Distinguished representations of $SO(n+1,1) \times SO(n,1)$, periods and branching laws

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Abstract

Given irreducible representations Π and π of the rank one special orthogonal groups $G = SO(n+1,1)$ and $G' = SO(n,1)$ with nonsingular integral infinitesimal character, we state in terms of θ-stable parameter necessary and sufficient conditions so that

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \neq \{0\}.$$

In the special case that both Π and π are tempered, this implies the Gross–Prasad conjectures for tempered representations of $SO(n+1,1) \times SO(n,1)$ which are nontrivial on the center.

We apply these results to construct nonzero periods and distinguished representations. If both Π and π have the trivial infinitesimal character ρ then we use a theorem that the periods are nonzero on the minimal K-type to obtain a nontrivial bilinear form on the $(\mathfrak{g}, K)$-cohomology of the representations.

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I Introduction

In the book [23] *The Classical Groups: Their Invariants and Representations* published in 1939, Hermann Weyl discusses the restriction of irreducible finite-dimensional representations of the orthogonal group $O(n)$ to an orthogonal subgroup $O(n-1)$. The restriction splits into a direct sum of irreducible representations and can be roughly described by the following *branching law.* Suppose $n \geq 2$. Denote by $[x]$ the greatest integer that does not exceed x. To each irreducible representation of $O(n)$ is assigned a highest weight

$$\mu = (\mu_1, \mu_2, \ldots, \mu_{[\frac{n}{2}]})$$

satisfying

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{[\frac{n}{2}]}$$

and a character ε of $O(n)/SO(n)$. We denote by this representation $F^{O(n)}(\mu)_\varepsilon$ where $\varepsilon \in \{+,-\}$ (ε is not unique when n is even and $\mu_{\frac{n}{2}} \neq 0$). H. Weyl obtained the "branching law" as

$$F^{O(n)}(\mu_1, \ldots, \mu_{[\frac{n}{2}]})_\varepsilon|_{O(n-1)} = \bigoplus F^{O(n-1)}(\nu_1, \ldots, \nu_{[\frac{n-1}{2}]})_\varepsilon \tag{1.1}$$

where the summation is taken over $(\nu_1, \ldots, \nu_{[\frac{n-1}{2}]}) \in \mathbb{Z}^{[\frac{n-1}{2}]}$ subject to

$$\begin{aligned} &\mu_1 \geq \nu_1 \geq \mu_2 \geq \cdots \geq \nu_{\frac{n-1}{2}} \geq 0 && \text{for } n \text{ odd,} \\ &\mu_1 \geq \nu_1 \geq \mu_2 \geq \cdots \geq \nu_{\frac{n-2}{2}} \geq \mu_{\frac{n}{2}} \geq 0 && \text{for } n \text{ even.} \end{aligned}$$

In this article we present similar branching laws for the restriction of irreducible *infinite-dimensional* representations of $SO(n+1,1)$ to the subgroup $SO(n,1)$. Since the restriction of an infinite-dimensional representation Π of $SO(n+1,1)$ to $SO(n,1)$ is not a direct sum of irreducible representations [7], we consider as in [9, 11] the representations Π and π of $SO(n+1,1)$, respectively $SO(n,1)$, realized as smooth representations of moderate growth [22, Chap. 11] and define the *multiplicity* by

$$m(\Pi, \pi) := \dim_{\mathbb{C}} \mathrm{Hom}_{SO(n,1)}(\Pi|_{SO(n,1)}, \pi).$$

The multiplicity is either 0 or 1 [17].

We consider in the article only representations of the special orthogonal group which have the same infinitesimal character as an irreducible finite-dimensional representation F of $SO(n+1)$. To simplify the notation and presentation we assume in this article that F is “self-dual”, *i.e.*, assume that the highest weight $\mu = (\mu_1, \cdots, \mu_{m+1})$ satisfies $\mu_{m+1} = 0$ when $n = 2m+1$. We do not impose any assumption on F when n is even. See Assumption A in Section II.2.1. For the general case, see [11] and [12].

For every irreducible representation π of $SO(n,1)$ we define in Section II.3 a *height*

$$h_\pi \in \{0, \dots m\} \quad \text{if} \quad n = 2m \text{ or } 2m+1$$

and in Section II.4 a *signature* $\delta \in \{+, -\}$. The signature is unique except for discrete series representations. If π has the same infinitesimal character as $F(\mu)$ we say that (μ, h_π, δ) are the *enhanced θ-stable parameters* of π. The representations with *enhanced θ-stable parameters* $((0, \dots, 0), i, \delta)$ are representations with trivial infinitesimal character ρ and are denoted by $\Pi_{i,\delta}$. See [11, Chap. 2, Sect. 4] for a description of the representations of $O(n+1,1)$ with trivial infinitesimal character ρ and [11, Chap. 14, Sect. 9] for their enhanced θ-parameters.

Remark I.1. In [11] we have treated mainly the full group $O(n+1,1)$ rather than the special orthogonal group $SO(n+1,1)$, and stated results for $SO(n+1,1)$ in Chapter 11 with “bar” for the corresponding objects. The relation between branching laws for $O(n+1,1) \downarrow O(n,1)$ and $SO(n+1,1) \downarrow SO(n,1)$ is discussed in [11, Chap. 15 (Appendix II)]. In this article, we treat mainly the special orthogonal group $SO(n+1,1)$, and use different convention in the point that we omit the “bar” for representations of $SO(n+1,1)$.

All results in this article are based on the following **branching theorem**:

Theorem I.2 (branching law)**.** *Let Π and π be irreducible representations of $SO(n+1,1)$ respectively $SO(n,1)$ with enhanced θ-stable parameters $(\mu, h_{\Pi,}\varepsilon)$, respectively (ν, h_π, δ).*

(1) *Suppose that $n = 2m$. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{SO(n,1)}(\Pi|_{SO(n,1)}, \pi) = 1$$

if and only if the enhanced θ-stable parameters of the representations Π and π satisfy

(a) $\varepsilon = \delta$,

(b) $h_\pi \in \{h_\Pi, h_\Pi - 1\}$ *when* $h_\pi < m$ *and* $h_\pi = h_\Pi$ *when* $h_\pi = m$,

(c) $\mu_0 \geq \nu_0 \geq \mu_1 \geq \cdots \geq \mu_m \geq \nu_m \geq 0$.

(2) *Suppose that $n = 2m-1$. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{SO(n,1)}(\Pi|_{SO(n,1)}, \pi) = 1$$

if and only if the enhanced θ-stable parameters of the representations satisfy

(a) $\varepsilon = \delta$,

(b) $h_\pi \in \{h_\Pi, h_\Pi - 1\}$,

(c) $\mu_0 \geq \nu_0 \geq \mu_1 \geq \cdots \geq \mu_m \geq \nu_{m-1} \geq \mu_m = 0$.

A detailed proof will be given in [12].

Remark I.3. A similar theorem was proved in [11, Thms. 4.1 and 4.2] for the restriction of irreducible representations of $O(n+1,1)$ to $O(n,1)$ with trivial infinitesimal character ρ.

Applications of the branching theorem

Gross–Prasad conjectures. The discussion in [11, Chap. 13, Sect. 3.3] shows that Theorem I.2 in the special case where $h_\Pi = \left[\frac{n+1}{2}\right]$ and $h_\pi = \left[\frac{n}{2}\right]$ implies the following.

Theorem I.4. *The Gross–Prasad conjectures are valid for all tempered representations with nonsingular infinitesimal character of $SO(n+1,1)$ and $SO(n,1)$, which are nontrivial on the center.*

Remark I.5. (1) For tempered principal series representations Π of $SO(n+1,1)$, and π of $SO(n,1)$ which are nontrivial on each center, it was proved in [11, Chap. 11, Sect. 4]. For irreducible tempered representations with trivial infinitesimal character ρ, this was announced in [10] and proved in [11, Chap. 11, Sect. 5].
(2) The branching law for nontempered representations (Theorem I.2) interpolates between the classical branching laws of finite-dimensional representations (1.1) and the branching laws of the conjecture by Gross and Prasad for tempered representations.

