{Irr}(P)_f$, and $\tau \in \mathrm{Irr}(H)_f$. In this section, we give an estimate of $\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}_P^G(\xi), \mathrm{Ind}_H^G(\tau))$ as a refinement of the bounded multiplicity theorems proved in [KO13, Thm. B] in terms of a pair of parabolic subgroups $Q \subset P_{\mathbb{C}}$. The main result of this section is Theorem 3.1, of which the first statement provides a uniform bound of the multiplicities ("QP estimate") under a geometric condition $\#(Q\backslash G_{\mathbb{C}}/H_{\mathbb{C}}) < \infty$, strengthening a formulation of Tauchi [Tu19]. In turn, this leads us to "QP estimates" for restriction in Section 4.

3.1 Geometric condition for bounded multiplicity

Let H be a closed subgroup of a Lie group G. For a finite-dimensional representation (η, V) of H, we write $\mathrm{Ind}_H^G(\eta)$ for the (unnormalized) induced representation of G on the Fréchet space $C^\infty(G/H, \mathcal{V})$ of smooth sections for the homogeneous G-vector bundle $\mathcal{V} := G \times_H V$ over G/H. If H is a parabolic subgroup P of G, then $\operatorname{Ind}_P^G(\eta)$ is of moderate growth.

Theorem 3.1 ("QP estimate" for induction)**.** *Let G be a real reductive algebraic Lie group, H an algebraic subgroup, P a parabolic subgroup of G, and $G_{\mathbb{C}} \supset H_{\mathbb{C}}$, $P_{\mathbb{C}}$ their complexifications. Suppose that Q is a (complex) parabolic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$ such that $Q \subset P_{\mathbb{C}}$.*

(1) *If $\#(Q \backslash G_{\mathbb{C}}/H_{\mathbb{C}}) < \infty$, then there exists $C > 0$ such that*

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_P^G(\eta), \operatorname{Ind}_H^G(\tau)) \leq C d_{\mathfrak{q}}(\eta) \dim \tau \tag{3.1}$$

for any $\eta \in \operatorname{Irr}(P)_f$ and any $\tau \in \operatorname{Irr}(H)_f$. In particular, one has

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_P^G(\eta), \operatorname{Ind}_H^G(\tau)) \leq C \dim \tau \tag{3.2}$$

for any $\eta \in \operatorname{Irr}(P; \mathfrak{q})_f$ and any $\tau \in \operatorname{Irr}(H)_f$.

(2) *Conversely, if there exists $C > 0$ such that (3.2) holds for any $\eta \in \operatorname{Irr}(P; \mathfrak{q})_f$ and any $\tau \in \operatorname{Irr}(H)_f$, then Q has an open orbit in $G_{\mathbb{C}}/H_{\mathbb{C}}$.*

See Definition 2.1 for the definition of $d_{\mathfrak{q}}(\eta)$ ($\leq \dim \eta$), and (2.1) for the definition of $\operatorname{Irr}(P; \mathfrak{q})_f$. The point of Theorem 3.1 is that Q is not necessarily defined over $\mathbb{R}$, which we shall see useful in the proof of Theorem 1.4 in Section 6. We also present a number of bounded multiplicity results for restriction in Section 4.

Remark 3.2. As the proof shows, one can relax the assumption of the second statement by the following condition: there exists $C > 0$ such that

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_P^G(\eta), \operatorname{Ind}_H^G(\tau)) \leq C$$

for any $\eta \in \operatorname{Irr}(P; \mathfrak{q})_f$ and any character τ of H.

Remark 3.3. (1) If Q is a Borel subgroup of $G_{\mathbb{C}}$, then $d_{\mathfrak{q}}(\eta) = 1$. In this case, the first statement of Theorem 3.1 was proved in [KO13, Thm. B].
(2) If $Q = P_{\mathbb{C}}$, then $d_{\mathfrak{q}}(\eta) = \dim \eta$. In this case, the first statement of Theorem 3.1 was proved in Tauchi [Tu19, Thm. 1.13].
(3) If $\#(P_{\mathbb{C}} \backslash G_{\mathbb{C}}/H_{\mathbb{C}}) < \infty$, then Theorem 3.1 (1) implies a *finite multiplicity theorem* of the induction

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_P^G(\eta), \operatorname{Ind}_H^G(\tau)) < \infty \quad {}^{\forall}\eta \in \operatorname{Irr}(P)_f, {}^{\forall}\tau \in \operatorname{Irr}(H)_f. \tag{3.3}$$

However, the converse statement is not true. For example, if P is a minimal parabolic subgroup, as one sees from [KO13, Thm. A] that

$$(3.3) \iff G/H \text{ is real spherical } \overset{\not\Rightarrow}{\Longleftarrow} \#(P_{\mathbb{C}}\backslash G_{\mathbb{C}}/H_{\mathbb{C}}) < \infty,$$

hence $\#(P_{\mathbb{C}}\backslash G_{\mathbb{C}}/H_{\mathbb{C}}) < \infty$ is *not* a necessary condition for the finite multiplicity property (3.3). For a general parabolic subgroup P, the following geometric necessary condition was proved in [K14, Cor. 6.8]:

$$(3.3) \Longrightarrow P \text{ has an open orbit in } G/H.$$

The second statement in Theorem 3.1 is a refinement of this statement for "QP estimate".

3.2 Proof of Theorem 3.1 (1)

In this subsection we give a proof of the first statement of Theorem 3.1. For a real analytic manifold M, we denote by $\mathcal{B}$ the sheaf of hyperfunctions à la Sato [S59]. We shall regard distributions as generalized functions à la Gelfand [GS64] rather than continuous linear forms on $C_c^\infty(M)$ so that one has a natural inclusion $C^\infty(M) \subset \mathcal{D}'(M) \subset \mathcal{B}(M)$.

For $M = G$ (group manifold), we consider two actions L and R of G on $C^\infty(G)$, $\mathcal{D}'(G)$, and $\mathcal{B}(G)$ by $L(g_1)R(g_2)f := f(g_1^{-1} \cdot g_2)$ for $g_1, g_2 \in G$ on the same spaces, which induce the actions of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$, to be denoted by dL and dR, respectively.

For a parabolic subgroup P of G, we denote by $\mathbb{C}_{2\rho}$ the one-dimensional representation of P defined by

$$P \to \mathbb{R}_{>0}, \quad p \mapsto \det|\operatorname{Ad}(p)\colon \mathfrak{g}/\mathfrak{p} \to \mathfrak{g}/\mathfrak{p}|^{-1}. \tag{3.4}$$

Then $\mathcal{L}_{2\rho} := G \times_P \mathbb{C}_{2\rho}$ is the volume bundle over G/P, and the integration yields a G-invariant linear form $C^\infty(G/P, \mathcal{L}_\lambda) \to \mathbb{C}$.

Let (η, V) be a finite-dimensional representation of P and $\mathcal{V} := G \times_P V$ the G-homogeneous vector bundle over G/P associated to (η, V). We set $\eta^* := \eta^\vee \otimes \mathbb{C}_{2\rho}$, where $\eta^\vee$ is the contragredient representation of η. The dualizing bundle $\mathcal{V}^*$ of $\mathcal{V} = G \times_P V$ is given as the G-homogeneous vector bundle over G/P associated to η^*. Then one has a canonical G-invariant perfect pairing between $\operatorname{Ind}_P^G(\eta)$ and $\operatorname{Ind}_P^G(\eta^*)$ by the composition of the two maps:

$$C^\infty(G/P, \mathcal{V}) \times C^\infty(G/P, \mathcal{V}^*) \to C^\infty(G/P, \mathcal{L}_{2\rho}) \to \mathbb{C}.$$

Suppose that (τ, W) is a finite-dimensional representation of a closed subgroup H. By the Schwartz kernel theorem, any continuous linear operator T from $\mathrm{Ind}_P^G(\eta)$ to $\mathrm{Ind}_H^G(\tau)$ can be obtained by a bundle-valued distribution kernel on $G/P \times G/H$. This distribution is G-invariant under the diagonal action on $G/P \times G/H$ if T intertwines G-actions. Moreover, since G/P is compact, it follows from [KS15, Prop. 3.2] that one has a natural bijection:

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G(\eta), \mathrm{Ind}_H^G(\tau)) \simeq (\mathcal{D}'(G) \otimes \eta^* \otimes \tau)^{P \times H}, \tag{3.5}$$

where we let P act on $\mathcal{D}'(G) \otimes V^\vee \otimes W$ by $R \otimes \eta^* \otimes \mathrm{id}$, and H by $L \otimes \mathrm{id} \otimes \tau$.

Let L be a Levi part of P, and $L_{\mathbb{C}}$ its complexification. Since $P_{\mathbb{C}}/Q \simeq L_{\mathbb{C}}/(L_{\mathbb{C}} \cap Q)$ is a (generalized) flag variety of the complex reductive group $L_{\mathbb{C}}$, one has $\#(P\backslash P_{\mathbb{C}}/Q) < \infty$ because $\#(L\backslash L_{\mathbb{C}}/(L_{\mathbb{C}} \cap Q)) < \infty$ by a result of Wolf [Wo69]. In particular, one finds $x \in P_{\mathbb{C}}$ such that PxQ/Q is closed in $P_{\mathbb{C}}/Q$. Obviously, the assumption of Theorem 3.1 is unchanged if we replace Q by $\mathrm{Ad}(x)Q$, and so is the conclusion of Theorem 3.1 by (2.4). Thus we may and do assume that $P/(P \cap Q)$ is closed in $P_{\mathbb{C}}/Q$ from now on.

Suppose $\eta \in \mathrm{Irr}(P)_f$. One observes $d_{\mathfrak{q}}(\eta) = d_{\mathfrak{q}}(\eta \otimes \mathbb{C}_{-2\rho})$. Take an irreducible $\mathfrak{q}$-submodule $\lambda^\vee$ of the $\mathfrak{p}_{\mathbb{C}}$-module $(\eta^*)^\vee \simeq \eta \otimes \mathbb{C}_{-2\rho}$ such that $\dim \lambda^\vee = d_{\mathfrak{q}}(\eta)$. We write λ ($\simeq \lambda^{\vee\vee}$) for the contragredient representation of $\lambda^\vee$. Clearly, $\dim \lambda = d_{\mathfrak{q}}(\eta)$. By Lemma 2.7, there is an injective P-homomorphism $\eta^* \hookrightarrow (C^\infty(P) \otimes \lambda)^{\mathfrak{q}}$. Hence, one has

$$(\mathcal{D}'(G) \otimes \eta^*)^P \hookrightarrow (\mathcal{D}'(G) \otimes C^\infty(P) \otimes \lambda)^{P \times \mathfrak{q}} \simeq (\mathcal{D}'(G) \otimes \lambda)^{\mathfrak{q}}.$$

By (3.5), the first statement of Theorem 3.1 is reduced to the following:

Proposition 3.4. *Assume $\#(Q\backslash G_{\mathbb{C}}/H_{\mathbb{C}}) < \infty$. Then there exists $C > 0$ such that for any $\mathfrak{q}$-module λ and for any $\tau \in \mathrm{Irr}(H)_f$*

$$\dim_{\mathbb{C}}(\mathcal{D}'(G) \otimes \lambda \otimes \tau)^{\mathfrak{q} \oplus \mathfrak{h}_{\mathbb{C}}} \leq C \dim \lambda \dim \tau, \tag{3.6}$$

where $\mathfrak{q}$ acts on the first and second factors by $dR \otimes \lambda$, and $\mathfrak{h}_{\mathbb{C}}$ on the first and third by $dL \otimes \tau$.

For the proof, we use the following result by Tauchi, which is based on the theory of (regular) holonomic $\mathcal{D}$-modules (see [Ka83, KK81] for instance).

Proposition 3.5 ([Tu19, Thm. 1.14])**.** *Suppose a complex Lie group B acts holomorphically on a complex manifold $X_{\mathbb{C}}$ with finite number of orbits. Let*

$X_\mathbb{R}$ be a real form of $X_\mathbb{C}$, and U a relatively compact semi-analytic open subset of $X_\mathbb{R}$. Then there exists a constant $C > 0$ such that $\dim_\mathbb{C}(\mathcal{B}(U) \otimes \mu)^\mathfrak{b} \leq C \dim \mu$ for any finite-dimensional representation μ of the Lie algebra $\mathfrak{b}$ of B. Here $(\mathcal{B}(U) \otimes \mu)^\mathfrak{b}$ denotes the space of $\mathfrak{b}$-invariant vector-valued hyperfunctions on U via the diagonal action.

Proof of Proposition 3.4. Let K be a maximal compact subgroup of G. We recall that we have chosen P such that $P/(P \cap Q)$ is closed in the flag variety $P_\mathbb{C}/Q$. In particular, the algebraic subgroup $P \cap Q$ is cocompact in P, hence one has $G = K(P \cap Q)$.

As in [KS15, Thm. 3.16], we capture all invariant distributions (or hyperfunctions) on G by those on a sufficiently large open subset U. For this, we fix a relatively compact semi-analytic open neighbourhood of $o = Ke \in K\backslash G$, and define U to be its full inverse via the quotient map $G \to K\backslash G$. Then U is a left K-invariant, relatively compact, semi-analytic open subset in G. Moreover the restriction map $(\mathcal{B}(G) \otimes \mu)^\mathfrak{q} \to (\mathcal{B}(U) \otimes \mu)^\mathfrak{q}$ is injective because $(G_\mathbb{C} \supset)\ UQ \supset K(P \cap Q) = G$. Hence one has natural inclusions:

$$(\mathcal{D}'(G) \otimes \lambda \otimes \tau)^{\mathfrak{q} \oplus \mathfrak{h}_\mathbb{C}} \subset (\mathcal{B}(G) \otimes \lambda \otimes \tau)^{\mathfrak{q} \oplus \mathfrak{h}_\mathbb{C}} \subset (\mathcal{B}(U) \otimes \lambda \otimes \tau)^{\mathfrak{q} \oplus \mathfrak{h}_\mathbb{C}}.$$

We take $X_\mathbb{R} := (G \times G)/\operatorname{diag} G$ ($\simeq G$) and apply Proposition 3.5 to the setting $(B, X_\mathbb{C}, \mu) := (Q \times H_\mathbb{C}, G_\mathbb{C}, \lambda \otimes \tau)$. Then $\dim_\mathbb{C}(\mathcal{B}_G(U) \otimes \lambda \otimes \tau)^{\mathfrak{q} \oplus \mathfrak{h}_\mathbb{C}} \leq C \dim \lambda \dim \tau$, whence the inequality (3.6) follows. ☐

3.3 Proof of Theorem 3.1 (2)

We recall our setting where G is a real form of $G_\mathbb{C}$. We take a positive system $\Delta^+(\mathfrak{g}_\mathbb{C}, \widetilde{\mathfrak{j}}_\mathbb{C})$ such that the corresponding Borel subgroup B is contained in the complex parabolic subgroup Q ($\subset P_\mathbb{C}$).

Lemma 3.6. *Let Π_λ be an irreducible holomorphic representation of $G_\mathbb{C}$ with highest weight $\lambda \in \Lambda_+$. Let η be the regular representation of $P_\mathbb{C}$ on $\mathcal{O}(P_\mathbb{C}/B, \mathcal{L}_{-\lambda})$, and define a representation of P by $\xi := \eta^\vee|_P \otimes \mathbb{C}_{2\rho}$.*

(1) *There is a surjective G-homomorphism $\operatorname{Ind}_P^G(\xi) \to \Pi_\lambda$.*

(2) *If $\Pi_\lambda \in \operatorname{Irr}(G; \mathfrak{q})_f$ then $\xi \in \operatorname{Irr}(P; \mathfrak{q})_f$.*

Proof. By the Borel–Weil theorem, the contragredient representation $\Pi_\lambda^\vee$ is realized as the regular representation on $\mathcal{O}(G_\mathbb{C}/B, \mathcal{L}_{-\lambda})$. We denote by $\mathcal{V}_\eta$

the $G_{\mathbb{C}}$-equivariant vector bundle over $G_{\mathbb{C}}/P_{\mathbb{C}}$ associated to the $P_{\mathbb{C}}$-module η. Induction by stages for $B \subset P_{\mathbb{C}} \subset G_{\mathbb{C}}$ shows a natural isomorphism as $G_{\mathbb{C}}$-modules:

$$(\Pi_\lambda^\vee \simeq)\ \mathcal{O}(G_{\mathbb{C}}/B, \mathcal{L}_{-\lambda}) \simeq \mathcal{O}(G_{\mathbb{C}}/P_{\mathbb{C}}, \mathcal{V}_\eta),$$

which yields an injective G-homomorphism $\Pi_\lambda^\vee \hookrightarrow \operatorname{Ind}_P^G(\eta|_P)$. Taking the dual, we see that Π_λ occurs as the quotient G-module of the induced representation $\operatorname{Ind}_P^G(\xi)$.

