

Temperedness criterion of the tensor product of parabolic induction for GL_n

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Abstract

We give a necessary and sufficient condition for a pair of parabolic subgroups P and Q of $G = GL_n(\mathbb{R})$ such that the tensor product of any two unitarily induced representations from P and Q are tempered. We also give an L^p -estimate of matrix coefficients of the regular representations on $L^2(G/L)$ when L is a Levi subgroup of G .

Key words and phrases: tempered representation, reductive group, tensor product, unitary representation, degenerate principal series representation

1 Statement of main results

For two unitary representations Π_j on Hilbert spaces \mathcal{H}_j ($j = 1, 2$) of a group G , the tensor product representation $\Pi_1 \otimes \Pi_2$ is a unitary representation of G defined on the Hilbert completion of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let σ and τ be unitary representations of parabolic subgroups P and Q of $G = GL_n$, respectively, and $\Pi_1 = \text{Ind}_P^G(\sigma)$ and $\Pi_2 = \text{Ind}_Q^G(\tau)$ the unitary induction, see Section 2.1. In this paper we address the following:

Problem 1.1. When is the tensor product representation $\Pi_1 \otimes \Pi_2$ tempered?

Let us explain some background of this problem.

Problem 1.1 asks a coarse information of the spectrum of the tensor product representation $\Pi_1 \otimes \Pi_2$. We note that the disintegration of $\Pi_1 \otimes \Pi_2$ is far from being understood even for $G = GL_n$ and even when σ and τ are the trivial one-dimensional representations. For the very special case where P is a maximal parabolic subgroup and Q is its opposite parabolic subgroup, the tensor product $\text{Ind}_P^G(\mathbf{1}) \otimes \text{Ind}_Q^G(\mathbf{1})$ is unitarily equivalent to the regular

representation for a reductive symmetric space of $G = GL_n$, for which the Plancherel-type theorem is known up to some complicated vanishing condition of cohomologically induced representations with singular parameters which may affect an answer to Problem 1.1 ([3, Sect. 1], [6, Rem. 1.4] and references therein). Slightly more generally, when both P and Q are arbitrary maximal parabolic subgroups, Problem 1.1 was solved recently in [4, Prop. 5.9] without the Plancherel-type formula. On the other hand, if P or Q is a Borel subgroup, Problem 1.1 has an affirmative answer by the general theory (Remark 2.3). However, for the general P and Q , an answer to Problem 1.1 has not been known. In this general setting, we note that the diagonal action of G on $(G \times G)/(P \times Q)$ is not necessarily (real) spherical, and that the multiplicity of irreducible unitary representations in the disintegration of $\text{Ind}_P^G(\mathbf{1}) \otimes \text{Ind}_Q^G(\mathbf{1})$ may be infinite, cf. [11].

Tempered representations of a locally compact group G are unitary representations that are weakly contained in $L^2(G)$ (Definition 2.1). For real reductive Lie groups G , irreducible ones were classified by Knapp and Zuckerman [10], and are cornerstones both in Harish-Chandra's theory of the Plancherel formula of $L^2(G)$ and in Langlands' classification theory of irreducible admissible representations, whereas the Selberg's 1/4 conjecture for congruence subgroups can be reformulated as the temperedness of certain unitary representations of $SL_2(\mathbb{R})$ and the Gan–Gross–Prasad conjecture is formulated as a branching problem for tempered representations. A complete description of pairs (G, H) of real reductive algebraic groups for which $L^2(G/H)$ is not tempered was accomplished in [5], but such a classification has not been known for non-reductive subgroups H except for a few cases [4, Cor. 5.8].

In this article, we give a solution to Problem 1.1. We shall prove that the solution depends only on the G -conjugacy classes of Levi parts of parabolic subgroups P and Q . We introduce the following notation: for a parabolic subgroup P of GL_n with the Levi subgroup $GL_{n_1} \times \cdots \times GL_{n_r}$ ($n_1 + \cdots + n_r = n$), we set

$$d(P) := \max_{1 \leq j \leq r} n_j.$$

Then $1 \leq d(P) \leq n$ with two extreme cases: $d(P) = 1 \iff P$ is a Borel subgroup, and $d(P) = n \iff P = G$. We prove:

Theorem 1.2. Let P and Q be parabolic subgroups of $G = GL_n(\mathbb{R})$. Then the following three conditions are equivalent:

- (i) The tensor product representation $\text{Ind}_P^G(\sigma) \otimes \text{Ind}_Q^G(\tau)$ is tempered for all unitary representations σ of P and τ of Q .
- (ii) The tensor product representation $\text{Ind}_P^G(\mathbf{1}) \otimes \text{Ind}_Q^G(\mathbf{1})$ is tempered.
- (iii) $d(P) + d(Q) \leq n + 1$.

An analogous statement holds also for $G = GL_n(\mathbb{C})$.

Theorem 1.2 is derived from the following results about the regular representation on $L^2(G/H)$ where H is not necessarily reductive:

Theorem 1.3. Let H be a closed subgroup of $G = GL_n(\mathbb{R})$ with finitely many connected components. Assume that the Lie algebra \mathfrak{h} is stable by a split Cartan subalgebra \mathfrak{a} of \mathfrak{g} . Let $\{e_1, \dots, e_n\}$ be the standard basis of \mathfrak{a}^* such that $\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$, and $\{E_1, \dots, E_n\}$ the dual basis of \mathfrak{a} . Then the following three conditions are equivalent:

- (i) $\text{Ind}_H^G(\sigma)$ is tempered for any unitary representation σ of H .
- (ii) $L^2(G/H)$ is tempered.
- (iii) $\dim \text{Image}(\text{ad}(E_i): \mathfrak{h} \rightarrow \mathfrak{h}) \leq n - 1$ for all i ($1 \leq i \leq n$).

In general $0 \leq \dim \text{Image}(\text{ad}(E_i): \mathfrak{h} \rightarrow \mathfrak{h}) \leq 2n - 1$ for any \mathfrak{a} -stable Lie algebra \mathfrak{h} and any i ($1 \leq i \leq n$). Theorem 1.3 justifies the ‘‘heuristic recipe’’ in [4, Rem. 5.7] for subgroups of three-by-three block matrix form.

Our proof relies on the temperedness criterion (Fact 2.5), which was established in [3, 4] by an analytic and dynamical approach in the general case. The criterion can be reduced to finitely many inequalities arising from the edges of convex polyhedral cones, actually 2^n inequalities in our setting. To solve Problem 1.1, we still need to analyze the 2^n inequalities. A number of combinatorial techniques were proposed in [4, 5], and among them, Theorem 1.3 was proved in the special setting where \mathfrak{h} is a subalgebra of three-by-three block matrix form ([4, Cor. 5.8]). The new technical ingredients in this article include yet another combinatorial trick which reduces 2^n inequalities to very simple n inequalities (the third condition in Theorem 1.3). The same technique also yields an L^p -estimate of the matrix coefficients of the regular representation $L^2(G/H)$ when H is reductive, see Theorem 4.4.