Periods. For representations Π, π of a real reductive Lie group G, respectively of a reductive subgroup G', the space of symmetry breaking operators

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi^\vee)$$

and the space of G'-invariant continuous linear functionals

$$\mathrm{Hom}_{G'}(\Pi \otimes \pi, \mathbb{C})$$

are naturally isomorphic to each other [11, Thm. 5.4], where $\pi^\vee$ denotes the contragredient representation of π in the category of admissible smooth representations (see Section III.2), and $\Pi \otimes \pi$ denotes the representation of G' acting on the outer tensor product representation $\Pi \boxtimes \pi$ of $G \times G'$ diagonally. Thus we may use symmetry breaking operators to determine G'-invariant continuous linear functionals on $\Pi \otimes \pi$, *i.e.*, periods. Hence, the branching theorem implies the following.

Theorem I.6. *Suppose that the representations* Π *and* π *of* G*, respectively of* G' *satisfy Assumption A* (*see Section II.2.1*) *with height* i *respectively* j*. Then the following statements on the pair* (Π, π) *are equivalent:*

(i) *The representation* $\Pi \boxtimes \pi$ *has a nontrivial* G'*-period;*

(ii) Π *and* π *have the same signature,* $j = i$ *or* $i - 1$ *and their enhanced* θ*-stable parameters satisfy the interlacing conditions of Theorem I.2.*

For representations with infinitesimal character ρ we proved in [11, Chap. 10] furthermore the following:

Theorem I.7. *If* Π *and* π *are representations with trivial infinitesimal character* ρ*, then any nonzero period does not vanish on the minimal* K*-type of the outer tensor product representation* $\Pi \boxtimes \pi$*.*

In a special case we determine in [11, Chap. 12] the value of the period on vectors in the minimal K-type of a representation with infinitesimal character ρ. See also Theorem V.11.

Distinguished representations. Let G be a reductive group and H a reductive subgroup. We regard H as a subgroup of the direct product group $G \times H$ via the diagonal embedding $H \hookrightarrow G \times H$.

Definition I.8. Let ψ be a one-dimensional representation of H. We say an admissible smooth representation Π of G is (H, ψ)-*distinguished* if

$$(\mathrm{Hom}_H(\Pi \boxtimes \psi^\vee, \mathbb{C}) \simeq) \, \mathrm{Hom}_H(\Pi|_H, \psi) \neq \{0\}.$$

If the character ψ is trivial, we say Π is H-*distinguished.*

Let $G = SO(n+1,1)$, and $\mathfrak{g} \simeq \mathfrak{so}(n+1,1)$ its Lie algebra. We fix a fundamental Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. For $0 \leq i \leq [\frac{n+1}{2}]$, there are θ-stable parabolic subalgebras $\mathfrak{q}_i \equiv \mathfrak{q}_i^+ = (\mathfrak{l}_i)_{\mathbb{C}} + \mathfrak{u}_i$ and $\mathfrak{q}_i^- = (\mathfrak{l}_i)_{\mathbb{C}} + \mathfrak{u}_i^-$ in $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{u}_i^- + (\mathfrak{l}_i)_{\mathbb{C}} + \mathfrak{u}_i$ such that $\mathfrak{q}_i$ and $\mathfrak{q}_i^-$ contain $\mathfrak{h}_\mathbb{C}$ and that both the Levi subgroups L_i of $\mathfrak{q} = \mathfrak{q}_i$ and $\mathfrak{q}_i^-$ are given by $L_i \simeq SO(2)^i \times SO(n-2i+1,1)$, see [11, Lem. 14.38].

There are two one-dimensional representations of $SO(n-2i+1,1)$ when $n \neq 2i-1$: we denote by χ_+ the trivial representation $\mathbf{1}$ and by χ_- the nontrivial one. For $n = 2i-1$, we consider only χ_+. We consider the differential $\lambda = (\lambda_1, \cdots, \lambda_i, 0, \ldots 0)$ of a one-dimensional representation of L_i and assume that λ satisfies the conditions in [11, Chap. 14, Sect. 9] and $\delta \in \{+,-\}$. We consider an irreducible one-dimensional L_i-module

$$\mathbb{C}_\lambda \boxtimes \chi_\delta$$

and define an admissible smooth representation of G of moderate growth denoted by $A_{\mathfrak{q}_i}(\lambda)_\delta$. Its underlying $(\mathfrak{g}, K)$-module is given by the cohomological parabolic induction from $\mathfrak{q}_i$, see [5, 21]. For $\delta = +$ we often omit the subscript.

Theorem I.9. *Suppose that $\Pi \in \mathcal{A}$ (see Section II.2.1 below for definition) is a representation of $SO(n+1,1)$ cohomologically induced from a one-dimensional representation of a θ-stable parabolic subalgebra $\mathfrak{q}_i$, i.e. that $\Pi = A_{\mathfrak{q}_i}(\lambda)$. Then the height (see Section II.3) of Π is i and Π is $SO(n+1-i,1)$-distinguished.*

Remark I.10. The proof of this theorem will be given in a subsequent paper, based on the work [11]. For a different proof and perspective of this theorem, see [6].

The irreducible representations with trivial infinitesimal character ρ are obtained through cohomological induction. We set $\Pi_i := A_{\mathfrak{q}_i}(0)$. Then Theorem I.9 may be regarded as a generalization of the following results in [11, Thm. 12.4 and Lem. 15.10]:

Theorem I.11. *Let $0 \leq i \leq n+1$. Then the representations Π_i of $G = SO(n+1,1)$ are $SO(n+1-i,1)$-distinguished.*

For details see [11, Chap. 12].

A bilinear form on the* $(\mathfrak{g}, K)$*-cohomology. In [11, Chap. 12, Sect. 3], we considered the morphism on $(\mathfrak{g}, K)$-cohomologies of representations induced by a symmetry breaking operator: Let $(G, G') = (SO(n+1,1), SO(n,1))$ and Π, π be irreducible representations and V, V' irreducible finite-dimensional representations of G and G', respectively. By abuse of notation, we use the same symbols Π and π to denote their underlying $(\mathfrak{g}, K)$-module and $(\mathfrak{g}', K')$-module, respectively, when we take $(\mathfrak{g}, K)$-cohomologies and $(\mathfrak{g}', K')$-cohomologies. Suppose that there exists a symmetry breaking operator $T\colon \Pi \otimes V \to \pi \otimes V'$. Then the symmetry breaking operator $T\colon \Pi \otimes V \to \pi \otimes V'$ induces for every j a morphism

$$T^j\colon H^j(\mathfrak{g}, K; \Pi \otimes V) \to H^j(\mathfrak{g}', K'; \pi \otimes V')$$

on the $(\mathfrak{g}, K)$-cohomologies and a bilinear form

$$B_T \colon H^j(\mathfrak{g}, K; \Pi \otimes V) \times H^{n-j}(\mathfrak{g}', K'; (\pi \otimes V')^\vee \otimes \chi_{(-1)^{n+1}}) \to \mathbb{C}$$

where $(\pi \otimes V')^\vee$ denotes the contragredient representation of $\pi \otimes V'$.

The induced morphism on the $(\mathfrak{g}, K)$-cohomologies may be zero, but in some special cases we conclude that it is nonzero.

For $0 \leq \ell \leq [\frac{n+1}{2}]$ and $\delta \in \{+, -\}$, we denote by $\Pi_{\ell,\delta}$ the irreducible admissible smooth representation of $G = SO(n+1, 1)$ with underlying $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}_\ell}(0)_\delta$ (see Theorem I.9 for notation). We also write simply Π_ℓ for $\Pi_{\ell,\delta}$ if $\delta = +$. We recall from [11, Thm. 2.20] that $\Pi_{\ell,\delta}$ is the unique submodule of the principal series representation $I_\delta(\Lambda^\ell(\mathbb{C}^n, \ell))$ for $\ell \neq \frac{n}{2}$ (see (2.6) below), and is isomorphic to $I_\delta(V_+, \frac{n}{2}) \simeq I_\delta(V_-, \frac{n}{2})$ for $\ell = \frac{n}{2}$ where V_+ and V_- are irreducible $SO(n)$-modules such that $\Lambda^{\frac{n}{2}}(\mathbb{C}^n) = V_+ \oplus V_-$. We also recall from [11, Prop. 15.11] that the set of irreducible admissible representations of $G = SO(n+1, 1)$ with the trivial $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character ρ is classified as follows:

$$\begin{aligned} &\{\Pi_{\ell,\delta} : 0 \leq \ell \leq \frac{n-1}{2}, \delta \in \{+, -\}\} \cup \{\Pi_{\frac{n+1}{2},+}\} && \text{if } n \text{ is odd,} \\ &\{\Pi_{\ell,\delta} : 0 \leq \ell \leq \frac{n}{2}, \delta \in \{+, -\}\} && \text{if } n \text{ is even.} \end{aligned}$$

Analogous notation $\pi_{j,\varepsilon}$ is applied to the subgroup $G' = SO(n, 1)$.

