

A hidden symmetry of a branching law

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Abstract

We consider branching laws for the restriction of some irreducible unitary representations Π of $G = O(p, q)$ to its subgroup $H = O(p-1, q)$. In Kobayashi (arXiv:1907.07994, [14]), the irreducible subrepresentations of $O(p-1, q)$ in the restriction of the unitary $\Pi|_{O(p-1, q)}$ are determined. By considering the restriction of packets of irreducible representations we obtain another very simple branching law, which was conjectured in Ørsted–Speh (arXiv:1907.07544, [17]).

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I Introduction

The restriction of a finite-dimensional irreducible representation Π^G of a connected compact Lie group G to a connected Lie subgroup H is a classical problem. For example, the restriction of irreducible representations of $SO(n+1)$ to the subgroup $SO(n)$ can be expressed as a combinatorial pattern satisfied by the highest weights of the irreducible representation Π^G of the large group and of the irreducible representations

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appearing in the restriction of π^H [20]. For the pair $(G, H) = (SO(n+1), SO(n))$, the branching law is always multiplicity-free, *i.e.*,

$$\dim \text{Hom}_H(\pi^H, \Pi^G|_H) \leq 1.$$

In this article we consider a family of infinite-dimensional irreducible representations $\Pi_{\delta, \lambda}^{p, q}$ with parameters $\lambda \in \mathbb{Z} + \frac{1}{2}(p+q)$, and $\delta \in \{+, -\}$ of noncompact orthogonal groups $G = O(p, q)$ with $p \geq 3$ and $q \geq 2$, which have the same infinitesimal character as a finite-dimensional representation and which are subrepresentations of $L^2(O(p, q)/O(p-1, q))$ for $\delta = +$, respectively of $L^2(O(p, q)/O(p, q-1))$ for $\delta = -$. We shall assume a *regularity condition* of the parameter λ (Definition III.7). Similarly we consider a family of infinite-dimensional irreducible unitary representations $\pi_{\varepsilon, \mu}^{p-1, q}$, $\varepsilon \in \{+, -\}$ of noncompact orthogonal groups $H = O(p-1, q)$.

Reviewing the results of [14] we see in Section IV that the restriction of these representations to the subgroup $H = O(p-1, q)$ is either of “finite type” (Convention IV.13) if $\delta = +$ or of “discretely decomposable type” (Convention IV.6) if $\delta = -$. If the infinitesimal characters of $\Pi_{\delta, \lambda}^{p, q}$ and of a direct summand of $(\Pi_{\delta, \lambda}^{p, q})|_H$ satisfy an interlacing condition (4.12) similar to that of the finite-dimensional representations of $(SO(n+1), SO(n))$, then $\delta = +$ and the restriction of a representations $\Pi_{\delta, \lambda}^{p, q}$ is of finite type. On the other hand, if the infinitesimal characters $\Pi_{\delta, \lambda}^{p, q}$ and of a direct summand of $(\Pi_{\delta, \lambda}^{p, q})|_H$ satisfy another interlacing condition (4.9) similar to those of the holomorphic discrete series representations of $(SO(p, 2), SO(p-1, 2))$, then $\delta = -$ and the restriction of a representations $\Pi_{\delta, \lambda}^{p, q}$ is of discretely decomposable type.

For each λ we define a packet $\{\Pi_{+, \lambda}^{p, q}, \Pi_{-, \lambda}^{p, q}\}$ of representations with the same infinitesimal character. For simplicity, we assume $p \geq 3$ and $q \geq 2$. Using the branching laws for the individual representations we show in Section V:

Theorem I.1. *Let $(G, H) = (O(p, q), O(p-1, q))$. Suppose that λ and μ are regular parameters.*

- (1) *Let Π_λ be a representation in the packet $\{\Pi_{+, \lambda}, \Pi_{-, \lambda}\}$. There exists exactly one representations π_μ in the packet $\{\pi_{+, \mu}, \pi_{-, \mu}\}$ so that*

$$\dim \text{Hom}_H(\Pi_\lambda|_H, \pi_\mu) = 1.$$

- (2) *Let π_μ be in the packet $\{\pi_{+, \mu}, \pi_{-, \mu}\}$. There exists exactly one representation Π_λ in the packet $\{\Pi_{+, \lambda}, \Pi_{-, \lambda}\}$ so that*

$$\dim \text{Hom}_H(\Pi_\lambda|_H, \pi_\mu) = 1.$$

Equivalently we may formulate the result as follows:

