Tempered homogeneous spaces IV

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Abstract

Let G be a complex semisimple Lie group and H a complex closed connected subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. We prove that the regular representation of G in $L^2(G/H)$ is tempered if and only if the orthogonal of $\mathfrak{h}$ in $\mathfrak{g}$ contains regular elements by showing simultaneously the equivalence to other striking conditions such as $\mathfrak{h}$ has a solvable limit algebra.

Contents

1 Introduction

Let $X = G/H$ be a homogeneous space of a Lie group G. This article is the fourth one in our series of papers [1, 2, 3] dealing with a question about when $L^2(X)$ is tempered, *i.e.*, to be weakly contained in the regular representation in $L^2(G)$. We proved in [1, 2] a criterion (1.1) below by an analytic and dynamical approach when G is real reductive, and accomplished in [3] a classification of all the pairs (G, H) of real reductive Lie groups for which $L^2(X)$ is non-tempered. We refer to the introduction of both [1] and [2] for some motivations and perspectives on this question.

In this article we find a striking relationship of this question with other disciplines such as a topological condition concerning the "limit subalgebras" and a geometric condition concerning coadjoint orbits. The relationship is perfect when G is complex reductive (Theorem 1.6). For the proof, we explore the temperedness of $L^2(X)$ beyond reductive setting (Theorem 1.1).

1.1 Real homogeneous spaces

> We extend the criterion in [1, 2] for the temperedness of $L^2(X)$ to the general setting where X is a homogeneous of a *real* Lie group which is *not* necessarily reductive.

In the first two papers [1] and [2], we first noticed that the property of $L^2(G/H)$ being tempered depends only on the pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras, and introduced for an $\mathfrak{h}$-module V and $Y \in \mathfrak{h}$, the quantity

$$\rho_V(Y) \quad := \quad \text{half the sum of the absolute values of the real part of the eigenvalues of } Y \text{ in } V.$$

We found the following temperedness criterion when G is a connected semisimple Lie group with finite center, and H is a connected closed subgroup:

$$L^2(G/H) \text{ is tempered} \quad \Longleftrightarrow \quad \rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}} \text{ on } \mathfrak{h}. \tag{1.1}$$

This criterion (1.1) was proven in [1] when $\mathfrak{h}$ is assumed to be semisimple by a dynamical approach, and was extended in [2] to arbitrary $\mathfrak{h}$ by an idea of "domination of G-spaces". Developing the techniques in a more general setting, we extend (1.1) without any reductive assumptions of $\mathfrak{g}$ and $\mathfrak{h}$:

Theorem 1.1 (see Theorem 3.2)**.** *Let G be a real algebraic Lie group, and H an algebraic subgroup. We fix maximal reductive subgroups G_{s} and H_{s} of G and H, respectively, such that $H_{\mathrm{s}} \subset G_{\mathrm{s}}$. Then one has the equivalence:*

$$L^2(G/H) \textit{ is } G_{\mathrm{s}}\textit{-tempered} \Leftrightarrow \rho_{\mathfrak{g}_{\mathrm{s}}} \leq 2\rho_{\mathfrak{g}/\mathfrak{h}} \textit{ on } \mathfrak{h}_{\mathrm{s}}.$$

Theorem 1.1 (and its further generalization to the Hilbert bundle valued case) serves as a "tool" in proving the relationship with other discipines, which is formulated in Theorem 1.6 below.

1.2 Temperedness condition and the orbit philosophy

> We discuss what the orbit philosophy suggests about the geometry of coadjoint orbits "corresponding to" the temperedness condition of $L^2(G/H)$.

Let $\mathfrak{g}$ be the Lie algebra of a Lie group G, and $\mathfrak{g}^*$ its dual. We denote by $\widehat{G}$ the unitary dual of G, *i.e.*, the set of equivalence classes of irreducible unitary representations of G. The orbit philosophy due to Kirillov–Kostant–Duflo expects an intimate connection of the unitary dual $\widehat{G}$ with the set of coadjoint orbits $\mathfrak{g}^*/\operatorname{Ad}^*(G)$. This works perfectly for simply connected nilpotent groups, but does not exactly for semisimple Lie groups. Nevertheless, $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ may be considered to be a fairly good approximation as a parameter set of $\widehat{G}$. As an expected functionality, the orbit philosophy also suggests that the disintegration of $L^2(G/H)$ would be supported on the subset of $\widehat{G}$ "corresponding to" the closure of $\mathrm{Ad}^*(G)\mathfrak{h}^\perp/\mathrm{Ad}^*(G)$ where $\mathfrak{h}^\perp := \mathrm{Ker}(\mathfrak{g}^* \to \mathfrak{h}^*)$. On the other hand, for a connected semisimple Lie group G, loosely speaking, irreducible tempered representations of G are supposed to be obtained as "geometric quantization" of semisimple coadjoint orbits having amenable isotropy subgroups. Thus one expects that the temperedness of the unitary representation $L^2(G/H)$ may be characterized by its "classical limit" in the geometry of coadjoint orbits via the orbit philosophy. When G is a complex Lie group, we formulate a precise criterion below from this viewpoint.

1.3 Complex homogeneous spaces

> In the third paper [3] and in this one, we extend and deepen the theory of tempered homogeneous spaces with focus on the complex setting.

Suppose $\mathfrak{g}$ is a complex semisimple algebra. Via the Killing form

$$K(X,Y) := \mathrm{tr}(\mathrm{ad}X\,\mathrm{ad}Y),$$

we identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and $\mathfrak{h}^\perp$ with the orthogonal subspace of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to K. An element $X \in \mathfrak{g}$ is called *regular* if its centralizer $\mathfrak{z}_{\mathfrak{g}}(X)$ in $\mathfrak{g}$ has minimal dimension, *i.e.*, $\dim \mathfrak{z}_{\mathfrak{g}}(X) = \mathrm{rank}\,\mathfrak{g}$. We denote by $\mathfrak{g}_{\mathrm{reg}}$ the set of regular elements X of $\mathfrak{g}$, and set

$$\mathfrak{h}^\perp_{\mathrm{reg}} := \mathfrak{h}^\perp \cap \mathfrak{g}_{\mathrm{reg}}.$$

In the third paper [3] we found yet another but more geometric tempered criterion for $L^2(G/H)$ when both $\mathfrak{g}$ and $\mathfrak{h}$ are assumed to be complex semisimple Lie algebras. As we see in Proposition 2.10 this geometric criterion can be reformulated as $\mathfrak{h}^\perp_{\mathrm{reg}} \neq \emptyset$. In the present paper, we extend this criterion to all complex Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$.

Theorem 1.2. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ be a complex Lie subalgebra. Then one has the equivalence :*

$$L^2(G/H) \text{ is tempered} \iff \mathfrak{h}^\perp_{\mathrm{reg}} \neq \emptyset. \tag{1.2}$$

Since the set $\mathfrak{h}^\perp_{\mathrm{reg}}$ is Zariski open in $\mathfrak{h}^\perp$, one always has the equivalence

$$\mathfrak{h}^\perp_{\mathrm{reg}} \neq \emptyset \iff \mathfrak{h}^\perp_{\mathrm{reg}} \text{ is dense in } \mathfrak{h}^\perp, \tag{1.3}$$

and thus Theorem 1.2 fits well into the aforementioned orbit philosophy.

One sees from [2, Cor. 5.6] that Theorem 1.2 for complex Lie groups yields the sufficiency of the temperedness in the real setting as well:

Corollary 1.3. *Let G be a real semisimple algebraic Lie group and H an algebraic subgroup. If $\mathfrak{h}^{\perp}_{\rm reg} \neq \emptyset$, then $L^2(G/H)$ is tempered.*

Remark 1.4. (1) The implications $\Longrightarrow$ in (1.2) and (1.5) are not always true for a real semisimple Lie group G. For instance, when G is not $\mathbb{R}$-split and H is a maximal compact subgroup, the representation $L^2(G/H)$ is tempered but $\mathfrak{h}^{\perp}_{\rm reg}$ is empty. Another example is given by $G/H = SL(3,\mathbb{H})/SL(2,\mathbb{H})$.
(2) Let $\mathfrak{g}_{\rm ame}$ denote the set of elements in $\mathfrak{g}$ with amenable stabilizer for the adjoint action of G. For reductive H, by [3, Thm. 1.5] and Lemma 2.14 below, one has the implication:

$$L^2(G/H) \text{ is tempered} \Longrightarrow \mathfrak{h}^{\perp} \cap \mathfrak{g}_{\rm ame} \text{ is dense in } \mathfrak{h}^{\perp}. \tag{1.4}$$

The converse implication (1.4) does not always hold even for semisimple symmetric spaces ([3, Sect. 8.5]).

By (1.1), our main task for Theorem 1.2 will be to prove the following.

Proposition 1.5. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a complex Lie subalgebra. Then one has the equivalence :*

$$2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}} \quad \Longleftrightarrow \quad \mathfrak{h}^{\perp}_{\rm reg} \neq \emptyset. \tag{1.5}$$

1.4 The equivalent conditions

We now introduce two other conditions that we will prove to be equivalent to (1.5).

Let us think of $\mathfrak{h}$ as a point in the variety $\mathcal{L}$ of all Lie subalgebras of $\mathfrak{g}$. One surprising feature of the equivalence (1.5) is that the left-hand side is a closed condition on $\mathfrak{h}$ while the right-hand side is an open condition on $\mathfrak{h}$. Since both conditions are invariant by conjugation by G, this remark suggests us to work with the adjoint orbit closure of $\mathfrak{h}$. As we will see, this new point of view will be very fruitful, first by suggesting new striking conditions equivalent to (1.5) and eventually by leading to a proof of (1.5).

Let $\mathrm{Ad}G$ be the adjoint group, let $\mathrm{Ad}G\,\mathfrak{h}$ be the $\mathrm{Ad}G$-orbit of $\mathfrak{h}$ in $\mathcal{L}$ and $\overline{\mathrm{Ad}G\,\mathfrak{h}}$ be the closure of this orbit. We introduce also the following two G-invariant algebraic subvarieties of $\mathcal{L}$:

$$\begin{aligned} \mathcal{L}_{sol} &:= \{\mathfrak{r} \in \mathcal{L} \mid \mathfrak{r} \text{ is solvable}\}, \\ \mathcal{L}_{mun} &:= \{\mathfrak{n} \in \mathcal{L} \mid \mathfrak{n} \text{ is maximal unipotent in } \mathfrak{g}\}. \end{aligned}$$

We recall that a Lie subalgebra is said to be unipotent if all its elements are nilpotent.

As we mentioned, we will prove the equivalence (1.5) by showing simultaneously the equivalence to other striking conditions that we introduce now. Let H be the closure of a connected subgroup of G with Lie subalgebra $\mathfrak{h}$.

$$\begin{aligned} Tem(\mathfrak{g},\mathfrak{h}) &: L^2(G/H) \text{ is tempered}, \\ Rho(\mathfrak{g},\mathfrak{h}) &: \rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}, \\ Sla(\mathfrak{g},\mathfrak{h}) &: \overline{\mathrm{Ad}G\,\mathfrak{h}} \cap \mathcal{L}_{sol} \neq \emptyset, \\ Tmu(\mathfrak{g},\mathfrak{h}) &: \text{there exists } \mathfrak{n} \in \mathcal{L}_{mun} \text{ such that } \mathfrak{h} \cap \mathfrak{n} = \{0\}, \\ Orb(\mathfrak{g},\mathfrak{h}) &: \mathfrak{h}^{\perp}_{\mathrm{reg}} \neq \emptyset. \end{aligned}$$

To refer to these conditions, we might say informally that
- $\mathfrak{h}$ is a tempered Lie subalgebra,
- $\mathfrak{h}$ satisfies the ρ-inequality,
- $\mathfrak{h}$ admits a solvable limit algebra,
- $\mathfrak{h}$ has a transversal maximal unipotent,
- $\mathfrak{h}^{\perp}$ meets a regular orbit.