(2) Using induction by stages for $B \subset Q \subset P_{\mathbb{C}}$ this time, one has $\eta \simeq \mathcal{O}(P_{\mathbb{C}}/Q, \mathcal{V}_{-\lambda})$, where $\mathcal{V}_{-\lambda}$ stands for the $P_{\mathbb{C}}$-equivariant holomorphic vector bundle over $P_{\mathbb{C}}/Q$ associated to the Q-module $\mathcal{O}(Q/B, \mathcal{L}_{-\lambda})$. By Lemma 2.6 (1), λ is orthogonal to all the roots in the Levi subalgebra of $\mathfrak{q}$, hence $\mathcal{O}(Q/B, \mathcal{L}_{-\lambda})$ is one-dimensional. In turn, one has $\eta^\vee \in \operatorname{Irr}(P_{\mathbb{C}}; \mathfrak{q})_{\mathrm{hol}}$ by Lemma 2.9 (2), and thus $\xi \in \operatorname{Irr}(P; \mathfrak{q})_f$ because $\eta^\vee|_P = \xi \otimes \mathbb{C}_{-2\rho}$. □

Proof of Theorem 3.1 (2). Suppose that Q does not have an open orbit in $G_{\mathbb{C}}/H_{\mathbb{C}}$. By Lemma 2.11, there exist $\lambda \in \Lambda_+$ and a character χ of H such that $\dim_{\mathbb{C}} \operatorname{Hom}_H(\Pi_{N\lambda}|_H, \chi^N) \geq N+1$ for all $N \in \mathbb{N}$. In turn, it follows from Lemma 3.6 that there exists $\xi_N \in \operatorname{Irr}(P; \mathfrak{q})_f$ for every $N \in \mathbb{N}$ such that the irreducible G-module $\Pi_{N\lambda}$ is a quotient of the degenerate principal series representation $\operatorname{Ind}_P^G(\xi_N)$, hence one has $\dim_{\mathbb{C}} \operatorname{Hom}_H(\operatorname{Ind}_P^G(\xi_N)|_H, \chi^N) \geq N+1$. The Frobenius reciprocity shows $\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_P^G(\xi_N), \operatorname{Ind}_H^G(\chi^N)) \geq N+1$, whence the second statement of Theorem 3.1. □

4 Bounded multiplicity results for restriction

In this section, we derive bounded multiplicity results ("QP estimates") for *restriction* from those for *induction* in Theorem 3.1 along the same line of the argument as in [K14, KO13]. The results here will be used in Section 6 for the proof of Theorem 1.4 by a specific choice of the parabolic subgroups $Q \subset P_{\mathbb{C}}$. Throughout this section, we suppose that $G_{\mathbb{C}} \supset G'_{\mathbb{C}}$ are connected reductive Lie groups and that $G \supset G'$ are their real forms.

4.1 Bounded multiplicity theorems for restriction

We begin with a "QP estimates" of the space of "symmetry breaking operators" between (degenerate) principal series representations of a group G and those of a subgroup G'. The results might be of their own interest because they indicate a nice broader framework for detailed study of such operators. See *e.g.*, [KS15, KS18] for the construction and the classification of symmetry breaking operators for principal series of two conformal groups, see also Examples 9.2, 9.3, and 9.4 by [CKØP11, KØP11, NØ18] for some of the most degenerate cases.

Theorem 4.1 ("QP estimate" for restriction)**.** *Let $G \supset G'$ be a pair of real reductive algebraic Lie groups, and P and P' are parabolic subgroups of G and G', respectively. Suppose that Q and Q' are (complex) parabolic subgroups of $G_{\mathbb{C}}$ and $G'_{\mathbb{C}}$, respectively, such that $\mathfrak{q} \subset \mathfrak{p}_{\mathbb{C}}$, $\mathfrak{q}' \subset \mathfrak{p}'_{\mathbb{C}}$, and $\#(Q'_{\mathrm{opp}} \backslash G_{\mathbb{C}}/Q) < \infty$. Then there exists $C > 0$ such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\mathrm{Ind}_P^G(\xi)|_{G'}, \mathrm{Ind}_{P'}^{G'}(\eta)) \leq C d_{\mathfrak{q}}(\xi) d_{\mathfrak{q}'}(\eta) \tag{4.1}$$

for any $\xi \in \mathrm{Irr}(P)_f$ and any $\eta \in \mathrm{Irr}(P')_f$. In particular, one has

$$\sup_{\xi \in \mathrm{Irr}(P;\mathfrak{q})_f} \sup_{\eta \in \mathrm{Irr}(P';\mathfrak{q}')_f} \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\mathrm{Ind}_P^G(\xi)|_{G'}, \mathrm{Ind}_{P'}^{G'}(\eta)) < \infty. \tag{4.2}$$

Here we recall from Definition 2.1 for the quantity $d_{\mathfrak{q}}(\xi)$, and from Definition 2.4 that Q'_{opp} is the opposite parabolic subgroup of Q' in $P'_{\mathbb{C}}$.

Proof of Theorem 4.1. For $\eta \in \mathrm{Irr}(P')_f$, we set $\eta^* := \eta^\vee \otimes \mathbb{C}_{2\rho'}$ where $\mathbb{C}_{2\rho'}$ is a character of P' defined as in (3.4). Then the induced representation $\mathrm{Ind}_{P'}^{G'}(\eta^*)$ is the contragredient representation of $\mathrm{Ind}_{P'}^{G'}(\eta)$ in the category $\mathcal{M}(G')$, and one has the following natural isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{G'}(\mathrm{Ind}_P^G(\xi)|_{G'}, \mathrm{Ind}_{P'}^{G'}(\eta)) &\xrightarrow{\sim} \mathrm{Hom}_{G'}(\mathrm{Ind}_P^G(\xi)|_{G'} \otimes \mathrm{Ind}_{P'}^{G'}(\eta^*), \mathbb{C}) \\ &\simeq \mathrm{Hom}_{G'}(\mathrm{Ind}_{P \times P'}^{G \times G'}(\xi \boxtimes \eta^*)|_{\mathrm{diag}\, G'}, \mathbb{C}) \\ &\xleftarrow{\sim} \mathrm{Hom}_{G \times G'}(\mathrm{Ind}_{P \times P'}^{G \times G'}(\xi \boxtimes \eta^*), \mathrm{Ind}_{\mathrm{diag}\, G'}^{G \times G'}(\mathbf{1})).\end{aligned}$$

Here the injectivity of the first isomorphism is easy, and for the proof of the surjectivity, see [KS18, Thm. 5.4] for instance. The last isomorphism is the Frobenius reciprocity.

Then the first assertion of Theorem 4.1 follows from Theorem 3.1 applied to the direct product group $G \times G'$ because

$$(Q \times Q'_{\mathrm{opp}}) \backslash (G_{\mathbb{C}} \times G'_{\mathbb{C}}) / \mathrm{diag}\, G'_{\mathbb{C}} \simeq Q'_{\mathrm{opp}} \backslash G_{\mathbb{C}}/Q \tag{4.3}$$

is a finite set. Moreover, if $\eta \in \mathrm{Irr}(P';\mathfrak{q}')_f$, then the contragredient representation $\eta^\vee$ belongs to $\mathrm{Irr}(P;\mathfrak{q}'_{\mathrm{opp}})_f$ by Lemma 2.5, hence so does η^*. Thus the second assertion also holds because the outer tensor product representation $\xi \boxtimes \eta^*$ belongs to $\mathrm{Irr}(P \times P';\mathfrak{q}\oplus\mathfrak{q}'_{\mathrm{opp}})_f$ if $\xi \in \mathrm{Irr}(P;\mathfrak{q})_f$ and $\eta \in \mathrm{Irr}(P';\mathfrak{q}')_f$. $\square$

When Q' is a Borel subgroup of $G'_{\mathbb{C}}$ in Theorem 4.1, one obtains the converse statement as follows. We recall from (1.10) that $\Omega_{P,\mathfrak{q}} = \{\mathrm{Ind}_P^G(\xi) : \xi \in \mathrm{Irr}(P;\mathfrak{q})_f\}$ $(\subset \mathcal{M}(G))$, and from (1.5) that $m(\Pi|_{G'}) = \sup_{\pi \in \mathrm{Irr}(G')} \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$.

Theorem 4.2. *Let $G \supset G'$ be a pair of real reductive algebraic Lie groups, and P a (real) parabolic subgroup of G. Suppose that Q is a parabolic subgroup of $G_{\mathbb{C}}$ such that $\mathfrak{q} \subset \mathfrak{p}_{\mathbb{C}}$. Then the following four conditions are equivalent:*

(i) *There exists $C > 0$ such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\mathrm{Ind}_P^G(\xi)|_{G'}, \pi) \leq C d_{\mathfrak{q}}(\xi) \tag{4.4}$$

for any $\xi \in \mathrm{Irr}(P)_f$ and any $\pi \in \mathrm{Irr}(G')$.

(ii) $\sup_{\Pi \in \Omega_{P,\mathfrak{q}}} m(\Pi|_{G'}) < \infty$.

(iii) *$G_{\mathbb{C}}/Q$ is $G'_{\mathbb{C}}$-spherical.*

(iv) *$G_{\mathbb{C}}/Q$ is G'_U-strongly visible ([K05, Def. 3.3.1]).*

Remark 4.3. (1) A distinguished feature of Theorem 4.2 is that the necessary and sufficient condition of the bounded multiplicity property is given only by the complexification $(G_{\mathbb{C}}, G'_{\mathbb{C}})$, which traces back to [K95, KO13].

(2) When $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ is a symmetric pair, the parabolic subgroups Q satisfying the sphericity condition (iii) were classified in [HNOO13]. See also [K05, Tn12] for some classification of strongly visible actions.

We present two extreme choices of the parabolic subgroup Q, namely, the smallest one $Q = B$ (Borel subgroup) in Example 4.4 and the largest one $Q = P_{\mathbb{C}}$ in Example 4.5 below. An intermediate choice of Q in Theorem 4.2 will be crucial in Section 6 for the proof of Theorem 1.4.

Example 4.4. When Q is a Borel subgroup B of $G_{\mathbb{C}}$, one has $\mathrm{Irr}(P;\mathfrak{b})_f = \mathrm{Irr}(P)_f$ by Lemma 2.2, hence the equivalence (ii) $\iff$ (iii) in Theorem 4.2 gives an alternative proof of [KO13, Thm. D].

Example 4.5. When $Q = P_{\mathbb{C}}$, Theorem 4.2 implies the equivalence of the following three conditions on the triple (G, P, G'):

(i) $[\mathrm{Ind}_P^G(\xi)|_{G'} : \pi] \leq C \dim \xi \quad$ for all $\xi \in \mathrm{Irr}(P)_f$ and all $\pi \in \mathrm{Irr}(G')$;

(ii) $\sup_{\chi \in \mathrm{Hom}(P, \mathbb{C}^\times)} m(\mathrm{Ind}_P^G(\chi)|_{G'}) < \infty$;

(iii) $G_{\mathbb{C}}/P_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.

See Example 9.2 for some concrete cases where the branching laws of the unitary representation $L^2\text{-}\mathrm{Ind}_P^G(\chi)|_{G'}$ are explicitly obtained in this framework.

Example 4.6. When $Q = P_{\mathbb{C}}$ and Q' is the complexification of a minimal parabolic subgroup $P'_{\min}$ of G' in Theorem 4.1, one has the following as in the proof of Theorem 4.2 below: *if P is a parabolic subgroup of G satisfying $\#(P'_{\min} \backslash G_{\mathbb{C}}/P_{\mathbb{C}}) < \infty$ then one has*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\mathrm{Ind}_P^G(\xi)|_{G'}, \pi) < \infty \textit{ for any } \xi \in \mathrm{Irr}(P)_f \textit{ and } \pi \in \mathrm{Irr}(G'). \tag{4.5}$$

Proof of Theorem 4.2. (iii) $\Rightarrow$ (i). Let P' be a minimal parabolic subgroup of G', and B' a Borel subgroup of $G'_{\mathbb{C}}$ such that $\mathfrak{b}' \subset \mathfrak{p}'_{\mathbb{C}}$. By Casselman's subrepresentation theorem (Example 5.8), for any $\pi \in \mathrm{Irr}(G')$ there exists $\eta \in \mathrm{Irr}(P')_f$ such that $\mathrm{Hom}_{G'}(\pi, \mathrm{Ind}_{P'}^{G'}(\eta)) \neq \{0\}$. Then one has an injection $\mathrm{Hom}_{G'}(\mathrm{Ind}_P^G(\xi)|_{G'}, \pi) \hookrightarrow \mathrm{Hom}_{G'}(\mathrm{Ind}_P^G(\xi)|_{G'}, \mathrm{Ind}_{P'}^{G'}(\eta))$. Thus the implication (iii) $\Rightarrow$ (i) follows as a special case of Theorem 4.1 because $\mathrm{Irr}(P')_f = \mathrm{Irr}(P'; \mathfrak{b}')_f$ and the number of B'-orbits in the $G'_{\mathbb{C}}$-spherical variety $G_{\mathbb{C}}/Q$ is finite by a result of Brion [B86] and Vinberg [V86].
(i) $\Rightarrow$ (ii). Obvious because $d_{\mathfrak{q}}(\xi) = 1$ if $\xi \in \mathrm{Irr}(P; \mathfrak{q})_f$ by definition.
(ii) $\Rightarrow$ (iii). This follows from Lemmas 2.10 and 3.6 as in the proof of Theorem 3.1 (2).
(iii) $\iff$ (iv). The equivalence (iii) $\iff$ (iv) is proved in [Tn21]. □

4.2 Bounded multiplicity for tensor product

The tensor product of two representations is regarded as the restriction of the outer tensor product representation of the direct product group $G \times G$ with respect to its subgroup $\mathrm{diag}\, G$. Thus the following theorem follows readily as a special case of Theorem 4.1.

Theorem 4.7 ("QP estimate" for tensor product)**.** *Let G be a real reductive algebraic Lie group, and P_j $(j = 1, 2, 3)$ (real) parabolic subgroups of G. Suppose that Q_j $(j = 1, 2, 3)$ are parabolic subgroups of $G_{\mathbb{C}}$ such that $Q_j \subset (P_j)_{\mathbb{C}}$ $(1 \leq j \leq 3)$ and $\#((Q_1 \times Q_2)\backslash(G_{\mathbb{C}} \times G_{\mathbb{C}})/\operatorname{diag}(Q_3)_{\mathrm{opp}}) < \infty$. Then there exists $C > 0$ such that*

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_{P_1}^G(\xi_1) \otimes \operatorname{Ind}_{P_2}^G(\xi_2), \operatorname{Ind}_{P_3}^G(\xi_3)) \leq C d_{\mathfrak{q}_1}(\xi_1) d_{\mathfrak{q}_2}(\xi_2) d_{\mathfrak{q}_3}(\xi_3)$$

for any $\xi_j \in \operatorname{Irr}(P_j)_f$ $(j = 1, 2, 3)$. In particular, one has

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_{P_1}^G(\xi_1) \otimes \operatorname{Ind}_{P_2}^G(\xi_2), \operatorname{Ind}_{P_3}^G(\xi_3)) \leq C \tag{4.6}$$

for any $\xi_j \in \operatorname{Irr}(P_j; \mathfrak{q}_j)_f$ $(1 \leq j \leq 3)$.

As in [K14] for instance, one may reformulate Theorem 4.7 as a bounded multiplicity theorem for invariant trilinear forms.

Theorem 4.8 (Invariant trilinear forms)**.** *Let G be a real reductive algebraic Lie group, and P_j $(j = 1, 2, 3)$ (real) parabolic subgroups of G. Suppose that Q_j $(j = 1, 2, 3)$ are parabolic subgroups of $G_{\mathbb{C}}$ such that $Q_j \subset (P_j)_{\mathbb{C}}$ $(1 \leq j \leq 3)$ and $\#((Q_1 \times Q_2)\backslash(G_{\mathbb{C}} \times G_{\mathbb{C}})/\operatorname{diag} Q_3) < \infty$. Then there exists $C > 0$ such that*

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_{P_1}^G(\xi_1) \otimes \operatorname{Ind}_{P_2}^G(\xi_2) \otimes \operatorname{Ind}_{P_3}^G(\xi_3), \mathbb{C}) \leq C d_{\mathfrak{q}_1}(\xi_1) d_{\mathfrak{q}_2}(\xi_2) d_{\mathfrak{q}_3}(\xi_3)$$

for any $\xi_j \in \operatorname{Irr}(P_j)_f$ $(j = 1, 2, 3)$. In particular, one has

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\operatorname{Ind}_{P_1}^G(\xi_1) \otimes \operatorname{Ind}_{P_2}^G(\xi_2) \otimes \operatorname{Ind}_{P_3}^G(\xi_3), \mathbb{C}) \leq C, \tag{4.7}$$

for any $\xi_j \in \operatorname{Irr}(P_j; \mathfrak{q}_j)_f$ $(j = 1, 2, 3)$.

See *e.g.,* [MWZ99, Mt15] for classification results of the triples (Q_1, Q_2, Q_3) satisfying $\#((Q_1 \times Q_2)\backslash(G_{\mathbb{C}} \times G_{\mathbb{C}})/\operatorname{diag} Q_3) < \infty$ for some classical complex simple Lie groups $G_{\mathbb{C}}$. See also Example 9.4 for some recent works on integral trilinear forms by Clerc et al. [CKØP11, C15] which fits well into the framework of Theorem 4.8.