This article is organized as follows. In Section 2, we review the Herz majoration principle and the temperedness criterion in a general setting. Section 3 provides a proof of Theorems 1.2 and 1.3, postponing a combinatorial proof of Lemma 3.2 until Section 4. In Section 5, we discuss Problem 1.1 for any simple groups under the assumption that Q is the opposite parabolic subgroup of P .

2 Preliminaries

In this section we fix some notations and recall the previous results on unitary representations that will be needed later.

2.1 Regular representations

For an m -dimensional manifold X , we denote by $\mathcal{L}_{vol} \equiv \mathcal{L}_{X,vol} := |\wedge^m(T^*X)|$ the density bundle of X , and by $L^2(X)$ the Hilbert space of square integrable sections for the half-density bundle $\mathcal{L}_{vol}^{1/2}$. Suppose a Lie group G acts continuously on X . Then G acts equivariantly on the half-density bundle $\mathcal{L}_{vol}^{1/2}$, and one has naturally a unitary representation λ_X of G on $L^2(X)$, referred to as the *regular representation*. Associated to a unitary representation (σ, W) of a closed subgroup H of G , the unitary induction $\text{Ind}_H^G(\sigma)$ is defined as a unitary representation of G on the Hilbert space of square integrable sections for the G -equivariant Hilbert bundle $(G \times_H W) \otimes \mathcal{L}_{vol}^{1/2}$ over the homogeneous space G/H . By definition, $\text{Ind}_H^G(\mathbf{1})$ is the regular representation $\lambda_{G/H}$ on $L^2(G/H)$, where $\mathbf{1}$ denotes the trivial one-dimensional representation of H .

2.2 Tempered representations

Let (π, \mathcal{H}) and (π', \mathcal{H}') be unitary representations of a locally compact group G . We say π is *weakly contained* in π' , to be denoted by $\pi \prec \pi'$ if for every $v \in \mathcal{H}$ the matrix coefficient $(\pi(g)v, v)$ can be approximated uniformly on compact subsets of G by a sequence of finite sums of functions $(\pi'(g)u_j, u_j)$ with $u_1, \dots, u_k \in \mathcal{H}'$.

Definition 2.1. A unitary representation π of G is called *tempered* if π is weakly contained in the (left) regular representation λ_G on $L^2(G)$.

When G is a semisimple Lie group, π is tempered if and only if π is almost L^2 , see [7]. Here we recall:

Definition 2.2. Let $p \geq 1$. A unitary representation (π, \mathcal{H}) of G is said to be *almost L^p* if there exists a dense subset D of \mathcal{H} for which the matrix coefficients $g \mapsto (\pi(g)u, v)$ are in $L^{p+\varepsilon}(G)$ for all $\varepsilon > 0$ and all $u, v \in D$.

Remark 2.3. Temperedness is closed under induction and restrictions of unitary representations. Moreover, if π is tempered, then the tensor product

representation $\pi \otimes \sigma$ is tempered for any unitary representation σ of G . In fact, if $\pi \prec \lambda_G$, then $\pi \otimes \sigma \prec \lambda_G \otimes \sigma$. Since $\lambda_G \otimes \sigma$ is a multiple of λ_G ([2, Cor. E.2.6]), one concludes $\pi \otimes \sigma \prec \lambda_G$.

We recall a classical lemma called ‘‘Herz majoration principle’’, see [1, Sect. 6]:

Lemma 2.4 ([4, Lem. 3.2]). Let G be a semisimple Lie group with finitely many connected components such that the identity component has finite center, and H a closed subgroup of G . If the regular representation $\lambda_{G/H}$ is tempered, then the induced representation $\text{Ind}_H^G(\sigma)$ is tempered for any unitary representation σ of H .

2.3 Temperedness criterion for $L^2(G/H)$

A Lie algebra is said to be *algebraic* if it is isomorphic to the Lie algebra of an affine algebraic group, or equivalently, the image of the adjoint representation $\text{ad}: \mathfrak{h} \rightarrow \text{End}(\mathfrak{h})$ is the Lie algebra of an algebraic subgroup of $\text{Aut}(\mathfrak{h})$, see [8]. A subalgebra \mathfrak{a} is said to be *split* if $\text{ad}(H) \in \text{End}(\mathfrak{h})$ is diagonalizable over \mathbb{R} for every $H \in \mathfrak{a}$. Let \mathfrak{a} be a maximal split abelian subalgebra in an (algebraic) Lie algebra \mathfrak{h} . Such \mathfrak{a} is unique up to conjugation, and we denote by $\text{rank}_{\mathbb{R}} \mathfrak{h}$ its dimension when \mathfrak{h} is a semisimple Lie algebra.

Let V be a finite-dimensional representation of \mathfrak{h} . Following [3, 4], we define a non-negative function ρ_V on \mathfrak{a} by

$$\rho_V(Y) := \frac{1}{2} \sum_{\lambda \in \Delta(V, \mathfrak{a})} m_\lambda |\lambda(Y)| \quad \text{for } Y \in \mathfrak{a},$$

where $\Delta(V, \mathfrak{a})$ is the set of weights of \mathfrak{a} in V and m_λ denotes the dimension of the corresponding weight space V_λ . The function ρ_V is continuous and is piecewise linear i.e. there exist finitely many convex polyhedral cones which covers \mathfrak{a} and on which ρ_V is linear, see [3, Sect. 4.7]. We set

$$(2.1) \quad p_V := \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}.$$

Fact 2.5. Let G be a linear semisimple Lie group, and H an algebraic subgroup.

(1) ([4, Thm. 2.9]) One has the equivalence:

$$L^2(G/H) \text{ is tempered} \iff 2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}} \text{ on } \mathfrak{a}.$$

- (2) ([3, Thm. 4.1]) Let p be a positive even integer. If H is reductive, one has the equivalence:

$$L^2(G/H) \text{ is almost } L^p \iff p_{\mathfrak{g}/\mathfrak{h}} \leq p - 1.$$

The inequality in Fact 2.5 can be checked only at finitely many points in \mathfrak{a} , namely, at the generators of the edges of the convex polyhedral cones, as we shall see in Lemma 3.1 below in the setting we need.

3 Proof of Theorems 1.2 and 1.3

In this section, we show the main results by using the temperedness criterion (Fact 2.5) and some combinatorial lemmas. We postpone the proof of Lemma 3.2 until Section 4.

Suppose $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$ and \mathfrak{h} is an \mathfrak{a} -invariant subalgebra as in the setting of Theorem 1.3. Since split Cartan subalgebras \mathfrak{a} are conjugate to each other by inner automorphisms, we may and do assume $\mathfrak{a} = \bigoplus_{i=1}^n \mathbb{R}E_{ii}$, where E_{ij} denotes the matrix unit.