Theorem 1.6. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a complex Lie subalgebra. Then the following five conditions are equivalent :*

$$Tem(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Rho(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Sla(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Tmu(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Orb(\mathfrak{g},\mathfrak{h}).$$

The proof of Theorem 1.6 will last up to Section 5.5.

Corollary 1.7. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. The set $\mathcal{L}_{sla}$ of Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ satisfying $Sla(\mathfrak{g},\mathfrak{h})$ is both closed and open in $\mathcal{L}$.*

Proof of Corollary 1.7. The condition $Rho(\mathfrak{g},\mathfrak{h})$ is closed, while the condition $Orb(\mathfrak{g},\mathfrak{h})$ is open. □

Corollary 1.8. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a complex Lie subalgebra. Choose $\mathfrak{h}' \in \overline{\mathrm{Ad}G\,\mathfrak{h}}$. Then one has the equivalence*

$$Sla(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Sla(\mathfrak{g},\mathfrak{h}'). \tag{1.6}$$

Proof of Corollary 1.8. This is a consequence of Corollary 1.7 □

The equivalence (1.6) can be reformulated as follows:

If the orbit closure $\overline{\mathrm{Ad}G\,\mathfrak{h}}$ contains at least one solvable $\mathfrak{h}''$, then any $\mathfrak{h}'$ in $\overline{\mathrm{Ad}G\,\mathfrak{h}}$ is solvable as far as $\mathrm{Ad}G\,\mathfrak{h}'$ is closed. (1.7)

Although the statement (1.6) is purely a structure theorem of Lie subalgebras, our proof of (1.6) relies on the theory of unitary representations via Theorem 1.6. We would like to point out that we are not aware of a more direct proof of (1.6).

Remark 1.9. We will explain in Theorem 5.1, how to extend the equivalence $Tem(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Rho(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Sla(\mathfrak{g},\mathfrak{h})$ to complex algebraic *non-semisimple* Lie algebras $\mathfrak{g}$. In particular, we will see in Corollary 5.2 that the equivalence (1.6) is true for any pair $\mathfrak{g} \supset \mathfrak{h}$ of complex Lie algebras.

1.5 Strategy of proof and organization

We now explain the strategy of the proof of Theorem 1.6. Since we already know from (1.1) the equivalence

$$Tem(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Rho(\mathfrak{g},\mathfrak{h})\,, \tag{1.8}$$

it remains to prove the equivalences

$$Rho(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Sla(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Tmu(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Orb(\mathfrak{g},\mathfrak{h}). \tag{1.9}$$

All these statements are purely algebraic and we will prove these implications by algebraic methods in Chapter 2 except for the implication

$$Sla(\mathfrak{g},\mathfrak{h}) \Longrightarrow Rho(\mathfrak{g},\mathfrak{h}). \tag{1.10}$$

The proof of this implication (1.10) is more delicate and will be given in Chapter 5. It will use an induction argument that reduces to the case where $\mathfrak{h}$ is semisimple. The induction argument will involve unitary representation theory and a parabolic subgroup G_0 of G containing H. This will force us to deal with algebraic groups G which are not semisimple.

The proof will also use the analytic interpretation of $Rho(\mathfrak{g},\mathfrak{h})$ as a temperedness criterion, and the disintegration of the unitary representation $L^2(G_0/H)$. Indeed we will spend Chapters 3 and 4 proving the extension of the temperedness criterion (1.1) that we need. This extension (Theorem 1.1) is valid for any real algebraic Lie group G and any real algebraic subgroup H. The proof of this extension will rely on the Hertz majoration principle for unitary representations.

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2 Sla, Tmu and Orb

In this chapter, we focus on the proof of the implications in (1.9) that uses only algebraic tools. That is all of them except for the implication (1.10).

2.1 Sla and Tmu

We begin with the easiest of all these equivalences.

Proposition 2.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}\subset\mathfrak{g}$ be a complex Lie subalgebra. Then, one has the equivalence*

$$Sla(\mathfrak{g},\mathfrak{h}) \iff Tmu(\mathfrak{g},\mathfrak{h})\,. \tag{2.1}$$

Proof of Proposition 2.1. $\Longrightarrow$ Since we assume $Sla(\mathfrak{g},\mathfrak{h})$, there exists a sequence $(g_n)_{n\geq 1}$ in G such that the limit $\mathfrak{r}=\lim\limits_{n\to\infty}\mathrm{Ad}g_n\,\mathfrak{h}$ exists and is a solvable Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{r}$ is solvable, there exists a Borel subalgebra $\mathfrak{b}^-$ of $\mathfrak{g}$ containing $\mathfrak{r}$. Let $\mathfrak{n}$ be a maximal unipotent subalgebra of $\mathfrak{g}$ which is opposite to $\mathfrak{b}^-$, so that one has $\mathfrak{b}^-\oplus\mathfrak{n}=\mathfrak{g}$. In particular, one has $\mathfrak{r}\cap\mathfrak{n}=\{0\}$ and, for n large, $\mathrm{Ad}g_n\,\mathfrak{h}\cap\mathfrak{n}=\{0\}$. This proves $Tmu(\mathfrak{g},\mathfrak{h})$.

$\Longleftarrow$ Since we assume $Tmu(\mathfrak{g},\mathfrak{h})$, there exists a maximal unipotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ such that $\mathfrak{h}\cap\mathfrak{n}=\{0\}$. Let $\mathfrak{b}$ be the Borel subalgebra containing $\mathfrak{n}$, let $\mathfrak{j}$ be a Cartan subalgebra of $\mathfrak{b}$ so that $\mathfrak{b}=\mathfrak{j}\oplus\mathfrak{n}$ and let $\mathfrak{n}^-$ be the maximal unipotent subalgebra of $\mathfrak{g}$ which is opposite to $\mathfrak{b}$ and normalized by $\mathfrak{j}$. Let

$\Delta = \Delta(\mathfrak{g}, \mathfrak{j})$ be the root system of $\mathfrak{j}$ in $\mathfrak{g}$. We write $\Delta = \Delta^+ \cup \Delta^-$ where Δ^+ and Δ^- are respectively the roots of $\mathfrak{j}$ in $\mathfrak{n}$ and $\mathfrak{n}^-$. Choose an element $X \in \mathfrak{j}$ in the positive Weyl chamber, this means that for all $\alpha \in \Delta^+$, one has $\mathbf{Re}(\alpha(X)) > 0$. Since $\mathfrak{h} \cap \mathfrak{n} = \{0\}$, the limit $\mathfrak{r} := \lim\limits_{n\to\infty} \mathrm{Ad} e^{-nX}\, \mathfrak{h}$ exists and is a subalgebra of $\mathfrak{b}^-$. In particular, this Lie algebra $\mathfrak{r}$ is solvable. This proves $Sla(\mathfrak{g}, \mathfrak{h})$. □

Corollary 2.2. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then, the set of subalgebras $\mathfrak{h}$ satisfying $Sla(\mathfrak{g}, \mathfrak{h})$ is open in $\mathcal{L}$.*

Proof. The condition $Tmu(\mathfrak{g}, \mathfrak{h})$ is clearly an open condition. □

2.2 Related Lie subalgebras

We now explain why we can often assume that $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$.

Lemma 2.3. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a complex Lie subalgebra. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $H_1 = H$ be the smallest closed subgroup of G whose Lie algebra contains $\mathfrak{h}$. Set $\mathfrak{h}_0 = [\mathfrak{h}, \mathfrak{h}]$ and $\mathfrak{h}_1 := Lie(H)$. Then, one has the equivalences*

$$(i)\quad Sla(\mathfrak{g}, \mathfrak{h}) \iff Sla(\mathfrak{g}, \mathfrak{h}_0)\,. \tag{2.2}$$

$$(ii)\quad Sla(\mathfrak{g}, \mathfrak{h}) \iff Sla(\mathfrak{g}, \mathfrak{h}_1)\,. \tag{2.3}$$

Proof of Lemma 2.3. $(i) \Longrightarrow$ This follows from the inclusion $\mathfrak{h}_0 \subset \mathfrak{h}$.

$(i) \Longleftarrow$ Since we assume $Sla(\mathfrak{g}, \mathfrak{h}_0)$, there exists a sequence $(g_n)_{n\geq 1}$ in G such that the limit $\mathfrak{r}_0 = \lim\limits_{n\to\infty} \mathrm{Ad} g_n\, \mathfrak{h}_0$ exists and is a solvable Lie subalgebra of $\mathfrak{g}$. Then, after extraction, the limit $\mathfrak{r} := \lim\limits_{n\to\infty} \mathrm{Ad} g_n\, \mathfrak{h}$ exists and satisfies $[\mathfrak{r}, \mathfrak{r}] \subset \lim\limits_{n\to\infty} [\mathrm{Ad} g_n\, \mathfrak{h}, \mathrm{Ad} g_n\, \mathfrak{h}] = \mathfrak{r}_0$. In particular, the limit $\mathfrak{r}$ is a solvable Lie subalgebra of $\mathfrak{g}$. This proves $Sla(\mathfrak{g}, \mathfrak{h})$.

(ii) This follows from (i) and the inclusions $[\mathfrak{h}_1, \mathfrak{h}_1] \subset \mathfrak{h} \subset \mathfrak{h}_1$. □

2.3 Sla and Orb

The proof of the following equivalence is still purely algebraic but slightly more tricky.

Proposition 2.4. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a complex Lie subalgebra. Then, one has the equivalence*

$$Sla(\mathfrak{g}, \mathfrak{h}) \iff Orb(\mathfrak{g}, \mathfrak{h})\,. \tag{2.4}$$

Proof of the implication $\Longrightarrow$ in Proposition 2.4. Since we assume $Sla(\mathfrak{g}, \mathfrak{h})$, there exists a sequence $(g_n)_{n\geq 1}$ in G such that the limit $\mathfrak{r} = \lim_{n\to\infty} \mathrm{Ad}g_n\, \mathfrak{h}$ exists and is a solvable Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{r}$ is solvable, there exists a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{r}$. Since the orthogonal of $\mathfrak{b}$ is the maximal unipotent subalgebra $\mathfrak{b}^\perp = \mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$, the orthogonal $\mathfrak{r}^\perp$ also contains $\mathfrak{n}$. By a result of Dynkin (see [6, Thm. 4.1.6]), the Lie algebra $\mathfrak{n}$ always contains regular elements of $\mathfrak{g}$, the orthogonal $\mathfrak{r}^\perp$ also contains regular elements of $\mathfrak{g}$. Since the set $\mathfrak{g}_{reg}$ is open, for n large, the orthogonal $\mathrm{Ad}g_n\, \mathfrak{h}^\perp$ contains regular elements and $\mathfrak{h}^\perp$ too. This proves $Orb(\mathfrak{g}, \mathfrak{h})$. □

The proof of the converse implication will rely on the following two lemmas.

Lemma 2.5. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra where $\mathfrak{l}$ is a reductive Lie subalgebra and $\mathfrak{u}$ is the unipotent radical of $\mathfrak{q}$.*

Let $X = X_{\mathfrak{l}} + X_{\mathfrak{u}}$ be an element of $\mathfrak{q}$ with $X_{\mathfrak{l}} \in \mathfrak{l}$ and $X_{\mathfrak{u}} \in \mathfrak{u}$. If X is regular in $\mathfrak{g}$, then $X_{\mathfrak{l}}$ is regular in $\mathfrak{l}$.