Then the following theorem follows from [11, Cor. 12.19 and Lem. 15.10] for the nonvanishing of B_T and [11, Thm. 15.19] for the uniqueness of T.

Theorem I.12. *Let $(G, G') = (SO(n+1, 1), SO(n, 1))$, $0 \leq i \leq \frac{n}{2}$, and $\delta \in \{\pm\}$. Let T be a nontrivial symmetry breaking operator $\Pi_{i,\delta} \to \pi_{i,\delta}$.*

(1) *T induces bilinear forms*

$$B_T \colon H^j(\mathfrak{g}, K; \Pi_{i,\delta}) \times H^{n-j}(\mathfrak{g}', K'; \pi_{i,(-1)^{n-1}\delta}) \to \mathbb{C} \quad \textit{for all } j.$$

(2) *The bilinear form B_T is nonzero if and only if $j = i$ and $\delta = (-1)^i$.*

In Section VI, we also state a nonvanishing theorem for bilinear forms on the $(\mathfrak{g}, K)$-cohomologies of principal series representations, see Theorem VI.1.

Remark I.13. If Π and π have trivial infinitesimal character ρ then V and V' are the trivial representations and the theorem follows from [11, Chaps. 9 and 12].

Detailed proofs of the results will be published elsewhere [12].

II Representations with nonsingular integral infinitesimal character

In this section we recall from [11] the results about principal series representations and irreducible representations of $G = SO(n+1,1)$ and introduce their height and signature.

II.1 Notation

We are using the same notation and assumptions as in the book [11] except that we do not use "bar" for subgroups of $SO(n+1,1)$, see Remark I.1. The proofs of most of the results stated in this section can be found in Chapter 2 and Appendices I, II, and III therein.

We first recall some notation which is up to small changes (see the comments in Remark I.1) the same as the notation in the Memoir article [9].

Consider the standard quadratic form

$$x_0^2 + x_1^2 + \cdots + x_n^2 - x_{n+1}^2 \tag{2.2}$$

of signature $(n+1,1)$. We define G to be the indefinite special orthogonal group $SO(n+1,1)$ that preserves the quadratic form (2.2) and the orientation. Let G' be the stabilizer of the vector $e_n = {}^t(0,0,\cdots,0,1,0)$. Then G' is isomorphic to $SO(n,1)$. We set

$$\begin{aligned} K &:= O(n+2) \cap G = \left\{\begin{pmatrix} B & \\ & \det B\end{pmatrix} : B \in O(n+1)\right\} \simeq O(n+1), \qquad (2.3)\\ K' &:= K \cap G' \qquad\quad = \left\{\begin{pmatrix} B & & \\ & 1 & \\ & & \det B\end{pmatrix} : B \in O(n)\right\} \simeq O(n). \end{aligned}$$

Then K and K' are maximal compact subgroups of G and G', respectively.

II.2 Principal series representations

Let $\mathfrak{g} = \mathfrak{so}(n+1,1)$ and $\mathfrak{g}' = \mathfrak{so}(n,1)$ be the Lie algebras of G and G', respectively. We take a hyperbolic element H as

$$H := E_{0,n+1} + E_{n+1,0} \in \mathfrak{g}', \tag{2.4}$$

and set

$$\mathfrak{a} := \mathbb{R}H \qquad \text{and } A := \exp \mathfrak{a}.$$

Then the centralizers of H in G and G' are given by MA and $M'A$, respectively, where

$$M := \left\{ \begin{pmatrix} \varepsilon & & \\ & B & \\ & & \varepsilon \end{pmatrix} : B \in SO(n),\ \varepsilon = \pm 1 \right\} \qquad \simeq SO(n) \times O(1),$$

$$M' := \left\{ \begin{pmatrix} \varepsilon & & & \\ & B & & \\ & & 1 & \\ & & & \varepsilon \end{pmatrix} : B \in SO(n-1) : \varepsilon = \pm 1 \right\} \qquad \simeq SO(n-1) \times O(1).$$

We observe that $\mathrm{ad}(H) \in \mathrm{End}_{\mathbb{R}}(\mathfrak{g})$ has eigenvalues -1, 0, and $+1$. Let

$$\mathfrak{g} = \mathfrak{n}_- + (\mathfrak{m} + \mathfrak{a}) + \mathfrak{n}_+$$

be the corresponding eigenspace decomposition, and P a minimal parabolic subgroup with Langlands decomposition $P = MAN_+$. Likewise, $P' := M'AN'_+$ is a compatible Langlands decomposition of a minimal (also maximal) parabolic subgroup P' of G' with Lie algebra

$$\mathfrak{p}' = \mathfrak{m}' + \mathfrak{a} + \mathfrak{n}'_+ = (\mathfrak{m} \cap \mathfrak{g}') + (\mathfrak{a} \cap \mathfrak{g}') + (\mathfrak{n}_+ + \mathfrak{g}'). \tag{2.5}$$

We note that we have chosen $H \in \mathfrak{g}'$ so that $P' = P \cap G'$ and $A = \exp(\mathbb{R}H)$ is a common maximally split abelian subgroup in P' and P.

The character group of $O(1)$ consists of two characters. We write $+$ for the trivial character $\mathbf{1}$, and $-$ for the nontrivial character. Since $M \simeq SO(n) \times O(1)$, any irreducible representation of M is the outer tensor product of an irreducible representation (σ, V) of $SO(n)$ and a character δ of $O(1)$.

Given $(\sigma, V) \in \widehat{SO(n)}$, $\delta \in \{\pm\} \simeq \widehat{O(1)}$, and a character $e_\lambda(\exp(tH)) = e^{\lambda t}$ of A for $\lambda \in \mathbb{C}$, we define the (unnormalized) principal series representation

$$I_\delta(V, \lambda) = \mathrm{Ind}_P^G(V \otimes \delta, \lambda) \tag{2.6}$$

of $G = SO(n+1, 1)$ on the Fréchet space of smooth maps $f\colon G \to V$ subject to

$$f(gmm'e^{tH}n) = \sigma(m)^{-1}\delta(m')e^{-\lambda t} f(g)$$
$$\text{for all } g \in G,\ mm' \in M \simeq SO(n) \times O(1),\ t \in \mathbb{R},\ n \in N_+.$$

By a result of R. Langlands [14] every irreducible nontempered representation with nonsingular integral infinitesimal character is isomorphic to the unique subrepresentation of a principal series representation $I_\delta(V,\lambda)$ with $\lambda < \frac{n}{2}$. We denote it by $\Pi_\delta(V,\lambda)$, see also [11, Chap. 15, Sect. 7].

Definition II.1. We call the triple (V,δ,λ) the *Langlands parameter* of the irreducible nontempered representation $\Pi_\delta(V,\lambda)$.

II.2.1 The set $\mathcal{A}$

Since we highlight in the article representations which are of interest in number theory, we consider from now on a subset of irreducible representations of special orthogonal groups. For results about irreducible representation of $SO(n+1,1)$ in the general case see [11, Chap. 15].

We start with irreducible finite-dimensional representations of $SO(n+1,1)$ that are self-dual, or equivalently, that are obtained as the restriction of irreducible representations of $O(n+1,1)$. So for $n = 2m$ we assume that the highest weight $(\mu_1,\dots,\mu_{m+1})$ of an irreducible finite-dimensional representation F of $G = SO(2m+1,1)$ is of the form $(\mu_1,\dots,\mu_m,0)$.