For $1 \leq i, j \leq n$, we set

$$(3.1) \quad \varepsilon_{ij} \equiv \varepsilon_{ij}(\mathfrak{h}) := \dim_{\mathbb{R}}(\mathfrak{h} \cap \mathbb{R}E_{ij}) \in \{0, 1\}.$$

By the weight decomposition of \mathfrak{h} with respect to \mathfrak{a} , one sees

$$(3.2) \quad \dim \text{Image}(\text{ad}(E_{ii}): \mathfrak{h} \rightarrow \mathfrak{h}) = \sum_{j \in \{1, \dots, n\} \setminus \{i\}} (\varepsilon_{ij} + \varepsilon_{ji}) = 2\rho_{\mathfrak{h}}(E_{ii}).$$

Since $\rho_{\mathfrak{g}}(E_{ii}) = n - 1$, the condition (iii) in Theorem 1.3 amounts to

$$2\rho_{\mathfrak{h}}(E_{ii}) \leq \rho_{\mathfrak{g}}(E_{ii}) \quad \text{for all } i \ (1 \leq i \leq n).$$

3.1 Reduction to finite inequalities

The temperedness criterion (Fact 2.5) is given by the inequality on \mathfrak{a} , which reduces to a finite number of inequalities on the generators of convex polyhedral cones. This is Lemma 3.1 below which reduces to 2^n inequalities. A further combinatorial argument reduces to n inequalities (Lemma 3.2).

For a non-empty subset $I \subset \{1, \dots, n\}$, we set $E_I := \sum_{i \in I} E_{ii}$. Then $E_I = E_{ii}$ if $I = \{i\}$; E_I generates the center $\mathfrak{z}(\mathfrak{g})$ of \mathfrak{g} if $I = \{1, 2, \dots, n\}$.

Lemma 3.1. The condition (ii) in Theorem 1.3 is equivalent to

$$(3.3) \quad 2\rho_{\mathfrak{h}}(E_I) \leq \rho_{\mathfrak{g}}(E_I) \quad \text{for all } I \subset \{1, \dots, n\}.$$

Proof. By the temperedness criterion (Fact 2.5), the condition (ii) in Theorem 1.3 is given by $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ on $\mathfrak{a}/\mathfrak{z}(\mathfrak{g})$. Thus it suffices to show

$$(3.4) \quad \max_{0 \neq Y \in \mathfrak{a}/\mathfrak{z}(\mathfrak{g})} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_{\mathfrak{g}}(Y)} = \max_{I \subset \{1, \dots, n\}} \frac{\rho_{\mathfrak{h}}(E_I)}{\rho_{\mathfrak{g}}(E_I)}.$$

To see the non-trivial inequality \leq , we begin with the dominant chamber $\bar{\mathfrak{a}}_+ = \{\text{diag}(x_1, \dots, x_n) : x_1 \geq \dots \geq x_n\}$. Since both $\rho_{\mathfrak{h}}$ and $\rho_{\mathfrak{g}}$ are linear on $\bar{\mathfrak{a}}_+$, the restriction of the function $\rho_{\mathfrak{h}}/\rho_{\mathfrak{g}}$ to the line segment $tY + (1-t)Z$ ($Y, Z \in \bar{\mathfrak{a}}_+ \setminus \mathfrak{z}(\mathfrak{g})$) is a linear fractional function of t ($0 \leq t \leq 1$), which attains its maximum either at $t = 0$ or $t = 1$. An iteration of the argument tells that the maximum of $\rho_{\mathfrak{h}}/\rho_{\mathfrak{g}}$ on $(\bar{\mathfrak{a}}_+/\mathfrak{z}(\mathfrak{g})) \setminus \{0\}$ is attained at one of the edges of the convex polyhedral cone $\bar{\mathfrak{a}}_+/\mathfrak{z}(\mathfrak{g})$, namely, at $\mathbb{R}_+ E_I$ for some $I = \{1, 2, \dots, k\}$ with $1 \leq k \leq n-1$.

Similar argument applies to the other Weyl chambers. \square

The following lemma tells that it suffices to use E_I with $\#I = 1$ for “witness vectors” ([5]) in our setting, and will be proved in Section 4.

Lemma 3.2. If $2\rho_{\mathfrak{h}}(E_{ii}) \leq \rho_{\mathfrak{g}}(E_{ii})$ for all i ($1 \leq i \leq n$), then (3.3) holds.

3.2 Proof of Theorem 1.3

The equivalence (i) \iff (ii) in Theorem 1.3 follows from the Herz majoration principle (Lemma 2.4). Let us verify the equivalence (ii) \iff (iii). We may and do assume that \mathfrak{h} contains $\mathfrak{a} = \sum_{i=1}^n \mathbb{R}E_{ii}$. In fact, if \mathfrak{h} is stable by \mathfrak{a} , then $\tilde{\mathfrak{h}} := \mathfrak{h} + \mathfrak{a}$ is a Lie subalgebra containing \mathfrak{a} . We write \tilde{H} for the connected subgroup of G with Lie algebra $\tilde{\mathfrak{h}}$. Then $L^2(G/H)$ is tempered if and only if $L^2(G/\tilde{H})$ is tempered by [4, Cor. 3.3]. Moreover, $\text{Image}(\text{ad}(E_{ii}) : \mathfrak{h} \rightarrow \mathfrak{h})$ remains the same if we replace \mathfrak{h} with $\tilde{\mathfrak{h}}$, hence the conditions (ii) and (iii) in Theorem 1.3 are unchanged. Now one has the equivalences:

$$\begin{aligned} \text{(ii)} &\iff 2\rho_{\mathfrak{h}}(Y) \leq \rho_{\mathfrak{g}}(Y) \quad (\forall Y \in \mathfrak{a}) && \text{by Fact 2.5} \\ &\iff 2\rho_{\mathfrak{h}}(E_I) \leq \rho_{\mathfrak{g}}(E_I) \quad (\forall I \subset \{1, \dots, n\}) && \text{by Lemma 3.1} \\ &\iff 2\rho_{\mathfrak{h}}(E_{ii}) \leq \rho_{\mathfrak{g}}(E_{ii}) \quad (1 \leq \forall i \leq n) && \text{by Lemma 3.2,} \end{aligned}$$

which is equivalent to (iii). Thus Theorem 1.3 is proved.

3.3 Proof of Theorem 1.2

Without loss of generality, we may and do assume that P and Q are standard parabolic subgroups with Levi subgroups $GL_{n_1} \times \cdots \times GL_{n_r}$ and $GL_{m_1} \times \cdots \times GL_{m_s}$, respectively. Let $w := \sum_{i=1}^n E_{i, n+1-i} \in G$, a representative of the longest element of the Weyl group $W(\mathfrak{g}, \mathfrak{a})$. Then $Q^\circ := w^{-1}Qw$ is a parabolic subgroup of G with Levi subgroup $GL_{m_s} \times \cdots \times GL_{m_1}$, and PQ° is open dense in G , hence the diagonal map $G \rightarrow G \times G$, $g \mapsto (g, g)$ induces an open dense embedding $\iota: G/H \hookrightarrow G/P \times G/Q^\circ$, where $H := P \cap Q^\circ$. Thus the tensor product representation $\text{Ind}_P^G(\mathbf{1}) \otimes \text{Ind}_{Q^\circ}^G(\mathbf{1}) \simeq \text{Ind}_P^G(\mathbf{1}) \otimes \text{Ind}_{Q^\circ}^G(\mathbf{1})$ is unitarily equivalent to $L^2(G/H)$ via the G -isomorphism of the equivariant line bundles $\iota^*(\mathcal{L}_{G/P, \text{vol}} \otimes \mathcal{L}_{G/Q^\circ, \text{vol}}) \simeq \mathcal{L}_{G/H, \text{vol}}$.