Theorem I.2 (Version 2). *Suppose that λ and μ are regular parameters. Then*

$$\dim \text{Hom}_H((\Pi_{+,\lambda} \oplus \Pi_{-,\lambda})|_H, (\pi_{+,\mu} \oplus \pi_{-,\mu})) = 1.$$

Another version of this theorem using interlacing properties of infinitesimal characters is stated in Section V.

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Notation: $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ = \{1, 2, \dots\}$.

II Generalities

We will use in this article the notation and conventions of [14] which we recall now. These conventions differ from those used in [17].

Consider the standard quadratic form on \mathbb{R}^{p+q}

$$Q(X, X) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2 \tag{2.1}$$

of signature (p, q) in a basis $e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q}$. We define $G = O(p, q)$ to be the indefinite special orthogonal group that preserves the quadratic form Q . Let H be the stabilizer of the vector e_1 . Then H is isomorphic to $O(p-1, q)$.

Consider another quadratic form on \mathbb{R}^{p+q}

$$Q_-(X, X) = x_1^2 + \dots + x_q^2 - x_{q+1}^2 - \dots - x_{p+q}^2 \tag{2.2}$$

of signature (q, p) with respect to a basis $e_{-,1}, \dots, e_{-,p}, e_{-,p+1}, \dots, e_{-,p+q}$. The orthogonal group $G_- = O(q, p)$ that preserves the quadratic form Q_- is conjugate to $O(p, q)$ in $GL(p+q, \mathbb{R})$. Thus we may consider representations of $G_- = O(q, p)$ as representations of $G = O(p, q)$.

Since G and G_- are conjugate, the subgroup H of G is also conjugate to a subgroup H_- of G_- which is isomorphic to $O(q, p-1)$. This group isomorphism induces an isomorphism of homogeneous spaces $G/H = O(p, q)/O(p-1, q)$ and $G_-/H_- = O(q, p)/O(q, p-1)$. On the other hand $O(p, q)/O(p-1, q)$ and $O(q, p)/O(q-1, p)$ are not even homeomorphic to each other if $p \neq q$. In the rest of the article we will assume that the subgroup H_- preserves the vector $e_{-,p+q}$.

The maximal compact subgroups of G , G_- and H , H_- are denoted by K , K_- respectively K_H , K_{H_-} . The Lie algebras of the groups are denoted by the corresponding lowercase Gothic letters.

To avoid considering special cases we make in this article the following:

Assumption \mathcal{O} :

$$p \geq 3 \text{ and } q \geq 2.$$

III Representations

We consider in this article a family of irreducible unitary representations introduced in [14]. Using the notation in [14] we recall their parametrization and some important properties in this section. The main reference is [14, Sect. 2].

The irreducible unitary subrepresentations of $L^2(O(p, q)/O(p-1, q))$ were considered by many authors after the pioneering work by I. M. Gelfand et. al. [6], T. Shintani, V. Molchanov, J. Faraut [4], and R. Strichartz [18]. For $p \geq 2$ and $q \geq 1$, they are parametrized by $\lambda \in \mathbb{Z} + \frac{1}{2}(p+q)$ with $\lambda > 0$. Following the notation of [14] we denote them by

$$\Pi_{+, \lambda}^{p, q}.$$

They have infinitesimal character

$$\left(\lambda, \frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \dots, \frac{p+q}{2} - \left[\frac{p+q}{2}\right]\right),$$

in the Harish-Chandra parametrization (see (4.8) below), and the minimal K -type

$$\begin{cases} \mathcal{H}^{b(\lambda)}(\mathbb{R}^p) \boxtimes \mathbf{1} & \text{if } b(\lambda) \geq 0, \\ \mathbf{1} \boxtimes \mathbf{1} & \text{if } b(\lambda) \leq 0, \end{cases} \quad (3.3)$$

where $b(\lambda) := \lambda - \frac{1}{2}(p-q-2) (\in \mathbb{Z})$ and $\mathcal{H}^b(\mathbb{R}^p)$ stands for the space of spherical harmonics of degree b . We note that $\Pi_{+, \lambda}^{p, q}$ are so called Flensted-Jensen representations discussed in [5] if $b(\lambda) \geq 0$, namely, if $\lambda \geq \frac{1}{2}(p-q-2)$. This is the case if