Let r be the rank of $\mathfrak{g}$. We recall that the set $\mathfrak{g}_{reg}$ of regular elements of $\mathfrak{g}$ is the set of elements $X \in \mathfrak{g}$ whose centralizer in $\mathfrak{g}$ has dimension $\dim \mathfrak{z}_{\mathfrak{g}}(X) = r$. Similarly, the set $\mathfrak{l}_{reg}$ of regular element of $\mathfrak{l}$ is the set of elements $X \in \mathfrak{l}$ whose centralizer in $\mathfrak{l}$ has dimension $\dim \mathfrak{z}_{\mathfrak{l}}(X) = r$. This set may not be equal to $\mathfrak{l} \cap \mathfrak{g}_{reg}$. For instance, when $\mathfrak{q}$ is a Borel subalgebra, then $\mathfrak{l}$ is a Cartan subalgebra of $\mathfrak{g}$ and one has $\mathfrak{l}_{reg} = \mathfrak{l}$.

Proof of Lemma 2.5. One computes

$$\begin{aligned}
\dim \mathfrak{g} - r &= \dim \mathrm{Ad}G\, X \\
&\leq \dim G/Q + \dim \mathrm{Ad}Q\, X \\
&\leq 2 \dim \mathfrak{u} + \dim(\mathrm{Ad}Q\, X + \mathfrak{u})/\mathfrak{u} \\
&= 2 \dim \mathfrak{u} + \dim \mathrm{Ad}L\, X_{\mathfrak{l}}.
\end{aligned}$$

This proves $\dim \mathrm{Ad}L\, X_{\mathfrak{l}} \geq \dim \mathfrak{l} - r$ and hence $X_{\mathfrak{l}}$ is regular in $\mathfrak{l}$. □

Lemma 2.6. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a complex Lie subalgebra, and $X \in \mathfrak{h}^\perp$. Then there exists $\mathfrak{h}' \in \overline{\mathrm{Ad}G\,\mathfrak{h}}$ such that $X \in \mathfrak{h}'^\perp$ and $[X, \mathfrak{h}'] \subset \mathfrak{h}'$.*

We recall that G is a connected complex Lie group with Lie algebra $\mathfrak{g}$. Such a Lie group has a unique structure of complex algebraic Lie group.

Proof of Lemma 2.6. Let $A \subset G$ be the Zariski closure of the one-parameter subgroup $\{e^{tX} \mid t \in \mathbb{C}\}$. This group A is abelian.

Note that, for all a in A, the Lie subalgebra $\mathrm{Ad}a\,\mathfrak{h}$ is orthogonal to X. Therefore, all Lie subalgebra $\mathfrak{h}'$ in the orbit closure $\overline{\mathrm{Ad}A\,\mathfrak{h}}$ are orthogonal to X. This orbit closure $\overline{\mathrm{Ad}A\,\mathfrak{h}}$ is a A-invariant subvariety of the projective algebraic variety $\mathcal{L}$. By Borel fixed point theorem [4, Theorem 10.6], the solvable group A has a fixed point in this subvariety. This means that there exists $\mathfrak{h}'$ in $\overline{\mathrm{Ad}A\,\mathfrak{h}}$ such that $\mathrm{Ad}A\,\mathfrak{h}' = \mathfrak{h}'$. In particular, $[X, \mathfrak{h}'] \subset \mathfrak{h}'$. □

Proof of the implication $\Longleftarrow$ in Proposition 2.4. We argue by induction on the dimension of $\mathfrak{g}$. We assume that $\mathfrak{h}^\perp$ contains a regular element X, and we want to prove $Sla(\mathfrak{g}, \mathfrak{h})$. By Corollary 2.2 and Lemma 2.6, we can also assume that X normalizes $\mathfrak{h}$, i.e. that $[X, \mathfrak{h}] \subset \mathfrak{h}$. In particular, the sum $\widetilde{\mathfrak{h}} := \mathbb{C}X \oplus \mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. By Lemma 2.3 (i), we may and do assume that $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ of minimal dimension containing $\widetilde{\mathfrak{h}}$, and $\mathfrak{u}$ the unipotent radical of $\mathfrak{q}$. By minimality of $\mathfrak{q}$, the image of $\widetilde{\mathfrak{h}}$ in $\mathfrak{q}/\mathfrak{u}$ is reductive. Therefore we can write $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{v}$ where $\mathfrak{s}$ is a semisimple Lie subalgebra and $\mathfrak{v} := \mathfrak{h} \cap \mathfrak{u}$ is the unipotent radical of $\mathfrak{h}$. We can then write $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{l}$ is a reductive Lie subalgebra containing $\mathfrak{s}$. We sum up this discussion by the inclusions:

$$\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{v} \ \subset \ \mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u} \ \subset \ \mathfrak{g}\,.$$

Since X is in $\widetilde{\mathfrak{h}} \subset \mathfrak{q}$, we can decompose X as $X = X_\mathfrak{l} + X_\mathfrak{u}$ with $X_\mathfrak{l} \in \mathfrak{l}$ and $X_\mathfrak{u} \in \mathfrak{u}$. By Lemma 2.5, the element $X_\mathfrak{l}$ is regular in $\mathfrak{l}$. Since $\mathfrak{u}$ is the orthogonal of $\mathfrak{q}$ with respect to the Killing form K, one has

$$K(X_\mathfrak{l}, \mathfrak{s}) = K(X_\mathfrak{l} + X_\mathfrak{u}, \mathfrak{s} \oplus \mathfrak{v}) = K(X, \mathfrak{h}) = 0.$$

This proves that $X_\mathfrak{l}$ is orthogonal to $\mathfrak{s}$.

We now claim that $\mathfrak{q} \neq \mathfrak{g}$. Indeed, if $\mathfrak{q} = \mathfrak{g}$, one has the equalities $\widetilde{\mathfrak{h}} = \mathfrak{h} = \mathfrak{s}$, and this Lie algebra is semisimple by the assumption that $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$.

Therefore the Killing form restricted to $\mathfrak{h}$ is nondegenerate. This contradicts the assumption $X \in \mathfrak{h}^{\perp}$.

Therefore one has $\mathfrak{q} \neq \mathfrak{g}$. The normalizer $L := N_G(\mathfrak{l})$ of $\mathfrak{l}$ in G has Lie algebra $\mathfrak{l}$. We have seen that the intersection $\mathfrak{s}^{\perp} \cap \mathfrak{l}_{reg}$ is non-empty. Therefore, by induction hypothesis, the orbit closure $\overline{\mathrm{Ad}L\,\mathfrak{s}}$ contains a solvable Lie algebra, and the orbit closure $\overline{\mathrm{Ad}L\,\mathfrak{h}}$ also contains a solvable Lie algebra. This proves $Sla(\mathfrak{g}, \mathfrak{h})$. □

2.4 Rho and Sla

In this section we will prove the following implication which is still purely algebraic. The proof of the converse will be much more delicate.

We will in fact prove a stronger statement

Proposition 2.7. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a complex Lie subalgebra. Then, one has the implication*

$$Rho(\mathfrak{g}, \mathfrak{h}) \implies Sla(\mathfrak{g}, \mathfrak{h})\,. \tag{2.5}$$

More precisely, if $\mathfrak{h}$ satisfies $Rho(\mathfrak{g}, \mathfrak{h})$, then every Lie algebra $\mathfrak{h}'$ in $\overline{\mathrm{Ad}G\mathfrak{h}}$ satisfies $Sla(\mathfrak{g}, \mathfrak{h})$.

It will be useful to introduce the following two G-invariant subsets of $\mathcal{L}$.

$$\mathcal{L}_{rho} := \{\mathfrak{h} \in \mathcal{L} \mid \rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}\}, \tag{2.6}$$
$$\mathcal{L}_{clo} := \{\mathfrak{h} \in \mathcal{L} \mid \mathrm{Ad}G\,\mathfrak{h} \text{ is closed in } \mathcal{L}\}. \tag{2.7}$$

Remark 2.8. We have the following nice characterisation of closed orbits in $\mathcal{L}$.

$$\mathfrak{h} \in \mathcal{L}_{clo} \iff \text{the normalizer } N_{\mathfrak{g}}(\mathfrak{h}) \text{ is a parabolic subalgebra of } \mathfrak{g} \tag{2.8}$$
$$\iff \mathfrak{h} \text{ is normalized by a Borel subalgebra of } \mathfrak{g} \tag{2.9}$$

Proof of Proposition 2.7. This follows from Lemma 2.9 below and from the fact that the orbit closure always contains a closed G-orbit. □

Lemma 2.9. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then,*
(i) $\mathcal{L}_{rho}$ is closed in $\mathcal{L}$.
(ii) Let $\mathfrak{h} \subset \mathfrak{g}$ be a complex Lie subalgebra with $\mathrm{Ad}G\,\mathfrak{h}$ closed. Then,

$$\mathfrak{h} \text{ is solvable} \iff Rho(\mathfrak{g}, \mathfrak{h})\,.$$

Proof of Lemma 2.9. (i) The map $(\mathfrak{h}, Y) \mapsto \rho_{\mathfrak{h}}(Y)$ is continuous on the set $\{(\mathfrak{h}, Y) \mid \mathfrak{h} \in \mathcal{L} \ , \ Y \in \mathfrak{h}\}$. Let $\mathfrak{h}_n \in \mathcal{L}_{rho}$ be a sequence that converges to a Lie algebra $\mathfrak{h}_\infty$. We want to prove that $\mathfrak{h}_\infty \in \mathcal{L}_{rho}$. Let $Y_\infty \in \mathfrak{h}_\infty$. We can find a sequence $Y_n \in \mathfrak{h}_n$ converging to Y_∞. Therefore, one has

$$\rho_{\mathfrak{g}}(Y_\infty) - 2\,\rho_{\mathfrak{h}_\infty}(Y_\infty) = \lim_{n\to\infty} \rho_{\mathfrak{g}}(Y_n) - 2\,\rho_{\mathfrak{h}_n}(Y_n) \geq 0\,.$$

This proves that $\mathfrak{h}_\infty$ is in $\mathcal{L}_{rho}$.

$(ii) \Longrightarrow$ Since $\mathfrak{h}$ is solvable, it is included in a Borel Lie subalgebra $\mathfrak{b}$. Note that $\mathfrak{b}$ satisfies the ρ-inequality, more precisely, one has the equality $\rho_{\mathfrak{b}}(Y) = \rho_{\mathfrak{g}/\mathfrak{b}}(Y)$, for all Y in $\mathfrak{b}$. Therefore, $\mathfrak{h}$ also satisfies $Rho(\mathfrak{g}, \mathfrak{h})$.