We define integers $N(a)$ ($0 \leq a \leq r$) and $M(b)$ ($0 \leq b \leq s$) by

$$N(a) := \sum_{j=1}^a n_j \quad (1 \leq a \leq r), \quad M(b) := \sum_{j=1}^b m_{s+1-j} \quad (1 \leq b \leq s),$$

and set $N(0) = M(0) = 0$. We note $N(r) = M(s) = n$. By definition, for each $1 \leq i \leq n$, there exist uniquely $a(i) \in \{1, \dots, r\}$ and $b(i) \in \{1, \dots, s\}$ such that

$$(3.5) \quad N(a(i) - 1) < i \leq N(a(i)) \text{ and } M(b(i) - 1) < i \leq M(b(i)).$$

By definition, one has for $1 \leq i, j \leq n$,

$$\begin{aligned} E_{ij} \in \mathfrak{p} &\iff N(a(i) - 1) < j, & E_{ij} \in \mathfrak{q}^\circ &\iff j \leq M(b(i)), \\ E_{ji} \in \mathfrak{p} &\iff j \leq N(a(i)), & E_{ji} \in \mathfrak{q}^\circ &\iff M(b(i) - 1) < j. \end{aligned}$$

Since the Lie algebra \mathfrak{h} of H is equal to $\mathfrak{p} \cap \mathfrak{q}^\circ$, (3.2) shows

$$\begin{aligned} 2\rho_{\mathfrak{h}}(E_{ii}) &= (M(b(i)) - N(a(i) - 1) - 1) + (N(a(i)) - M(b(i) - 1) - 1) \\ &= n_{a(i)} + m_{s-b(i)+1} - 2. \end{aligned}$$

Since \mathfrak{h} contains \mathfrak{a} , we can apply Theorem 1.3, and conclude that $L^2(G/H)$ is tempered if and only if

$$(3.6) \quad n_{a(i)} + m_{s-b(i)+1} \leq n + 1 \quad \text{for all } i \text{ (} 1 \leq i \leq n \text{)}.$$

We claim (3.6) holds if and only if

$$(3.7) \quad d(P) + d(Q) \leq n + 1.$$

The implication (3.7) \Rightarrow (3.6) is obvious. To see the converse implication, we take $a \in \{1, \dots, r\}$ and $b \in \{1, \dots, s\}$ such that $n_a = d(P)$ and $m_{s+1-b} = d(Q)$. Then the subsets $\{N(a-1)+1, \dots, N(a)\}$ and $\{M(b-1)+1, \dots, M(b)\}$ of $\{1, 2, \dots, n\}$ have $d(P)$ and $d(Q)$ elements, respectively. If (3.7) fails, then one finds a common element, say i . By (3.5), $a = a(i)$ and $b = b(i)$, hence (3.6) fails. Thus Theorem 1.2 is proved.

4 Proof of Lemma 3.2

In this section, we show Lemma 3.2, hence complete the proof of Theorems 1.2 and 1.3. Actually, we prove a generalization of Lemma 3.2 (see Lemma 4.1 below) which will be used also in an L^p estimate of matrix coefficients (Theorem 4.4).

4.1 Reduction to quadratic inequalities

We recall that \mathfrak{h} is a Lie subalgebra of $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$ containing the Lie algebra \mathfrak{a} of diagonal matrices. We also recall the notation $E_I = \sum_{i \in I} E_{ii} \in \mathfrak{a}$ for a subset I of $\{1, \dots, n\}$. We prove the following.

Lemma 4.1. Suppose p is an even integer ≥ 2 . Then the inequality

$$p\rho_{\mathfrak{h}}(E_I) \leq (p-1)\rho_{\mathfrak{g}}(E_I)$$

is true for all subsets I as soon as it is true when I is a singleton.

Remark 4.2. An analogous statement to Lemma 4.1 fails for $p = 3$, for instance, when $n = 4$ and \mathfrak{h} is a maximal parabolic subalgebra of dimension 12.

Let $\{f_1, \dots, f_n\}$ be the standard basis of \mathbb{R}^n , and $W_j = \mathbb{R}f_j$ ($1 \leq j \leq n$). By definition, \mathfrak{a} is a subalgebra of \mathfrak{h} which is of the form $\mathfrak{gl}(W_1) \oplus \dots \oplus \mathfrak{gl}(W_n)$. Let \mathfrak{l} be a maximal reductive subalgebra of \mathfrak{h} of this type, namely, maximal among all reductive subalgebras of \mathfrak{h} containing \mathfrak{a} which is of the form $\mathfrak{gl}(V_1) \oplus \dots \oplus \mathfrak{gl}(V_r)$ for some direct sum decomposition $\mathbb{R}^n = V_1 \oplus \dots \oplus V_r$ where each V_j is spanned by a subset of the standard basis. We set

$$n_k := \dim V_k \text{ and } m_k \equiv m_k(I) := \#\{i \in I \mid f_i \in V_k\} \text{ so that}$$

$$n_1 + \dots + n_r = n, \quad m_1 + \dots + m_r = \#I \text{ and } 0 \leq m_k \leq n_k, \text{ for all } k \leq r.$$

Similarly to (3.1), we set $\varepsilon_{k\ell} := 1$ if $\text{Hom}_{\mathbb{R}}(V_{\ell}, V_k) \subset \mathfrak{h}$, and $\varepsilon_{k\ell} := 0$ otherwise. One has $\varepsilon_{kk} = 1$ ($1 \leq k \leq r$) and $\varepsilon_{k\ell} + \varepsilon_{\ell k} \in \{0, 1\}$ by the maximality of \mathfrak{l} . To compute $\rho_{\mathfrak{g}}(E_I)$ and $\rho_{\mathfrak{h}}(E_I)$, we first observe

$$\text{ad}(E_{aa})E_{ij} = (\delta_{ai} - \delta_{aj})E_{ij},$$

where δ_{ab} denotes the Kronecker delta. Hence one has

$$\text{ad}(E_I)E_{ij} = \begin{cases} E_{ij} & \text{if } i \in I, j \notin I, \\ -E_{ij} & \text{if } i \notin I, j \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Summing up the absolute values of the eigenvalues of $\text{ad}(E_I)$ on \mathfrak{g} , one has

$$\rho_{\mathfrak{g}}(E_I) = \#I(n - \#I) = \sum_{1 \leq k, \ell \leq r} m_k(n_{\ell} - m_{\ell}).$$