It also deserves to discuss Theorem 4.7 in the special setting where one of Q_1, Q_2 or Q_3 is a Borel subgroup of $G_{\mathbb{C}}$:

Theorem 4.9. *Let G be a real reductive algebraic Lie group, Q_1, Q_2 be parabolic subgroups of $G_{\mathbb{C}}$, and G_U a maximal compact subgroup of $G_{\mathbb{C}}$. Suppose that P_1 and P_2 are real parabolic subgroups of G such that $Q_j \subset (P_j)_{\mathbb{C}}$ $(j = 1, 2)$. Then the following four conditions are equivalent:*

(i) *There exists* $C > 0$ *such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}_{P_1}^G(\xi_1) \otimes \mathrm{Ind}_{P_2}^G(\xi_2), \pi) \leq C d_{\mathfrak{q}_1}(\xi_1) d_{\mathfrak{q}_2}(\xi_2) \tag{4.8}$$

for any $\xi_j \in \mathrm{Irr}(P_j)_f$ $(j = 1, 2)$ *and any* $\pi \in \mathrm{Irr}(G)$.

(ii) *There exists* $C > 0$ *such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}_{P_1}^G(\xi_1) \otimes \mathrm{Ind}_{P_2}^G(\xi_2), \pi) \leq C \tag{4.9}$$

for any $\xi_j \in \mathrm{Irr}(P_j; \mathfrak{q}_j)_f$ $(j = 1, 2)$ *and any* $\pi \in \mathrm{Irr}(G)$.

(iii) $G_{\mathbb{C}}/Q_1 \times G_{\mathbb{C}}/Q_2$ *is* $G_{\mathbb{C}}$*-spherical via the diagonal action.*

(iv) $G_{\mathbb{C}}/Q_1 \times G_{\mathbb{C}}/Q_2$ *is* G_U*-strongly visible via the diagonal action.*

A special case with $Q_j = (P_j)_{\mathbb{C}}$ $(j = 1, 2)$ implies the following:

Corollary 4.10. *Let* G *be a real reductive algebraic Lie group, and* P_j $(j = 1, 2)$ *parabolic subgroups. Then the following five conditions on the triple* (G, P_1, P_2) *are equivalent:*
(i) *There exists* $C > 0$ *such that*

$$m(\mathrm{Ind}_{P_1}^G(\xi_1) \otimes \mathrm{Ind}_{P_2}^G(\xi_2)) \leq C \dim \xi_1 \dim \xi_2, \quad {}^\forall \xi_j \in \mathrm{Irr}(P_j)_f \, (j = 1, 2).$$

(ii) *There exists* $C > 0$ *such that*

$$m(\mathrm{Ind}_{P_1}^G(\xi_1) \otimes \mathrm{Ind}_{P_2}^G(\xi_2)) \leq C \tag{4.10}$$

for all characters ξ_j *of* P_j $(j = 1, 2)$.
(iii) $\mathcal{O}(G_{\mathbb{C}}/P_{1\mathbb{C}}, \mathcal{L}_1) \otimes \mathcal{O}(G_{\mathbb{C}}/P_{2\mathbb{C}}, \mathcal{L}_2)$ *is a multiplicity free* $G_{\mathbb{C}}$*-module for any* $G_{\mathbb{C}}$*-equivariant holomorphic line bundles* $\mathcal{L}_j$ *on* $G_{\mathbb{C}}/P_{j\mathbb{C}}$ $(j = 1, 2)$.
(iv) $G_{\mathbb{C}}/P_{1\mathbb{C}} \times G_{\mathbb{C}}/P_{2\mathbb{C}}$ *is* $\mathrm{diag}(G_U)$*-strongly visible.*
(v) $G_{\mathbb{C}}/P_{1\mathbb{C}} \times G_{\mathbb{C}}/P_{2\mathbb{C}}$ *is* $\mathrm{diag}(G_{\mathbb{C}})$*-spherical.*

Proof. The equivalence (i) $\iff$ (ii) $\iff$ (iv) $\iff$ (v) is a special case of Theorem 4.9. The equivalence (iii) $\iff$ (v) for the finite-dimensional representation theory follows from [VK78] and Lemma 2.10. □

Littelmann [Li94] classified the pairs of parabolic subgroups $(P_{1\mathbb{C}}, P_{2\mathbb{C}})$ such that $G_{\mathbb{C}}/P_{1\mathbb{C}} \times G_{\mathbb{C}}/P_{2\mathbb{C}}$ are $G_{\mathbb{C}}$-spherical under the assumption that $P_{1\mathbb{C}}$ and $P_{2\mathbb{C}}$ are maximal, whereas Tanaka [Tn12] classified all the pairs $(P_{1\mathbb{C}}, P_{2\mathbb{C}})$ such that $G_{\mathbb{C}}/P_{1\mathbb{C}} \times G_{\mathbb{C}}/P_{2\mathbb{C}}$ is G_U-strongly visible.

For later applications it would deserve to mention a further special case:

Corollary 4.11. *Let G be a real reductive algebraic Lie group, and P_1, P_2 parabolic subgroups with abelian unipotent radical. Then the uniform bounded estimate* (4.10) *holds for any characters ξ_j of P_j $(j = 1, 2)$.*

Proof. If the unipotent radicals of P_1 and P_2 are abelian, then $G_{\mathbb{C}}/P_{1\mathbb{C}} \times G_{\mathbb{C}}/P_{2\mathbb{C}}$ is strongly visible via the diagonal G_U-action [K07a, Thm. 1.7]. $\square$

Example 4.12. Let $G = GL_n(\mathbb{R})$ and P_1, P_2 any two maximal parabolic subgroups of G. Then Corollary 4.11 applies.

Example 4.13. Let $G = SO(n, 1)$ and P be a minimal parabolic subgroup of G. Then Corollary 4.11 applies to $P_1 = P_2 = P$. We note that the uniform bounded estimate (4.10) fails if we allow $\xi_j \in \mathrm{Irr}(P_j)_f$ when $n \geq 4$ by [K14].

5 Oshima's embedding theorem — revisited

In this section, we analyze irreducible H-distinguished representations of G for reductive symmetric spaces G/H.

The classical Casselman's subrepresentation theorem asserts that any irreducible representation $\Pi \in \mathrm{Irr}(G)$ can be realized as a subrepresentation of a principal series representation, or equivalently, as a quotient of another principal series representation. If Π is H-distinguished, there should be some constraints on the parameter of the principal series representations depending on H. The main result of this section is Theorem 5.4, which asserts that Π is a quotient of some degenerate principal series $\mathrm{Ind}_{P_{G/H}}^{G}(\xi)$ where $P_{G/H}$ is a "minimal parabolic subgroup" for G/H (Definition 5.1). We discover a useful description of the constraints of ξ by using the notion "$\mathrm{Irr}(P; \mathfrak{q})_f$" introduced in Definition 2.1 with $\mathfrak{q}$ being a Borel subalgebra for G/H (Definition 1.3). The results of this section will be used to deduce Theorem 1.4 from Theorem 3.1.

Throughout this section, we retain our setting that G is contained in a connected complexification $G_{\mathbb{C}}$. We also assume G is connected.

5.1 Quotient theorem for $\mathrm{Irr}(G)_H$

Suppose that G/H is a symmetric space given by an involutive automorphism σ of a connected real reductive Lie group G. We take a Cartan involution θ of G which commutes with σ, and write K for the corresponding maximal compact subgroup of G.

Definition 5.1 (Minimal parabolic subgroup for G/H)**.** Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{g}^{-\sigma,-\theta} := \{X \in \mathfrak{g} : \sigma(X) = \theta(X) = -X\}$, and L_σ the centralizer of $\mathfrak{a}$ in G. We fix a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, and write $P_{G/H}$ or P_σ for the corresponding (real) parabolic subgroup of G. We say $P_{G/H}$ is a *minimal parabolic subgroup for* G/H. We write $P_{G/H} = L_\sigma N_\sigma$ and $\mathfrak{p}_{G/H} = \mathfrak{l}_\sigma + \mathfrak{n}_\sigma$ for the Levi decomposition with $\mathfrak{l}_\sigma \supset \mathfrak{a}$.

Remark 5.2. (1) We note that $\mathfrak{a}$ may be strictly smaller than the split abelian subspace of $\mathfrak{l}_\sigma$.
(2) It seems that the terminologies "Borel subalgebras for reductive symmetric spaces G/H" (Definition 1.3) and "minimal parabolic subgroups for G/H" (Definition 5.1) are not commonly used, cf. [K19b, Ex. 4.4]. As Theorem 5.4 below suggests, we believe that these terminologies are natural generalizations of the classical ones for the group manifold.

As we see in Section 5.2 below, the following statement holds:

Lemma 5.3. *One can take a Borel subalgebra* $\mathfrak{b}_{G/H}$ *for* G/H *(Definition 1.3) such that* $\mathfrak{b}_{G/H} \subset (\mathfrak{p}_{G/H})_{\mathbb{C}}$.

The goal of this section is to prove the following reformulation of Oshima's embedding theorem [O88, Thm. 4.15].

Theorem 5.4 (Quotient representation theorem for H-distinguished representations)**.** *Let* G/H *be a reductive symmetric space, and* $P_{G/H}$ *a minimal parabolic subgroup for* G/H. *For any* $\Pi \in \mathrm{Irr}(G)_H$, *there exists* $\xi \in \mathrm{Irr}(P_{G/H}; \mathfrak{b}_{G/H})_f$ *such that* Π *is a quotient of the (degenerate) principal series representation* $\mathrm{Ind}_{P_{G/H}}^G(\xi)$ *of* G, *where* $\mathfrak{b}_{G/H}$ *is a Borel subalgebra for* G/H *with* $\mathfrak{b}_{G/H} \subset (\mathfrak{p}_{G/H})_{\mathbb{C}}$.

It should be noted that the argument of this section gives a "coarse estimate of the size" of $\Pi \in \mathrm{Irr}(G)_H$ as follows:

Proposition 5.5. *For any irreducible subquotient* Π *in* $C^\infty(G/H)$, *the Gelfand–Kirillov dimension, to be denoted by* $\mathrm{DIM}(\Pi)$, *satisfies* $\mathrm{DIM}(\Pi) \leq \dim G/P_{G/H}$. *In particular,* $\mathrm{DIM}(\Pi) \leq \dim G/P_{G/H}$ *for any* $\Pi \in \mathrm{Irr}(G)_H$.

The proof of Theorem 5.4 and Proposition 5.5 will be given in Section 5.3.

5.2 Oshima's embedding theorem

In order to state Oshima's embedding theorem, we need to prepare some notations and structure theorems of reductive symmetric spaces G/H, which we recall now from [OS84].

Retain the setting as in Section 5.1. We extend $\mathfrak{a}$ ($\subset \mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\theta}$) to two maximal abelian subspaces $\mathfrak{j}$ in $\mathfrak{g}^{-\sigma}$ and $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{g}^{-\theta}$. Then one has $[\mathfrak{j}, \mathfrak{a}_{\mathfrak{p}}] = \{0\}$, hence there exists a σ-split and θ-split Cartan subalgebra $\widetilde{\mathfrak{j}}$ of $\mathfrak{g}$ containing both $\mathfrak{j}$ and $\mathfrak{a}_{\mathfrak{p}}$. As in Section 2.6, one has a direct sum decomposition $\widetilde{\mathfrak{j}} = \mathfrak{t} \oplus \mathfrak{j}$ with $\mathfrak{t} := \widetilde{\mathfrak{j}} \cap \mathfrak{h}$. Moreover one can take positive systems $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$, $\Sigma^+(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$ and $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ such that they are compatible with a fixed positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ in the sense that the restriction maps induce the following commutative diagram.

$$\begin{array}{ccccc} & & \Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}}) \cup \{0\} & & \\ & \nearrow & & \searrow & \\ \Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}}) & & & & \Sigma^+(\mathfrak{g}, \mathfrak{a}) \cup \{0\} \\ & \searrow & & \nearrow & \\ & & \Sigma^+(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}}) \cup \{0\} & & \end{array}$$

We define a Borel subalgebra $\mathfrak{b}_{G/H}$ for the symmetric space G/H by using the positive system $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. Then one has $\mathfrak{b}_{G/H} \subset (\mathfrak{p}_{G/H})_{\mathbb{C}}$.

We denote by $\mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; \lambda)$ the weight space for a linear form λ on $\mathfrak{a}_{\mathfrak{p}}$, and set $\mathfrak{m} := \mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; 0) \cap \mathfrak{k}$. Since $G_{\mathbb{C}}$ is connected, it follows from a result of Satake [Sa60] (also [He78, p. 435]) that the centralizer M of $\mathfrak{a}_{\mathfrak{p}}$ in K equals $M_0 K(\mathfrak{a}_{\mathfrak{p}})$, where M_0 is the identity component of M and $K(\mathfrak{a}_{\mathfrak{p}})$ is the finite group defined by $K \cap \exp(\sqrt{-1}\mathfrak{a}_{\mathfrak{p}})$.

We fix a G-invariant non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the Lie algebra such that $\langle \cdot, \cdot \rangle$ is negative definite on $\mathfrak{k} \equiv \mathfrak{g}^{\theta}$, positive definite on $\mathfrak{g}^{-\theta}$ and $\mathfrak{g}^{\theta} \perp \mathfrak{g}^{-\theta}$. Then $\langle \cdot, \cdot \rangle$ is non-degenerate on any θ-stable subspace of $\mathfrak{g}$, in particular, on the centralizer $\mathfrak{l}_{\sigma}$ of $\mathfrak{a}$ in $\mathfrak{g}$. Let $\mathfrak{z}(\mathfrak{l}_{\sigma})$ be the center of the Lie algebra $\mathfrak{l}_{\sigma}$. Obviously $\mathfrak{a} \subset \mathfrak{z}(\mathfrak{l}_{\sigma})$, but we need a more detailed description of $\mathfrak{l}_{\sigma}$. We set

$\mathfrak{a}_{\sigma} := \mathfrak{z}(\mathfrak{l}_{\sigma})^{-\theta} \equiv \{Y \in \mathfrak{z}(\mathfrak{l}_{\sigma}) : \theta Y = -Y\}$,

$\mathfrak{g}(\sigma)$: the semisimple ideal of $\mathfrak{l}_{\sigma}$ generated by $\mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; \lambda)$ with $\lambda|_{\mathfrak{a}} \not\equiv 0$.

$\mathfrak{m}(\sigma)$: the orthogonal complement of $\mathfrak{g}(\sigma) \oplus \mathfrak{a}_{\sigma}$ in $\mathfrak{l}_{\sigma}$.

Then one has $\mathfrak{g}(\sigma) \subset \mathfrak{h}$, $\mathfrak{m}(\sigma) \subset \mathfrak{m}$, and $\mathfrak{a} \subset \mathfrak{a}_\sigma \subset \mathfrak{a}_\mathfrak{p}$. The Levi subalgebra $\mathfrak{l}_\sigma$ is decomposed into the direct sum of σ-stable ideals:

$$\mathfrak{l}_\sigma = \mathfrak{m}(\sigma) \oplus \mathfrak{g}(\sigma) \oplus \mathfrak{a}_\sigma. \tag{5.1}$$

Let $G(\sigma)$, $M(\sigma)_0$, A_σ, and N_σ be the analytic subgroups of G with Lie algebra $\mathfrak{g}(\sigma)$, $\mathfrak{m}(\sigma)$, $\mathfrak{a}_\sigma$, and $\mathfrak{n}_\sigma$, respectively. We set $M(\sigma) := M(\sigma)_0 K(\mathfrak{a}_\mathfrak{p})$. Accordingly to the direct decomposition (5.1), one has $P_{G/H} = M(\sigma)G(\sigma)A_\sigma N_\sigma$ by [OS84, Lem. 8.12], see also [O88, Lem. 1.5]. We introduce a subset of $\mathrm{Irr}(P_{G/H})_f$ as follows.

Definition 5.6. Let Ξ be the collection of $\zeta \in \mathrm{Irr}(P_{G/H})_f$ of the form

$$\zeta(mxe^Y n) = \eta(m)e^{\mu(Y)} \quad \text{for } m \in M(\sigma),\ x \in G(\sigma),\ Y \in \mathfrak{a},\ n \in N_\sigma, \tag{5.2}$$

for some $\eta \in \mathrm{Irr}(M(\sigma))_f$ and $\mu \in \mathfrak{a}_\mathbb{C}^*$ such that η has a non-zero $(\mathfrak{m}(\sigma) \cap \mathfrak{h})$-fixed vector.

We are ready to state Oshima's embedding theorem in a slightly different formulation from the original:

Proposition 5.7 (T. Oshima)**.** *If* $\Pi \in \mathrm{Irr}(G)$ *occurs as a subquotient in* $C^\infty(G/H)$*, then there exists* $\zeta \in \Xi$ *such that* $\mathrm{Hom}_G(\Pi, \mathrm{Ind}_{P_{G/H}}^G(\zeta)) \neq \{0\}$.