Assumption A. Suppose that Π is an irreducible representation of $G = SO(n+1,1)$ with regular integral infinitesimal character, see [11, Chap. 2, Sect. 1.4]. We say that a representation Π of G satisfies *Assumption* A if it has the same infinitesimal character as a self-dual irreducible finite-dimensional representation of G. When n is odd, Assumption A is automatically satisfied.

Notation II.2. The set of irreducible representations of $G = SO(n+1,1)$ satisfying Assumption A is denoted by $\mathcal{A}$.

For the convenience of the reader, we give a description of irreducible admissible representations of $G = SO(n+1,1)$ satisfying Assumption A in Section II.2.2 for n even and in Section II.2.3 for n odd.

II.2.2 Classification of the set $\mathcal{A}$ for n even

Suppose $n = 2m$. By using the highest weight, we write $V \in \widehat{SO(n)}$ as $V = F^{SO(n)}(\sigma)$ with $\sigma = (\sigma_1,\cdots,\sigma_m) \in \mathbb{Z}^m$ satisfying $\sigma_1 \geq \cdots \geq \sigma_{m-1} \geq |\sigma_m|$. Then the contragredient representation of V is given as

$$V^\vee \simeq F^{SO(n)}(\sigma_1,\cdots,\sigma_{m-1},-\sigma_m).$$

Hence V is self-dual if and only if $\sigma_m = 0$.

Proposition II.3. *For $V = F^{SO(2m)}(\sigma)$, we consider the following condition on λ:*

$$\lambda \in \mathbb{Z},\ \lambda < m,\ \lambda \notin \{1-\sigma_1, \cdots, m-\sigma_m\}. \tag{2.7}$$

Then irreducible admissible representations of $SO(2m+1,1)$ in $\mathcal{A}$ are classified as

- (nontempered case)

$$\{\Pi_\delta(V,\lambda) : \delta = \pm, \sigma_m = 0,\ \textit{and } \lambda \textit{ satisfies } (2.7)\},$$

  or

- (tempered case)

$$\{I_\delta(V,m) : \delta = \pm, \sigma_m > 0\}.$$

We note that in the tempered case $V \not\simeq V^\vee$ as $SO(n)$-modules because $\sigma_m > 0$, whereas there is a G-isomorphism, see [11, Prop.15.5]:

$$I_\delta(V,m) \simeq I_\delta(V^\vee, m).$$

II.2.3 Classification of the set $\mathcal{A}$ for n odd

Suppose $n = 2m-1$. We write $V \in \widehat{SO(n)}$ as $V = F^{SO(n)}(\sigma)$ with $\sigma = (\sigma_1, \cdots, \sigma_{m-1}) \in \mathbb{Z}^{m-1}$ satisfying $\sigma_1 \geq \cdots \geq \sigma_{m-1} \geq 0$.

Proposition II.4. *For $V = F^{SO(2m-1)}(\sigma)$, we consider the following conditions on λ:*

$$\lambda \in \mathbb{Z},\ \lambda \leq m-1, \quad \textit{and } \lambda \notin \{1-\sigma_1, \cdots, m-1-\sigma_{m-1}\}. \tag{2.8}$$

$$\lambda \in \mathbb{Z},\ m \leq \lambda \leq m-1+\sigma_{m-1}. \tag{2.9}$$

All irreducible admissible representations of $SO(2m,1)$ belong to $\mathcal{A}$, which are classified as

- (nontempered case)

$$\{\Pi_\delta(V,\lambda) : \delta = \pm,\ \lambda \textit{ satisfies } (2.8)\},$$

  or

- (discrete series)

$$\{\Pi_+(V,\lambda) : \sigma_{m-1} > 0,\ \lambda \textit{ satisfies } (2.9)\}.$$

We note that in the discrete series case there is a G-isomorphism

$$\Pi_+(V,\lambda) \simeq \Pi_-(V,\lambda).$$

II.3 The height of representations in $\mathcal{A}$

In this section, we define a *height*

$$h\colon \mathcal{A} \to \{0, 1, \cdots, [\frac{n+1}{2}]\}$$

for irreducible representations of $G = SO(n+1,1)$ that belong to $\mathcal{A}$ (see Section II.2.1).

We recall that the i-th exterior representation $\bigwedge^i(\mathbb{C}^n)$ of $SO(n)$ is irreducible if $2i \neq n$, and splits into two irreducible representations if $2i = n$, which may be written as

$$\bigwedge^{\frac{n}{2}}(\mathbb{C}^n) \simeq \bigwedge^{\frac{n}{2}}(\mathbb{C}^n)^{(+)} \oplus \bigwedge^{\frac{n}{2}}(\mathbb{C}^n)^{(-)}.$$

We define a finite family of principal series representations of $G = SO(n+1,1)$ by

$$I_\delta(i,i) := \begin{cases} I_\delta(\bigwedge^i(\mathbb{C}^n), i) & \text{for } i \neq \frac{n}{2}, \\ I_\delta(\bigwedge^i(\mathbb{C}^n)^{(\varepsilon)}, i) & \text{for } i = \frac{n}{2}. \end{cases}$$

Then $I_\delta(i,i)$ has the trivial infinitesimal character ρ, and it does not depend on the choice $\varepsilon = +$ or $-$ when $i = \frac{n}{2}$, see [11, (15.5)].

Suppose that Π is an irreducible nontempered representation of $SO(n+1,1)$ in $\mathcal{A}$ with Langlands parameter (V, λ, δ). The principal series representation $I_\delta(V,\lambda)$ can be obtained by using a translation functor from exactly one principal series representation $I_\delta(i,i)$, $i \in \{0, 1, \ldots, [\frac{n+1}{2}]\}$ without crossing a wall. See [11, Thm. 16.24].

Following [11, Chap. 14, Sect. 5 and Thm. 16.17], we say that the principal series representation $I_\delta(V,\lambda)$ has *height* i if it can be obtained from a principal series representations $I_\delta(i,i)$ by a translation functor without crossing walls.

Definition II.5 (height). (1) Suppose that the representation $\Pi \in \mathcal{A}$ is not tempered and has Langlands parameter (V, λ, δ). If $I_\delta(V,\lambda)$ has *height* i, we say that Π has height i.

(2) If $n = 2m-1$ we say that a discrete series representation $\Pi \in \mathcal{A}$ has *height* m.

(3) If $n = 2m$ and $\Pi \in \mathcal{A}$ is a tempered representation, then we say that it has *height* m.

An explicit formula of the height can be derived from the case for $O(n+1,1)$ in [11, Def. 14.26 and Chap. 15]:

Proposition II.6 (height for nontempered representation). (1) *Suppose* $n = 2m$. *With notation as in Proposition II.3, the height* i *of the nontempered representation* $\Pi_\delta(V,\lambda)$ *takes the value in* $\{0, 1, \cdots, m-1\}$, *and is determined by the following inequalities:*

$$i - \sigma_i < \lambda < i + 1 - \sigma_{i+1}.$$

(2) *Suppose* $n = 2m-1$. *With notation as in Proposition II.4, the height of the nontempered representation* $\Pi_\delta(V,\lambda)$ *takes the value in* $\{0, 1, \cdots, m-1\}$, *and is determined by the following condition:*

- *for* $\lambda < m - 1 - \sigma_{m-1}$, *we have* $0 \le i \le m-2$ *and*

$$i - \sigma_i < \lambda < i + 1 - \sigma_{i+1};$$

- *for* $m - 1 - \sigma_{m-1} < \lambda < m$, *we have* $i = m-1$.

Example II.7. For $\Pi \in \mathcal{A}$, the height of Π is zero if and only if Π is finite-dimensional.

II.4 Signatures of representations in $\mathcal{A}$

In this section, we define a *signature*

$$\operatorname{sgn}\colon \mathcal{A} \to \{+, -, \pm\}$$

for irreducible representations of $G = SO(n+1,1)$ that belong to $\mathcal{A}$. We shall impose the condition

$$\operatorname{sgn}(\Pi \otimes \chi_-) = -\operatorname{sgn}\Pi. \tag{2.10}$$

II.4.1 Tempered Representations

We recall from Propositions II.3 and II.4 (see also [11, Thms. 13.7 and 13.9]) that for every irreducible finite-dimensional representation F in $\mathcal{A}$ there exist irreducible tempered representations with the same infinitesimal character.