λ is regular (Definition III.7). The underlying (\mathfrak{g}, K) -module of $\Pi_{+, \lambda}^{p, q}$ is given by a Zuckerman derived functor module. See [9, Thm. 3] or [14, Sect. 2.2].

Remark III.1. When $p = 1$ and $q \geq 1$, there are *no* irreducible subrepresentations in $L^2(O(p, q)/O(p-1, q))$, and we regard $\pi_{+, \lambda}^{p, q}$ as zero in this case.

Remark III.2. (1) For any $p \geq 2$, $q \geq 1$ and $\mathbb{Z} + \frac{1}{2}(p+q) \ni \lambda > 0$, the representation $\Pi_{+, \lambda}^{p, q}$ of $G = O(p, q)$ stays irreducible when restricted to $SO(p, q)$, see also Remark III.6.

(2) If $p = 2$ and $\lambda \geq \frac{1}{2}(p+q-2)$, then the representation $\Pi_{+, \lambda}^{p, q}$ is a direct sum of a holomorphic discrete series representation and an anti-holomorphic discrete series representation when restricted to the identity component $G_0 = SO_0(p, q)$ of G .

Similarly there exist a family of irreducible unitary subrepresentations

$$\Pi_{+, \lambda}^{q, p} \quad (\lambda \in \mathbb{Z} + \frac{1}{2}(p+q), \lambda > 0)$$

of $G_- = O(q, p)$ in $L^2(G_-/H_-) = L^2(O(q, p)/O(q-1, p))$ when $p \geq 1$ and $q \geq 2$, with the same infinitesimal character and the same properties. Via the isomorphism between (G_-, H_-) and (G, H) , we may consider them as representations of $G = O(p, q)$ and irreducible subrepresentations of $L^2(G/H) = L^2(O(p, q)/O(p, q-1))$.

If no confusion is possible we use the simplified notation

$$\Pi_{+, \lambda} = \Pi_{+, \lambda}^{p, q}$$

and

$$\Pi_{-, \lambda} \simeq \Pi_{+, \lambda}^{q, p} \quad (\text{via } G_- \simeq G),$$

to denote representations of $G = O(p, q)$.

Remark III.3. The irreducible representation $\Pi_{+, \lambda}$ are nontempered if $p \geq 3$, and $\Pi_{-, \lambda}$ are nontempered if $q \geq 3$.

Lemma III.4. *Assume that $\lambda \geq \frac{1}{2}(p+q-2)$. The representations $\Pi_{+, \lambda}$, $\Pi_{-, \lambda}$ are inequivalent, but have the same infinitesimal character.*

Proof. The representation $\Pi_{+, \lambda}$ and $\Pi_{-, \lambda}$ are irreducible representations of $G =$

$O(p, q)$ with respective minimal K -types

$$\begin{aligned} \mathcal{H}^b(\mathbb{R}^p) \boxtimes \mathbf{1}, & \quad b := \lambda - \frac{1}{2}(p - q - 2), \\ \mathbf{1} \boxtimes \mathcal{H}^{b'}(\mathbb{R}^q), & \quad b' := \lambda - \frac{1}{2}(q - p - 2), \end{aligned}$$

because the assumption $\lambda \geq \frac{1}{2}(p + q - 2)$ implies both $b \geq 0$ and $b' \geq 0$ by (3.3). \square

Remark III.5. Lemma III.4 holds in the more general setting where $\lambda \geq 0$, see [9, Thm. 3 (4)] for the proof.