Proof. By taking the "hyperfunction boundary-valued maps" [O88, Thm. 4.15], see also [KO13], one sees that there exists $\zeta \in \mathrm{Irr}(P_{G/H})_f$ such that Π is a subrepresentation of the (degenerate) principal series representation $\mathrm{Ind}_{P_{G/H}}^G(\zeta)$ with ζ satisfying certain additional constraints, which imply $\zeta \in \Xi$ by Proposition 4.1 and (4.5), loc. cit. $\square$

Proposition 5.7 generalizes Harish-Chandra's subquotient theorem and Casselman's subrepresentation theorem:

Example 5.8 (Casselman's subrepresentation theorem)**.** The group manifold $(G \times G)/\operatorname{diag} G$ is an example of a symmetric space. For any $\Pi \in \mathrm{Irr}(G)$, one has a natural embedding of the outer tensor product representation $\Pi \boxtimes \Pi^\vee$ into $C^\infty(G) \simeq C^\infty((G \times G)/\operatorname{diag} G)$ by taking matrix coefficients.

Let P_+ be a minimal parabolic subgroup of the group G, and P_- the opposite parabolic subgroup. Then $P_+ \times P_-$ is a minimal parabolic subgroup for the symmetric space $(G \times G)/\operatorname{diag} G$ in the sense of Definition 5.1. Therefore Proposition 5.7 tells that $\Pi \boxtimes \Pi^\vee$ is realized as a subrepresentation of some principal series representation of the direct product group $G \times G$, hence so is Π for G.

5.3 Proof of Theorem 5.4 and Proposition 5.5

A key lemma to derive Theorem 5.4 from Proposition 5.7 is the following. We recall that the Levi subgroup L_σ of $P_{G/H}$ is not necessarily connected.

Lemma 5.9. *Let $\zeta = \zeta_1 \oplus \cdots \oplus \zeta_r$ be the irreducible decomposition of $\zeta \in \Xi$ when restricted to the identity component $(L_\sigma)_0$ of the Levi subgroup L_σ. Let λ_i $(\in \widetilde{\mathfrak{j}}_{\mathbb{C}}^*)$ be the highest weight of ζ_i with respect to the positive system $\Delta^+(\mathfrak{l}_{\sigma\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}}) := \Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}}) \cap \Delta(\mathfrak{l}_{\sigma\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$. Then for any i $(1 \le i \le \lambda)$, λ_i vanishes on the subspace $\mathfrak{t} = \widetilde{\mathfrak{j}} \cap \mathfrak{h}$.*

Proof of Lemma 5.9. Accordingly to the ideal decomposition (5.1), $\mathfrak{t}$ is given as a direct sum:

$$\mathfrak{t} = (\mathfrak{t} \cap \mathfrak{m}(\sigma)) \oplus (\mathfrak{t} \cap \mathfrak{g}(\sigma)) \oplus (\mathfrak{a}_\sigma \cap \mathfrak{h}).$$

One sees readily from (5.2) that $\lambda_1, \cdots, \lambda_r$ vanish on $(\mathfrak{t} \cap \mathfrak{g}(\sigma)) \oplus (\mathfrak{a}_\sigma \cap \mathfrak{h})$. Let us prove that they also vanish on $\mathfrak{t} \cap \mathfrak{m}(\sigma)$. We denote by η_i $(1 \le i \le r)$ the restriction of the differential representation of ζ_i to $\mathfrak{m}(\sigma)$, which is still irreducible. Since $\zeta \in \Xi$, at least one of $\eta_j \in \mathrm{Irr}(\mathfrak{m}(\sigma))_f$ has an $(\mathfrak{m}(\sigma) \cap \mathfrak{h})$-fixed vector. Since $\mathfrak{m}(\sigma)$ is σ-stable, $(\mathfrak{m}(\sigma), \mathfrak{m}(\sigma) \cap \mathfrak{h})$ forms a symmetric pair, and $\widetilde{\mathfrak{j}} \cap \mathfrak{m}(\sigma) = (\mathfrak{t} \cap \mathfrak{m}(\sigma)) \oplus (\mathfrak{j} \cap \mathfrak{m}(\sigma))$ is a σ-split Cartan subalgebra of $\mathfrak{m}(\sigma)$ such that $\mathfrak{j} \cap \mathfrak{m}(\sigma)$ is a maximal abelian subspace in $\mathfrak{m}(\sigma)^{-\sigma}$. Applying the Cartan–Helgason theorem (Lemma 2.16) to the semisimple symmetric pair $(\mathfrak{m}(\sigma), \mathfrak{m}(\sigma) \cap \mathfrak{h})$, one obtains $\lambda_j|_{\mathfrak{t} \cap \mathfrak{m}(\sigma)} = 0$. Hence one has $\lambda_j|_{\mathfrak{t}} \equiv 0$.

Since $P_{G/H} = (P_{G/H})_0 K(\mathfrak{a}_\mathfrak{p})$ and since the finite group $K(\mathfrak{a}_\mathfrak{p}) = K \cap \exp(\sqrt{-1}\mathfrak{a}_\mathfrak{p})$ acts trivially on $\widetilde{\mathfrak{j}}$, one has $\lambda_1 = \cdots = \lambda_r$. Thus Lemma 5.9 is proved. □

We recall that $\mathfrak{b}_{G/H}$ is the parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ associated to $\sum^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$.

Lemma 5.10. *For any $\zeta \in \Xi$, both ζ and $\zeta^\vee$ belongs to* $\mathrm{Irr}(P_{G/H}; \mathfrak{b}_{G/H})_f$.

Proof. We observe that the σ-split Cartan subalgebra $\widetilde{\mathfrak{j}} = \mathfrak{t} + \mathfrak{j}$ of $\mathfrak{g}$ is also that of $\mathfrak{l}_\sigma$, and that $\mathfrak{b}_{G/H} \cap \mathfrak{l}_{\sigma\mathbb{C}}$ is a minimal parabolic subalgebra for the smaller symmetric pair $(\mathfrak{l}_\sigma, \mathfrak{l}_\sigma \cap \mathfrak{h})$, as $\mathfrak{b}_{G/H} \cap \mathfrak{l}_{\sigma\mathbb{C}}$ is the parabolic subalgebra of $\mathfrak{l}_{\sigma\mathbb{C}}$ associated to $\sum^+(\mathfrak{l}_{\sigma\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$.

Suppose $\zeta \in \Xi$. With the notation as in Lemma 5.9, one has $\lambda_i|_{\mathfrak{t}_{\mathbb{C}}} \equiv 0$ for all i $(1 \le i \le r)$. In turn, it follows from Lemma 2.6 that ζ_i belongs to $\mathrm{Irr}(\mathfrak{l}_\sigma; \mathfrak{b}_{G/H} \cap \mathfrak{l}_{\sigma\mathbb{C}})_f$, which is naturally identified with $\mathrm{Irr}(\mathfrak{p}_{G/H}; \mathfrak{b}_{G/H})_f$ by (2.3), hence $\zeta \in \mathrm{Irr}(P_{G/H}; \mathfrak{b}_{G/H})_f$.

By Lemma 2.17, if ζ_j has an $(\mathfrak{m}(\sigma) \cap \mathfrak{h})$-fixed vector for some j, then so does the contragredient representation $\zeta_j^\vee \in \mathrm{Irr}(\mathfrak{l}_\sigma)_f$. Thus the same argument as in Lemma 5.9 works, and one concludes $\zeta^\vee \in \mathrm{Irr}(P_{G/H}; \mathfrak{b}_{G/H})_f$, too. □

Proof of Theorem 5.4. Suppose $\Pi \in \mathrm{Irr}(G)_H$. Then the contragredient representation $\Pi^\vee$ in the category $\mathcal{M}(G)$ can be realized as a subrepresentation of $C^\infty(G/H)$ by the Frobenius reciprocity. By Oshima's embedding theorem (Proposition 5.7), one finds $\zeta \in \Xi$ such that $\mathrm{Hom}_G(\Pi^\vee, \mathrm{Ind}_{P_{G/H}}^G(\zeta)) \neq \{0\}$. Taking the dual in the category $\mathcal{M}(G)$, one has $\mathrm{Hom}_G(\mathrm{Ind}_{P_{G/H}}^G(\zeta^*), \Pi) \neq \{0\}$ where we set $\zeta^* := \zeta^\vee \otimes \mathbb{C}_{2\rho} \in \mathrm{Irr}(P_{G/H})_f$ as in (3.4). Since $\zeta^\vee \in \mathrm{Irr}(P_{G/H}; \mathfrak{b}_{G/H})_f$ by Lemma 5.10, so does ζ^*. Thus Theorem 5.4 is proved. □

Proof of Proposition 5.5. Since any irreducible subquotient $\Pi \in C^\infty(G/H)$ occurs in $\mathrm{Ind}_{P_{G/H}}^G(\zeta)$ for some $\zeta \in \mathrm{Irr}(P_{G/H})_f$ by Proposition 5.7, the conclusion follows from [K14, Lem. 6.7]. □

6 Proof of Theorems 1.4 and 1.5

In this section we complete the proof of Theorems 1.4 and 1.5. We retain the assumption that $G_\mathbb{C} \supset G'_\mathbb{C}$ are connected complex Lie groups and that G and G' are real forms of $G_\mathbb{C}$ and $G'_\mathbb{C}$, respectively. Since the equivalence between the sphericity (ii) and the strong visibility (iii) is known in this setting by [Tn21], we shall prove the equivalence between the bounded multiplicity property (i) and the sphericity (ii).

6.1 Proof of (ii) ⇒ (i) in Theorem 1.4

Suppose we are in the setting of Theorem 1.4. Let $B_{G/H}$ ($\subset G_\mathbb{C}$) be a Borel subgroup for the symmetric space G/H (Definition 1.3).

Assume that $G_\mathbb{C}/B_{G/H}$ is $G'_\mathbb{C}$-spherical. We take a minimal parabolic subgroup $P_{G/H}$ for the symmetric space G/H (Definition 5.1) such that $\mathfrak{b}_{G/H} \subset (\mathfrak{p}_{G/H})_\mathbb{C}$ as in Lemma 5.3.

Suppose $\Pi \in \mathrm{Irr}(G)_H$ and $\pi \in \mathrm{Irr}(G')$. By Theorem 5.4, one finds $\xi \in \mathrm{Irr}(P_{G/H}; \mathfrak{b}_{G/H})_f$ such that $\mathrm{Hom}_G(\mathrm{Ind}_{P_{G/H}}^G(\xi), \Pi) \neq \{0\}$, which induces an inclusion:

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \hookrightarrow \mathrm{Hom}_{G'}(\mathrm{Ind}_{P_{G/H}}^G(\xi)|_{G'}, \pi).$$

Thus the implication (ii) ⇒ (i) in Theorem 1.4 is deduced from the implication (iii) ⇒ (ii) in Theorem 4.2.

6.2 Proof of (i) ⇒ (ii) in Theorem 1.4

In this section, we give a proof for the implication (i) ⇒ (ii) in Theorem 1.4 by reducing it to the finite-dimensional case as formulated in Theorem 6.1 below.

Theorem 6.1. *Let (G, H) be a reductive symmetric pair, and $B_{G/H}$ a Borel subgroup for G/H (Definition 1.3). Suppose G' is an algebraic reductive subgroup of G. If $G_{\mathbb{C}}/B_{G/H}$ is not $G'_{\mathbb{C}}$-spherical, then one has*

$$\sup_{\Pi \in \mathrm{Irr}(G)_{H,f}} m(\Pi|_{G'}) = \infty.$$

We should remark that under the same assumption one has

$$\sup_{\Pi \in \mathrm{Irr}(G;\mathfrak{b}_{G/H})_f} m(\Pi|_{G'}) = \infty,$$

as was seen in Lemma 2.10. However, the set $\mathrm{Irr}(G)_{H,f}$ may be much smaller than $\mathrm{Irr}(G; \mathfrak{b}_{G/H})_f$, see Remark 2.15 and Lemma 2.16. Thus, the proof of Theorem 6.1 needs some further argument.

We recall from (2.5) that Π_λ denotes an irreducible finite-dimensional holomorphic representation of $G_{\mathbb{C}}$ having highest weight λ with respect to $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$. The same letter Π_λ is used to denote the restriction to the real form G. We begin with an elementary but useful observation:

Lemma 6.2. *Let $G_{\mathbb{C}} \supset G'_{\mathbb{C}}$ be a pair of connected complex reductive Lie groups. Suppose $\Pi_\lambda \in \mathrm{Irr}(G_{\mathbb{C}})_{\mathrm{hol}}$. Then one has*

$$m(\Pi_{\lambda+\nu}|_{G'_{\mathbb{C}}}) \geq m(\Pi_\lambda|_{G'_{\mathbb{C}}}) \tag{6.1}$$

for any $\Pi_\nu \in \mathrm{Irr}(G_{\mathbb{C}})_{\mathrm{hol}}$. Here we recall (2.7) for the definition of $m(\Pi_\lambda|_{G'_{\mathbb{C}}})$.

Proof. Without loss of generality, we may assume that $G_{\mathbb{C}}$ is simply connected. Let B be the Borel subgroup of $G_{\mathbb{C}}$ corresponding to the positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$.

We lift $\lambda \in \Lambda_+$ to a holomorphic character of the Cartan subgroup of $G_{\mathbb{C}}$, extend it to the opposite Borel subgroup B_- by letting the unipotent radical act trivially, and then form a $G_{\mathbb{C}}$-equivariant holomorphic line bundle $\mathcal{L}_\lambda = G_{\mathbb{C}} \times_{B_-} \mathbb{C}_\lambda$ over the flag variety $G_{\mathbb{C}}/B_-$ so that the Borel–Weil theorem gives a realization of Π_λ on $\mathcal{O}(G_{\mathbb{C}}/B_-, \mathcal{L}_\lambda)$. Likewise, we realize Π_ν on $\mathcal{O}(G_{\mathbb{C}}/B_-, \mathcal{L}_\nu)$.

For the subgroup $G'_{\mathbb{C}}$, we take a Cartan subalgebra of $\mathfrak{g}'_{\mathbb{C}}$, fix a positive system, write the Cartan–Weyl bijection (2.5) as $\mathrm{Irr}(\mathfrak{g}')_f \simeq \Lambda'_+$, $\pi_\mu \leftrightarrow \mu$, and denote by B' the Borel subgroup of $G'_{\mathbb{C}}$. We take $\pi_\mu \in \mathrm{Irr}(G')_f$ such that the multiplicity $[\Pi_\lambda|_{G'_{\mathbb{C}}} : \pi_\mu]$ attains its maximum $k := m(\Pi_\lambda|_{G'_{\mathbb{C}}})$. We also take $\pi_\tau \in \mathrm{Irr}(G')_f$ such that $[\Pi_\nu|_{G'_{\mathbb{C}}} : \pi_\tau] \neq 0$. Via the Borel–Weil realization, one finds holomorphic functions h_j $(1 \leq j \leq k)$ and $h \in \mathcal{O}(G_{\mathbb{C}})$ corresponding to highest weight vectors for the subgroup G' satisfying

$$\begin{aligned} h_j(b^{-1}gq) =& \mu(b)\lambda^{-1}(q)h_j(g), \qquad 1 \leq j \leq k, \\ h(b^{-1}gq) =& \tau(b)\nu^{-1}(q)h(g) \end{aligned}$$

for all $b \in B'$, $g \in G_{\mathbb{C}}$ and $q \in B_-$, where we use the same letters μ and τ to denote the holomorphic characters of B'. This shows that $h_j h$ $(1 \leq j \leq k)$ belong to $\mathcal{O}(G_{\mathbb{C}}/B_-, \mathcal{L}_{\lambda+\nu})$ on which $G_{\mathbb{C}}$ acts as the irreducible representation $\Pi_{\lambda+\nu}$, and they are highest weight vectors for the subgroup $G'_{\mathbb{C}}$ with the some highest weight $\mu + \tau$. Since $h_j h$ $(1 \leq j \leq k)$ are linearly independent, one concludes $[\Pi_{\lambda+\nu}|_{G'_{\mathbb{C}}} : \pi_{\mu+\tau}] \geq k$ $(= m(\Pi_\lambda|_{G'_{\mathbb{C}}}))$. Thus the lemma is proved. $\square$

Proof of Theorem 6.1. Since neither of the assumption nor the conclusion changes if we replace G by the identity component of the derived subgroup $[G, G]$, we may and do assume G is a connected semisimple Lie group. Moreover, we can assume G' is connected. We take compatible positive systems $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$ and $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ as in Section 2.5, and define a parabolic subalgebra $\mathfrak{q}$ associated to $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ as in Definition 1.3. Then $\mathfrak{q}$ is a Borel subalgebra for the symmetric space G/H. We denote by $\mathfrak{l}$ the centralizer of $\mathfrak{j}$ in $\mathfrak{g}$.

We begin with the case where G is contained in a simply connected complexification $G_{\mathbb{C}}$. Let Q be the parabolic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$, which is a Borel subgroup for G/H.