Suppose $G = SO(2m+1,1)$ and $\lambda = m$ $(= \frac{n}{2})$. The unitary principal series representation $I_\delta(V,m)$ is tempered, and it has the same infinitesimal character as an irreducible finite-dimensional representation of G if and only if V is not self-dual. In this case, there is an isomorphism $I_\delta(V,m) \simeq I_\delta(V^\vee,m)$ as $SO(2m+1,1)$-modules as we saw in Section II.2.2. We define the signature of $I_\delta(V,m)$ to be δ.

For $G = SO(2m, 1)$, there is exactly one discrete series representation Π of G having the same infinitesimal character as F (see Proposition II.4). Furthermore there is an isomorphism

$$\Pi \otimes \chi_- \simeq \Pi$$

as G-modules, hence we define the signature of a discrete series representation Π to be $\pm$.

II.4.2 Nontempered representations

An irreducible representation Π in $\mathcal{A}$ of $G = SO(n+1, 1)$, which is not tempered, is isomorphic to a subrepresentation of a principal series representation $I_\delta(V, \lambda)$ of height $i \in \{0, 1, \cdots, [\frac{n-1}{2}]\}$, see Proposition II.6. It has a Langlands parameter (V, δ, λ) where V is a self-dual representation of $SO(n)$, $\lambda < \frac{n}{2}$ and $\delta \in \{+, -\}$. We refer to $\delta(-1)^{i-\lambda}$ as the signature of the representation Π. Since the Langlands parameter of a nontempered representation is unique, the signature is unique as far as Π is not tempered.

Remark II.8. If a representation Π has signature δ, then one sees $\Pi \otimes \chi_-$ has signature $-\delta$.

Example II.9 (one-dimensional representations)**.** There are two one-dimensional representations of $G = SO(n+1, 1)$: we denote the trivial representation $\mathbf{1}$ by χ_+ and the nontrivial one by χ_-. The Langlands parameter of χ_+ and χ_- is $(+, \mathbf{1}, 0)$ and $(-, \mathbf{1}, 0)$, respectively, and their height is 0. Hence the representation χ_+ has signature $+$ and the representation χ_- has signature $-$.

Example II.10 (irreducible representations with trivial infinitesimal character ρ)**.** If V is the representation of $SO(n)$ on the i-th exterior tensor space $\bigwedge^i(\mathbb{C}^n)$ $(2i \neq n)$, we write for simplicity $I_\delta(i, \lambda)$ instead of $I_\delta(V, \lambda)$. Then the $SO(n)$-isomorphism on the exterior representations $\bigwedge^i(\mathbb{C}^n) \simeq \bigwedge^{n-i}(\mathbb{C}^n)$ leads us to the following G-isomorphism:

$$I_\delta(i, \lambda) \simeq I_\delta(n-i, \lambda).$$

If n is even and $n = 2i$, the exterior representation $\bigwedge^i(\mathbb{C}^n)$ splits into two irreducible representations of $SO(n)$:

$$\bigwedge^{\frac{n}{2}}(\mathbb{C}^n) \simeq \bigwedge^{\frac{n}{2}}(\mathbb{C}^n)_+ \oplus \bigwedge^{\frac{n}{2}}(\mathbb{C}^n)_-$$

with highest weights $(1, \cdots, 1, 1)$ and $(1, \cdots, 1, -1)$, respectively, with respect to a fixed positive system for $\mathfrak{so}(n, \mathbb{C})$. Accordingly, we have a direct sum decomposition of the induced representation:

$$\mathrm{Ind}_P^G(\bigwedge^{\frac{n}{2}}(\mathbb{C}^n)\otimes\delta,\lambda)=I_\delta(\bigwedge^{\frac{n}{2}}(\mathbb{C}^n)_+,\lambda)\oplus I_\delta(\bigwedge^{\frac{n}{2}}(\mathbb{C}^n)_-,\lambda),$$

which we shall write as

$$I_\delta\left(\frac{n}{2},\lambda\right)=I_\delta^{(+)}\left(\frac{n}{2},\lambda\right)\oplus I_\delta^{(-)}\left(\frac{n}{2},\lambda\right). \tag{2.11}$$

The representations $I_\delta^{(+)}\left(\frac{n}{2},\lambda\right)$ and $I_\delta^{(-)}\left(\frac{n}{2},\lambda\right)$ are isomorphic to each other and have the signature δ.

Let $\mathfrak{Z}(\mathfrak{g})$ be the center of the enveloping algebra $U(\mathfrak{g})$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}\simeq\mathfrak{so}(n+2,\mathbb{C})$. Via the Harish-Chandra isomorphism, the $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character of the trivial one-dimensional representation $\mathbf{1}$ is given by

$$\rho=\left(\frac{n}{2},\frac{n}{2}-1,\cdots,\frac{n}{2}-[\frac{n}{2}]\right)$$

in the standard coordinates of the Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}=\mathfrak{so}(n+2,\mathbb{C})$, whereas up to conjugation by the Weyl group the infinitesimal character of $I_\delta^{(\pm)}(i,\lambda)$ (when $2i\le n$) is given by

$$\left(\frac{n}{2},\frac{n}{2}-1,\cdots,\frac{n}{2}-i+1,\widehat{\frac{n}{2}-i},\frac{n}{2}-i-1,\cdots,\frac{n}{2}-[\frac{n}{2}],\lambda-\frac{n}{2}\right). \tag{2.12}$$

The irreducible representation $\Pi_{i,\delta}$ has height $\min(i,n-i)$. Since $\lambda=i$, the signature is equal to δ if $2i\le n$ and to $\delta(-1)^n$ if $2i\ge n$. The irreducible tempered representations are denoted by Π_m if $n=2m-1$ and $\Pi_{m,\delta}$ if $n=2m$. See [11, Chap. 2, Sect. 4.5] for $O(n+1,1)$ and [11, Chap. 15, Sect. 5] for $SO(n+1,1)$ in detail.

For the group $G'=SO(n,1)$, we shall use the notation $J_\varepsilon(j,\nu)$ for the unnormalized parabolic induction $\mathrm{Ind}_{P'}^{G'}(\bigwedge^j(\mathbb{C}^{n-1})\otimes\varepsilon,\nu)$ for $0\le j\le n-1$, $\varepsilon\in\{+,-\}$, and $\nu\in\mathbb{C}$. The irreducible representations are denoted by $\pi_{j,\varepsilon}$ respectively π_j.

II.4.3 Hasse and Standard sequences

The notion of the height of representation in $\mathcal{A}$ is motivated by the Hasse sequences in [11, Chap. 13], which were defined for the full orthogonal group $O(n+1,1)$. We adapt the definition for the special orthogonal group $G=SO(n+1,1)$ as follows. Let $n=2m$ or $2m-1$. For every irreducible finite-dimensional representation F of the group G, there exists a unique sequence

$$U_0 \quad , \quad \dots \quad , \quad U_{m-1} \quad , \quad U_m$$

of irreducible admissible smooth representations $U_i \equiv U_i(F)$ of G such that

1. $U_0 \simeq F$;
2. consecutive representations are composition factors of a principal series representation;
3. U_i $(0 \leq i \leq m)$ are pairwise inequivalent as G-modules.

Definition II.11 (Hasse sequence and standard sequence)**.** We refer to the sequence

$$U_0 \quad , \quad \dots \quad , \quad U_{m-1} \quad , \quad U_m$$

as the *Hasse sequence* of irreducible representations starting with the finite-dimensional representation $U_0 = F$. We shall write $U_j(F)$ for U_j if we emphasize the sequence $\{U_j(F)\}$ starts with $U_0 = F$, and we refer to

$$\Pi_0 := U_0 \quad , \quad \dots \quad , \quad \Pi_{m-1} := U_{m-1} \otimes (\chi_-)^{m-1} \quad , \quad \Pi_m := U_m \otimes (\chi_-)^m$$

as the *standard sequence* of irreducible representations $\Pi_i = \Pi_i(F)$ starting with $\Pi_0 = U_0 = F$, where χ_- is the nontrivial one-dimensional representation of G defined in Example II.9.

More details about the standard sequence for $O(n+1,1)$ can be found in [11, Chap. 13], from which the case for the normal subgroup $G = SO(n+1,1)$ is derived as follows.