$(ii) \Longleftarrow$ Let $\mathfrak{h}$ be a Lie subalgebra with $\mathrm{Ad}G\,\mathfrak{h}$ closed and which satisfies $Rho(\mathfrak{g}, \mathfrak{h})$. We want to prove that $\mathfrak{h}$ is solvable. Replacing a few times $\mathfrak{h}$ by its derived subalgebra $[\mathfrak{h}, \mathfrak{h}]$ if necessary, we may assume that $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$. Let $\mathfrak{q}$ be the normalizer of $\mathfrak{h}$ and $\mathfrak{u}$ be the unipotent radical of $\mathfrak{q}$. By assumption $\mathfrak{q}$ is a parabolic Lie subalgebra. The projection of $\mathfrak{h}$ in the reductive Lie algebra $\mathfrak{q}/\mathfrak{u}$ is an ideal and hence is a semisimple Lie algebra. Therefore we can write $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{v}$, where $\mathfrak{s}$ is a semisimple Lie subalgebra and $\mathfrak{v} := \mathfrak{h} \cap \mathfrak{u}$ is the unipotent radical of $\mathfrak{h}$. We then write $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{l}$ is a reductive Lie subalgebra containing $\mathfrak{s}$. Let $\mathfrak{u}^-$ be the opposite unipotent subalgebra which is opposite to $\mathfrak{q}$ and normalized by $\mathfrak{l}$ so that $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}$. Fix Y in $\mathfrak{s}$. Since $\mathfrak{q}$ normalizes $\mathfrak{h}$ one has

$$\rho_{\mathfrak{h}}(Y) \;\; = \;\; \rho_{\mathfrak{l}}(Y) + \rho_{\mathfrak{u}}(Y)\,. \tag{2.10}$$

Since $\mathfrak{u}^-$ is dual to $\mathfrak{u}$ as an $\mathfrak{l}$-module, one has

$$\rho_{\mathfrak{g}}(Y) \;\; = \;\; \rho_{\mathfrak{l}}(Y) + 2\,\rho_{\mathfrak{u}}(Y)\,. \tag{2.11}$$

Combining (2.10) and (2.11), and using the ρ-inequality, one gets

$$\rho_{\mathfrak{s}}(Y) \;\leq\; \rho_{\mathfrak{l}}(Y) \;\; = \;\; 2\,\rho_{\mathfrak{h}}(Y) - \rho_{\mathfrak{g}}(Y) \;\leq\; 0\,.$$

Since this is true for all Y in the semisimple Lie algebra $\mathfrak{s}$, one must have $\mathfrak{s} = 0$. This proves that $\mathfrak{h}$ is solvable. □

2.5 Reductive homogeneous spaces

In this section we check Theorem 1.6 for $\mathfrak{h}$ reductive by relying on the previous papers of this series. We will prove:

Proposition 2.10. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a complex reductive Lie subalgebra. The following conditions are equivalent :*

$$Tem(\mathfrak{g}, \mathfrak{h}) \iff Rho(\mathfrak{g}, \mathfrak{h}) \iff Sla(\mathfrak{g}, \mathfrak{h}) \iff Tmu(\mathfrak{g}, \mathfrak{h}) \iff Orb(\mathfrak{g}, \mathfrak{h}).$$

Remark 2.11. Since $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is reductive, one has a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ with respect to the Killing form, and the orthogonal complement $\mathfrak{h}^\perp$ is isomorphic to the quotient $\mathfrak{g}/\mathfrak{h}$ as an $\mathfrak{h}$-module.

The proof uses the condition $Ags(\mathfrak{g}, \mathfrak{h})$ that we introduced in [3] and proven to be equivalent to $Rho(\mathfrak{g}, \mathfrak{h})$. It is defined by:

$$Ags(\mathfrak{g}, \mathfrak{h}) \quad : \quad \text{the set } \{X \in \mathfrak{h}^\perp \mid \mathfrak{z}_\mathfrak{h}(X) \text{ is abelian } \} \text{ is dense in } \mathfrak{h}^\perp.$$

Proof of Proposition 2.10.
⋆ The equivalence $Tem(\mathfrak{g}, \mathfrak{h}) \iff Rho(\mathfrak{g}, \mathfrak{h})$ is proven in [1, Thm. 4.1] for all real semisimple Lie algebra $\mathfrak{g}$ and all real reductive Lie subalgebra $\mathfrak{h}$.
⋆ The equivalence $Sla(\mathfrak{g}, \mathfrak{h}) \iff Tmu(\mathfrak{g}, \mathfrak{h}) \iff Orb(\mathfrak{g}, \mathfrak{h})$ has been proven in the previous sections for all complex Lie subalgebra $\mathfrak{h}$.
⋆ The equivalence $Rho(\mathfrak{g}, \mathfrak{h}) \iff Ags(\mathfrak{g}, \mathfrak{h})$ is proven in [3, Thm. 1.6] for all complex semisimple Lie algebra $\mathfrak{g}$ and all complex reductive Lie subalgebra $\mathfrak{h}$.
⋆ The equivalence $Ags(\mathfrak{g}, \mathfrak{h}) \iff Orb(\mathfrak{g}, \mathfrak{h})$ is proven in Proposition 2.12 below. □

Proposition 2.12. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a complex reductive Lie subalgebra. Then, one has the equivalence*

$$Ags(\mathfrak{g}, \mathfrak{h}) \iff Orb(\mathfrak{g}, \mathfrak{h})\,. \tag{2.12}$$

We will need the following lemma which relates centralizer in $\mathfrak{g}$ and centralizer in $\mathfrak{h}$.

Lemma 2.13. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{h}$ a real reductive Lie subalgebra, and regard $\mathfrak{h}^\perp \subset \mathfrak{g}$ via the Killing form as before. Let*

$$\mathfrak{h}^\perp_{\min} := \{X \in \mathfrak{h}^\perp \mid \dim \mathfrak{z}_\mathfrak{g}(X) = r_{\mathfrak{g},\mathfrak{h}}\} \text{ where } r_{\mathfrak{g},\mathfrak{h}} := \min_{X \in \mathfrak{h}^\perp} \dim \mathfrak{z}_\mathfrak{g}(X)$$

Then, for every X_0 in $\mathfrak{h}^\perp_{\min}$, one has $[\mathfrak{z}_\mathfrak{g}(X_0), \mathfrak{z}_\mathfrak{g}(X_0)] \subset \mathfrak{z}_\mathfrak{h}(X_0)$.

Note that Lemma 2.13 applied to $\mathfrak{h} = \{0\}$ implies that $\mathfrak{z}_{\mathfrak{g}}(X_0)$ is abelian if $X_0 \in \mathfrak{g}_{\rm reg}$. Indeed, when $\mathfrak{h} = \{0\}$, one has $r_{\mathfrak{g},\mathfrak{h}} = \operatorname{rank} \mathfrak{g}$ and $\mathfrak{h}^{\perp}_{\rm min} = \mathfrak{g}_{\rm reg}$.

This lemma is a special case of the following general lemma for coadjoint orbits of real Lie algebras which is well-known when $\mathfrak{h} = \{0\}$.

Lemma 2.14. *Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a real Lie subalgebra. Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$ and $\mathfrak{h}^{\perp} := \{f \in \mathfrak{g}^* \mid f(\mathfrak{h}) = \{0\}\}$. We set*

$$\mathfrak{h}^{\perp}_{\rm min} := \{f \in \mathfrak{h}^{\perp} \mid \dim \mathfrak{g}_f = r_{\mathfrak{g},\mathfrak{h}}\} \;\text{ where }\; r_{\mathfrak{g},\mathfrak{h}} := \min_{f \in \mathfrak{h}^{\perp}} \dim \mathfrak{g}_f .$$

Then, for every f_0 in $\mathfrak{h}^{\perp}_{\rm min}$, one has $[\mathfrak{g}_{f_0}, \mathfrak{g}_{f_0}] \subset \mathfrak{h}_{f_0}$.

Here $\mathfrak{g}_f := \{Y \in \mathfrak{g} \mid Yf = 0\}$ denotes the stabilizer of f in $\mathfrak{g}$ and $\mathfrak{h}_f := \mathfrak{g}_f \cap \mathfrak{h}$ its stabilizer in $\mathfrak{h}$.

Proof of Lemma 2.14. Fix $f_0 \in \mathfrak{h}^{\perp}_{\rm min}$ and two elements Y_0 and Z_0 in $\mathfrak{g}_{f_0}$. We want to prove that $[Y_0, Z_0] \in \mathfrak{h}$. We write

$$\mathfrak{g} = \mathfrak{g}_{f_0} \oplus \mathfrak{m}$$

where $\mathfrak{m}$ is a complementary vector subspace.

For all $f \in \mathfrak{h}^{\perp}$, for $t \in \mathbb{R}$ small enough the element $f_t := f_0 + tf$ is also in the open set $\mathfrak{h}^{\perp}_{\rm min}$. Choose a linear projection $\pi_0 \colon \mathfrak{g}^* \to \mathfrak{g} f_0$. By the local inversion theorem, the map

$$\begin{array}{rcl} \Phi \colon (Y_0 + \mathfrak{m}) \times \mathbb{R} & \to & \mathfrak{g} f_0 \times \mathbb{R} \\ (Y, t) & \mapsto & (\pi_0(Y f_t), t) \end{array}$$

is a local diffeomorphism near $(Y_0, 0)$. Let $t \mapsto Y_t$ be the differentiable curve near 0 starting from Y_0 given by $\Phi(Y_t, t) = (0, t)$. Since for t small the linear map $\pi_0 \colon \mathfrak{g} f_t \to \mathfrak{g} f_0$ is an isomorphism, it satisfies

$$Y_t \in Y_0 + \mathfrak{m} \quad \text{and} \quad Y_t f_t = 0 \,.$$

For the same reason, there exists a differentiable curve $t \mapsto Z_t$ near 0 starting from Z_0 such that

$$Z_t \in Z_0 + \mathfrak{m} \quad \text{and} \quad Z_t f_t = 0 \,.$$

They satisfy the equality $f_t([Y_t, Z_t]) = 0$ whose derivative at $t = 0$ gives

$$f([Y_0, Z_0]) + f_0([Y_0', Z_0]) + f_0([Y_0, Z_0']) = 0$$

Since both Y_0 and Z_0 stabilize f_0 the last two terms are zero. One deduces

$$f([Y_0, Z_0]) = 0 \text{ for all } f \text{ in } \mathfrak{h}^\perp.$$

This proves that $[Y_0, Z_0]$ is in $\mathfrak{h}$ as required. $\square$

The following lemma will also be useful.

Lemma 2.15. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a complex reductive Lie subalgebra. Then the set*

$$\mathfrak{h}^\perp_{ss} := \{X \in \mathfrak{h}^\perp \mid X \text{ is semisimple}\}.$$

is Zariski dense in $\mathfrak{h}^\perp$.

Proof of Lemma 2.15. There exists a compact real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$ such that $\mathfrak{h}$ is defined over $\mathbb{R}$. Since $\mathfrak{g}_\mathbb{R} = \mathfrak{h}_\mathbb{R} \oplus \mathfrak{h}^\perp_\mathbb{R}$, the vector space $\mathfrak{h}^\perp_\mathbb{R}$ is Zariski dense in $\mathfrak{h}^\perp$. Since all elements of $\mathfrak{g}_\mathbb{R}$ are semisimple, this proves our claim. $\square$

Proof of Proposition 2.12. $\Longleftarrow$ Since the Zariski open set $\mathfrak{g}_{\rm reg}$ meets the orthogonal $\mathfrak{h}^\perp$ for the Killing form, the intersection $\mathfrak{h}^\perp_{\rm reg}$ is dense in $\mathfrak{h}^\perp$. By Lemma 2.13 applied with the zero subalgebra, every X_0 in $\mathfrak{g}_{\rm reg}$ has an abelian centralizer in $\mathfrak{g}$. In particular, every X_0 in $\mathfrak{g}_{\rm reg}$ has an abelian centralizer in $\mathfrak{h}$. This proves $Ags(\mathfrak{g}, \mathfrak{h})$.