Suppose $G_{\mathbb{C}}/Q$ is not $G'_{\mathbb{C}}$-spherical. By Lemma 2.10, one finds $\lambda \in \Lambda_+$ such that $\Pi_{N\lambda} \in \mathrm{Irr}(G_{\mathbb{C}}; \mathfrak{q})_{\mathrm{hol}}$ and $m(\Pi_{N\lambda}|_{G'_{\mathbb{C}}}) \geq N + 1$ for all $N \in \mathbb{N}$. We recall $\widetilde{\mathfrak{j}} = \widetilde{\mathfrak{j}}^{\sigma} \oplus \widetilde{\mathfrak{j}}^{-\sigma} = \mathfrak{t} + \mathfrak{j}$ is a σ-split Cartan subalgebra and that the positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$ is compatible with $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. We also recall that any element $\nu \in \Lambda_+(\mathfrak{g}_{\mathbb{C}}; \mathfrak{h}_{\mathbb{C}})$ satisfies $\sigma\nu = -\nu$ by the definition (2.10). We take $\nu \in \Lambda_+(\mathfrak{g}_{\mathbb{C}}; \mathfrak{h}_{\mathbb{C}})$ which is regular enough, namely, $2\langle \nu, \alpha \rangle \geq \langle \sigma\lambda, \alpha \rangle$ for all $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$ such that $\alpha|_{\mathfrak{j}} \neq 0$. On the other hand, for $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$ with $\alpha|_{\mathfrak{j}} = 0$, one has $\langle \sigma\lambda, \alpha \rangle = \langle \lambda, \alpha \rangle = 0$ because such α is regarded as an element of $\Delta(\mathfrak{l}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$ whereas λ vanishes on $[\mathfrak{l}_{\mathbb{C}}, \mathfrak{l}_{\mathbb{C}}]$ as the differential of a character of Q. Thus $\langle 2\nu - \sigma\lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}}_{\mathbb{C}})$. We define a dominant integral weight by

$$\Lambda := \lambda + \nu - \sigma(\lambda + \nu) = \lambda + (2\nu - \sigma\lambda)$$

which vanishes on $\mathfrak{t}$ $(= \widetilde{\mathfrak{j}}^{\sigma})$. Thus one has $2\Lambda \in \Lambda_+(\mathfrak{g}_{\mathbb{C}}; \mathfrak{h}_{\mathbb{C}})$. Moreover the irreducible representation $\Pi_{2N\Lambda}$ of $G_{\mathbb{C}}$ satisfies $m(\Pi_{2N\Lambda}|_{G'_{\mathbb{C}}}) \geq m(\Pi_{2N\lambda}|_{G'_{\mathbb{C}}}) \geq 2N + 1$ for any $N \in \mathbb{N}$ when restricted to the subgroup $G'_{\mathbb{C}}$ by Lemma 6.2.

When $G_{\mathbb{C}}$ is not simply connected, we replace Λ by $k\Lambda$ where k is a positive integer given in Lemma 2.16. Then by the Cartan–Helgason theorem (Lemma 2.16), $\Pi_{2N\Lambda} \in \mathrm{Irr}(G)_H$. Thus Theorem 6.1 is shown. $\square$

6.3 Proof of Theorem 1.5

Suppose we are in the setting of Theorem 1.5. We observe that the direct product group $B_{G/H_1} \times B_{G/H_2}$ is a Borel subgroup for the reductive symmetric space $(G \times G)/(H_1 \times H_2)$, see Definition 1.3.

We also observe that the restriction of the outer tensor product representation $\Pi_1 \boxtimes \Pi_2$ of $G \times G$ to its subgroup $\mathrm{diag}\, G$ $(\simeq G)$ is nothing but the tensor product representation $\Pi_1 \otimes \Pi_2$. Hence Theorem 1.5 follows as a special case of Theorem 1.4 for the restriction $(G \times G) \downarrow \mathrm{diag}\, G$.

7 Classification of the triples (G, H, G')

Problem 1.2 asks a criterion for the triple $H \subset G \supset G'$ having the bounded multiplicity property (1.7) of the restriction, namely,

$$\sup_{\Pi \in \mathrm{Irr}(G)_H} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty, \tag{7.1}$$

where we recall $[\Pi|_{G'} : \pi] = \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$ from (1.1).

In this section we give a classification of the triples $H \subset G \supset G'$ in the setting where (G, H) and (G, G') are symmetric pairs. In order to state the classification, let us consider what will be a natural equivalence relation on triples (G, H, G') for this problem. First we observe that the bounded multiplicity property (7.1) does not change by taking finite coverings or the identity components of the groups. Second, it does not change also by replacing the two subgroups H and G' simultaneously with σH and $\sigma G'$, respectively, by an automorphism σ of the group G. Third, we observe a G-equivalence of the regular representations on $C^\infty(G/H) \simeq C^\infty(G/aHa^{-1})$ for any $a \in G$ and a G'-equivalence of the restriction $\Pi|_{G'} \simeq \Pi|_{bG'b^{-1}}$ via the inner automorphism $G' \simeq bG'b^{-1}$ for any $b \in G$. Hence we shall adopt the following definition of the infinitesimal equivalence in our classification of the triples (G, H, G') satisfying the bounded multiplicity property (7.1).

Definition 7.1 (equivalence of the triple (G, H, G'))**.** We say the triples $H \subset G \supset G'$ and $\widetilde{H} \subset \widetilde{G} \supset \widetilde{G}'$ are *infinitesimally equivalent* if there is an isomorphism $\mathfrak{g} \simeq \widetilde{\mathfrak{g}}$ between the Lie algebras of G and $\widetilde{G}$, and if via this identification, $\mathfrak{h}$ is conjugate to $\widetilde{\mathfrak{h}}$ by an inner automorphism and $\mathfrak{g}'$ is conjugate to $\widetilde{\mathfrak{g}}'$ by another inner automorphism.

The above definition says, in particular, that the triples $H_1 \subset G \supset G'_1$ and $H_2 \subset G \supset G'_2$ are infinitesimally equivalent if there exist $a, b \in G$ and $\sigma \in \operatorname{Aut}(\mathfrak{g})$ such that $\mathfrak{h}_2 = \sigma \operatorname{Ad}(a)\mathfrak{h}_1$ and $\mathfrak{g}'_2 = \sigma \operatorname{Ad}(b)\mathfrak{g}'_1$. We note that σ may be an outer automorphism.

In what follows, we shall assume that at least one of the symmetric spaces G/H or G/G' is irreducible. This means that we shall treat the following cases:

Case I. G is simple,
Case II. G/H is a group manifold $({}^\backprime G \times {}^\backprime G)/\operatorname{diag}{}^\backprime G$ for simple ${}^\backprime G$,
Case III. G/G' is a group manifold $({}^\backprime G \times {}^\backprime G)/\operatorname{diag}{}^\backprime G$ for simple ${}^\backprime G$.

Case II deals with the representation $\Pi = \tau \boxtimes \tau^\vee$ of $G = {}^\backprime G \times {}^\backprime G$ for some $\tau \in \operatorname{Irr}({}^\backprime G)$, whereas the restriction of representations $\Pi|_{G'}$ in Case III is nothing but the tensor product of two irreducible representations Π_1 and Π_2 of ${}^\backprime G$ when $\Pi \in \operatorname{Irr}(G)$ is of the form $\Pi_1 \boxtimes \Pi_2$. We note that there is an overlap between Case II and Case III.

Finally, our criterion in Theorem 1.4 tells that it suffices to classify the triples of the *complexified Lie algebras* $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$. With these observations, the classification of the triples (G, H, G') will be given in Theorems 7.2 and 7.6 for Case I, in Theorem 7.8 for Case II, and in Theorem 7.9 for Case III.

The proof of these theorems will be given in Section 8.

7.1 Classification of (G, H, G') with G simple

In this subsection we deal with the case that G is a simple Lie group. The classification of the triples (G, H, G') having the bounded multiplicity property is given in Theorem 7.2 when $\mathfrak{g}$ is not a complex Lie algebra (equivalently, the complexification $\mathfrak{g}_{\mathbb{C}}$ is simple) and Theorem 7.6 when $\mathfrak{g}$ is a complex simple Lie algebra up to the infinitesimal equivalence in Definition 7.1.

Theorem 7.2. *Suppose that $\mathfrak{g}_{\mathbb{C}}$ is simple and that (G, H) and (G, G') are symmetric pairs. Then the bounded multiplicity property* (7.1) *holds for the triple (G, H, G') if and only if the complexified Lie algebras $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ are in Table 7.1. In the table, p, q are arbitrary subject to $n = p + q$.*

$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{h}_{\mathbb{C}}$	$\mathfrak{g}'_{\mathbb{C}}$
$\mathfrak{sl}_n$	$\mathfrak{gl}_{n-1}$	$\mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus \mathbb{C}$
$\mathfrak{sl}_{2m}$	$\mathfrak{gl}_{2m-1}$	$\mathfrak{sp}_m$
$\mathfrak{sl}_6$	$\mathfrak{sp}_3$	$\mathfrak{sl}_4 \oplus \mathfrak{sl}_2 \oplus \mathbb{C}$
$\mathfrak{so}_n$	$\mathfrak{so}_{n-1}$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$
$\mathfrak{so}_{2m}$	$\mathfrak{so}_{2m-1}$	$\mathfrak{gl}_m$
$\mathfrak{so}_{2m}$	$\mathfrak{so}_{2m-2} \oplus \mathbb{C}$	$\mathfrak{gl}_m$
$\mathfrak{sp}_n$	$\mathfrak{sp}_{n-1} \oplus \mathfrak{sp}_1$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_q$
$\mathfrak{sp}_n$	$\mathfrak{sp}_{n-2} \oplus \mathfrak{sp}_2$	$\mathfrak{sp}_{n-1} \oplus \mathfrak{sp}_1$
$\mathfrak{e}_6$	$\mathfrak{f}_4$	$\mathfrak{so}_{10} \oplus \mathbb{C}$
$\mathfrak{f}_4$	$\mathfrak{so}_9$	$\mathfrak{so}_9$

$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{h}_{\mathbb{C}}$	$\mathfrak{g}'_{\mathbb{C}}$
$\mathfrak{sl}_n$	$\mathfrak{so}_n$	$\mathfrak{gl}_{n-1}$
$\mathfrak{sl}_{2m}$	$\mathfrak{sp}_m$	$\mathfrak{gl}_{2m-1}$
$\mathfrak{sl}_n$	$\mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus \mathbb{C}$	$\mathfrak{gl}_{n-1}$
$\mathfrak{so}_n$	$\mathfrak{so}_p \oplus \mathfrak{so}_q$	$\mathfrak{so}_{n-1}$
$\mathfrak{so}_{2m}$	$\mathfrak{gl}_m$	$\mathfrak{so}_{2m-1}$

Table 7.1: Triples $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ with $\mathfrak{g}_{\mathbb{C}}$ simple in Theorem 1.4

Remark 7.3. Since the classification is given by the infinitesimal equivalence given in Definition 7.1, we have omitted some cases such as

$$
\begin{aligned}
(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}) &= (\mathfrak{so}_8, \mathfrak{spin}_7) && \sim (\mathfrak{so}_8, \mathfrak{so}_7),\\
(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}) &= (\mathfrak{so}_8, \mathfrak{gl}_4, \mathfrak{so}_6 \oplus \mathfrak{so}_2) && \sim (\mathfrak{so}_8, \mathfrak{so}_6 \oplus \mathfrak{so}_2, \mathfrak{gl}_4),\\
(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}) &= (\mathfrak{sl}_4, \mathfrak{sp}_2, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathbb{C}) && \sim (\mathfrak{so}_6, \mathfrak{so}_5, \mathfrak{so}_4 \oplus \mathfrak{so}_2),
\end{aligned}
$$

because the right-hand sides appear as special cases of the more general family in Table 7.1.

From the classification in Theorem 7.2, one obtain the following:

Corollary 7.4. *Suppose G/H is a symmetric space of rank one, and $\mathfrak{g}_{\mathbb{C}}$ is simple. Then for any symmetric pair (G, G') the bounded multiplicity property* (7.1) *holds except for the following two cases:*

$$(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}) = (\mathfrak{sl}_n, \mathfrak{gl}_{n-1}, \mathfrak{so}_n),\ (\mathfrak{sp}_n, \mathfrak{sp}_{n-1} \oplus \mathfrak{sp}_1, \mathfrak{gl}_n),\ \textit{or}\ (\mathfrak{f}_4, \mathfrak{so}_9, \mathfrak{sp}_3 \oplus \mathfrak{sl}_2).$$

We shall examine in Example 9.5 the failure of (7.1) in the simplest exceptional case $(\mathfrak{sl}_3, \mathfrak{gl}_2, \mathfrak{so}_3)$ by an explicit computation of multiplicities.

Theorem 7.2 was announced in [K21, Cor. 7.8]. The right-hand side of Table 7.1 collects the case (1.4) where a stronger bounded multiplicity property (1.2) holds for the restriction $G \downarrow G'$ independently of H, see [KO13, Thm. D]. The left-hand side includes:

Example 7.5. Suppose $p_1 + p_2 = p$ and $q_1 + q_2 = q$. We set $G/H := O(p,q)/O(p,q-1)$, $G :'= O(p_1, q_1) \times O(p_2, q_2)$. Then the triple (G, H, G') satisfies the bounded multiplicity property (1.7) as the triple $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ of the complexified Lie algebras occurs in the fourth row of the left-hand column in Table 7.1. The branching laws of the restriction $\Pi|_{G'}$ of a discrete series representation Π for G/H have been studied in [K93, K21, MO15, ØS19], see Example 9.1 for some more details.

Next we consider the classification when G has a complex structure.

Theorem 7.6. *Suppose that G is a complex simple Lie group, and that both (G, H) and (G, G') are symmetric pairs. Then the bounded multiplicity property* (7.1) *holds if and only if one of the following conditions hold:*

- **a.** *Both $\mathfrak{h}$ and $\mathfrak{g}'$ are complex Lie subalgebras and the triple $(\mathfrak{g}, \mathfrak{h}, \mathfrak{g}')$ are in Table 7.1.*
- **b.** $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}_n(\mathbb{C}), \mathfrak{so}_{n-1}(\mathbb{C}))$ *and $\mathfrak{g}'$ is any real form of $\mathfrak{g}$.*
- **c.** $(\mathfrak{g}, \mathfrak{g}')$ *is* $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{gl}_{n-1}(\mathbb{C}))$ $(n \geq 2)$ *or* $(\mathfrak{so}_n(\mathbb{C}), \mathfrak{so}_{n-1}(\mathbb{C}))$ $(n \geq 5)$, *and $\mathfrak{h}$ is any real form of $\mathfrak{g}$.*
- **d.** $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, *and both $\mathfrak{h}$ and $\mathfrak{g}'$ are any real forms of $\mathfrak{g}$.*

Remark 7.7. As in Remark 7.3, we omit some triples (G, H, G') for which the infinitesimally equivalent ones are in the list.

7.2 Classification of (G, H, G'): group manifold case

In this subsection we give a classification of the triples (G, H, G') having the bounded multiplicity property (1.7) in the setting where G/H is a group manifold, namely, G is the direct product group ${}^{\backprime}G \times {}^{\backprime}G$ for some simple Lie group ${}^{\backprime}G$ and H is of the form $\mathrm{diag}\,{}^{\backprime}G$. We refer to this as the "group manifold case".

In what follows, we use the letter G to denote a simple Lie group instead of ${}^{\backprime}G$. Our task now is to classify the symmetric pair $(G \times G, G')$ such that the following bounded multiplicity property holds:

$$\sup_{\Pi \in \mathrm{Irr}(G\times G)_{\mathrm{diag}\,G}} \ \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty, \tag{7.2}$$

where we recall $[\Pi|_{G'} : \pi] = \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$. Here we note that $\Pi \in \mathrm{Irr}(G \times G)_{\mathrm{diag}\,G}$ if and only if Π is of the form $\tau \boxtimes \tau^{\vee}$ for some $\tau \in \mathrm{Irr}(G)$ where $\tau^{\vee}$ is the contragredient representation of τ in the category $\mathcal{M}(G)$.

Up to the infinitesimal equivalence as in Definition 7.1, there are two possibilities for the subgroup G' in $G \times G$:

Case II-1. $G' = G_1 \times G_2$ where (G, G_j) $(j = 1, 2)$ are symmetric pairs.

Case II-2. $G' = \mathrm{diag}_{\sigma}(G)$ where σ is an involutive automorphism ${}^{\backprime}G$.

We shall see that Case II-2 occurs in the classification below only when either $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Theorem 7.8 (group manifold case)**.** *Let G be a simple Lie group.*

(1) *Suppose $G' = G_1 \times G_2$ where (G, G_j) $(j = 1, 2)$ are symmetric pairs. Then the bounded multiplicity property* (7.2) *holds if and only if* $(\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2)$ *or* $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{1\mathbb{C}}, \mathfrak{g}_{2\mathbb{C}})$ *are isomorphic to* $(\mathfrak{sl}_n, \mathfrak{gl}_{n-1}, \mathfrak{gl}_{n-1})$ *or* $(\mathfrak{so}_n, \mathfrak{so}_{n-1}, \mathfrak{so}_{n-1})$.