Similarly, summing up the absolute values of the eigenvalues of $\text{ad}(E_I)$ on \mathfrak{h} , one has from the definition of $\varepsilon_{k\ell}$ the following formula:

$$2\rho_{\mathfrak{h}}(E_I) = \sum_{1 \leq k, \ell \leq r} \varepsilon_{k\ell}(m_k(n_{\ell} - m_{\ell}) + m_{\ell}(n_k - m_k)) = \sum_{1 \leq k, \ell \leq r} b_{k\ell}m_k(n_{\ell} - m_{\ell}),$$

where $b_{kk} = 2$ and $b_{k\ell} = \varepsilon_{k\ell} + \varepsilon_{\ell k}$ ($k \neq \ell$). Hence, setting $a_{kk} = 1$ and $a_{k\ell} = 1 + \frac{p}{2}(\varepsilon_{k\ell} + \varepsilon_{\ell k} - 2)$, one has

$$p\rho_{\mathfrak{h}}(E_I) - (p-1)\rho_{\mathfrak{g}}(E_I) = \sum_{1 \leq k, \ell \leq r} a_{k\ell}m_k(n_{\ell} - m_{\ell}).$$

Since $\varepsilon_{k\ell} + \varepsilon_{\ell k} \in \{0, 1\}$, we see $a_{k\ell} \in \{1-p, 1-\frac{p}{2}\}$ for all $k \neq \ell$, in particular, $a_{k\ell}$ are non-positive integers when p is even. Hence Lemma 4.1 follows from Lemma 4.3 below.

4.2 Quadratic inequalities

This section is independent of the previous one. We forget about Lie algebras. We fix integers $r \geq 1$, $n_1, \dots, n_r \geq 1$ and $(a_{k\ell})_{1 \leq k, \ell \leq r}$ a symmetric matrix with integer coefficients which are equal to 1 on the diagonal and are non-positive outside the diagonal:

$$a_{k\ell} = a_{\ell k} \in -\mathbb{N} \text{ for all } k \neq \ell \text{ and } a_{\ell\ell} = 1 \text{ for all } \ell.$$

Here, we used the notation $\mathbb{N} = \{0, 1, 2, \dots\}$. We denote by $\mathbf{e}_\ell \in \mathbb{N}^r$ the r -tuple $\mathbf{e}_\ell = (\delta_{k,\ell})_{1 \leq k \leq r}$. We fix $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, and set

$$f(\mathbf{m}) = \sum_{1 \leq k, \ell \leq r} a_{k\ell} m_k (n_\ell - m_\ell) \text{ for } \mathbf{m} = (m_1, \dots, m_r) \in \mathbb{N}^r.$$

For instance, one has $f(\mathbf{e}_{\ell_0}) = n_{\ell_0} - 1 + \sum_{\ell \neq \ell_0} a_{\ell_0 \ell} n_\ell$.

Lemma 4.3. Assume that $f(\mathbf{e}_\ell) \leq 0$ for all $1 \leq \ell \leq r$. Then one has $f(\mathbf{m}) \leq 0$ for all \mathbf{m} in \mathbb{N}^r with $\mathbf{n} - \mathbf{m} \in \mathbb{N}^r$.

Proof. We argue by induction on $s := \sum_k m_k$. Our assumption tells us that the conclusion is true for $s \leq 1$. We assume $s \geq 2$ and distinguish two cases.

Case 1 : there exists $1 \leq \ell \leq r$ such that $\sum_k a_{k\ell} m_k \geq 1$.

In this case, we fix such an ℓ . Since $a_{k\ell} \leq 0$ for all $k \neq \ell$ and $a_{\ell\ell} > 0$, we can write $\mathbf{m} = \mathbf{m}' + \mathbf{e}_\ell$ with $\mathbf{m}' \in \mathbb{N}^r$. Since $a_{\ell\ell} = 1$ and $a_{\ell k} = a_{k\ell}$, one has

$$f(\mathbf{m}) = f(\mathbf{m}') + f(\mathbf{e}_\ell) + 2 - 2 \sum_k a_{k\ell} m_k.$$

Using our assumptions and the induction hypothesis, we get $f(\mathbf{m}) \leq 0$.

Case 2 : For all $1 \leq \ell \leq r$, one has $\sum_k a_{k\ell} m_k \leq 0$.

In this case, since $n_\ell - m_\ell \geq 0$ for all ℓ , the inequality $f(\mathbf{m}) \leq 0$ follows directly from the definition of $f(\mathbf{m})$.

Since the coefficients $a_{k\ell}$ are integers, these two cases are the only possibilities and this ends the proof of Lemma 4.3 and hence of Lemma 4.1. \square

4.3 L^p -estimate of matrix coefficients

When H is reductive, Lemma 4.1 determines an explicit bound of p such that $L^2(G/H)$ is almost L^p . We end this section with the following:

Theorem 4.4. Let $n_1 + \dots + n_r \leq n$ and $p \in 2\mathbb{N}$. We set $m := \max(n_1, \dots, n_r)$. Then one has the equivalence:

- (i) $L^2(GL_n(\mathbb{R})/(GL_{n_1}(\mathbb{R}) \times \dots \times GL_{n_r}(\mathbb{R})))$ is almost L^p .
- (ii) $m \leq n - \frac{n-1}{p}$.

The case $p = 2$ was proved in [5, Thms. 1.4 and 3.1].

Proof. For $\mathfrak{h} = \mathfrak{gl}_{n_1}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{gl}_{n_r}(\mathbb{R})$, we set

$$(4.1) \quad c \equiv c(\mathfrak{h}) := \min_{1 \leq i \leq n} \frac{\rho_{\mathfrak{g}}(E_{ii})}{\rho_{\mathfrak{h}}(E_{ii})} = \frac{2(n-1)}{\max_{1 \leq i \leq n} \dim \text{Image}(\text{ad}(E_{ii}) : \mathfrak{h} \rightarrow \mathfrak{h})}.$$

By definition, $c(\mathfrak{h}) = \frac{n-1}{m-1}$, and therefore $p_{\mathfrak{g}/\mathfrak{h}} = \frac{1}{c(\mathfrak{h})-1} = \frac{m-1}{n-m}$ by (3.4) and Lemma 4.1. Then Theorem 4.4 follows from the criterion given in Fact 2.5 (2). \square

5 Appendix — the opposite parabolic case

So far we have discussed the temperedness of the tensor product representations $\Pi_1 \otimes \Pi_2$ when Π_1 and Π_2 are induced from unitary representations of parabolic subgroups P and Q of $G = GL_n$, respectively, see Problem 1.1. In this appendix, we discuss Problem 1.1 for other reductive groups G under the assumption that Q is the opposite parabolic subgroup of P . In this case $P \cap Q$ is a reductive subgroup, and we can utilize the list of pairs (G, H) of real reductive algebraic groups for which $L^2(G/H)$ is non-tempered [5]. The main result of this section is the classification of the pairs (G, P) for which $\text{Ind}_P^G(\mathbf{1}) \otimes \text{Ind}_Q^G(\mathbf{1})$ is tempered, see Theorem 5.1.

To describe the classification, we fix some notation. For a reductive Lie group L , there is a unique maximal connected normal non-compact semisimple subgroup, denoted by L_{ns} , to which we refer as the *non-compact semisimple part* of L . Its Lie algebra \mathfrak{l}_{ns} is an ideal of \mathfrak{l} contained in $[\mathfrak{l}, \mathfrak{l}]$.