Theorem II.12. *Let $G = SO(n+1,1)$.*
(1) *Suppose that $\Pi \in \mathcal{A}$ is not a discrete series representation of G, having height j and signature δ. Then there exists exactly one irreducible finite-dimensional representation F of G with signature δ so that Π is the j-th representation in the standard sequence starting with F.*
(2) *Suppose that $\Pi \in \mathcal{A}$ is a discrete series representation with signature $\delta \in \{\pm\}$. Then n is odd, and there exists a unique irreducible finite-dimensional representation F of G with signature $+$, so that Π is the $\frac{n+1}{2}$-th representation in the standard sequence starting with F and with $F \otimes \chi_-$.*

Remark II.13. The last representation in the standard sequence starting at an irreducible finite-dimensional representation $F \in \mathcal{A}$ is tempered.

II.4.4 The θ-stable and enhanced θ-stable parameters of irreducible representations

We summarize the results in [11, Chap. 14] and give a parametrization of irreducible subquotients of the principal series representations $I_\delta(V,\lambda)$ of the group $G = SO(n+1,1)$ in terms of cohomological parabolic induction.

We recall quickly cohomological parabolic induction. A basic reference is Vogan [19] and Knapp–Vogan [5]. We begin with a *connected* real reductive Lie group G. Let K be a maximal compact subgroup, and θ the corresponding Cartan involution. Given an element $X \in \mathfrak{k}$, the complexified Lie algebra

$$\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$$

is decomposed into the eigenspaces of $\sqrt{-1}\,\mathrm{ad}(X)$, and we write

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}_- + \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$$

for the sum of the eigenspaces with negative, zero, and positive eigenvalues. Then $\mathfrak{q} := \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ is a θ-stable parabolic subalgebra with Levi subgroup

$$L = \{g \in G : \mathrm{Ad}(g)\mathfrak{q} = \mathfrak{q}\}. \tag{2.13}$$

The homogeneous space G/L is endowed with a G-invariant complex manifold structure such that its holomorphic cotangent bundle is given as $G \times_L \mathfrak{u}$. As an algebraic analogue of Dolbeault cohomology groups for G-equivariant holomorphic vector bundle over G/L, Zuckerman introduced a cohomological parabolic induction functor $\mathcal{R}^j_{\mathfrak{q}}(\cdot \otimes \mathbb{C}_{\rho(\mathfrak{u})})$ ($j \in \mathbb{N}$) from the category of $(\mathfrak{l}, L \cap K)$-modules to the category of $(\mathfrak{g}, K)$-modules. We adopt here the normalization of the cohomological parabolic induction $\mathcal{R}^j_{\mathfrak{q}}$ from a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ so that the $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character of the $(\mathfrak{g}, K)$-module $\mathcal{R}^j_{\mathfrak{q}}(F)$ equals

$$\text{the } \mathfrak{Z}(\mathfrak{l})\text{-infinitesimal character of the } \mathfrak{l}\text{-module } F$$

modulo the Weyl group via the Harish-Chandra isomorphism.

For each i with $0 \le i \le [\frac{n+1}{2}]$, there are θ-stable parabolic subalgebras $\mathfrak{q}_i \equiv \mathfrak{q}_i^+ = (\mathfrak{l}_i)_{\mathbb{C}} + \mathfrak{u}_i$ and $\mathfrak{q}_i^- = (\mathfrak{l}_i)_{\mathbb{C}} + \mathfrak{u}_i^-$ in $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$ such that $\mathfrak{q}_i$ and $\mathfrak{q}_i^-$ contain a fundamental Cartan subalgebra $\mathfrak{h}$. The Levi subgroup $L_i = N_G(\mathfrak{q}_i)$ of the θ-stable parabolic subalgebra $\mathfrak{q}_i$ and $\mathfrak{q}_i^-$ is isomorphic to $L_i = SO(2)^i \times SO(n-2i+1,1)$.

We set

$$\Lambda^+(N) := \{(\lambda_1, \cdots, \lambda_N) \in \mathbb{Z}^N : \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N \ge 0\}.$$

For $\nu = (\nu_1, \cdots, \nu_i) \in \mathbb{Z}^i$, $\mu \in \Lambda^+([\frac{n}{2}] - i + 1)$, and $\delta \in \{+, -\}$, we consider an irreducible finite-dimensional L_i-module

$$F^{O(n-2i+1,1)}(\mu)_\delta \otimes \mathbb{C}_\nu$$

and define an admissible smooth representation of G of moderate growth, whose underlying $(\mathfrak{g}, K)$-module is given by the cohomological parabolic induction

$$\mathcal{R}^{S_i}_{\mathfrak{q}_i}(F^{SO(n-2i+1,1)}(\mu)_\delta \otimes \mathbb{C}_{\nu+\rho(\mathfrak{u}_i)}) \tag{2.14}$$

of degree S_i, where we set

$$S_i := \dim_{\mathbb{C}}(\mathfrak{u}_i \cap \mathfrak{k}_{\mathbb{C}}) = i(n-i). \tag{2.15}$$

It is denoted by

$$(\nu_1, \cdots, \nu_i \mid\mid \mu_1, \cdots, \mu_{[\frac{n}{2}]-i+1})_\delta.$$

We note that if $i = 0$ then $(\mid\mid \mu_1, \cdots, \mu_{[\frac{n}{2}]+1})_\delta$ is finite-dimensional.

Definition II.14. We call $(\nu_1, \cdots, \nu_i \mid\mid \mu_1, \cdots, \mu_{[\frac{n}{2}]-i+1})_\delta$ the **θ-stable parameter** of the representation $\mathcal{R}^{S_i}_{\mathfrak{q}_i}(F^{O(n-2i+1,1)}(\mu)_\delta \otimes \mathbb{C}_{\nu+\rho(\mathfrak{u}_i)})$.

Remark II.15. By [11, Chaps. 14 and 16], the double bars $\mid\mid$ in the θ-stable parameter of a representation in $\mathcal{A}$ of height i are before the $i+1$-th entry.

Remark II.16. In the introduction we refer to (μ, i, δ) as the **enhanced θ-stable parameter** of the representation with θ-stable parameter $(\mu_1, \cdots, \mu_i \mid\mid \mu_{i+1}, \cdots, \mu_{[\frac{n}{2}]+1})_\delta$

Example II.17. (1) An irreducible finite-dimensional representation $F^G(\mu)_\delta$ has the θ-stable parameter $(\mid\mid \mu_1, \mu_2, \ldots, \mu_{[\frac{n}{2}]+1})_\delta$.

(2) The θ-stable parameter of a representation of height i with trivial infinitesimal character ρ is

$$(0, 0, \ldots, 0 \mid\mid 0, \ldots, 0)_\delta$$

where the double bars $\mid\mid$ are before the $i+1$-th zero (see [11, Chap. 14, Sect. 9.3]).

(3) The representations Π in $\mathcal{A}$ with θ-stable parameter $(\lambda_1, \lambda_2, \ldots, \lambda_i \mid\mid 0, \ldots, 0)_\delta$ are unitary and are often referred to as $A_{\mathfrak{q}}(\lambda)_\delta$. There exists a finite-dimensional representation V of G so that $H^*(\mathfrak{g}, K; \Pi \otimes V) \neq \{0\}$, see [21].

II.4.5 The Hasse and Standard sequences in θ-stable parameters.

We set $m := [\frac{n+1}{2}]$, namely $n = 2m - 1$ or $2m$. Let $F = F^G(s_0, \ldots, s_{[\frac{n}{2}]})_\delta$ be an irreducible finite-dimensional representation of $G = SO(n + 1, 1)$, and $U_i \equiv U_i(F)$ $(0 \le i \le [\frac{n+1}{2}])$ be the Hasse sequence with $U_0 \simeq F$. In [11, Chap. 14] we show:

Theorem II.18. *Let $n = 2m$ and $0 \le i \le m$.*

(1) (Hasse sequence) $U_i(F) \simeq (s_0, \cdots, s_{i-1} || s_i, \cdots, s_m)_{(-1)^{i-s_i}\delta}$.