(2) *Suppose $G' := \mathrm{diag}_{\sigma}(G)$ where σ is an involutive automorphism of G. Then* (7.2) *holds if and only if $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ or $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$ (σ is arbitrary).*

Theorem 7.8 (1) includes the following assertion: *when G is a complex Lie group and $G' = G_1 \times G_2$ with at least one of G_j being a real form of G, then the bounded multiplicity property* (7.2) *holds if and only if* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

7.3 Classification of (G, H, G'): tensor product case

In this subsection we give a classification of the triples (G, H, G') having the bounded multiplicity property (1.7) in Case III, that is, in the setting where $(G, G') = ({}^\backprime G \times {}^\backprime G, \operatorname{diag} {}^\backprime G)$ for some simple Lie group ${}^\backprime G$. In this case, the restriction of an irreducible representation of G to the subgroup G' is nothing but the tensor product of two irreducible representations of ${}^\backprime G$. We refer to this as the "tensor product case".

In what follows, we use the letter G instead of ${}^\backprime G$. Our task now is to classify the symmetric pairs $(G \times G, H)$ having the following bounded multiplicity property: $\sup_{(\Pi_1,\Pi_2)} m(\Pi_1 \otimes \Pi_2) < \infty$, see (1.8), namely,

$$\sup_{(\Pi_1,\Pi_2)} \sup_{\Pi \in \operatorname{Irr}(G)} \dim_{\mathbb{C}} \operatorname{Hom}_G(\Pi_1 \otimes \Pi_2, \Pi) < \infty \tag{7.3}$$

where the first supremum is taken over all the pairs (Π_1, Π_2) with $\Pi_1, \Pi_2 \in \operatorname{Irr}(G)$ subject to $\Pi_1 \boxtimes \Pi_2 \in \operatorname{Irr}(G \times G)_H$.

Up to the infinitesimal equivalence given as in Definition 7.1, there are two possibilities of the subgroup H in $G \times G$:

Case III-1. $H = H_1 \times H_2$ where (G, H_j) $(j = 1, 2)$ are symmetric pairs,

Case III-2. H is of the form $\operatorname{diag}_\sigma(G)$ where σ is an involution of G.

We note that there is an overlap between Case II-2 and Case III-2. We shall see in Theorem 7.9 below that Case III-2 occurs in the classification only when either $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Theorem 7.9 (Tensor product case)**.** *Suppose that G is a simple Lie group, and $(G \times G, H)$ is a symmetric pair. Then the bounded multiplicity property* (7.3) *holds if and only if one of the following conditions hold:*

(1) *Either $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}}$ is $\mathfrak{sl}_2(\mathbb{C})$. No condition on $\mathfrak{h}$.*

(2) *Suppose $\mathfrak{g}_{\mathbb{C}}$ is simple and $\mathfrak{g}_{\mathbb{C}} \neq \mathfrak{sl}_2(\mathbb{C})$. Then the bounded multiplicity property* (7.3) *holds if and only if H is of the form $H = H_1 \times H_2$ and the triple $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ is in the following table.*

$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{h}_{1\mathbb{C}}$	$\mathfrak{h}_{2\mathbb{C}}$
$\mathfrak{so}_n$	$\mathfrak{so}_{n-1}$	$\mathfrak{so}_{n-1}$
$\mathfrak{so}_8$	$\mathfrak{so}_7$	$\mathfrak{spin}_7$
$\mathfrak{so}_8$	$\mathfrak{so}_7$	$\mathfrak{gl}_4$
$\mathfrak{sl}_4$	$\mathfrak{gl}_3$	$\mathfrak{sp}_2$

Table 7.2: Tensor product with bounded multiplicities

(3) *Suppose* $\mathfrak{g}$ *is a complex simple Lie algebra and* $\mathfrak{g} \neq \mathfrak{sl}_2(\mathbb{C})$. *Then the bounded multiplicity property* (7.3) *holds if and only if* $\mathfrak{h}$ *is a complex subalgebra and* H *is of the form* $H = H_1 \times H_2$ *and the triple* $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ *is in Table 7.2.*

Remark 7.10. (1) The infinitesimal equivalence in Definition 7.1 includes the switch of factors H_1 and H_2 because it is induced by an outer automorphism of the direct product group.
(2) Inside the Lie algebra $\mathfrak{so}_8$, there are three subalgebras that are isomorphic to $\mathfrak{so}_7$ up to inner automorphisms, to which we may refer as $\mathfrak{so}_7$, $\mathfrak{spin}_7^+$, and $\mathfrak{spin}_7^-$, and these are conjugate to each other by an outer automorphism of $\mathfrak{so}_8$. By the equivalence in Definition 7.1, there are two equivalence classes of the triples $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{1\mathbb{C}}, \mathfrak{h}_{2\mathbb{C}})$ where $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_8$ and $\mathfrak{h}_{j\mathbb{C}}$ are isomorphic to $\mathfrak{so}_7$ $(j = 1, 2)$, according to whether $\mathfrak{h}_{1\mathbb{C}}$ is conjugate to $\mathfrak{h}_{2\mathbb{C}}$ by an inner automorphism or not. In Table 7.2, we write them as $(\mathfrak{so}_8, \mathfrak{so}_7, \mathfrak{so}_7)$ and $(\mathfrak{so}_8, \mathfrak{so}_7, \mathfrak{spin}_7)$, respectively.

8 Proof of classification results

By applying the geometric criteria in Theorems 1.4 and 1.5, we complete the proof of Theorems 7.2, 7.6, and 7.8 about the classification of the triples (G, H, G') having the bounded multiplicity property (1.7) of the restriction $\Pi|_{G'}$ for all $\Pi \in \mathrm{Irr}(G)_H$ and also the proof of Theorem 7.9 for the tensor product case.

8.1 Preliminary lemmas

In this subsection we collect some lemmas that we shall use in the proof.

Given two G-spaces X_j ($j = 1, 2$) and an automorphism σ of G, we let $g \in G$ act on the direct product space $X_1 \times X_2$ by $(x, y) \mapsto (gx, \sigma(g)y)$, and call it the *$\sigma$-twisted diagonal action* of G.

Lemma 8.1. *Let $G_{\mathbb{C}}$ be a complex simple Lie group, and G_U a maximal compact subgroup of $G_{\mathbb{C}}$. Suppose that Q_1, Q_2 are parabolic subgroups of $G_{\mathbb{C}}$, and we set $X = G_{\mathbb{C}}/Q_1 \times G_{\mathbb{C}}/Q_2$. Let σ be an automorphism of $G_{\mathbb{C}}$. Then the following four conditions are equivalent:*

(i) *X is $G_{\mathbb{C}}$-spherical via the diagonal action;*

(i)′ *X is strongly G_U-visible via the diagonal action;*

(ii) *X is $G_{\mathbb{C}}$-spherical via the σ-twisted diagonal action;*

(ii)′ *X is strongly G_U-visible via the σ-twisted diagonal action.*

Proof. See [Tn21] for the equivalences (i) $\iff$ (i)′ and (ii) $\iff$ (ii)′. The equivalences (i) $\iff$ (ii) and (i)′ $\iff$ (ii)′ are obvious when σ is an inner automorphism, and follow from the classification of strongly visible actions [Tn12] for (i)′ $\iff$ (ii)′, or alternatively that of spherical varieties [HNOO13, Thm. 5.2] for (i) $\iff$ (ii) when σ is an outer automorphism. □

Since our criteria in Theorems 1.4 and 1.5 are formulated by the complexified Lie group $G_{\mathbb{C}}$, it is convenient to fix our convention when G itself has a complex structure. Suppose G is a complex Lie group. We write J for the complex structure on $\mathfrak{g}$, and decompose $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ into the direct sum of the eigenspaces $\mathfrak{g}^{\mathrm{hol}}$ and $\mathfrak{g}^{\mathrm{anti}}$ of J with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Then one has a direct sum decomposition:

$$\mathfrak{g} \oplus \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^{\mathrm{hol}} \oplus \mathfrak{g}^{\mathrm{anti}} = \mathfrak{g}_{\mathbb{C}}, \quad (X, Y) \mapsto \frac{1}{2}(X - \sqrt{-1}JX, Y + \sqrt{-1}JY).$$

Accordingly, the complexification $G_{\mathbb{C}}$ of the complex Lie group G is given by the totally real embedding

$$\mathrm{diag}\colon G \hookrightarrow G \times G =: G_{\mathbb{C}}, \tag{8.1}$$

where the second factor equipped with the reverse complex structure.

For a connected complex simple Lie group G, there are two types for symmetric pairs (G, H) defined by an involution σ of G:

(1) (σ is holomorphic) H is a complex subgroup of G,

(2) (σ is anti-holomorphic) H is a real form of G.

For simplicity, suppose that the subgroup H is connected. Then via the identification $G_{\mathbb{C}} \simeq G \times G$ in (8.1), one has

$$\begin{aligned} H_{\mathbb{C}} &\simeq H \times H && \text{for (1)}, \\ H_{\mathbb{C}} &\simeq \operatorname{diag}_\sigma(G) := \{(g, \sigma g) : g \in G\} && \text{for (2)}. \end{aligned}$$

Correspondingly, the Borel subgroup $B_{G/H}$ for the symmetric space G/H (Definition 1.3) is given as follows.

Lemma 8.2. *Suppose G is a complex simple Lie group, and G/H a symmetric space defined by an involutive automorphism σ of G. Let $B_{G/H}$ be a Borel subgroup for G/H as a (real) symmetric space, which is regarded as a subgroup of $G \times G$ via the identification $G_{\mathbb{C}} \simeq G \times G$ in* (8.1)*.*

(1) *If σ is holomorphic, then G/H is a complex symmetric space. We write $B^c_{G/H}$ for the Borel subgroup for the complex symmetric space G/H, see below. Then $B_{G/H}$ is isomorphic to $B^c_{G/H} \times B^c_{G/H}$.*

(2) *If σ is anti-holomorphic, then H is a real form of G and $B_{G/H}$ is isomorphic to $B \times B$ where B is a Borel subgroup of the complex Lie group G.*

Here, a Borel subgroup $B^c_{G/H}$ for the *complex* symmetric space G/H is defined as a parabolic subgroup of G, rather than that of the complexification $G_{\mathbb{C}}$. That is, when G itself is a complex reductive Lie group and σ is a holomorphic involution, we define $B^c_{G/H}$ to be the parabolic subgroup of G associated to $\Sigma^+(\mathfrak{g}, \mathfrak{j})$ instead of that of $G_{\mathbb{C}}$ associated to $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ in Definition 1.3.

The case $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$ is distinguished in the classification from other cases, for which we formulate in the following two lemmas.

Lemma 8.3. *Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$, and B a Borel subgroup of G. Then the following three conditions are equivalent:*

(i) *$(G \times G)/(B \times B)$ is G-spherical via the diagonal action.*

(ii) *$(G \times G \times G)/\operatorname{diag}(G)$ is spherical as a $(G \times G \times G)$-space.*

(iii) $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

Proof. The first equivalence (i) $\iff$ (ii) is clear from the bijection:

$$\operatorname{diag} B\backslash(G \times G)/(B \times B) \simeq (B \times B \times B)\backslash(G \times G \times G)/\operatorname{diag} G.$$

See [K95, Ex. 2.8.6] or [KM14, Prop. 4.3] *e.g.*, for the equivalence (ii) $\iff$ (iii). □

Lemma 8.4. *Let G be a semisimple Lie group, $G_{\mathbb{C}}$ a complexification of G, and B a Borel subgroup of $G_{\mathbb{C}}$. Assume that $\mathfrak{g}_{\mathbb{C}}$ is a direct sum of copies of $\mathfrak{sl}_2(\mathbb{C})$. Then for any symmetric pair (G, G'), $G_{\mathbb{C}}/B$ is $G'_{\mathbb{C}}$-spherical.*

Proof. It suffices to show when the symmetric pair (G, G') is irreducible, namely, either $\mathfrak{g}$ is simple or $(\mathfrak{g}, \mathfrak{g}') = ({}'\mathfrak{g} \oplus {}'\mathfrak{g}, \operatorname{diag}{}'\mathfrak{g})$ for simple ${}'\mathfrak{g}$. The assertion is straightforward in the first case where $G_{\mathbb{C}}/B \simeq \mathbb{P}^1\mathbb{C}$, and follows from the implication (iii) $\Rightarrow$ (i) and (ii) of Lemma 8.3 in the second case. □

8.2 Proof of Theorem 7.2 ($\mathfrak{g}_{\mathbb{C}}$ simple)

In this subsection, we give a proof of Theorem 7.2 which deals with the case that G is a simple Lie group and G is not a complex Lie group, namely, the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is simple.

Let G/H and G/G' be symmetric spaces, and $B_{G/H}$ ($\subset G_{\mathbb{C}}$) a Borel subgroup for G/H. The proof of Theorem 7.2 is based on the criterion in Theorem 1.4 for the bounded multiplicity property (1.7), which we observe is determined only by the complexified Lie algebras $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{h}_{\mathbb{C}}$, and $\mathfrak{g}'_{\mathbb{C}}$. Then our strategy to classify the triple $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ is to fix a complex symmetric pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ and to classify $\mathfrak{h}_{\mathbb{C}}$ such that $G_{\mathbb{C}}/B_{G/H}$ is $G'_{\mathbb{C}}$-spherical, which is divided into two steps.

Step 1. Classify parabolic subgroups P of a complex simple Lie group $G_{\mathbb{C}}$ such that $G_{\mathbb{C}}/P$ is $G'_{\mathbb{C}}$-spherical or equivalently, is G'_U-strongly visible.

Step 2. Classify the complex symmetric pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ such that the Borel subgroup $B_{G/H}$ appears in the list obtained in Step 1.

Step 1 is done in [HNOO13, Thm. 5.2]. See also [K05, Tn12] for some classification results of strongly visible actions.

Proof of Theorem 7.2. By the above argument, it suffices is to carry out a computation in Step 2 for each complex symmetric pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ with $\mathfrak{g}_{\mathbb{C}}$ simple. We illustrate this computation in the following setting:

$$(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}) = (\mathfrak{sl}_n, \mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus \mathbb{C}) \text{ with } n = p + q. \tag{8.2}$$

Take a subset Θ of the set $\{\alpha_1, \cdots, \alpha_{n-1}\}$ of simple roots for $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \widetilde{\mathfrak{j}_{\mathbb{C}}})$ labelled as in Bourbaki [Br02]. We write Θ^c for the complement of Θ, and P^Θ for the parabolic subgroup of $G_{\mathbb{C}}$ corresponding to Θ. We recall our convention that P^Θ is a Borel subgroup of $G_{\mathbb{C}}$ if Θ is empty. Then $G_{\mathbb{C}}/P^\Theta$ is $G'_{\mathbb{C}}$-spherical ([HNOO13, Thm. 5.2]), or equivalently, G'_U-strongly visible ([K07b, Thm. A], see also [K05, Thm. 16]) if and only if Θ satisfies one of the following conditions:

Case 1. $\#\Theta^c \leq 1$,

Case 2. $\Theta^c = \{\alpha_1, \alpha_i\}$, $\{\alpha_i, \alpha_{i+1}\}$, $\{\alpha_i, \alpha_{n-2}\}$ for some i,

Case 3. $\min(p, q) = 2$ and $\#\Theta^c = 2$,

Case 4. $\min(p, q) = 1$ and Θ is arbitrary.

As a second step, we now examine if the corresponding parabolic subalgebra $\mathfrak{p}^\Theta$ is isomorphic to the Borel subalgebra of some symmetric pair $(\mathfrak{g}, \mathfrak{h})$. Suppose $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair. Let $\mathfrak{g}_{\mathbb{R}}$ be the real form of $\mathfrak{g}_{\mathbb{C}}$ corresponding to the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ as in (2.9), and Θ the set of the black circles in the Satake diagram of $\mathfrak{g}_{\mathbb{R}}$, see [He78, Ch. X, Table VI], for instance. By Lemma 2.13, the complex parabolic subalgebra $\mathfrak{p}^\Theta$ is a Borel subalgebra $\mathfrak{b}_{G/H}$ for the symmetric pair $(\mathfrak{g}, \mathfrak{h})$. Among real forms $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{su}(p, n-p)$, and $\mathfrak{su}^*(n)$ (n: even) of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_n(\mathbb{C})$, the number of the white circles ($= \#\Theta^c$) in the Satake diagram is equal to 1 or 2 if and only if the real form $\mathfrak{g}_{\mathbb{R}}$ is isomorphic to one of the following:

$$\mathfrak{sl}(n, \mathbb{R}) \ (n = 2, 3); \ \mathfrak{su}^*(2m) \ (m = 2, 3); \ \text{or } \mathfrak{su}(1, n-1), \tag{8.3}$$

and correspondingly, the set of the white circles is given by

$$\{\alpha_1\}, \{\alpha_1, \alpha_2\}; \ \{\alpha_2\}, \{\alpha_2, \alpha_4\}; \ \{\alpha_1, \alpha_{n-1}\},$$

respectively. Therefore $G_{\mathbb{C}}/P^\Theta$ is $G'_{\mathbb{C}}$-spherical if and only if one of the following conditions holds:

Case 1. $\mathfrak{g}_{\mathbb{R}} \simeq \mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{su}^*(4)$,

Case 2. $\mathfrak{g}_{\mathbb{R}} \simeq \mathfrak{sl}(3, \mathbb{R})$ or $\mathfrak{su}(1, n-1)$,

Case 3. $\min(p, q) = 2$ and $\mathfrak{g}_{\mathbb{R}}$ is one of (8.3),

Case 4. $\min(p, q) = 1$ and $\mathfrak{g}$ is arbitrary.