Remark III.6. For p and q positive and even, the restriction of the representations $\Pi_{+, \lambda}, \Pi_{-, \lambda}$ to $SO(p, q)$ are in an Arthur packet as discussed in [3, 16]. Global versions of Arthur packets were introduced by J. Arthur in the theory of automorphic representations and are inspired by the trace formula [1, 2]. Our considerations of Arthur packets of representations of the orthogonal groups which are discrete series representations for symmetric spaces are inspired by Arthur's considerations as well as by the conjectures of B. Gross and D. Prasad. In this article we will refer to $\{\Pi_{+, \lambda}, \Pi_{-, \lambda}\}$ as a **packet** of irreducible representations.

Similarly we have $\mu \in \mathbb{Z} + \frac{1}{2}(p + q - 1)$ satisfying $\mu \geq \frac{1}{2}(p + q - 3)$ a packet $\{\pi_{+, \mu}, \pi_{-, \mu}\}$ of unitary irreducible representations of $G' = O(p - 1, q)$.

Definition III.7. We say $\lambda \in \mathbb{Z} + \frac{1}{2}(p + q)$ respectively $\mu \in \mathbb{Z} + \frac{1}{2}(p + q - 1)$ are **regular** if $\lambda \geq \frac{1}{2}(p + q - 2)$ respectively $\mu \geq \frac{1}{2}(p + q - 3)$.

Remark III.8. The irreducible representation $\Pi_{+, \lambda}$ (or $\Pi_{-, \lambda}$) has the same infinitesimal character as a finite-dimensional irreducible representation of $G = O(p, q)$ if and only if $\lambda \geq \frac{1}{2}(p + q - 2)$, namely, λ is regular. Similarly, $\pi_{+, \mu}$ (or $\pi_{-, \mu}$) has the same infinitesimal character with a finite-dimensional representation of $G' = O(p - 1, q)$ if and only if $\mu \geq \frac{1}{2}(p + q - 3)$, namely, μ is regular.

For later use we define for regular λ and μ the reducible representations

$$U(\lambda) = \Pi_{+, \lambda} \oplus \Pi_{-, \lambda} \tag{3.4}$$

and

$$V(\mu) = \pi_{+, \mu} \oplus \pi_{-, \mu}. \tag{3.5}$$

of $G = O(p, q)$ respective of $H = O(p - 1, q)$.

IV Branching laws

In this section we summarize the results of [14]. For simplicity, we suppose that the assumption \mathcal{O} is satisfied, namely, we assume $p \geq 3$ and $q \geq 2$. We note that the results in Section IV.2 hold in the same form for $p \geq 2$ and $q \geq 2$, and those in Section IV.3 hold for $p \geq 3$ and $q \geq 1$.

IV.1 Quick introduction to branching laws

Consider the restriction of a unitary representation Π of G to a subgroup G' . We say that an irreducible unitary representation π of H is in the discrete spectrum of the restriction $\Pi|_H$ if there exists an isometric H -homomorphism $\pi \rightarrow \Pi|_H$, or equivalently, if

$$\mathrm{Hom}_H(\pi, \Pi|_H) \neq \{0\}$$

where $\mathrm{Hom}_H(,)$ denotes the space of continuous H -homomorphisms. We define the multiplicity for the unitary representations by

$$m(\Pi, \pi) := \dim \mathrm{Hom}_H(\pi, \Pi|_H) = \dim \mathrm{Hom}_H(\Pi|_H, \pi).$$

Remark IV.1. As in [7, 15], we also may consider the multiplicity $m(\Pi^\infty, \pi^\infty)$ for smooth admissible representations Π^∞ of G and π^∞ of G' by

$$m(\Pi^\infty, \pi^\infty) := \dim \mathrm{Hom}_H(\Pi^\infty, \pi^\infty).$$

In general, one has

$$m(\Pi^\infty, \pi^\infty) \geq m(\Pi, \pi).$$

Besides the discrete spectrum there may be also continuous spectrum. Here are two interesting cases:

1. There is no continuous spectrum and the representation Π is a direct sum of irreducible representations of H , *i.e.*, the underlying Harish-Chandra module is a direct sum of countably many Harish-Chandra modules of (\mathfrak{h}, K_H) . We say that the restriction $\Pi|_H$ is *discretely decomposable*.
2. There is continuous spectrum and there are only finitely many representations in the discrete spectrum in the irreducible decomposition of the restriction $\Pi|_H$.