$\Longrightarrow$ Let $r' := \min\{\dim \mathfrak{z}_\mathfrak{h}(X) \mid X \in \mathfrak{h}^\perp\}$. The set

$$\mathfrak{h}^\perp_{\rm gen} := \{X \in \mathfrak{h}^\perp_{\rm min} \mid \dim \mathfrak{z}_\mathfrak{h}(X) = r'\}$$

is nonempty and Zariski open in $\mathfrak{h}^\perp$. By assumption the set

$$\mathfrak{h}^\perp_{\rm abe} := \{X \in \mathfrak{h}^\perp_{\rm gen} \mid \mathfrak{z}_\mathfrak{h}(X) \text{ is abelian}\}$$

is dense in $\mathfrak{h}^\perp_{\rm gen}$. Since it is also closed in $\mathfrak{h}^\perp_{\rm gen}$, one has $\mathfrak{h}^\perp_{\rm abe} = \mathfrak{h}^\perp_{\rm gen}$. Therefore by Lemma 2.15 the set $\mathfrak{h}^\perp_{\rm abe}$ contains a semisimple element X_0. The centralizer $\mathfrak{z}_\mathfrak{g}(X_0)$ is then a reductive Lie algebra. By Lemma 2.13, the Lie algebra $[\mathfrak{z}_\mathfrak{g}(X_0), \mathfrak{z}_\mathfrak{g}(X_0)]$ is included in $\mathfrak{z}_\mathfrak{h}(X_0)$ which is an abelian Lie algebra. Therefore the Lie algebra $\mathfrak{z}_\mathfrak{g}(X_0)$ itself is abelian. Since X_0 is semisimple, this centralizer is a Cartan subalgebra and X_0 is regular in $\mathfrak{g}$. This proves $Orb(\mathfrak{g}, \mathfrak{h})$. $\square$

3 Real algebraic homogeneous spaces

The proof of the last remaining implication (1.10) will last up to the end of this paper. Because of the induction method which involves parabolic subgroups, we need to extend the temperedness criterion of [2] to non-semisimple algebraic groups G. This extension will be valid for all real algebraic group.

3.1 Notations

Let G be a real algebraic Lie group, H be an algebraic Lie subgroup. We write $G = LU$ and $H = SV$ where $S \subset L$ are reductive subgroups and where V and U are the unipotent radical of H and G. Note that, in general one does not have the inclusion $V \subset U$. We denote by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{l}$, $\mathfrak{u}$, etc... the corresponding Lie algebras.

We consider the following conditions:

$$\begin{aligned} Tem(\mathfrak{g},\mathfrak{h}) &: \ L^2(G/H) \text{ is } L\text{-tempered.} \\ Rho(\mathfrak{g},\mathfrak{h}) &: \ \rho_{\mathfrak{l}} \leq 2\,\rho_{\mathfrak{g}/\mathfrak{h}} \text{ as functions on } \mathfrak{s}. \\ Sla(\mathfrak{g},\mathfrak{h}) &: \ \overline{\mathrm{Ad}G\,\mathfrak{h}} \text{ contains a solvable Lie algebra.} \end{aligned}$$

Remark 3.1. By L-tempered, we mean tempered as a representation of L, or, equivalently, tempered as a representation of the semisimple Lie group $[L, L]$. When G is not semisimple this notion happens to be much more useful than the temperedness as a representation of G.

Theorem 3.2. *Let G be a real algebraic Lie group, H be an algebraic Lie subgroup. One has the equivalence,*

$$Tem(\mathfrak{g},\mathfrak{h}) \Longleftrightarrow Rho(\mathfrak{g},\mathfrak{h}).$$

Remark 3.3. For real algebraic groups, the last condition $Sla(\mathfrak{g},\mathfrak{h})$ is not always equivalent to the first two, but it is often the case. For instance, we will see in Theorem 5.1, that this is true for complex algebraic Lie groups.

In the induction process, we will have to work with slightly more general representations than the regular representation $L^2(G/H)$. Let W be a finite-dimensional algebraic representation of H. We will have to deal with the (L^2-)induced representation $\mathrm{Ind}_H^G(L^2(W)) \simeq L^2(G \times_H W)$, where $G \times_H W$ is the G-equivariant bundle over G/H with fiber W, see [2, Section 2.1] for more precise definition. This is why we also introduce the following two conditions.

$$
\begin{aligned}
Tem(\mathfrak{g},\mathfrak{h},W) &\ :\ \mathrm{Ind}_H^G(L^2(W)) \ \text{ is } L\text{-tempered.}\\
Rho(\mathfrak{g},\mathfrak{h},W) &\ :\ \rho_{\mathfrak{l}} \leq 2\,\rho_{\mathfrak{g}/\mathfrak{h}} + 2\,\rho_W \ \text{as a functions on } \mathfrak{s}.
\end{aligned}
$$

The following theorem is a generalization of our Theorem 3.6 in [2] where we assumed that G is semisimple.

Theorem 3.4. *Let G be a real algebraic Lie group, H be an algebraic Lie subgroup and W a finite-dimensional algebraic representation of H. One has the equivalence,*

$$Tem(\mathfrak{g},\mathfrak{h},W) \Longleftrightarrow Rho(\mathfrak{g},\mathfrak{h},W).$$

We have assumed here that G and H are algebraic only to avoid uninteresting technicalities. It is not difficult to get rid of this assumption.

Proof of Theorem 3.2. It is a special case of Theorem 3.4 with $W = 0$. □

The proof of Theorem 3.4 follows the same line as in [2, Theorem 3.6].
In this Chapter 3 we will prove the direct implication $\Longrightarrow$.
In the next Chapter 4, we will prove the converse implication $\Longleftarrow$.

3.2 The Herz majoration principle

We first recall a few lemmas on tempered representations and on induced representations.

The first lemma is a variation on Herz majoration principle.

Lemma 3.5. *Let G be a real algebraic Lie group, L be a reductive algebraic Lie subgroup of G and H be a closed subgroup of G. If the regular representation in $L^2(G/H)$ is L-tempered then the induced representation $\Pi = \mathrm{Ind}_H^G(\pi)$ is also L-tempered for any unitary representation π of H.*

Proof. See for instance [2, Lemma 3.2]. □

The second lemma will prevent us to worry about connected components of H and will allow us to assume that $H = [H,H]$.

Lemma 3.6. *Let G be a real algebraic Lie group, L be a reductive algebraic subgroup of G and $H' \subset H$ be two closed subgroup of G.*
1) If $L^2(G/H)$ is L-tempered then $L^2(G/H')$ is L-tempered.
2) The converse is true when H' is normal in H and H/H' is amenable (for instance finite, compact, or abelian).

Proof. See for instance [2, Proposition 3.1]. □

The third lemma is good to keep in mind.

Lemma 3.7. *Let $Q = LU$ be a real algebraic Lie group which is a semidirect product of a reductive subgroup L and its unipotent radical U. Let π_0 be a unitary representation of Q which is L-tempered and trivial on U. Then the representation π_0 is also Q-tempered.*

Proof. See for instance [2, Lemma 4.3]. □

This lemma is useful for a parabolic subgroup Q of a semisimple Lie group G. In this case the induced representation $\mathrm{Ind}_Q^G(\pi_0)$ is also G-tempered.

3.3 Decay of matrix coefficients

We now recall the control of the matrix coefficients of tempered representations of a reductive Lie group.

In the sequel, it will be more comfortable to deal with a reductive group L than just with a semisimple group even though, in the temperedness condition, the center Z_L of L plays no role.

So, let L be a real reductive algebraic Lie group. We fix a maximal compact subgroup K of L and denote by Ξ the Harish-Chandra spherical function on L. By definition, Ξ is the matrix coefficient of a normalized K-invariant vector v_0 of the spherical unitary principal representation $\pi_0 = \mathrm{Ind}_P^L(\mathbf{1}_P)$ where P is a minimal parabolic subgroup of L. That is

$$\Xi(\ell) = \langle \pi_0(\ell)v_0, v_0\rangle \ , \ \text{ for all } \ell \text{ in } L. \tag{3.1}$$

Since P is amenable, the representation π_0 is L-tempered.

Proposition 3.8 (Cowling, Haagerup and Howe [7])**.** *Let L be a real algebraic reductive Lie group and π be a unitary representation of L. The following are equivalent:*

(i) the representation π is tempered,
(ii) for every K-finite vector v in $\mathcal{H}_\pi$, for every ℓ in L, one has

$$|\langle \pi(\ell)v, v\rangle| \leq \Xi(\ell)\, \|v\|^2 \dim\langle Kv\rangle.$$

See [7, Thms. 1, 2 and Cor.]. See also [8], [10] for other applications of Proposition 3.8.

For the regular representation in an L-space, this proposition becomes:

Corollary 3.9. *Let L be a real algebraic reductive Lie group and X be a locally compact space endowed with a continuous action of L preserving a Radon measure* vol. *The regular representation of L in $L^2(X)$ is L-tempered if and only if, for any K-invariant compact subset C of X, one has*

$$\mathrm{vol}(\ell\, C \cap C) \leq \mathrm{vol}(C)\ \Xi(\ell)\ , \quad \text{for all } \ell \text{ in } L. \tag{3.2}$$

Recall that the notation $\ell\, C$ denotes the set $\ell\, C := \{\ell x \in X : x \in C\}$.

3.4 The function ρ_V

We now explain, following [2, Section 2.3] how to deal with the functions ρ_V occurring in the temperedness criterion.

Let H be a real algebraic Lie group, $\mathfrak{h}$ its Lie algebra and V be a real algebraic finite-dimensional representation of H. For all element Y in $\mathfrak{h}$, we consider the eigenvalues of Y in V and we denote by V_+ and V_- the largest vector subspaces of V on which the real part of all the eigenvalues of Y are respectively positive and negative, and we set

$$\rho_V(Y) \ := \ \tfrac{1}{2}\,\mathrm{Tr}(Y|_{V_+}) - \tfrac{1}{2}\,\mathrm{Tr}(Y|_{V_-}).$$

Let $\mathfrak{a} = \mathfrak{a}_{\mathfrak{h}}$ be a maximal split abelian Lie subalgebra of $\mathfrak{h}$ *i.e.* the Lie subalgebra of a maximal split torus A of H. The function ρ_V on $\mathfrak{h}$ is completely determined by its restriction to $\mathfrak{a}$. Let P_V be the set of weights of $\mathfrak{a}$ in V and, for all α in P_V, let $m_\alpha := \dim V_\alpha$ be the dimension of the corresponding weight space. Then one has the equality

$$\rho_V(Y) = \tfrac{1}{2} \sum_{\alpha \in P_V} m_\alpha |\alpha(Y)| \quad \text{for all } Y \text{ in } \mathfrak{a}. \tag{3.3}$$

For example, when $\mathfrak{h}$ is semisimple and $V = \mathfrak{h}$ via the adjoint action, our function $\rho_\mathfrak{h}$ is equal on each positive Weyl chamber $\mathfrak{a}_+$ of $\mathfrak{a}$ to the sum of the corresponding positive roots *i.e.* to twice the usual "ρ" linear form.

The functions ρ_V occurs in the volume estimate of Corollary 3.9 through the following Lemma.