This exhausts the list in Table 7.1 with $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}) = (\mathfrak{sl}_n, \mathfrak{sl}_p \oplus \mathfrak{sl}_q \oplus \mathbb{C})$ except for the triple $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}) = (\mathfrak{sl}_4, \mathfrak{sp}_2, \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathbb{C})$, which we have omitted from the table because it is isomorphic to a special case of the triple $(\mathfrak{so}_{a+b}, \mathfrak{so}_{a+b-1}, \mathfrak{so}_a \oplus \mathfrak{so}_b)$ with $(a, b) = (2, 4)$.

For other symmetric pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$, the proof of the classification $\mathfrak{h}_{\mathbb{C}}$ is similar (and often simpler). $\square$

8.3 Proof of Theorem 7.6 ($\mathfrak{g}$ complex simple)

In this section, we give a proof of Theorem 7.6 which deals with the case that G is a complex simple Lie group. Since the Lie algebra $\mathfrak{g}$ has a complex structure, there are four possibilities for the symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}, \mathfrak{g}')$:

Case I^c-a. Both $\mathfrak{h}$ and $\mathfrak{g}'$ are complex subalgebras.

Case I^c-b. $\mathfrak{h}$ is a complex subalgebra and $\mathfrak{g}'$ is a real form of $\mathfrak{g}$.

Case I^c-c. $\mathfrak{h}$ is a real form of $\mathfrak{g}$, and $\mathfrak{g}'$ is a complex subalgebra.

Case I^c-d. Both $\mathfrak{h}$ and $\mathfrak{g}'$ are real forms of $\mathfrak{g}$.

Proof of Theorem 7.6. The classification in Case I^c-a goes similarly to the aforementioned proof of Theorem 7.2. By Lemma 8.1, the classification in Case I^c-b is equivalent to the special case of Theorem 7.9 (2) with $\mathfrak{h}_1 = \mathfrak{h}_2$, which will be proved in Section 8.5. The classification in Case I^c-c is reduced to the classification of the pairs $(\mathfrak{g}, \mathfrak{g}')$ such that $(G_{\mathbb{C}} \times G_{\mathbb{C}})/(B \times B)$ is $(G'_{\mathbb{C}} \times G'_{\mathbb{C}})$-spherical, or equivalently, $G_{\mathbb{C}}/B$ is $G'_{\mathbb{C}}$-spherical. This is classified as in (1.4) by Krämer [Kr76]. The pair $(\mathfrak{so}_8, \mathfrak{spin}_7)$ is not listed in Theorem 7.6 because the classification is listed up to the equivalence in Definition 7.1. The classification in Case I^c-d is equivalent to that of $\mathfrak{g}$ such that $(G_{\mathbb{C}} \times G_{\mathbb{C}})/(B \times B)$ is $G_{\mathbb{C}}$-spherical under the diagonal action of $G_{\mathbb{C}}$. Then $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{sl}_2(\mathbb{C})$ by Lemma 8.3. $\square$

8.4 Proof of Theorem 7.8 for group manifold case

Let G be a simple Lie group. In this section we complete the proof of Theorem 7.8 by classifying symmetric pairs $(G \times G, G')$ having the bounded multiplicity property (1.7). We need to treat the following cases.

Case II-1. $\mathfrak{g}' = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2$ such that $(\mathfrak{g}, \mathfrak{g}'_j)$ $(j = 1, 2)$ are symmetric pairs.

  Case II^r-1. $\mathfrak{g}_{\mathbb{C}}$ is simple.

  Case II^c-1. $\mathfrak{g}$ is a complex simple Lie algebra.

Case II-2. $\mathfrak{g}' = \mathrm{diag}_\sigma(\mathfrak{g})$ for some involutive automorphism σ of $\mathfrak{g}$.

  Case II^r-2. $\mathfrak{g}_{\mathbb{C}}$ is simple.

  Case II^c-2. $\mathfrak{g}$ is a complex simple Lie algebra.

We recall that for $\mathfrak{g}$ a simple Lie algebra, $\mathfrak{g}$ is a complex simple Lie algebra if and only if $\mathfrak{g}_{\mathbb{C}}$ is not simple.

Proof of Theorem 7.8. We note that the Borel subgroup for the symmetric space $(G \times G)/\operatorname{diag} G$ is given by $B \times B_-$, where B is a Borel subgroup of $G_{\mathbb{C}}$ and B_- its opposite Borel subgroup. Since B_- is conjugate by B, we shall simply use $B \times B$ instead of $B \times B_-$. Then the bounded multiplicity property (7.2) holds if and only if $(G_{\mathbb{C}} \times G_{\mathbb{C}})/(B \times B)$ is $G'_{\mathbb{C}}$-spherical by Theorem 1.4.
(1) In Case II-1, it suffices to classify the triple $(\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2)$ for which the flag variety $(G_{\mathbb{C}} \times G_{\mathbb{C}})/(B \times B)$ is $(G_{1\mathbb{C}} \times G_{2\mathbb{C}})$-spherical. By a classical result of Krämer [Kr76], this happens if and only if $(\mathfrak{g}, \mathfrak{g}_j)$ or $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{j\mathbb{C}})$ are in the list (1.4). By the equivalence relation for triples in Definition 7.1, the classification in Case II-1 follows.
(2) In Case II-2, it suffices to classify (G, σ) for which $(G_{\mathbb{C}} \times G_{\mathbb{C}})/(B \times B)$ is $G_{\mathbb{C}}$-spherical via the σ-twisted diagonal action. Then the classification follows from Lemma 8.3 in Case II^r-2. Similarly for Case II^c-2. □

8.5 Proof of Theorem 7.9 for tensor product case

Let G be a simple Lie group. In this section we complete the proof of Theorem 7.9 by classifying the symmetric pairs $(G \times G, H)$ having the bounded multiplicity property (7.3) for the tensor product representations by using the criterion in Theorem 1.5. According to whether the simple Lie algebra $\mathfrak{g}$ has a complex structure or not, we treat separately in the following subcases:

Case III-1. $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ such that $(\mathfrak{g}, \mathfrak{h}_j)$ $(j = 1, 2)$ are symmetric pairs.
- **Case III^r-1.** $\mathfrak{g}_{\mathbb{C}}$ is simple.
- **Case III^c-1.** $\mathfrak{g}$ is a complex simple Lie algebra.

Case III-2. $\mathfrak{h} = \operatorname{diag}_\sigma(\mathfrak{g})$ for some involutive automorphism σ of $\mathfrak{g}$.
- **Case III^r-2.** $\mathfrak{g}_{\mathbb{C}}$ is simple.
- **Case III^c-2.** $\mathfrak{g}$ is a complex simple Lie algebra.

We first prove the following proposition:

Proposition 8.5. *Let B_{G/H_j} $(\subset G_{\mathbb{C}})$ be the Borel subgroup for symmetric spaces G/H_j $(j = 1, 2)$. Suppose $\mathfrak{g}_{\mathbb{C}}$ is simple. Then the following three conditions are equivalent:*

(i) $(G_{\mathbb{C}} \times G_{\mathbb{C}})/(B_{G/H_1} \times B_{G/H_2})$ *is $G_{\mathbb{C}}$-spherical.*

(ii) $(G_{\mathbb{C}} \times G_{\mathbb{C}})/(B_{G/H_1} \times B_{G/H_2})$ *is* G_U*-strongly visible.*

(iii) *The triple* $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{1\mathbb{C}}, \mathfrak{h}_{2\mathbb{C}})$ *is in Table 7.2.*

In Case III-1, we recall from Theorem 1.5 that the bounded multiplicity property (1.9) of the tensor product representation $\Pi_1 \otimes \Pi_2$ is equivalent to one of (therefore, any of) (i) and (ii) in Proposition 8.5.

Proof. The complete list of the pairs of parabolic subgroups (Q_1, Q_2) satisfying (ii) (or equivalently (i)) is given in [Tn12], see also [K07b] for type A, and [Li94] for maximal parabolic subgroups. In order to prove (ii) $\iff$ (iii) (or (i) $\iff$ (iii)), it suffices to determine when Q_j $(j = 1, 2)$ are isomorphic to Borel subgroups for some symmetric pairs (G, H_j). We illustrate the proof for (ii) $\iff$ (iii) with two cases: $\mathfrak{g}_{\mathbb{C}}$ is of type A and of type D, in particular, of D_4. Other cases are similar.

Suppose $\mathfrak{g}_{\mathbb{C}}$ is of type A. We first observe from [K07b] or [Tn12] that if (ii) holds then $\#\Theta_1^c + \#\Theta_2^c \leq 3$ when $\mathfrak{g}_{\mathbb{C}}$ is of type A.

In view of the Satake diagram for real forms of $\mathfrak{g}_{\mathbb{C}}$ of type A, $\#\Theta_1^c \leq 2$ if and only if one of the following holds:

$$\begin{aligned} &\#\Theta_1^c = 1 \quad (\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{1\mathbb{C}}) = (\mathfrak{sl}_4(\mathbb{C}), \mathfrak{sp}_2(\mathbb{C})), \\ &\#\Theta_1^c = 2 \quad (\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{1\mathbb{C}}) = (\mathfrak{sl}_n(\mathbb{C}), \mathfrak{gl}_{n-1}(\mathbb{C})). \end{aligned}$$

Hence $\#\Theta_1^c + \#\Theta_2^c \leq 3$ only if $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{1\mathbb{C}}, \mathfrak{h}_{2\mathbb{C}})$ is either $(\mathfrak{sl}_4(\mathbb{C}), \mathfrak{sp}_2(\mathbb{C}), \mathfrak{sp}_2(\mathbb{C}))$ or $(\mathfrak{sl}_4(\mathbb{C}), \mathfrak{sp}_2(\mathbb{C}), \mathfrak{gl}_3(\mathbb{C}))$ up to switch of factors. Conversely, the condition (ii) holds in this case by [K07b].

Suppose $\mathfrak{g}$ is of type D. Then (ii) holds only if $\#\Theta_1^c + \#\Theta_2^c \leq 3$. Then a similar argument to the type A case tells that the equality is not attained if $n \geq 5$ when Θ_j^c arise from symmetric pairs $(\mathfrak{g}, \mathfrak{h}_j)$ and that (ii) holds if and only if $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{1\mathbb{C}}, \mathfrak{h}_{2\mathbb{C}}) = (\mathfrak{so}_{2n}(\mathbb{C}), \mathfrak{so}_{2n-1}(\mathbb{C}), \mathfrak{so}_{2n-1}(\mathbb{C}))$ for the type D case if $n \geq 5$.

The remaining case is when $\mathfrak{g}_{\mathbb{C}}$ is of type D_4 and $\#\Theta_1^c + \#\Theta_2^c = 3$. By [Tn12], the list of such pairs (Θ_1^c, Θ_2^c) satisfying (ii) is of three types

$$\begin{aligned} &(\{\alpha_i\}, \{\alpha_j, \alpha_k\}) \quad \{i, j, k\} = \{1, 3, 4\} \quad (3 \text{ cases}), \\ &(\{\alpha_i\}, \{\alpha_i, \alpha_j\}) \quad \{i, j\} \subset \{1, 3, 4\} \quad (6 \text{ cases}), \\ &(\{\alpha_i\}, \{\alpha_2, \alpha_j\}) \quad \{i, j\} \subset \{1, 3, 4\} \quad (6 \text{ cases}), \end{aligned}$$

up to switch of factors. The first two types do not arise from symmetric pairs, whereas the third types arises from

$$(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{1\mathbb{C}}, \mathfrak{h}_{2\mathbb{C}}) = (\mathfrak{so}_8(\mathbb{C}), \mathfrak{so}_7(\mathbb{C}), \mathfrak{gl}_4(\mathbb{C})) \tag{8.4}$$

up to outer automorphisms. (We recall that the group of outer automorphisms of D_4 is of order 6.)

The cases when $\mathfrak{g}_{\mathbb{C}}$ is not of type A or D, the proof is similar. □

Proof of Theorem 7.9. Let Q $(\subset G_{\mathbb{C}} \times G_{\mathbb{C}})$ be a Borel subgroup for the symmetric space $(G \times G)/H$. By Theorem 1.4, it suffices to determine when the flag variety $(G_{\mathbb{C}} \times G_{\mathbb{C}})/Q$ is $G'_{\mathbb{C}}$-spherical.
(1) For $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$, the assertion follows from Lemma 8.4 for any $\mathfrak{h}$ $(\subset \mathfrak{g} \oplus \mathfrak{g})$.
(2) Suppose that $\mathfrak{g}_{\mathbb{C}}$ is simple. In Case III^r-1, the classification is given by Proposition 8.5.
In Case III^r-2, Q is the direct product of Borel subgroup of $G_{\mathbb{C}}$, hence the classification follows from Lemma 8.3.
(3) Suppose that $\mathfrak{g}$ is a complex simple Lie algebra. In this case, the complexification $G_{\mathbb{C}}$ of G is given as $G \times G$ by (8.1).
In Case III^c-1, the proof is the same with Case III^r-1 if both $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are complex subalgebras of $\mathfrak{g}$.

If at least one of $\mathfrak{h}_1$ or $\mathfrak{h}_2$ is a real form of $\mathfrak{g}$ in Case III^c-1, then one has $Q = Q_1 \times Q_2$ where Q_1 or Q_2 must be a Borel subgroup of $G_{\mathbb{C}}$ by Lemma 8.2. Therefore, if $(G_{\mathbb{C}} \times G_{\mathbb{C}})/(Q_1 \times Q_2)$ is $G_{\mathbb{C}}$-spherical by the diagonal action, then $\mathfrak{g}_{\mathbb{C}}$ must be $\mathfrak{sl}_2(\mathbb{C})$ from the implication (i) ⇒ (iii) in Lemma 8.3.

Similarly in Case III^c-2, the Borel subgroup Q for the symmetric space $(G \times G)/\operatorname{diag}_\sigma(G)$ is the direct product of Borel subgroups of $G_{\mathbb{C}}$, hence the bounded multiplicity criterion in Theorem 1.5 amounts to the sphericity of $(G_{\mathbb{C}} \times G_{\mathbb{C}})/(B \times B)$ by the diagonal $G_{\mathbb{C}}$-action, which forces $\mathfrak{g}$ to be $\mathfrak{sl}_2(\mathbb{C})$ again by (i) ⇒ (iii) in Lemma 8.3. □

9 Examples and some perspectives

By the branching problems we mean the broad problem of understanding how irreducible representations of a group behave when restricted to a subgroup. As viewed in [K15], we may divide the branching problems into the following three stages:

Stage A. Abstract features of the restriction;
Stage B. Branching law;
Stage C. Construction of symmetry breaking/holographic operators.

The role of Stage A is to develop an abstract theory on the restriction of representations as generally as possible. In turn, we could expect a detailed study of the restriction in Stages B (decomposition of representations) and C (decomposition of vectors) in the specific settings that are *a priori* guaranteed to be "nice" in Stage A.

Solving Problem 1.1 or Problem 1.2 on the bounded multiplicity property may be considered as in Stage A. In this section, we discuss some promising examples in Stages B and C in the new settings that fit well into the framework of the present article. We also mention some examples which are "outside" this framework, to clarify the limitation as well.

When the pair (G, G') satisfies the bounded multiplicity property (1.2), or when the complexified pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ is essentially $(\mathfrak{sl}_n, \mathfrak{gl}_{n-1})$ or $(\mathfrak{so}_n, \mathfrak{so}_{n-1})$ up to outer automorphisms, see (1.4), there have been active and rich study of the branching problems in Stages B and C in recent years, such as the Gan–Gross–Prasad conjecture (Stage B) and the construction of symmetry breaking operators (Stage C), see *e.g.,* an exposition [K19a] and references therein. Let us now focus on the **new settings** when the pair (G, G') does not satisfy (1.2), but the triple $H \subset G \supset G'$ satisfies the bounded multiplicity property (1.7), the triple (Π, G, G') with $\Pi \in \mathrm{Irr}(G)$ satisfies $m(\Pi|_{G'}) < \infty$ (see (1.5)), or the triple (G, Π_1, Π_2) with $\Pi_1, \Pi_2 \in \mathrm{Irr}(G)$ satisfies $m(\Pi_1 \otimes \Pi_2) < \infty$ (see (1.8)). We shall see in Examples 9.1–9.4 below some previous successful results on the analysis of the branching laws $\Pi|_{G'}$ in Stages B and C in these settings. One may observe that the existing results treated only a small part of this new framework in comparison with the complete list in Section 7. This observation indicates possible new avenues of the rich study of the branching problems in Stages B and C.