We refer to the necessary and sufficient conditions of the parameters of the irreducible representations Π, π so that $m(\Pi, \pi) \neq 0$ (or $m(\Pi^\infty, \pi^\infty) \neq 0$) as a *branching law*. In the examples below, $m(\Pi^\infty, \pi^\infty), m(\Pi, \pi) \in \{0, 1\}$ for all Π and π .

Examples of branching laws:

1. Finite-dimensional representations of semisimple Lie groups are parametrized by highest weights. The classical branching law of the restriction of finite-dimensional representations of $SO(n)$ to $SO(n-1)$ is phrased as an interlacing pattern of highest weights, see Weyl [20].
2. The Gross–Prasad conjectures for the restriction of discrete series representations of $SO(2m, 2n)$ to $SO(2m-1, 2n)$ are expressed as interlacing properties of their parameters, see [7].
3. The branching laws for the restriction of irreducible self-dual representations Π^∞ of $SO(n+1, 1)$ to $SO(n, 1)$ are expressed by using *signatures*, *heights* and interlacing properties of weights, see [15].

If $\Pi \in \{\Pi_{+, \lambda}, \Pi_{-, \lambda}\}$, and

$$\mathrm{Hom}_H(\pi_H, \Pi|_H) \neq \{0\}$$

then for a character χ of $O(1)$

$$\mathrm{Hom}_{H \times O(1)}(\pi_H \boxtimes \chi, \Pi|_{H \times O(1)}) \neq \{0\}.$$

Moreover, by [14, Thm. 1.1] there exists a regular μ so that $\pi_H \in \{\pi_{+, \mu}, \pi_{-, \mu}\}$.

If Π is in the packet $\{\Pi_{+, \lambda}, \Pi_{-, \lambda}\}$ and π in the packet $\{\pi_{+, \mu}, \pi_{-, \mu}\}$ the branching laws discussed in the next part will involve the parameters $\lambda, \mu, \varepsilon, \delta$.

IV.2 Branching laws for the restriction of $\Pi_{-, \lambda}$ to $H = O(p-1, q)$ — discretely decomposable type

This section treats the restriction $\Pi_{-, \lambda}|_H$, which is discretely decomposable. We use the explicit branching law given in [14, Example 1.2 (1)]. The results were also obtained in [10] by using different techniques, see [12, 13] for details.

We begin with the pair $(G_-, H_-) = (O(q, p), O(q, p-1))$. The restriction of the representation $\Pi_{+, \lambda}^{q, p}$ of G_- to the subgroup $H_- \times O(1) = O(q, p-1) \times O(0, 1)$ is a direct sum of irreducible representations, and is isomorphic to the Hilbert direct sum of countably many Hilbert spaces:

$$\bigoplus_{n \in \mathbb{N}} \pi_{+, \lambda + n + \frac{1}{2}}^{q, p-1} \boxtimes (\mathrm{sgn})^n$$

where sgn stands for the nontrivial character of $O(1) = O(0, 1)$. Then via the identification $(G_-, H_-) \simeq (G, H) = (O(p, q), O(p-1, q))$ and $\Pi_{+, \lambda}^{q, p} \simeq \Pi_{-, \lambda}$ as a

representation of $G_- \simeq G$, we see the restriction of $\Pi_{-, \lambda}$ to $H \times O(1) = O(p-1, q) \times O(1, 0)$ is discretely decomposable, and we have an isomorphism

$$\Pi_{-, \lambda}|_H \simeq \bigoplus_{n \in \mathbb{N}} \pi_{-, \lambda + n + \frac{1}{2}} \boxtimes (\text{sgn})^n.$$

Hence

Proposition IV.2 (Version 1). *The restriction of $\Pi_{-, \lambda}$ to $H = O(p-1, q)$ is a Hilbert direct sum*

$$\bigoplus_{n \in \mathbb{N}} \pi_{-, \lambda + n + \frac{1}{2}}$$

and each representation has multiplicity one.