(2) (standard sequence) $U_i(F) \otimes \chi_-^i \simeq (s_0, \cdots, s_{i-1} || \; s_i, \cdots, s_m)_{(-1)^{s_i}\delta}$.

The case n odd is given similarly as follows.

Theorem II.19. *Let $n = 2m - 1$, and $0 \le i \le m - 1$.*

(1) (Hasse sequence) $U_i(F) \simeq (s_0, \cdots, s_{i-1} || \; s_i, \cdots, s_{m-1})_{(-1)^{i-s_i}\delta}$.

(2) (standard sequence) $U_i(F) \otimes \chi_-^i \simeq \; (s_0, \cdots, s_{i-1} || \; s_i, \cdots, s_{m-1})_{(-1)^{s_i}\delta}$.

III The restriction of representations of $SO(n+1, 1)$ in $\mathcal{A}$ to the subgroup $SO(n, 1)$

In this section we discuss the branching law for the restriction of irreducible representations $\Pi \in \mathcal{A}$ of $SO(n + 1, 1)$ to the subgroup $SO(n, 1)$. We state it for infinite-dimensional representations in Langlands parameter and θ-stable parameters as well in the language of height and signature.A branching law for irreducible representations without the assumption $\Pi \in \mathcal{A}$ will appear in [12].

III.1 Branching laws for finite-dimensional representations

We first recall the branching laws for finite-dimensional representations. As in the classical branching law for $SO(N) \downarrow SO(N - 1)$ the irreducible decomposition of finite-dimensional representations of $SO(N, 1)$ when restricted to the subgroup $SO(N - 1, 1)$ is as follows:

Theorem III.1 (branching rule for $SO(N,1) \downarrow SO(N-1,1)$)**.** *Let $N \geq 2$. Suppose that $(\lambda_1, \cdots, \lambda_{[\frac{N+1}{2}]}) \in \Lambda^+([\frac{N+1}{2}])$ and $\delta \in \{+,-\}$. Then the irreducible finite-dimensional representation $F^{O(N,1)}(\lambda_1, \cdots, \lambda_{[\frac{N+1}{2}]})_\delta$ of $SO(N,1)$ decomposes into a multiplicity-free sum of irreducible representations of $SO(N-1,1)$ as follows:*

$$F^{SO(N,1)}(\lambda_1, \cdots, \lambda_{[\frac{N+1}{2}]})_\delta|_{SO(N-1,1)} \simeq \bigoplus F^{SO(N-1,1)}(\nu_1, \cdots, \nu_{[\frac{N}{2}]})_\delta,$$

where the summation is taken over $(\nu_1, \cdots, \nu_{[\frac{N}{2}]}) \in \mathbb{Z}^{[\frac{N}{2}]}$ subject to

$$\begin{aligned} &\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \cdots \geq \nu_{\frac{N}{2}} \geq 0 && \textit{for } N \textit{ even},\\ &\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \cdots \geq \nu_{\frac{N-1}{2}} \geq \lambda_{\frac{N+1}{2}} && \textit{for } N \textit{ odd}. \end{aligned}$$

Example III.2. We proved the branching rule for the restriction of the representations of $SO(n,1)$ on the space of harmonic polynomials in [9, Prop. 2.3].

III.2 Symmetry breaking operators

Irreducible infinite-dimensional representations of G typically do not decompose into a direct sum of irreducible representations of G when restricted to a noncompact subgroup G', see [7] for details. To obtain information about the restriction and the branching laws we have to proceed differently.

For a continuous representation Π of G on a complete, locally convex topological vector space $\mathcal{H}$, the space $\mathcal{H}^\infty$ of C^∞-vectors of $\mathcal{H}$ is naturally endowed with a Fréchet topology, and $(\Pi, \mathcal{H})$ induces a continuous representation Π^∞ of G on $\mathcal{H}^\infty$. If Π is an admissible representation of finite length on a Banach space $\mathcal{H}$, then the Fréchet representation $(\Pi^\infty, \mathcal{H}^\infty)$, which we refer to as an *admissible smooth representation*, depends only on the underlying $(\mathfrak{g}, K)$-module $\mathcal{H}_K$. In the context of asymptotic behaviour of matrix coefficients, these representations are also referred to as an admissible representations of moderate growth [22, Chap. 11]. We shall work with these representations and write simply Π for Π^∞. We denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible admissible smooth representations. We also sometimes call these representations "irreducible admissible representations" for simplicity.

Given another admissible smooth representation π of a reductive subgroup G', we consider the space of continuous G'-intertwining operators (*symmetry breaking operators*)

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi).$$

If $G = G'$ then these operators include the Knapp–Stein operators [4] and the differential intertwining operators studied by B. Kostant [13]. Including the general case where $G \neq G'$, we define now the multiplicity of π occurring in the restriction $\Pi|_{G'}$ as follows.

Definition III.3 (multiplicity)**.** For $G \supset G'$, we say

$$m(\Pi, \pi) := \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi)$$

the *multiplicity* of π occurring in the restriction $\Pi|_{G'}$.

A finiteness criterion and a uniformly boundedness criterion are proved in [8, Thms. C and D]. Moreover, by a result of B. Sun and C. B. Zhu [17], our assumptions imply that the multiplicities are either 0 or 1. These multiplicities yield important information of the restriction of Π to G', as we will see in the applications in the next part of the article.

III.3 Branching laws for representations in $\mathcal{A}$: First formulation

Theorem III.4. *Let F be an irreducible finite-dimensional representations of $G = SO(n+1, 1)$, and $\{\Pi_i(F)\}$ be the standard sequence starting at $\Pi_0(F) = F$. Let F' be an irreducible finite-dimensional representation of the subgroup $G' = SO(n, 1)$, and $\{\pi_j(F')\}$ the standard sequence starting at $\pi_0(F') = F'$. Assume that*

$$\mathrm{Hom}_{G'}(F|_{G'}, F') \neq \{0\}.$$

Then symmetry breaking for the representations $\Pi_i(F)$, $\pi_j(F')$ in the standard sequences is presented graphically in Diagrams III.1 and III.2. In the first row are representations of G, in the second row are representations of G'. Nontrivial symmetry breaking operators are represented by arrows, namely, there exist nonzero symmetry breaking operators between 2 representations if and only if there are arrows in the Diagrams III.1 and III.2.

III.4 Branching laws for representations in $\mathcal{A}$: Second formulation

Let $F^G(\mu)_\delta$ and $F^{G'}(\nu)_\delta$ be irreducible finite-dimensional representations in $\mathcal{A}$ of $G = SO(n+1, 1)$, respectively of the subgroup $G' = SO(n, 1)$, where $\mu \in \Lambda^+([\frac{n+2}{2}])$, $\nu \in \Lambda^+([\frac{n+1}{2}])$, and $\delta \in \{+, -\}$.

Diagram III.1: Symmetry breaking for $SO(2m+1,1) \downarrow SO(2m,1)$

$$\begin{array}{ccccccccc}
\Pi_0(F) & & \Pi_1(F) & & \ldots & & \Pi_{m-1}(F) & & \Pi_m(F) \\
\downarrow & \swarrow & \downarrow & \swarrow & & \swarrow & \downarrow & \swarrow & \downarrow \\
\pi_0(F') & & \pi_1(F') & & \ldots & & \pi_{m-1}(F') & & \pi_m(F')
\end{array}$$

Diagram III.2: Symmetry breaking for $SO(2m+2,1) \downarrow SO(2m+1,1)$

$$\begin{array}{cccccccccc}
\Pi_0(F) & & \Pi_1(F) & & \ldots & & \Pi_{m-1}(F) & & \Pi_m(F) & & \Pi_{m+1}(F) \\
\downarrow & \swarrow & \downarrow & \swarrow & & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \\
\pi_0(F') & & \pi_1(F') & & \ldots & & \pi_{m-1}(F') & & \pi_m(F') & &
\end{array}$$

Suppose that
$$\mathrm{Hom}_{G'}(F^G(\mu)_\delta|_{G'}, F^{G'}(\nu)_\delta) \neq \{0\}.$$

If $n = 2m$, then $\mu = (\mu_0, \cdots, \mu_{m+1}) \in \Lambda^+(m+1)$ and $\nu = (\nu_0, \cdots, \nu_m) \in \Lambda^+(m)$ and
$$\mu_0 \geq \nu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \nu_n \geq \mu_{n+1} = 0. \tag{3.16}$$

If $n = 2m+1$, then $\mu = (\mu_0, \cdots, \mu_{m+1}) \in \Lambda^+(m+1)$ and $\nu = (\nu_0, \cdots, \nu_m) \in \Lambda^+(m+1)$ and
$$\mu_0 \geq \nu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \nu_n \geq 0. \tag{3.17}$$

We represent the result graphically in the following theorem by representing nontrivial symmetry breaking operators by arrows connecting the θ-stable parameters of the representations.