Example 9.1 (Restriction of discrete series representations for G/H)**.**
Suppose that $(G, H, G') = (O(p, q), O(p-1, q), O(p_1, q_1) \times O(p_2, q_2))$ where $p_1 + p_2 = p$ and $q_1 + q_2 = q$. For simplicity, suppose $p + q \geq 5$. In this case the pair (G, G') does not satisfy the bounded multiplicity property (1.2) if $p_1 + q_1 > 1$ and $p_2 + q_2 > 1$, and not the finite multiplicity property (1.3) if $p > 1$ and $q > 1$ in addition, see [KM14], however, the triple (G, H, G') always satisfies the finer bounded multiplicity property (1.7) as was seen in Example 7.5. The branching problem for the restriction $\Pi|_{G'}$ of $\Pi \in \mathrm{Irr}(G)_H$ relates

harmonic analysis involving three groups G, H, and G'. Discrete series representations Π were classified by Faraut [F79] and Strichartz [St83], which can be expressed also in algebraic terms of Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$ [KØ02]. The branching laws $\Pi|_{G'}$ to the subgroup G' were determined in the discretely decomposable case ($p_2 = 0$) in [K93], and also in the case containing continuous spectrum under the assumption that $(q_1, q_2) = (1, 0)$ by Frahm and Y. Oshima [MO15]. For general (p_1, p_2, q_1, q_2), full discrete spectrum of the restriction $\Pi|_{G'}$ occurs in a multiplicity-free fashion and is constructed and classified in [K21]. See also a recent work of Ørsted and Speh [ØS19] for another approach to capture a generic part of discrete spectrum in the branching law of $\Pi|_{G'}$. The triple (G, H, G') in this example is a real form of the complexified triple appearing in the fourth row of the left column in Table 7.1.

Example 9.2 (Unitary branching laws for mirabolic)**.** Let $G = GL_n(\mathbb{R})$ and P a mirabolic, *i.e.,* a maximal parabolic subgroup with Levi factor $GL_1(\mathbb{R}) \times GL_{n-1}(\mathbb{R})$. Then for any symmetric pair (G, G'), namely, $G' = O(p, n-p)$, $GL_p(\mathbb{R}) \times GL_{n-p}(\mathbb{R})$, $Sp_m(\mathbb{R})$ or $GL_m(\mathbb{C})$ when $n = 2m$, the generalized flag variety $G_{\mathbb{C}}/P_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical, hence the triple (G, P, G') is an example that fulfills the geometric assumption in Theorem 4.2 with $Q = P_{\mathbb{C}}$. In this case the branching laws of the unitary representation $\Pi|_{G'}$ is explicitly found in [KØP11] for all the symmetric pairs (G, G') when Π is a unitarily induced representation of a unitary character of P. The multiplicity in the (unitary) branching laws of the restriction $\Pi|_{G'}$ is guaranteed to be uniformly bounded by Theorem 4.2, even though the symmetric pairs (G, G') do not satisfy the general finite multiplicity condition (1.3) for most of the cases, see [KM14].

Example 9.3 (Symmetry breaking operators)**.** For the symmetric pair $(G, G') = (Sp_n(\mathbb{R}), GL_n(\mathbb{R}))$, the general finite multiplicity property (1.3) fails [KM14]. However, if we take P to be the Siegel parabolic subgroup of G, and P' to be a maximal parabolic subgroup of G', then one has $\#(P'_{\mathbb{C}} \backslash G_{\mathbb{C}}/P_{\mathbb{C}}) < \infty$, and therefore the geometric assumption in Theorem 4.1 is satisfied with $Q = P_{\mathbb{C}}$ and $Q' = P'_{\mathbb{C}}$, and thus the bounded multiplicity property (4.1) holds for the space of symmetry breaking operators. Nishiyama–Ørsted [NØ18] has constructed explicitly (integral) symmetry breaking operators between the corresponding degenerate principal series representations of G and G' generalizing [KS15].

Example 9.4 (Invariant trilinear form)**.** For a noncompact simple Lie group G, the space of invariant trilinear forms $\mathrm{Hom}_G(\Pi_1 \otimes \Pi_2 \otimes \Pi_3, \mathbb{C})$ is finite-dimensional for *all* Π_1, Π_2, $\Pi_3 \in \mathrm{Irr}(G)$, or equivalently, the pair $(G \times G, \mathrm{diag}\, G)$ satisfies the finite multiplicity condition (1.3), if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(n,1)$ ([K95], see also [K14, Cor. 4.2]). Beyond this case, one may consider the setting where Π_1, Π_2, Π_3 are "small representations" such as degenerate principal series representations. For instance, if $G = Sp_n(\mathbb{R})$ and P is a Siegel parabolic subgroup, then $G_{\mathbb{C}}/P_{\mathbb{C}} \times G_{\mathbb{C}}/P_{\mathbb{C}}$ is $G_{\mathbb{C}}$-spherical via the diagonal action [Li94], or equivalently, it is G_U-strongly visible via the diagonal action [Tn12], and therefore one has the bounded property (4.7) of the space of invariant trilinear forms by Theorem 4.8 and the one (4.10) of the tensor product by Corollary 4.10. Construction of trilinear forms and explicit evaluations of spherical vectors by the generalized Bernstein–Reznikov integrals in these cases have been studied in Clerc *et. al.* [CKØP11] and Clare [C15], for instance.

Finally, we mention a couple of examples for which the bounded multiplicity property fails by explicit computations.

Example 9.5 (Compact symmetric pairs of rank one)**.** Let us consider the triple $(G, H, G') = (SU(3), U(2), SO(3))$. Note that the triple of the complexified Lie algebras $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}}) = (\mathfrak{sl}_3, \mathfrak{gl}_2, \mathfrak{so}_3)$ is not in Table 7.1, and thus the bounded multiplicity property (1.7) of the branching should fail. In fact, let Π_n be the irreducible representations of $G = SU(3)$ with highest weight $(n, 0, -n)$ in the standard coordinates, and π_n the $(2n+1)$-dimensional irreducible representation of $G' = SO(3)$. Then $\Pi_n \in \mathrm{Irr}(G)_H$ and $[\Pi_n|_{G'} : \pi_n] = [\frac{n}{2}] + 1$, hence $\sup_{n \in \mathbb{N}} m(\Pi_n|_{G'}) = \infty$ showing the failure of the bounded multiplicity property (1.7) for the triple (G, H, G'). We note that this triple is one of the three exceptional cases when $\mathrm{rank}\, G/H = 1$ indicated in Corollary 7.4.

The following example is a reformulation of [K08, Sect. 6.3].

Example 9.6 (Restriction of Harish-Chandra's discrete series representations)**.** Let $(G, H, G') = (SO(5, \mathbb{C}), SO(3, 2), SO(3, 2))$. If Π is a discrete series representation Π for the symmetric space G/H, then there exists a (Harish-Chandra) discrete series representation π of G' such that π occurs in the restriction $\Pi|_{G'}$ as discrete spectrum of infinite multiplicity, and in particular, $[\Pi|_{G'} : \pi] = \infty$. In fact, this triple (G, H, G') does not appear in the classification given in Theorem 7.6.

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References

[BK21] Y. Benoist, T. Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31**, (2021), 833–869.

[Bo61] A. Borel, Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes. Tôhoku Math. J., **13**, (1961), 216–240.

[Br02] N. Bourbaki, *Lie Groups and Lie Algebras. Chapters 4–6.* Springer, 2002.

[B86] M. Brion, Quelques propriétés des espaces homogenès sphériques, Manuscripta Math. **55**, (1986), 191–198.

[C15] P. Clare, Invariant trilinear forms for spherical degenerate principal series of complex symplectic groups, Internat. J. Math., **26**, (2015), 1550107, 16pp.

[CKØP11] J.-L. Clerc, T. Kobayashi, B. Ørsted, and M. Pevzner, *Generalized Bernstein–Reznikov integrals*, Math. Ann., **349** (2011), pp. 395–431.

[F79] J. Faraut, Distributions sphériques sur les espaces hyperboliques, J. Math. Pures Appl. **58** (1979), 369–444.

[GS64] I. M. Gel'fand and G. E. Shilov, Generalized functions Vol. I: Properties and operations, Translated by Eugene Soletan, Academic Press, New York-London, 1964.

[HNOO13] X. He, K. Nishiyama, H. Ochiai, Y. Oshima, On orbits in double flag varieties for symmetric pairs, Transform. Groups, **18**, (2013), 1091–1136.

[He78] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Space, Academic Press, 1978.

[He94] S. Helgason, Geometric Analysis on Symmetric Spaces, Amer. Math. Soc., 1994.

[HT93] R. Howe, E. Tan, Homogeneous functions on light cones, Bull. Amer. Math. Soc., **28** (1993), 1–74.

[Ka83] M. Kashiwara, Systems of Microdifferential Equations, Progr. Math., **34**, Birkhäuser, Boston, 1983, xv+159pp.

[KK81] M. Kashiwara, T. Kawai, On holonomic systems of microdifferential equations. III. Systems with regular singularities, Publ. Res. Inst. Math. Sci. **17**, (1981), 813–979.

[K93] T. Kobayashi, *The restriction of* $A_{\mathfrak{q}}(\lambda)$ *to reductive subgroups*, Proc. Japan Acad., **69** (1993), 262–267.

[K94] T. Kobayashi, *Discrete decomposability of the restriction of* $A_{\mathfrak{q}}(\lambda)$ *with respect to reductive subgroups and its applications*, Invent. Math., **117** (1994), 181–205.

[K95] T. Kobayashi, *Introduction to harmonic analysis on real spherical homogeneous spaces*, In F. Sato, editor, Proceedings of the 3rd Summer School on Number Theory "Homogeneous Spaces and Automorphic Forms" in Nagano, 1995, pp. 22–41.

[K98a] T. Kobayashi, *Discrete decomposability of the restriction of* $A_{\mathfrak{q}}(\lambda)$ *with respect to reductive subgroups* II*—micro-local analysis and asymptotic* K*-support*, Ann. of Math., **147** (1998), pp. 709–729.

[K98b] T. Kobayashi, *Discrete decomposability of the restriction of* $A_{\mathfrak{q}}(\lambda)$ *with respect to reductive subgroups* III*—restriction of Harish-Chandra modules and associated varieties*, Invent. Math., **131** (1998), 229–256.

[K05] T. Kobayashi, *Multiplicity-free representations and visible actions on complex manifolds*, Publ. Res. Inst. Math. Sci. **41** (2005), pp. 497–549, special issue commemorating the fortieth anniversary of the founding of RIMS.

[K07a] T. Kobayashi, Visible actions on symmetric spaces, Transform. Groups, **12**, (2007), 671–694.

[K07b] T. Kobayashi, *A generalized Cartan decomposition for the double coset space* $(U(n_1) \times U(n_2) \times U(n_3))\backslash U(n)/(U(p) \times U(q))$, J. Math. Soc. Japan **59** (2007), pp. 669–691.

[K08] T. Kobayashi, *Multiplicity-free theorems of the restrictions of unitary highest weight modules with respect to reductive symmetric pairs*, Progr. Math., **255**, 45–109, 45–109. Birkhäuser, 2008.

[K14] T. Kobayashi, *Shintani functions, real spherical manifolds, and symmetry breaking operators*, Dev. Math., **37**, (2014), 127–159. Springer.

[K15] T. Kobayashi, *A program for branching problems in the representation theory of real reductive groups*, In: Representations of Reductive Groups—In Honor of the 60th Birthday of David A. Vogan, Jr., (eds. M. Nevins and P. E. Trapa), Progr. Math., **312**, Birkhäuser, 2015, 277–322.

[K19a] T. Kobayashi, Recent advances in branching laws of representations, Sugaku **71** (2019), 388–416, Math. Soc. Japan; English translation to appear in Sugaku Exposition, Amer. Math. Soc.

[K19b] T. Kobayashi, Admissible restrictions of irreducible representations of reductive Lie groups: Symplectic geometry and discrete decomposability, Pure and Applied Mathematics Quarterly, in special issue in memory of Bertram Kostant (in press). Available also at arXiv:1907.12964.

[K21] T. Kobayashi, Branching laws of unitary representations associated to minimal elliptic orbits for indefinite orthogonal group $O(p, q)$, Adv. Math., **388**, (2021), Paper No. 107862, 38pp.

[KM14] T. Kobayashi, T. Matsuki, *Classification of finite-multiplicity symmetric pairs*, Transform. Groups, **19** (2014), 457–493, Special issue in honor of Dynkin for his 90th birthday.

[KØ02] T. Kobayashi, B. Ørsted, Analysis on the minimal representation of $O(p, q)$, II, Branching laws, Adv. Math., **180** (2003), 513–550.

[KØP11] T. Kobayashi, B. Ørsted, M. Pevzner, Geometric analysis on small unitary representations of $GL(N,\mathbb{R})$, J. Funct. Anal., **260**, (2011), 1682–1720.

[KO13] T. Kobayashi, T. Oshima, *Finite multiplicity theorems for induction and restriction*, Adv. Math., **248** (2013), 921–944.

[KO12] T. Kobayashi, Y. Oshima, *Classification of discretely decomposable $A_{\mathfrak{q}}(\lambda)$ with respect to reductive symmetric pairs*, Adv. Math., **231**, (2012), 2013–2047.

[KO15] T. Kobayashi, Y. Oshima, *Classification of symmetric pairs with discretely decomposable restrictions of $(\mathfrak{g}, K)$-modules*, J. Reine Angew. Math. **703**, (2015), 201–223.

[KS15] T. Kobayashi, B. Speh, Symmetry Breaking for Representations of Rank One Orthogonal Groups, (2015), Mem. Amer. Math. Soc. **238** no.1126, 118 pages.

[KS18] T. Kobayashi, B. Speh, Symmetry Breaking for Representations of Rank One Orthogonal Groups, Part II, Lecture Notes in Math., **2234** Springer, 2018. xv+342 pages.

[Kr76] M. Krämer, *Multiplicity free subgroups of compact connected Lie groups*, Arch. Math. (Basel) **27**, (1976), 28–36.

[Li94] P. Littelmann, On spherical double cones, J. Algebra **166**, (1994), 142–157.

[MWZ99] P. Magyar, J. Weyman, A. Zelevinsky, Multiple flag varieties of finite type, Adv. Math., **141**, (1999), 97–118.

[Mt15] T. Matsuki, Orthogonal multiple flag varieties of finite type I: Odd degree case, J. Algebra, **425**, (2015), 450–523.

[MO15] J. Möllers, Y. Oshima, Restriction of most degenerate representations of $O(1, N)$ with respect to symmetric pairs. J. Math. Sci. Univ. Tokyo **22**, (2015), 279–338.

[NØ18] K. Nishiyama, B. Ørsted, Real double flag varieties for the symplectic group, J. Funct. Anal., **274**, (2018), 573–604.

[ØS19] B. Ørsted, B. Speh, Branching laws for discrete series of some affine symmetric spaces, preprint, arXiv:1907.07544.

[O88] T. Oshima, A realization of semisimple symmetric spaces and construction of boundary value maps. Advanced Studies in Pure Mathematics, **14**, (1988), 603–650.

[OS84] T. Oshima, J. Sekiguchi, The restricted root systems of a semisimple symmetric pair, Advanced studies in Pure Math., **4**, (1984), 433–497.

[Sa60] I. Satake, On representations and compactifications of symmetric Riemannian spaces, Ann. Math., **71**, (1960), 77–110.

[S59] M. Sato, Theory of hyperfunctions I, II, J. Fac Sci. Univ. Tokyo, Sect. I, **8**, (1959), 133–193; ibid, (1960), 387–431.

[St83] R. S. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold. J. Funct. Anal., **52**, (1983), 48–79.

[SZ12] B. Sun, C.-B. Zhu, Multiplicity one theorems: the Archimedean case, Ann. of Math., **175**, (2012), pp. 23–44.

[Tn12] Y. Tanaka, Classification of visible actions on flag varieties, Proc. Japan Acad. Ser. A Math. Sci., **88**, (2012), 91–96.

[Tn21] Y. Tanaka, Visible actions of compact Lie groups on complex spherical varieties, to appear in J. Diff. Geom.

[Tu19] T. Tauchi, A generalization of the Kobayashi–Oshima uniformly bounded multiplicity theorem, to appear in Internat. J. Math. Available also at arXiv:2108.02139 (Ph.D. thesis, Chap. 4, Univ. Tokyo, 2019).

[V86] É. B. Vinberg, Complexity of action of reductive groups, Func. Anal. Appl. **20**, (1986), 1–11.

[VK78] É. B. Vinberg and B. N. Kimel'fel'd, Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups, Functional Anal. Appl., **12**, (1978), 168–174.

[Wal92] N. R. Wallach, Real reductive groups. I, II, Pure Appl. Math. **132** Academic Press, Inc., Boston, MA, 1988; **132**-II, ibid, 1992.

[War72] G. Warner, Harmonic Analysis on Semi-Simple Lie Groups I, Grundlehren Math. Wiss., vol. **188**, Springer-Verlag, New York, 1972.

[Wo69] J. A. Wolf, The action of a real semisimple group on a complex flag manifold. I. Orbit structure and holomorphic arc components. Bull. Amer. Math. Soc. **75**, (1969), 1121–1237.

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