Remark IV.3. If λ is regular, then μ is regular whenever $\text{Hom}_H(\pi_{-, \mu}, \Pi_{-, \lambda}|_H) \neq \{0\}$. In contrast, an analogous statement fails for the restriction $\Pi_{+, \lambda}|_H$, see Remark IV.10 below.

Remark IV.4. If $G = SO_0(p, 2)$ the representation $\Pi_{-, \lambda}$ with λ regular is a holomorphic discrete series representation. In this case, this result follows from the work of H. Plesner-Jacobson and M. Vergne [8, Cor. 3.1] or as a special case of the general formula proved in [11, Thm. 8.3].

We define $\kappa: \mathbb{N} \rightarrow \{0, \frac{1}{2}\}$ by

$$\kappa(n) = 0 \quad \text{for } n \text{ even; } = \frac{1}{2} \quad \text{for } n \text{ odd.}$$

Then the infinitesimal character of the representation $\Pi_{-, \lambda}$ of G is

$$\left(\lambda, \frac{p+q-4}{2}, \dots, \kappa(p+q)\right), \tag{4.6}$$

and the infinitesimal character of the representations in $\pi_{-, \mu}$ of H is

$$\left(\mu, \frac{p+q-5}{2}, \dots, \kappa(p+q-1)\right). \tag{4.7}$$

Here we note that the groups G and H are not of Harish-Chandra class, but the infinitesimal characters of the centers $\mathfrak{Z}_G(\mathfrak{g}) := U(\mathfrak{g})^G$ and $\mathfrak{Z}_H(\mathfrak{h}) := U(\mathfrak{h})^H$ of the enveloping algebras can be still described by elements of \mathbb{C}^M with $M := \lfloor \frac{1}{2}(p+q) \rfloor$

and \mathbb{C}^N with $N := \lfloor \frac{1}{2}(p+q-1) \rfloor$ modulo finite groups via the Harish-Chandra isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}_G(\mathfrak{g}), \mathbb{C}) &\simeq \mathbb{C}^M / \mathfrak{S}_M \times (\mathbb{Z}/2\mathbb{Z})^M, \\ \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}_H(\mathfrak{h}), \mathbb{C}) &\simeq \mathbb{C}^N / \mathfrak{S}_N \times (\mathbb{Z}/2\mathbb{Z})^N. \end{aligned} \tag{4.8}$$

In our normalization, the infinitesimal character of the trivial one-dimensional representation of $G = O(p, q)$ is given by

$$\left(\frac{p+q-2}{2}, \frac{p+q-4}{2}, \dots, \kappa(p+q) \right).$$

Hence we may also reformulate the branching laws in Proposition IV.2 as follows.

Proposition IV.5 (Version 2). *Suppose λ is a regular parameter (Definition III.7). Then an irreducible representation π of $H = O(p-1, q)$ in the discrete spectrum of the restriction of $\Pi_{-, \lambda}^{p, q}$ must be isomorphic to $\pi_{-, \mu}$ for some regular parameter μ , and the infinitesimal characters have the interlacing property*

$$\mu > \lambda > \frac{p+q-4}{2} > \dots > \frac{1}{2} > 0. \tag{4.9}$$

Conversely, $\pi = \pi_{-, \mu}$ occurs in the discrete spectrum of the restriction $\Pi_{-, \lambda}^{p, q}|_H$ if the interlacing property (4.9) is satisfied.

Convention IV.6. *We say that the restriction of the representation $\Pi_{-, \lambda}$ of G to $H = O(p-1, q)$ is of discretely decomposable type.*

IV.3 Branching laws for the restriction of $\Pi_{+, \lambda}$ to $H = O(p-1, q)$ — finite type

This section treats the restriction $\Pi_{+, \lambda}|_H$ which is *not* discretely decomposable. We use [14, Example 1.2 (2)] which determines the whole discrete spectrum in the restriction $\Pi_{+, \lambda}|_H$. A large part of discrete summands are also obtained in [17] using different techniques.