In what follows, we assume that the real simple Lie group G has at most finitely many connected components and that the identity component has finite center.

Theorem 5.1. Let G be a non-compact real simple Lie group, P a proper parabolic subgroup, and Q the opposite parabolic. We set $L := P \cap Q$, which is a Levi subgroup of P (and also of Q). We write \mathfrak{l} for the Lie algebra of L . Then the following three conditions on the pair (G, P) are equivalent:

- (i) The tensor product representation $\text{Ind}_P^G(\sigma) \otimes \text{Ind}_Q^G(\tau)$ is tempered for all unitary representations σ of P and τ of Q .
- (ii) The tensor product representation $\text{Ind}_P^G(\mathbf{1}) \otimes \text{Ind}_Q^G(\mathbf{1})$ is tempered.
- (iii) One of the following conditions holds:

Case (a). P is any proper parabolic subgroup when $\text{rank}_{\mathbb{R}} \mathfrak{g} = 1$.

- Case (b). P is any proper parabolic subgroup when $\mathfrak{g} = \mathfrak{su}(p, q)$ ($p+q \leq 5$), $\mathfrak{so}(p, q)$ ($p+q \leq 6$), $\mathfrak{sp}(p, q)$ ($p+q \leq 4$), $\mathfrak{e}_{6(2)}$, $\mathfrak{e}_{6(-14)}$, $\mathfrak{e}_{6(-26)}$, $\mathfrak{f}_{4(4)}$, $\mathfrak{f}_{4, \mathbb{C}}$, $\mathfrak{g}_{2(2)}$, or $\mathfrak{g}_{2, \mathbb{C}}$.
- Case (c). \mathfrak{g} is complex simple or split. The semisimple part $[\mathfrak{l}, \mathfrak{l}]$ of \mathfrak{l} is not in the list of Table 1.
- Case (d). \mathfrak{g} is neither complex nor split. The semisimple part $[\mathfrak{l}, \mathfrak{l}]$ or its non-compact semisimple part \mathfrak{l}_{n_s} is not in the list of Table 2.

Table 1: \mathfrak{g} is complex or split

| \mathfrak{g} | $[\mathfrak{l}, \mathfrak{l}]$ |
|------------------|--|
| \mathfrak{a}_n | $\mathfrak{a}_{n_1} \oplus \cdots \oplus \mathfrak{a}_{n_k}$ $2 \max_{1 \leq j \leq k} n_j \geq n + 1$ |
| \mathfrak{b}_n | $\mathfrak{a}_{n_1} \oplus \cdots \oplus \mathfrak{a}_{n_k} \oplus \mathfrak{b}_m$ $2m \geq n + 1$ |
| \mathfrak{c}_n | $\mathfrak{a}_{n_1} \oplus \cdots \oplus \mathfrak{a}_{n_k} \oplus \mathfrak{c}_m$ $2m \geq n + 1$ |
| \mathfrak{d}_n | $\mathfrak{a}_{n_1} \oplus \cdots \oplus \mathfrak{a}_{n_k} \oplus \mathfrak{d}_m$ $2m \geq n + 2$ |
| \mathfrak{d}_n | \mathfrak{a}_{n-1} $n \geq 3$ |
| \mathfrak{e}_6 | \mathfrak{d}_5 |
| \mathfrak{e}_7 | \mathfrak{d}_6 or \mathfrak{e}_6 |
| \mathfrak{e}_8 | \mathfrak{e}_7 |

Table 2: \mathfrak{g} is neither complex nor split

| \mathfrak{g} | \mathfrak{l}_{ns} | |
|-------------------------|--|---|
| $\mathfrak{su}(p, q)$ | $\mathfrak{su}(p - k, q - k)$ | $1 \leq k \leq \min(p - 1, q - 1, \frac{p+q-2}{4})$ |
| $\mathfrak{so}(p, q)$ | $\mathfrak{so}(p - k, q - k)$ | $1 \leq k \leq \min(p - 1, q - 1, \frac{p+q-3}{4})$ |
| $\mathfrak{sp}(p, q)$ | $\mathfrak{sp}(p - k, q - k)$ | $1 \leq k \leq \min(p - 1, q - 1, \frac{p+q-1}{4})$ |
| \mathfrak{g} | $[\mathfrak{l}, \mathfrak{l}]$ | |
| $\mathfrak{su}^*(2n)$ | $\bigoplus_{j=1}^k \mathfrak{su}^*(2m_j)$ | $2 \max_{1 \leq j \leq k} m_j \geq n + 2$ |
| $\mathfrak{so}^*(4n)$ | $\mathfrak{su}^*(2n)$ | $n \geq 2$ |
| $\mathfrak{so}^*(2n)$ | $\mathfrak{so}^*(2m) \oplus \bigoplus_{j=1}^k \mathfrak{su}^*(2m_j)$ | $m \geq n + 2$ |
| $\mathfrak{e}_{7(-5)}$ | $\mathfrak{so}^*(12)$ | |
| $\mathfrak{e}_{7(-25)}$ | $\mathfrak{so}(2, 10)$ or $\mathfrak{e}_{6(-26)}$ | |
| $\mathfrak{e}_{8(-24)}$ | $\mathfrak{e}_{7(-25)}$ | |

Proof. Since the diagonal map $G \rightarrow G \times G$ induces an open dense embedding $\iota: G/L \hookrightarrow G/P \times G/Q$, the tensor product representation $\text{Ind}_P^G(\sigma) \otimes \text{Ind}_Q^G(\tau)$ is unitarily equivalent to $\text{Ind}_L^G(\sigma \otimes \tau)$ via the pullback ι^* . Then the equivalence (i) \iff (ii) follows from the Herz majoration principle (Lemma 2.4) as in Theorem 1.2.

To see the equivalence (ii) \iff (iii), we may and do assume that G is an algebraic Lie group without loss of generality by [4, Cor. 3.3 and Rem. 3.4]. We shall write $G_{\mathbb{C}}$ for a complex Lie group which contains G as a real form.

The tensor product representation $\text{Ind}_P^G(\mathbf{1}) \otimes \text{Ind}_Q^G(\mathbf{1})$ is unitarily equivalent to $L^2(G/L)$ via the pullback ι^* . So our main task is to give a classification of the Levi subgroups L of G such that the regular representation on $L^2(G/L)$ is non-tempered.

We divide the proof into the following cases.

Case I. G is complex or split.

Case II. G is neither complex nor split.

Case II-a. $\mathfrak{g} \neq \mathfrak{sl}(2n - 1, \mathbb{H}), \mathfrak{e}_{6(-26)}, \text{ or } \mathfrak{e}_{6(-14)}$.

Case II-b. $\mathfrak{g} = \mathfrak{sl}(2n - 1, \mathbb{H}), \mathfrak{e}_{6(-26)}, \text{ or } \mathfrak{e}_{6(-14)}$.