Lemma 3.10. *Let $V = \mathbb{R}^d$. Let $\mathfrak{a}$ be an abelian split Lie subalgebra of* $\mathrm{End}(V)$ *and C be a compact neighborhood of* 0 *in V. Then there exist constants $m_C > 0$, $M_C > 0$ such that*

$$m_C e^{-\rho_V(Y)} \le e^{-\mathrm{Tr}(Y)/2} \operatorname{vol}(e^Y C \cap C) \le M_C e^{-\rho_V(Y)} \text{ for all } Y \in \mathfrak{a}.$$

Proof. This is [2, Lemma 2.8]. □

3.5 The direct implication

We first prove the direct implication in Theorem 3.4 which is :

Proposition 3.11. *Let G be a real algebraic Lie group, H an algebraic Lie subgroup of G and W an algebraic representation of H. Let L be a maximal reductive subgroup of G containing a maximal reductive subgroup S of H. If $\Pi := \mathrm{Ind}_H^G(L^2(W))$ is L-tempered then one has $\rho_\mathfrak{l} \le 2\,\rho_{\mathfrak{g}/\mathfrak{h}} + 2\,\rho_W$ on $\mathfrak{s}$.*

Proof. This representation Π is also the regular representation of the G-space $X := G \times_H W$. Let A be a maximal split torus of S and $\mathfrak{a}$ be the Lie algebra of A. We choose an A-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and small closed balls $B_0 \subset \mathfrak{m}$ and $B_W \subset W$ centered at 0. We can see B_W as a subset of X and the map

$$B_0 \times B_W \longrightarrow G \times_H W, \qquad (u, v) \mapsto \exp(u)v$$

is a homeomorphism onto its image C. Since Π is L-tempered one has a bound as in (3.2)

$$\langle \Pi(\ell) 1_C, 1_C \rangle \le M_C\, \Xi(\ell) \quad \text{for all } \ell \text{ in } L. \tag{3.4}$$

We will exploit this bound for elements $\ell = e^Y$ with Y in $\mathfrak{a}$. In our coordinate system (3.4) we can choose the measure ν_X to coincide with the Lebesgue measure on $\mathfrak{m} \oplus W$. Taking into account the Radon–Nikodym derivative and the A-invariance of $\mathfrak{m}$, one computes as in [2, Section 3.3],

$$\langle \Pi(e^Y) 1_C, 1_C \rangle \;\ge\; e^{-Tr_\mathfrak{m}(Y)/2 - Tr_W(Y)/2} \operatorname{vol}_\mathfrak{m}(e^Y B_0 \cap B_0) \operatorname{vol}_W(e^Y B_W \cap B_W),$$

and therefore, using Lemma 3.10, one deduces

$$\langle \Pi(e^Y)1_C, 1_C\rangle \;\geq\; m_{_C}\, e^{-\rho_{\mathfrak{m}}(Y)} e^{-\rho_W(Y)} \quad \text{for all } Y \text{ in } \mathfrak{a}. \tag{3.5}$$

Combining (3.4) and (3.5) with known bounds for the spherical function Ξ as in [9, Prop 7.15], one gets, for suitable positive constants d, M_0,

$$\frac{m_C}{M_C} e^{-\rho_{\mathfrak{m}}(Y)-\rho_W(Y)} \leq \Xi(e^Y) \leq M_0\, (1+\|Y\|)^d e^{-\rho_{\mathfrak{l}}(Y)/2} \quad \text{for all } Y \text{ in } \mathfrak{a}.$$

Therefore one has $\rho_{\mathfrak{l}} \leq 2\,\rho_{\mathfrak{m}} + 2\,\rho_W$ as required. □

4 Proof of temperedness for real groups

In this Chapter, we prove the converse implication in Theorem 3.4 which is :

Proposition 4.1. *Let G be a real algebraic Lie group, H an algebraic Lie subgroup of G and W an algebraic representation of H. Let L be a maximal reductive subgroup of G containing a maximal reductive subgroup S of H. If $\rho_{\mathfrak{l}} \leq 2\,\rho_{\mathfrak{g}/\mathfrak{h}} + 2\,\rho_{_W}$ on $\mathfrak{s}$, then $\Pi := \mathrm{Ind}_H^G(L^2(W))$ is L-tempered.*

Recall that, when $W = 0$, one has $\Pi = L^2(G/H)$.

4.1 Domination of G-spaces

The proof relies on the notion of domination of a G-action that we have introduced in [2] without giving it a name.

Here is the definition. Let G be a locally compact group. Let X and X_0 be two locally compact spaces endowed with a continuous action of G, and with a G-invariant class of measures vol_X and vol_{X_0}. Let π and π_0 be the unitary regular representations of G in the Hilbert spaces of square-integrable half-densities $L^2(X)$ and $L^2(X_0)$.

Definition 4.2 (Domination of a G-space)**.** We say that X is *G-dominated by X_0* if for every compactly supported bounded half-density v on X, there exists a compactly supported bounded half-density v_0 on X_0 such that, for all g in G,

$$|\langle \pi(g)v, v\rangle| \leq \langle \pi_0(g)v_0, v_0\rangle. \tag{4.1}$$

Remark 4.3. When both measures vol_X and vol_{X_0} are G-invariant, the bound (4.1) means that, for every compact set $C \subset X$, there exists a constant $\lambda > 0$ and a compact set $C_0 \subset X_0$ such that, for all g in G,

$$\mathrm{vol}(g\,C \cap C) \leq \lambda\,\mathrm{vol}(g\,C_0 \cap C_0)$$

This definition is very much related to our temperedness question because of the following lemma.

Lemma 4.4. *Let G be a real algebraic reductive Lie group and P be a minimal parabolic subgroup of G, and let X be a G-space. The regular representation of G in $L^2(X)$ is G-tempered if and only if X is G-dominated by the flag variety $X_0 = G/P$.*

Proof. This lemma is a direct consequence of Corollary 3.9. □

The following proposition gives us a nice situation where an action is dominating another one.

Proposition 4.5. *Let $F = SU$ be a real algebraic Lie group which is a semidirect product of a reductive subgroup S and its unipotent radical U. Let $H = SV$ be an algebraic subgroup of F containing S where $V = U \cap H$. Let Z be the F-space $Z = F/H = U/V$. Let $Z_0 := Z$ endowed with another F-action where the S-action is the same but the U-action is trivial.*

Then Z is F-dominated by Z_0.

Proof. This is [2, Corollary 4.6]. □

4.2 Inducing a dominated action

The following proposition tells us that the induction of actions preserves the domination.

Proposition 4.6. *Let G be a locally compact group, and F a closed subgroup of G. Let Z and Z_0 be two locally compact F-spaces with G-invariant class of measures. Let $X := G \times_F Z$ and $X_0 := G \times_F Z_0$ be the two induced G-spaces.*

If Z is F-dominated by Z_0 then X is G-dominated by X_0.

Proof of Proposition 4.6. The proof is an adaptation of [2, Proposition 4.9] where G was an algebraic semisimple group. We assume to simplify that the measures on Z and Z_0 are G-invariant. This avoids to complicate the formulas with square roots of Radon-Nikodym derivative. The projection

$$G \to X' := G/F$$

is a G-equivariant principal bundle with structure group F. We fix a Borel measurable trivialization of this principal bundle

$$G \simeq X' \times F \tag{4.2}$$

which sends relatively compact subsets to relatively compact subsets. The action of G by left multiplication through this trivialization can be read as

$$g\,(x', f) = (gx', \sigma_F(g, x')f) \quad \text{for all } g \in G,\ x' \in X' \text{ and } f \in F,$$

where $\sigma_F \colon G \times X' \to F$ is a Borel measurable cocycle. This trivialization (4.2) induces a trivialization of the associated bundles

$$\begin{array}{rcl} X = G \times_F Z & \simeq & X' \times Z\,, \\ X_0 = G \times_F Z_0 & \simeq & X' \times Z_0\,. \end{array}$$

We start with a compact set C of X. Through the first trivialization, this compact set is included in a product of two compact sets $C' \subset X'$ and $D \subset Z$

$$C \subset C' \times D\,. \tag{4.3}$$

Since Z is F-dominated by Z_0 there exists $\lambda > 0$ and a compact subset $D_0 \subset Z_0$ such that, for all f in F ,

$$\mathrm{vol}_Z(f\,D \cap D) \leq \lambda\,\mathrm{vol}_{Z_0}(f\,D_0 \cap D_0)$$

We compute, for g in G,

$$\begin{array}{rcl} \mathrm{vol}_X(g\,C \cap C) & \leq & \displaystyle\int_{gC' \cap C'} \mathrm{vol}_Z(\sigma_F(g, g^{-1}x')D \cap D)\,\mathrm{d}x' \\ & \leq & \lambda \displaystyle\int_{gC' \cap C'} \mathrm{vol}_{Z_0}(\sigma_F(g, g^{-1}x')D_0 \cap D_0)\,\mathrm{d}x' \\ & \leq & \lambda\,\mathrm{vol}_{X_0}(g\,C_0 \cap C_0), \end{array}$$

where $\mathrm{d}x'$ is a G-invariant measure on X' and C_0 is a compact subset of $X_0 \simeq X' \times Z_0$ which contains $C' \times D_0$. □

4.3 The converse implication

> We conclude the proof of the converse implication in Theorem 3.4, by reducing it to the case where G is *reductive* which was proven in [2, Theorem 3.6]

We will need the following lemma on the structure of nilpotent homogeneous spaces. See [2, Lemma 4.7], for a similar statement. We recall that a unipotent Lie group is an algebraic nilpotent Lie group with no torus factor.

Lemma 4.7. *Let U be a real unipotent Lie group, V a unipotent subgroup and $\mathfrak{v} \subset \mathfrak{u}$ their Lie algebra.*
(1) *There exists a real vector subspace $\mathfrak{m} \subset \mathfrak{u}$ such that $\mathfrak{u} = \mathfrak{m} \oplus \mathfrak{v}$ and the exponential map induces a polynomial bijection* $\exp\colon \mathfrak{m} \xrightarrow{\sim} U/V$.
(2) *Moreover, if $\mathfrak{v}$ is invariant by a reductive subgroup $S \subset \mathrm{Aut}(\mathfrak{u})$, one can choose $\mathfrak{m}$ to be S-invariant.*

Proof of Lemma 4.7. We proceed by induction on $\dim U$. Let Z be the center of U and $\mathfrak{z}$ its Lie algebra.

First case : $\mathfrak{z} \cap \mathfrak{v} \neq \{0\}$. In this case we apply the induction assumption to the Lie algebra $\mathfrak{u}' := \mathfrak{u}/(\mathfrak{z} \cap \mathfrak{v})$ and its Lie subalgebra $\mathfrak{v}' := \mathfrak{v}/(\mathfrak{z} \cap \mathfrak{v})$. This gives us an S-invariant subspace $\mathfrak{m}'$ of $\mathfrak{u}'$ such that $\mathfrak{u}' = \mathfrak{m}' \oplus \mathfrak{v}'$ and $\exp\colon \mathfrak{m}' \to U'/V' \simeq U/V$ is a bijection. We denote by $\pi\colon \mathfrak{u} \to \mathfrak{u}'$ the projection and choose $\mathfrak{m}$ to be any S-invariant vector subspace of $\pi^{-1}\mathfrak{m}'$ such that $\mathfrak{m} \oplus (\mathfrak{z} \cap \mathfrak{v}) = \pi^{-1}\mathfrak{m}'$.

Second case : $\mathfrak{z} \cap \mathfrak{v} = \{0\}$. In this case we apply the induction assumption to the Lie algebra $\mathfrak{u}' := \mathfrak{u}/\mathfrak{z}$ and its subalgebra $\mathfrak{v}' := (\mathfrak{v} \oplus \mathfrak{z})/\mathfrak{z}$. This gives us an S-invariant subspace $\mathfrak{m}'$ of $\mathfrak{u}'$ such that $\mathfrak{u}' = \mathfrak{m}' \oplus \mathfrak{v}'$ and $\exp\colon \mathfrak{m}' \to U'/V'$ is a bijection. We denote by $\pi\colon \mathfrak{u} \to \mathfrak{u}'$ the projection and choose $\mathfrak{m} := \pi^{-1}\mathfrak{m}'$. The identifications $\mathfrak{m}' \simeq \mathfrak{m}/\mathfrak{z}$ and $U'/V' \simeq U/VZ$ prove that the exponential map $\exp\colon \mathfrak{m} \to U/V$ is bijective. □

Proof of Proposition 4.1. We distinguish two cases.