The restriction $\Pi_{+, \lambda}|_H$ contains at most finitely many irreducible summands. We recall from [14, Thm. 1.1] (or [14, Ex. 1.2 (2)]), an irreducible representation π of $H \times O(1, 0) = O(p-1, q) \times O(1)$ occurs in the discrete spectrum of the restriction of $\Pi_{+, \lambda}$ if and only if it is of the form

$$\pi_{+, \lambda - n - \frac{1}{2}}^{p-1, q} \boxtimes (\mathrm{sgn})^n \quad \text{for some } 0 \leq n < \lambda - \frac{1}{2},$$

where sgn stands for the nontrivial character of $O(1)$.

Proposition IV.7 (Version 1). *An irreducible representation π of $H = O(p-1, q)$ occurs in the discrete spectrum of the restriction of $\Pi_{+, \lambda}$ of $G = O(p, q)$ when restricted to H if and only if it is of the form*

$$\pi_{+, \lambda - \frac{1}{2} - n}^{p-1, q} \text{ where } \lambda - \frac{1}{2} - n \text{ for } 0 \leq n < \lambda - \frac{1}{2}.$$

Remark IV.8. There does not exist discrete spectrum in the restriction $\Pi_{+, \lambda}|_H$ if $p = 2$. In fact $\pi_{+, \mu}^{1, q}$ is zero for all μ if $q \geq 1$, see Remark III.1.

Remark IV.9. The representation $\pi_{+, \lambda - \frac{1}{2} - n}^{p-1, q}$ has a regular parameter, or equivalently, has the same infinitesimal character as a finite-dimensional representation iff

$$\lambda - \frac{1}{2} - n > \frac{p+q-5}{2}.$$

Remark IV.10. In contrast to the discretely decomposable case (Remark IV.3), Proposition IV.7 tells that the implication

$$\lambda \text{ regular} \Rightarrow \mu \text{ regular}$$

does not necessarily hold when $\text{Hom}_H(\pi_{+, \mu}, \Pi_{+, \lambda}|_H) \neq \{0\}$, see Remark IV.9 above.

We observe that for these representations the condition in the proposition depends only on $p+q$ and thus the proposition for these representations does not depend on the inner form $SO(r, s)$ of $SO(p+q, \mathbb{C})$ when $r+s = p+q$ with $r \geq 3$.

Recall that the infinitesimal character of the representation $\Pi_{+, \lambda}$ is

$$\left(\lambda, \frac{p+q-4}{2}, \dots, \kappa(p+q)\right) \tag{4.10}$$

and the infinitesimal character of the representations in $\pi_{+, \mu}$

$$\left(\mu, \frac{p+q-5}{2}, \dots, \kappa(p+q-1)\right) \tag{4.11}$$

as in (4.6) and (4.7).

Proposition IV.11 (Version 2). *Suppose π is an irreducible unitary representation of $H = O(p-1, q)$. If π occurs in the discrete spectrum of the restriction of $\Pi_{+, \lambda}$ to H , then π must be isomorphic to $\pi_{+, \mu}$ for some $\mu > 0$ with $\mu \in \mathbb{Z} + \frac{1}{2}(p+q-1)$. Assume further that λ and μ are regular. Then $\pi_{+, \mu}$ occurs in the discrete spectrum of the restriction $\Pi_{+, \lambda}|_H$ if and only if the two infinitesimal characters (4.10) and (4.11) have the interlacing property*

$$\lambda > \mu > \frac{p+q-4}{2} > \dots > \frac{1}{2} > 0. \quad (4.12)$$

Remark IV.12. Consider the example: $q = 0$ and so G is compact. The representation $\Pi_{-, \lambda}$ is finite-dimensional and has highest weight

$$\left(\lambda - \frac{p}{2}, 0, \dots, 0\right)$$

for an integer λ . A representation $\pi_{-, \mu}$ is a summand of the restriction to $H = SO(p-1)$ if it has highest weight

$$\left(\mu - \frac{p-1}{2}, 0, \dots, 0\right)$$

for $\mu \in \mathbb{N} + \frac{1}{2}$ with $\mu \geq \frac{p-1}{2}$ and $\lambda - \frac{p}{2} \geq \mu - \frac{p-1}{2} \geq 0$, *i.e.*, if there exists an integer $n \in \mathbb{N}$ so that $\mu = \lambda - \frac{1}{2} - n \geq \frac{1}{2}(p-1)$.