Case I. G is complex or split. In this case one can read the list of the pairs (G, L) such that $L^2(G/L)$ is non-tempered from the classification results of tempered reductive homogeneous spaces in [5, Thms. 3.1 and 4.1] and from a description of Levi subgroups L of complex Lie algebras by the Dynkin diagram. We illustrate the argument by taking \mathfrak{g} to be $\mathfrak{a}_n^{\mathbb{C}}, \mathfrak{e}_8^{\mathbb{C}}$ or their split real forms as examples. For instance, let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ and \mathfrak{l} be any Levi subalgebra. Then the semisimple part $[\mathfrak{l}, \mathfrak{l}]$ of \mathfrak{l} is of the form $\mathfrak{sl}_{m_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{sl}_{m_k}(\mathbb{C})$ for some $m_1, \dots, m_k (\geq 2)$ with $m_1 + \cdots + m_k \leq n + 1$. By [4, Cor. 3.3], $L^2(G/L)$ is non-tempered if and only if $L^2(G/[L, L])$ is non-tempered. By [5, Thm. 3.1], this happens if and only if $[\mathfrak{l}, \mathfrak{l}]$ contains $\mathfrak{sl}_p(\mathbb{C})$ as an ideal for some p with $p \geq (n + 1 - p) + 2$ or coincides with $\mathfrak{sp}_p(\mathbb{C})$ ($2p = n + 1$). The former happens when $2 \max m_j \geq n + 3$ and the latter never happens in our setting. Putting $n_j = m_j - 1$, we conclude that $\text{Ind}_P^G(\mathbf{1}) \otimes \text{Ind}_Q^G(\mathbf{1})$ is non-tempered if and only if $2 \max n_j \geq n + 1$. The same conclusion holds if \mathfrak{g} is the split real form $\mathfrak{sl}_{n+1}(\mathbb{R})$ of $\mathfrak{sl}_{n+1}(\mathbb{C})$ by [3, Prop. 5.2]. This shows the first row in the Table 1. Of course, the conclusion matches Theorem 1.2 because $d(P) + d(Q) = 2 \max m_j = 2 \max(n_j + 1)$ (We note that n_j and n in Theorem 1.2 are $n_j + 1$ and $n + 1$ here.)

If \mathfrak{g} is a complex simple Lie algebra $\mathfrak{e}_8^{\mathbb{C}}$, then the classification in [5, Thm. 4.1] tells us that $L^2(G/H)$ is non-tempered if and only if $\mathfrak{e}_7^{\mathbb{C}} \subset \mathfrak{h} \subset \mathfrak{e}_7^{\mathbb{C}} \oplus \mathfrak{sl}_2(\mathbb{C})$ when H is a (proper) complex reductive subgroup of G . On the other hand, the Dynkin diagram of type E_8 shows that a Levi subalgebra \mathfrak{l} of $\mathfrak{e}_8^{\mathbb{C}}$ containing $\mathfrak{e}_7^{\mathbb{C}}$ is either $\mathfrak{e}_7^{\mathbb{C}} \oplus \mathbb{C}$ or $\mathfrak{e}_8^{\mathbb{C}}$ itself. This gives the last row in Table 1. The same conclusion holds for the split real form by [3, Prop. 5.2]. Table 1 for other (complex or split) simple Lie algebras is obtained similarly by using the Dynkin diagram and [5, Thms. 3.1 and 4.1].

Case II. G is neither complex nor split. We recall from [4, Prop. 3.1] that $L^2(G/L)$ is tempered if and only if $L^2(G/L_{ns})$ is tempered. Thus the condition (ii) is equivalent to that $L^2(G/L_{ns})$ is tempered. We note that the non-compact semisimple factor L_{ns} may be much smaller than L in Case II. Accordingly, it may well happen that $L^2(G/L)$ is tempered but $L^2(G_{\mathbb{C}}/L_{\mathbb{C}})$ is not tempered. This means that the tensor product representation $\text{Ind}_P^G(\mathbf{1}) \otimes \text{Ind}_Q^G(\mathbf{1})$ in Case II is more likely to be tempered than $\text{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}}(\mathbf{1}) \otimes \text{Ind}_{Q_{\mathbb{C}}}^{G_{\mathbb{C}}}(\mathbf{1})$ which was treated in Case I.

For example, if $\text{rank}_{\mathbb{R}} G = 1$, then any (proper) parabolic subgroup is a minimal parabolic subgroup, hence $L_{ns} = \{e\}$ and thus $L^2(G/L)$ is tempered. For the computation of L_{ns} in the general case, we can use the Satake diagram, which we recall now, see [9, Chap. 10] for example.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition, \mathfrak{a} a maximal abelian subspace in \mathfrak{p} , and extend \mathfrak{a} to a Cartan subalgebra \mathfrak{j} of \mathfrak{g} . We take compatible positive systems $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ and $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ such that $\alpha|_{\mathfrak{a}} \in \Sigma^+(\mathfrak{g}, \mathfrak{a}) \cup \{0\}$, whenever $\alpha \in \Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. Then one has a surjective map $r: \Psi \rightarrow \Phi \cup \{0\}$ where Ψ and Φ are the sets of simple roots of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ and $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, respectively. The Satake diagram is an enriched Dynkin diagram for Ψ by coloring $r^{-1}(\{0\})$ black and by connecting two white nodes $\alpha \neq \beta$ by arrows if $r(\alpha) = r(\beta) \neq 0$. Any Levi subalgebra \mathfrak{l} of a real semisimple Lie algebra \mathfrak{g} is conjugate to $\bigoplus \mathfrak{g}(\mathfrak{a}; \lambda)$ by an inner automorphism of \mathfrak{g} , where the sum is taken over all λ in the \mathbb{Z} -span of S for some subset S of Φ . Then the Dynkin diagram for $\Delta(\mathfrak{l}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ of the complexified Lie algebra $\mathfrak{l}_{\mathbb{C}}$ is given by $r^{-1}(S \cup \{0\})$. Let V be the union of the connected components in $r^{-1}(S \cup \{0\})$ that consist of black nodes in the Satake diagram. Then the Dynkin diagram for $(\mathfrak{l}_{ns})_{\mathbb{C}}$ is given by $r^{-1}(S \cup \{0\}) \setminus V$. With this in mind, we apply the classification theory in [5] as follows.

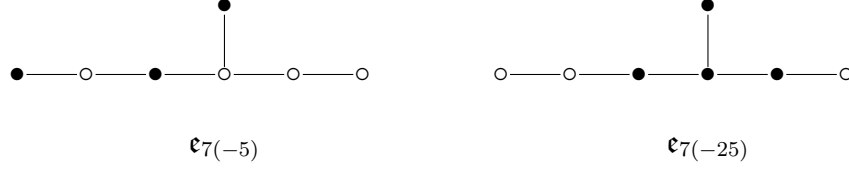
Case II-a. Assume \mathfrak{g} is not isomorphic to $\mathfrak{sl}(2n-1, \mathbb{H}) (\simeq \mathfrak{su}^*(4n-2))$, $\mathfrak{e}_{6(-26)}$, or $\mathfrak{e}_{6(-14)}$. Then [5, Thm. 1.4] shows that the following conditions are equivalent:

- $L^2(G/L_{ns})$ is a tempered representation of G ;
- $L^2(G_{\mathbb{C}}/(L_{ns})_{\mathbb{C}})$ is a tempered representation of $G_{\mathbb{C}}$.