First case : $W = \{0\}$. In this case, one has $\Pi = L^2(G/H)$. We denote by U and V the unipotent radical of G and H, so that we have the equalities $G = LU$ and $H = SV$. We have the inclusion $S \subset L$, but the group V might not be included in U. We introduce the unipotent group $V' := VU \cap L$ and the algebraic groups $F := HU$ and $F' := F \cap L$ so that we have the equality $F' = SV'$ and the inclusions

$$H = SV \subset F = F'U \subset G = LU\,.$$

Let

$$Z := F/H$$

and let Z_0 be the F-space Z endowed with the same S-action but with a trivial VU-action. One can easily describe Z_0. Indeed, let $\mathfrak{u}$, $\mathfrak{v}$,... be the Lie algebras of U, V,... By Lemma 4.7, Z_0 can be identified with the S-module $W' := \mathfrak{u}/(\mathfrak{u}\cap\mathfrak{v})$, as is seen from the following isomorphisms:

$$F/H \simeq VU/U \simeq U/(U\cap V) \simeq \mathfrak{u}/(\mathfrak{u}\cap\mathfrak{v})\,.$$

According to Proposition 4.5, the F-space Z is dominated by Z_0. We introduce now the two induced G-spaces

$$X := G\times_F Z = G/H \quad\text{and}\quad X_0 := G\times_F Z_0\,.$$

According to Proposition 4.6, the G-space X is dominated by X_0. Hence

the L-space $X = G/H$ is dominated by the L-space $X_0 = L\times_{F'} W'$

By assumption one has

$$\rho_{\mathfrak{l}} \leq 2\,\rho_{\mathfrak{g}/\mathfrak{h}}.$$

Since $\rho_{\mathfrak{g}/\mathfrak{h}} = \rho_{\mathfrak{g}/\mathfrak{f}} + \rho_{\mathfrak{f}/\mathfrak{h}} = \rho_{\mathfrak{l}/\mathfrak{f}'} + \rho_{\mathfrak{u}/(\mathfrak{u}\cap\mathfrak{v})}$, this can be rewritten as

$$\rho_{\mathfrak{l}} \leq 2\,\rho_{\mathfrak{l}/\mathfrak{f}'} + 2\,\rho_{W'}\,.$$

Since L is reductive, we can apply [2, Theorem 3.6]. This tells us that the representation $L^2(L\times_{F'} W')$ is L-tempered.

Therefore since the L-space X is L-dominated by X_0 the representation of L in $L^2(G/H)$ is L-tempered, as required.

Second case : $W \neq \{0\}$. In this case, one has $\Pi = L^2(G\times_H W)$. For w in W, we denote by H_w the stabilizer of w in H. We write $H_w = S_w U_w$ with S_w reductive and U_w the unipotent radical. Since the action of H on W is algebraic, there exists a Borel measurable subset $T\subset W$ which meets each of these H-orbits in exactly one point. We can assume that for each w in T, one has $S_w \subset S$. Let μ be a probability measure on W with positive density and ν be the probability measure on $T \simeq S\backslash W$ given as the image of μ. One has an integral decomposition of the regular representation

$$L^2(G\times_H W) = \int_T^{\oplus} L^2(G/H_w)\,\mathrm{d}\nu(w). \qquad (4.4)$$

Since the direct integral of tempered representations is tempered, we only need to prove that, for ν-almost all w in T,

$$L^2(G/H_w) \text{ is } L\text{-tempered.} \tag{4.5}$$

We can choose w in the Zariski open set where $\dim H_w$ is minimal. According to [2, Lemma 3.9], for such a w,

$$\text{the action of } H_w \text{ on } W/(\mathfrak{h}\, w) \text{ is trivial.} \tag{4.6}$$

Our assumption implies that one has the inequality on $\mathfrak{s}_w$

$$\rho_{\mathfrak{l}} \leq 2\,\rho_{\mathfrak{g}/\mathfrak{h}} + 2\,\rho_W\,.$$

Thanks to (4.6), this can be rewritten as

$$\rho_{\mathfrak{l}} \leq 2\,\rho_{\mathfrak{g}/\mathfrak{h}} + 2\,\rho_{\mathfrak{h}/\mathfrak{h}_w} \;=\; 2\,\rho_{\mathfrak{g}/\mathfrak{h}_w}\,.$$

Then the first case tells us that for such w, the representation of L in $L^2(G/H_w)$ is tempered. This proves (4.5) as required. □

4.4 Using parabolic subgroups

> The aim of this section is to explain how, when dealing with a quotient G/H of real algebraic groups, one can, using parabolic subgroups, reduce to the case where the unipotent radical V of H is included in the unipotent radical U of G. This reduction method will be used in Chapter 5 for complex Lie groups.

Let G be a real algebraic Lie group and H a real algebraic subgroup of G. We write $G = LU$ and $H = SV$ where U and V are the unipotent radicals of G and H, and where S and L are reductive algebraic subgroups. We can manage so that $S \subset L$ but we cannot always assume that V is included in U. For instance this is not possible when G is reductive and H is not. We fix a parabolic subgroup G_0 of G that contains H and which is minimal with this property. We denote by $U_0 \supset U$ the unipotent radical of G_0.

Lemma 4.8. *One has the inclusion $V \subset U_0$. Moreover, we can choose a reductive subgroup $L_0 \subset G_0$ such that $G_0 = L_0 U_0$ and $S \subset L_0$.*

Proof. The group $V_0 := U_0 \cap H$ is a unipotent normal subgroup of H. The quotient $S' := H/V_0$ is an algebraic subgroup of the reductive group G_0/U_0 which is not contained in any proper parabolic subgroup of G_0/U_0. Therefore, by [5, Sec. VIII.10] this group S' is reductive and the group V_0 is the unipotent radical V of H. This proves the inclusion $V \subset U_0$.

Since maximal reductive subgroups L_0 of G_0 are U_0-conjugate, one can choose L_0 containing S. $\square$

We introduce the L_0-module $W_0 := \mathfrak{u}_0/\mathfrak{v}$. The following two lemmas will be useful in our induction process.

Proposition 4.9. *Keep this notation. The following are equivalent:*
(*i*) $L^2(G/H)$ *is* L*-tempered;*
(*ii*) $\rho_{\mathfrak{l}} \leq 2\,\rho_{\mathfrak{g}/\mathfrak{h}}$ *as a function on* $\mathfrak{s}$*;*
(*iii*) $L^2(G_0/H)$ *is* L_0*-tempered;*
(*iv*) $\rho_{\mathfrak{l}_0} \leq 2\,\rho_{\mathfrak{g}_0/\mathfrak{h}}$ *as a function on* $\mathfrak{s}$*;*
(*v*) $L^2(L_0 \times_S W_0)$ *is* L_0*-tempered.*

Proof of Proposition 4.9. (*i*) $\Leftrightarrow$ (*ii*) and (*iii*) $\Leftrightarrow$ (*iv*). This is Theorem 3.2.

(*ii*) $\Leftrightarrow$ (*iv*) Write $\mathfrak{u}_0 = \mathfrak{u}_0' \oplus \mathfrak{u}$ where $\mathfrak{u}_0' := \mathfrak{u}_0 \cap \mathfrak{l}$. The equivalence follows from the equalities $\rho_{\mathfrak{l}} = \rho_{\mathfrak{l}_0} + 2\,\rho_{\mathfrak{u}_0'}$ and $\rho_{\mathfrak{g}} = \rho_{\mathfrak{g}_0} + \rho_{\mathfrak{u}_0'}$.

(*iv*) $\Leftrightarrow$ (*v*) This follows from Theorem 3.4 if one notices the equality $\rho_{\mathfrak{g}_0/\mathfrak{h}} = \rho_{\mathfrak{l}_0/\mathfrak{s}} + \rho_{W_0}$. $\square$

The following lemma will also be useful in this reduction process.

Lemma 4.10. *Keep this notation. The following are equivalent:*
(i) *the orbit closure* $\overline{\mathrm{AdG}\,\mathfrak{h}}$ *contains a solvable Lie algebra;*
(ii) *the orbit closure* $\overline{\mathrm{AdG_0}\,\mathfrak{h}}$ *contains a solvable Lie algebra.*

Proof of Lemma 4.10. This follows from the compactness of G/G_0. $\square$

5 Complex algebraic homogeneous spaces

The aim of this chapter is to prove the last remaining implication in Theorem 1.6 which is the converse of Proposition 2.7. We keep the notation of the previous Chapters 3 and 4. We assume in this chapter that both G and H are complex algebraic Lie group, but do not assume G to be semisimple.

5.1 The equivalence for G algebraic

We first state the extension of Theorem 1.6, which relates temperedness to the existence of solvable limit algebras for a general algebraic group G. This extension will be useful because of the induction process in the proof. We still use the notation in Section 3.1.

Theorem 5.1. *Let G be a complex algebraic Lie group and H be a complex algebraic subgroup. Then one has the equivalences,*

$$Tem(\mathfrak{g}, \mathfrak{h}) \Longleftrightarrow Rho(\mathfrak{g}, \mathfrak{h}) \Longleftrightarrow Sla(\mathfrak{g}, \mathfrak{h}).$$

Proof of Theorem 5.1. The first equivalence follows from Theorem 3.2. We split the proof of the second equivalence into Propositions 5.4 and 5.7. □

Corollary 5.2. *Let G be a complex algebraic Lie group, H be a complex algebraic subgroup, and $\mathfrak{h}' \in \overline{\mathrm{Ad}G\,\mathfrak{h}}$. Then one has the equivalence,*

$$Sla(\mathfrak{g}, \mathfrak{h}) \iff Sla(\mathfrak{g}, \mathfrak{h}').$$

This equivalence says that if a Lie subalgebra admits one solvable limit, then all its limit Lie algebras also admit a solvable limit.

Proof of Corollary 5.2. More precisely it is a corollary of Propositions 5.4 and 5.7. Indeed, if $\mathfrak{h}$ satisfies $Sla(\mathfrak{g}, \mathfrak{h})$, then by Proposition 5.7, it satisfies $Rho(\mathfrak{g}, \mathfrak{h})$. Then by Proposition 5.4, all limit subalgebras $\mathfrak{h}' \in \overline{\mathrm{Ad}G\,\mathfrak{h}}$ also satisfy $Sla(\mathfrak{g}, \mathfrak{h}')$. □

Remark 5.3. The set of Lie subalgebras $\mathfrak{h}$ in $\mathfrak{g}$ satisfying $Sla(\mathfrak{g}, \mathfrak{h})$ is closed. Indeed, this follows from the *Rho*-condition in Theorem 5.1.

5.2 Rho and Sla

We extend Proposition 2.7 to general algebraic groups G.

Proposition 5.4. *Let $\mathfrak{g}$ be an algebraic complex Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a complex Lie subalgebra. Then, one has the implication*

$$Rho(\mathfrak{g}, \mathfrak{h}) \implies Sla(\mathfrak{g}, \mathfrak{h}).$$

More precisely, if $\mathfrak{h}$ satisfies $Rho(\mathfrak{g}, \mathfrak{h})$, then every Lie algebra $\mathfrak{h}'$ in $\overline{\mathrm{Ad}G\mathfrak{h}}$ satisfies $Sla(\mathfrak{g}, \mathfrak{h})$.

Remark 5.5. In Propositions 5.4 and 5.7, the assumption that $\mathfrak{g}$ is algebraic, i.e. is the Lie algebra of a complex algebraic Lie group can easily be removed. We will not need it.

Proof of Proposition 5.4. This follows from Lemma 5.6 below and from the fact that the orbit closure always contains a closed G-orbit. □

We denote again by $\mathcal{L}_{rho}$ the set of Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ that satisfy $Rho(\mathfrak{g}, \mathfrak{h})$.

Lemma 5.6. *Let $\mathfrak{g}$ be an algebraic complex Lie algebra. Then,*
(i) $\mathcal{L}_{rho}$ is closed in $\mathcal{L}$.
(ii) Let $\mathfrak{h} \subset \mathfrak{g}$ be a complex Lie subalgebra with $\mathrm{Ad}G\,\mathfrak{h}$ *closed. Then,*

$$\mathfrak{h} \textit{ is solvable } \iff Rho(\mathfrak{g}, \mathfrak{h}) .$$

Proof of Lemma 5.6. This is a straightforward extension of Lemma 2.9. We write $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u}$ with $\mathfrak{l}$ reductive and $\mathfrak{u}$ the unipotent radical.

(i) Same as for Lemma 2.9.