This motivates the following:

Convention IV.13. *We say that the restriction of the representation $\Pi_{-, \lambda}$ to $H = SO(p-1, q)$ is of finite type.*

V The main theorems

We retain Assumption \mathcal{O} , namely, $p \geq 3$ and $q \geq 2$. Combing the branching laws in the previous section proves the conjectures in [17, Sect. V] and suggests a generalization of a conjecture by B. Gross and D. Prasad [7], which was formulated for tempered representations.

V.1 Results for pairs $(O(p, q), O(p-1, q))$

Theorem V.1 (Version 1). *Suppose that λ and μ are regular parameters (Definition III.7).*

1. Let Π_λ be a representations in the packet $\{\Pi_{+,\lambda}, \Pi_{-,\lambda}\}$. There exists exactly one representations π_μ in the packet $\{\pi_{+,\mu}, \pi_{-,\mu}\}$ so that

$$\dim \text{Hom}_H(\Pi_\lambda|_H, \pi_\mu) = 1.$$

2. Let π_μ be in the packet $\{\pi_{+,\mu}, \pi_{-,\mu}\}$. There exists exactly one representation Π_λ in the packet $\{\Pi_{+,\lambda}, \Pi_{-,\lambda}\}$ so that

$$\dim \text{Hom}_H(\Pi_\lambda|_H, \pi_\mu) = 1.$$

Equivalently we may formulate the results in terms of reducible representations $U(\lambda)$ and $V(\mu)$ defined in (3.4) and (3.5) as follows:

Theorem V.2 (Version 2). *Suppose that λ and μ are regular parameters. Then*

$$\dim \text{Hom}_H(U(\lambda)|_H, V(\mu)) = 1.$$

We may formulate the results in interlacing properties of parameter the infinitesimal characters similar to the results in [7].

Recall that the infinitesimal character of the representations of G in the packet $\{\Pi_{+,\lambda}, \Pi_{-,\lambda}\}$ is

$$\left(\lambda, \frac{p+q-4}{2}, \dots, \kappa(p+q)\right)$$

and the infinitesimal character of the representations of the subgroup H in the packet $\{\pi_{+,\mu}, \pi_{-,\mu}\}$ is

$$\left(\mu, \frac{p+q-5}{2}, \dots, \kappa(p+q-1)\right),$$

where we recall $(\kappa(p+q), \kappa(p+q-1)) = (0, \frac{1}{2})$ if $p+q$ is even, $= (\frac{1}{2}, 0)$ if $p+q$ is odd.

Theorem V.3 (Version 3). *Suppose that λ and μ are regular parameters.*

1. *If the two infinitesimal characters satisfy the following interlacing property:*

$$\mu > \lambda > \frac{p+q-4}{2} > \dots > \frac{1}{2} > 0$$

then

$$\dim \text{Hom}_H(\Pi_{-,\lambda}|_H, \pi_{-,\mu}) = 1.$$

2. If the two infinitesimal characters satisfy the following interlacing property:

$$\lambda > \mu > \frac{p+q-4}{2} > \dots > \frac{1}{2} > 0$$

then

$$\dim \operatorname{Hom}_H(\Pi_{+, \lambda}|_H, \pi_{+, \mu}) = 1.$$

Remark V.4. The trivial representation $\mathbf{1}$ of $H = O(p-1, q)$ is in the dual of the smooth representation $\Pi_{+, \lambda}^\infty$ but not in the dual of $\Pi_{-, \lambda}^\infty$. There is no other representation in the “packet” of the trivial representation of H and so we deduce

$$\dim \operatorname{Hom}_H(U(\lambda)^\infty|_H, \mathbf{1}) = 1,$$

or equivalently there is exactly one representation Π_λ in the set $\{\Pi_{+, \lambda}^\infty, \Pi_{-, \lambda}^\infty\}$ so that

$$\dim \operatorname{Hom}_H(\Pi_\lambda^\infty|_H, \mathbf{1}) = 1.$$

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