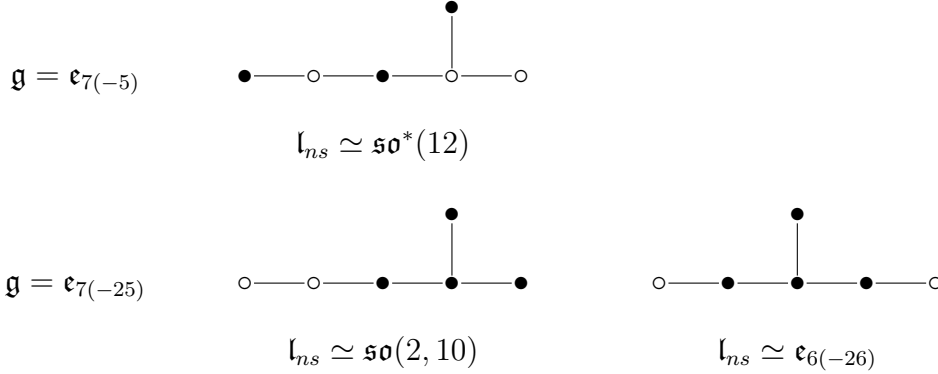
In this case, we apply the classification result in [5, Thms. 3.1 and 4.1] to the complex homogeneous space $G_{\mathbb{C}}/(L_{ns})_{\mathbb{C}}$. We illustrate the argument by taking $\mathfrak{g} = \mathfrak{su}(p, q)$ and real forms of $\mathfrak{e}_7^{\mathbb{C}}$ as examples. First, let us consider $\mathfrak{g} = \mathfrak{su}(p, q)$ ($p \geq q$). Then any Levi subalgebra \mathfrak{l} of \mathfrak{g} is of the form $\mathfrak{l} \simeq \bigoplus_{j=1}^{\ell} \mathfrak{gl}_{m_j}(\mathbb{C}) \oplus \mathfrak{su}(p-k, q-k)$ where $m_1 + \dots + m_{\ell} = k \leq q$. Accordingly, the complexification of \mathfrak{l}_{ns} is given as $(\mathfrak{l}_{ns})_{\mathbb{C}} \simeq \bigoplus_{j=1}^{\ell} (\mathfrak{sl}_{m_j}(\mathbb{C}) \oplus \mathfrak{sl}_{m_j}(\mathbb{C})) \oplus \mathfrak{sl}_{p+q-2k}(\mathbb{C})$ if $q > k$ and $(\mathfrak{l}_{ns})_{\mathbb{C}} \simeq \bigoplus_{j=1}^{\ell} (\mathfrak{sl}_{m_j}(\mathbb{C}) \oplus \mathfrak{sl}_{m_j}(\mathbb{C}))$ if $q = k$. Thus [5, Thm. 3.1] implies that $L^2(G_{\mathbb{C}}/(L_{ns})_{\mathbb{C}})$ is non-tempered if and only if $2(p+q-2k) \geq p+q+2$ and $(p \geq)q > k$. This shows the first row in Table 2.

Next let us treat real forms of $\mathfrak{e}_7^{\mathbb{C}}$. By the classification [5, Thm. 4.1], for any real form \mathfrak{g} of $\mathfrak{e}_7^{\mathbb{C}}$, $L^2(G_{\mathbb{C}}/(L_{ns})_{\mathbb{C}})$ is non-tempered if and only if $(\mathfrak{l}_{ns})_{\mathbb{C}}$ contains $\mathfrak{d}_6^{\mathbb{C}}$ or $\mathfrak{e}_6^{\mathbb{C}}$. There are four real forms of $\mathfrak{e}_7^{\mathbb{C}}$, namely, a compact real form, $\mathfrak{e}_{7(7)}$ (= EV), $\mathfrak{e}_{7(-5)}$ (= EVI), and $\mathfrak{e}_{7(-25)}$ (= EVII). The second one is split, and was treated in Case I. For the remaining two real forms, the Satake

diagrams are given as below.



Then the non-compact semisimple part \mathfrak{l}_{ns} of a (real) Levi subalgebra \mathfrak{l} having the property $(\mathfrak{l}_{ns})_{\mathbb{C}} \supset \mathfrak{d}_6^{\mathbb{C}} \simeq \mathfrak{so}_{12}(\mathbb{C})$ or $\mathfrak{e}_6^{\mathbb{C}}$ is listed as follows.



This shows the last two and three rows in Table 2.

Case II-b. $\mathfrak{g} = \mathfrak{su}^*(4m - 2)$, $\mathfrak{e}_{6(-26)}$, or $\mathfrak{e}_{6(-14)}$. In this case, it may happen that $L^2(G/H)$ is tempered but $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$ is not tempered for some reductive subgroup H even when $H = H_{ns}$, see [5, Thm. 1.4 (ii)-(iv)] for the list of such H . We need to take this exceptional case into account if such H arises as the non-compact semisimple part L_{ns} of a Levi subgroup L of G . For example, suppose $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{H}) (\simeq \mathfrak{su}^*(2n))$. Then any Levi subalgebra \mathfrak{l} of \mathfrak{g} is of the form

$$\mathfrak{gl}(m_1, \mathbb{H}) \oplus \cdots \oplus \mathfrak{gl}(m_k, \mathbb{H}) \simeq \bigoplus_{j=1}^k \mathfrak{su}^*(2m_j) \oplus \mathbb{R}^k,$$

where $m_1 + \cdots + m_k = n$. We may and do assume that $m_j > 1$ for $1 \leq j \leq \ell$ and $m_j = 1$ for $\ell + 1 \leq j \leq k$. Then $\mathfrak{l}_{ns} \simeq \bigoplus_{j=1}^{\ell} \mathfrak{su}^*(2m_j)$ because $\mathfrak{su}^*(2) \simeq \mathfrak{su}(2)$. By [5, Thm. 1.4 (iii)], the exceptional case occurs when $\max_{1 \leq j \leq \ell} m_j = \frac{1}{2}(n + 1)$, namely, $L^2(G/L_{ns})$ is non-tempered if and only if $\max_{1 \leq j \leq \ell} m_j \neq \frac{1}{2}(n + 1)$ and $L^2(G_{\mathbb{C}}/(L_{ns})_{\mathbb{C}})$ is non-tempered. The latter

condition amounts to $\max_{1 \leq j \leq \ell} m_j \geq \frac{1}{2}(n+1)$ by [5, Thm. 3.1] (see also [5, Ex. 8.8]) because $(\mathfrak{I}_{ns})_{\mathbb{C}} \simeq \bigoplus_{j=1}^{\ell} \mathfrak{sl}_{2m_j}(\mathbb{C})$. Hence $L^2(G/L)$ is non-tempered if and only if $2 \max_{1 \leq j \leq \ell} m_j > n+1$, or equivalently, $2 \max_{1 \leq j \leq k} m_j > n+1$, as listed in Table 2. Other cases are similar and easier.

This completes the proof of Theorem 5.1. □

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