$(ii) \Longrightarrow$ Same as for Lemma 2.9, but note that for $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{u}$ with $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{l}$, one has $\rho_{\mathfrak{l}} = 2\,\rho_{\mathfrak{l}/\mathfrak{b}} = 2\,\rho_{\mathfrak{g}/\mathfrak{h}}$.

$(ii) \Longleftarrow$ We may assume that $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$. Let $\mathfrak{q}$ be the normalizer of $\mathfrak{h}$. By assumption $\mathfrak{q}$ is a parabolic Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ is an ideal of $\mathfrak{q}$. Let $\mathfrak{g}_0$ be a parabolic subalgebra of $\mathfrak{q}$ containing $\mathfrak{h}$ and which is minimal with this property. We can write $\mathfrak{g}_0 = \mathfrak{l}_0 \oplus \mathfrak{u}_0$ and $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{v}$, where $\mathfrak{l}_0$ is a reductive Lie algebra, where $\mathfrak{u}_0$ is the unipotent radical of $\mathfrak{g}_0$, where $\mathfrak{s} := \mathfrak{h} \cap \mathfrak{l}_0$ is an ideal of $\mathfrak{l}_0$ and where $\mathfrak{v} := \mathfrak{h} \cap \mathfrak{u}_0$. By assumption one has $Rho(\mathfrak{g}, \mathfrak{h})$. Then, by the equivalence $(ii) \Leftrightarrow (iv)$ in Proposition 4.9 one also has $Rho(\mathfrak{g}_0, \mathfrak{h})$ i.e.

$$\rho_{\mathfrak{l}_0} \leq 2\,\rho_{\mathfrak{g}_0/\mathfrak{h}} \quad \text{as a function on } \mathfrak{s}.$$

But since $\mathfrak{h}$ is an ideal in $\mathfrak{g}_0$, the right hand side is null and this inequality can be rewritten as $\rho_{\mathfrak{s}} \leq 0$. This tells us that $\mathfrak{s}$ is abelian and $\mathfrak{h}$ is solvable. □

5.3 Sla and Rho

We are now able to prove the last remaining implication (1.10) by proving the following stronger Proposition 5.7 which is the converse to Proposition 5.4.

Proposition 5.7. *Let* $\mathfrak{g}$ *be a complex algebraic Lie algebra and* $\mathfrak{h} \subset \mathfrak{g}$ *be a complex Lie subalgebra. Then, one has the implication*

$$Sla(\mathfrak{g}, \mathfrak{h}) \implies Rho(\mathfrak{g}, \mathfrak{h}) .$$

Beginning of proof of Proposition 5.7. The proof of Proposition 5.7 will be by induction on the dimension of $\mathfrak{g}$, reducing to the case where both $\mathfrak{g}$ and $\mathfrak{h}$ are semisimple that we discussed in Proposition 2.10. Using Lemma 3.6 and Theorem 3.2, we can assume that $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$. In Proposition 4.9 and Lemma 4.10, we have introduced an intermediate algebraic complex Lie algebra $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$ such that the unipotent radical $\mathfrak{v}$ of $\mathfrak{h}$ is included in the unipotent radical $\mathfrak{u}_0$ of $\mathfrak{g}_0$, and for which we have the equivalences :

$$Rho(\mathfrak{g}, \mathfrak{h}) \Longleftrightarrow Rho(\mathfrak{g}_0, \mathfrak{h}) \quad \text{and} \quad Sla(\mathfrak{g}, \mathfrak{h}) \Longleftrightarrow Sla(\mathfrak{g}_0, \mathfrak{h}).$$

The proof will go on for two more sections. □

5.4 Pushing down the Sla condition

We sum up the previous notation.

Notation

Let $G_0 = L_0 U_0$ be an algebraic complex Lie group,
where L_0 is reductive and U_0 is the unipotent radical of G_0.
Let $H = SV$ be a connected algebraic complex Lie subgroup,
where S is reductive and V is the unipotent radical of H.
Assume that $S \subset L_0$ and $V \subset U_0$, and let $W_0 := U_0/V$.
For w in W_0, we denote by S_w the stabilizer of w in S.
Let $\mathfrak{g}_0$, $\mathfrak{h}$,..., $\mathfrak{s}_w$ be the corresponding Lie algebras.

Lemma 5.8. *Keep this notation. If* $\mathfrak{h}$ *satisfies* $Sla(\mathfrak{g}_0, \mathfrak{h})$*, then there exists a non-empty Zariski open set* $W_0' \subset W_0$ *such that for all* w *in* W_0'*,* $\mathfrak{s}_w$ *satisfies* $Sla(\mathfrak{l}_0, \mathfrak{s}_w)$

Proof of Lemma 5.8. By Lemma 4.7, there exists an S-invariant vector subspace $\mathfrak{m} \subset \mathfrak{u}_0$ such that $\mathfrak{u}_0 = \mathfrak{m} \oplus \mathfrak{v}$ and the map $\exp\colon \mathfrak{m} \to W_0 = U_0/V$ is a bijection.

By assumption, there exists a sequence $g_n \in G_0$ such that the limit

$$\mathfrak{h}_\infty \;:=\; \lim_{n\to\infty} \mathrm{Ad} g_n \, \mathfrak{h} \tag{5.1}$$

exists and is a solvable Lie subalgebra of $\mathfrak{g}_0$.

Since V normalizes $\mathfrak{h}$, we can assume that

$$g_n = \ell_n e^{X_n} \quad \text{with} \quad \ell_n \in L_0 \text{ and } X_n \in \mathfrak{m}. \tag{5.2}$$

We denote by $w_n \in W_0$ the image $w_n := \exp(X_n)$. The stabilizer $\mathfrak{s}_{w_n}$ of w_n in $\mathfrak{s}$ is also the centralizer of X_n in $\mathfrak{s}$. Therefore, one has the equality

$$\mathrm{Ad} e^{X_n} \, \mathfrak{s}_{w_n} \;=\; \mathfrak{s}_{w_n}. \tag{5.3}$$

Therefore, after extraction the limit $\mathfrak{s}_\infty := \lim\limits_{n\to\infty} \mathrm{Ad}\ell_n \, \mathfrak{s}_{w_n}$ exists and is a Lie subalgebra of $\mathfrak{h}_\infty$. In particular, this limit $\mathfrak{s}_\infty$ is solvable. Therefore there exists a maximal unipotent Lie algebra $\mathfrak{n}_0$ of $\mathfrak{l}_0$ such that

$$\mathfrak{s}_\infty \cap \mathfrak{n}_0 = \{0\},$$

and, for n large, one also has $\mathrm{Ad}\ell_n \mathfrak{s}_{w_n} \cap \mathfrak{n}_0 = \{0\}$. We have found at least one point w_0 in W_0 whose stabilizer $\mathfrak{s}_{w_0}$ is transversal to a maximal unipotent subalgebra $\mathfrak{n}$ of $\mathfrak{l}_0$. For such a subalgebra $\mathfrak{n}$ the set

$$W_0' := \{w \in W_0 \mid \mathfrak{s}_w \cap \mathfrak{n} = \{0\}\}$$

is a non-empty Zariski open subset of W_0.

By the equivalence of *Sla* and *Tmu* proven in Proposition 2.1, and since $\mathfrak{l}_0$ is reductive, for all w in W_0', the stabilizer $\mathfrak{s}_w$ satisfies $Sla(\mathfrak{l}_0, \mathfrak{s}_w)$. □

5.5 Pushing up the Rho condition

We now explain how a disintegration argument allows us to push the *Rho*-condition from $(\mathfrak{l}_0, \mathfrak{s}_w)$ up to $(\mathfrak{g}_0, \mathfrak{h})$. It is very surprising that we need this analytic argument to relate these two algebraic conditions.

End of proof of Proposition 5.7. We keep the notation of Sections 4.4 and 5.4, and we go on the proof by induction on the dimension of G.

First case : $L_0 \neq G$. We want to prove the condition $Rho(\mathfrak{g}, \mathfrak{h})$. We first check that the regular representation of L_0 in $L^2(L_0 \times_S W_0)$ is tempered. We argue as in the second case of Section 4.3. As in (4.4), we write the representation $L^2(L_0 \times_S W_0)$ as an integral of $L^2(L_0/S_w)$ so that we only need to prove that, for Lebesgue almost all w in W_0, the representation

$$L^2(L_0/S_w) \text{ is } L_0\text{-tempered.} \tag{5.4}$$

Note that the non-empty Zariski open set W_0' introduced in Lemma 5.8 has full Lebesgue measure. We have seen in Lemma 5.8 that

$$\mathfrak{s}_w \text{ satisfies } Sla(\mathfrak{l}_0, \mathfrak{s}_w), \quad \text{for all } w \text{ in } W_0'.$$

Since $\dim L_0 < \dim G$, our induction assumption implies that

$$\mathfrak{s}_w \text{ satisfies } Rho(\mathfrak{l}_0, \mathfrak{s}_w), \quad \text{for all } w \text{ in } W_0'.$$

And therefore by Theorem 3.2,

$$\mathfrak{s}_w \text{ satisfies } Tem(\mathfrak{l}_0, \mathfrak{s}_w), \quad \text{for all } w \text{ in } W_0'.$$

This proves (5.4) and the representation of L_0 in $L^2(L_0 \times_S W_0)$ is tempered.

Finally, using Proposition 4.9, one deduces that $L^2(G/H)$ is L_0-tempered, or equivalently $\mathfrak{h}$ satisfies $Rho(\mathfrak{g}, \mathfrak{h})$.

Second case : $L_0 = G$. In this case both G and H must be reductive. As we have seen in Lemma 3.6, we can assume that $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}]$. We can also assume that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Therefore one is reduced to the case where both $\mathfrak{g}$ and $\mathfrak{h}$ are semisimple which was settled in Proposition 2.10. This ends the proof of Proposition 5.7. □

This also ends simultaneously the proofs of Theorems 1.2, 1.6 and 5.1.

5.6 Comments and perspectives

We conclude by a few remaining questions

5.6.1 Openness of the Sla condition

Question 5.9. Let $\mathfrak{g}$ be a complex Lie algebra. Is the set of Lie subalgebras $\mathfrak{h}$ satisfying $Sla(\mathfrak{g}, \mathfrak{h})$ an open set?

We have seen that this set is closed in Remark 5.3 and we have seen that this set is open when $\mathfrak{g}$ is semisimple in Corollary 1.7.

5.6.2 Regular finite-dimensional representation

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ be a complex Lie subalgebra. We denote by $Irr(\mathfrak{g})_{reg}$ the set of finite-dimensional irreducible representations V of $\mathfrak{g}$ whose highest weight is regular. We now consider the condition

$$Rep(\mathfrak{g},\mathfrak{h}) \ : \ \text{there exists } \ V \in Irr(\mathfrak{g})_{reg} \text{ such that } \mathbb{P}(V)^{\mathfrak{h}} \neq \emptyset.$$

Question 5.10. Does one have the equivalence $Rep(\mathfrak{g},\mathfrak{h}) \Leftrightarrow Orb(\mathfrak{g},\mathfrak{h})$?

We know that the implication $\Longrightarrow$ is true.
We also know that the converse $\Longleftarrow$ is true when $\mathfrak{h}$ is reductive.

5.6.3 Parabolic induction of tempered representation

The strategy we followed in this series of paper could be simplified if we knew the answer to the following

Conjecture 5.11. Let G be a real algebraic semisimple group, $Q = LU$ be a parabolic subgroup, and π be a unitary representation of Q. Does one have

$$\pi \text{ is } L\text{-tempered} \Longleftrightarrow \mathrm{Ind}_Q^G \pi \text{ is } G\text{-tempered}.$$

We know that the implication $\Longleftarrow$ is true.
We have seen the implication $\Longrightarrow$ when $\pi|_U$ is trivial in Lemma 3.7.
We know the implication $\Longrightarrow$ when $G = \mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})$.

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