Theorem III.5. *Two representations in the standard sequences of $F^G(\mu)_\delta$ respectively $F^{G'}(\nu)_\delta$ have a nontrivial symmetry breaking operator if and only if the θ-stable parameters of the representations satisfy one of the following conditions:*
First case: $n = 2m$.

Case A

$$\begin{array}{c}
(\mu_0, \ldots, \mu_i \mid\mid \mu_{i+1}, \ldots, \mu_{m+1})_\delta \\
\Downarrow \\
(\nu_0, \ldots, \nu_i \mid\mid \nu_{i+1}, \ldots, \nu_m)_\delta
\end{array}$$

or Case B

$$\begin{gathered}(\mu_0,\ldots,\mu_i \mid\mid \mu_{i+1},\ldots,\mu_{m+1})_\delta\\ \Downarrow\\ (\nu_0,\ldots,\nu_{i-1} \mid\mid \nu_i,\nu_{i+1},\ldots,\nu_m)_\delta\end{gathered}$$

Second case: $n = 2m+1$.

Case A

$$\begin{gathered}(\mu_0,\ldots,\mu_i \mid\mid \mu_{i+1},\ldots,\mu_{m+1})_\delta\\ \Downarrow\\ (\nu_0,\ldots,\nu_i \mid\mid \nu_{i+1},\ldots,\nu_{m+1})_\delta\end{gathered}$$

or Case B

$$\begin{gathered}(\mu_0,\ldots,\mu_i \mid\mid \mu_{i+1},\ldots,\mu_{m+1})_\delta\\ \Downarrow\\ (\nu_0,\ldots,\nu_{i-1} \mid\mid \nu_i,\ldots,\nu_{m+1})_\delta\end{gathered}$$

III.5 Branching laws for representations in $\mathcal{A}$: Third formulation

We summarize the results as follows:

Theorem III.6 (branching law)**.** *Let* Π *and* π *be irreducible representations in* $\mathcal{A}$ *of* $SO(n+1,1)$ *respectively* $SO(n,1)$.

(1) *Suppose first that* $n = 2m$. *Then*

$$\mathrm{Hom}_{SO(n,1)}(\Pi|_{SO(n,1)},\pi) \neq \{0\}$$

if and only if the enhanced θ*-stable parameters of the representations satisfy the following conditions:*

(a) Π *and* π *have the same signature* δ*;*

(b) $h_\pi \in \{h_\Pi, h_\Pi - 1\}$*;*

(c) $\mu_0 \geq \nu_0 \geq \mu_1 \geq \cdots \geq \mu_n \geq \nu_n \geq \mu_{n+1}$.

(2) *Suppose that* $n = 2m+1$*. Then*

$$\mathrm{Hom}_{SO(n,1)}(\Pi|_{SO(n,1)}, \pi) \neq \{0\}$$

if and only if the enhanced θ*-stable parameters of the representations satisfy the following conditions:*

(a) *they have the same signature* δ*;*

(b) $h_\pi \in \{h_\Pi, h_\Pi - 1\}$*;*

(c) $\mu_0 \geq \nu_0 \geq \mu_1 \geq \cdots \geq \mu_n \geq \nu_n \geq 0$.

IV Gross–Prasad conjectures for tempered representations of $(SO(n+1,1), SO(n,1))$

In this section we discuss the Gross–Prasad conjectures for irreducible tempered representations in $\mathcal{A}$ which are nontrivial on the center. This is a generalization of the results for irreducible tempered representations with infinitesimal character ρ which are nontrivial on the center in [11, Chap. 11]. For simplicity we discuss here only the case $n = 2m$.

Recall that for $n = 2m$, tempered representations Π_m of $G = SO(n+1,1)$ in $\mathcal{A}$ are irreducible unitary principal series representations [11, Prop. 15.5], with height m, signature δ and θ-stable parameter

$$(\mu_1, \mu_2, \ldots, \mu_m \mid\mid 0)_\delta.$$

The tempered representations π_m of $G' = SO(n,1)$ in $\mathcal{A}$ are discrete series representations with height m. Their signature is not unique and their θ-stable parameter is

$$(\nu_1, \nu_2, \ldots, \nu_m \mid\mid)_\delta.$$

We now assume that the representation $\Pi = \Pi_m$ is nontrivial on the center. These data determine Vogan packets $VP(\Pi_m)$ and $VP(\pi_m)$ of the representations Π of G, respectively π of the subgroup G'. By Theorem III.6, there is a nontrivial symmetry breaking operator

$$B\colon \Pi \to \pi$$

if and only if the interlacing conditions are satisfied. Following exactly the steps of the algorithm by Gross and Prasad outlined in [10], see also [11, Chap. 11], we conclude that the Gross–Prasad conjecture predicts correctly that the pair (Π, π) in $(VP(\Pi_m), VP(\pi_m))$ has a nontrivial symmetry breaking operator.

Together with the results about tempered principal series representations in [11, Chap. 11, Sect. 4] this completes the proof of the following:

Theorem IV.1. *The Gross–Prasad conjectures are correct for tempered representations of the pair* $(G, G') = (SO(n+1, 1), SO(n, 1))$ *which are nontrivial on the center of* G *and* G'.

Remark IV.2. The third formulation of the branching laws (Theorem III.6) shows that branching for representations in $\mathcal{A}$ "interpolate" between the classical branching laws of finite-dimensional representations and the branching laws of Gross–Prasad for tempered representations.

V Distinguished representations and periods.

We discuss periods of a pair of irreducible representations Π of $G = SO(n+1, 1)$ and π of the subgroup $G' = SO(n, 1)$. Using the branching law (Theorem III.4), we see in this section that we can prove that the representations $A_{\mathfrak{q}}(\lambda) \in \mathcal{A}$ are distinguished for some orthogonal group H.

V.1 Periods

We recall from [11, Thm. 5.4] that for representations Π, π of a real reductive Lie group G, respectively of a reductive subgroup G', the space of symmetry breaking operators

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi^{\vee})$$

and the space of G'-invariant continuous linear functionals

$$\mathrm{Hom}_{G'}(\Pi \boxtimes \pi, \mathbb{C})$$

are naturally isomorphic to each other. Thus we may use symmetry breaking operators to construct G'-invariant continuous linear functionals. This technique allows us to obtain G'-invariant continuous linear functionals not only for unitary representations but also for nonunitary representations.

Definition V.1. A nontrivial linear functional $\mathcal{F}$ on $\Pi \boxtimes \pi$ is called a *period* of $\Pi \boxtimes \pi$ if $\mathcal{F}$ is invariant under the diagonal G'-action, *i.e.*, $\mathcal{F} \in \mathrm{Hom}_{G'}(\Pi \boxtimes \pi, \mathbb{C})$.

We say that vector $\Phi \otimes \phi \in \Pi \boxtimes \pi$ is a *test vector* for the period $\mathcal{F}$ if $\Phi \otimes \phi$ is not in the kernel of $\mathcal{F}$. If the period is nontrivial on a test vector $\Phi \otimes \phi$, we refer to its image as the *value of the period* on $\Phi \otimes \phi$.

Remark V.2. If (Π, π) is a pair of discrete series representations for the symmetric pairs $(G_1(\mathbb{R}), G_2(\mathbb{R}))$ we may consider a realization of $\Pi \boxtimes \pi$ in $L^2(G_1(\mathbb{R}) \times G_2(\mathbb{R}))$. The integral

$$\int_{G_2(\mathbb{R})} \Phi(h)\phi(h)dh$$

converges for some smooth vectors $(\Phi, \phi) \in \Pi \boxtimes \pi$ in the minimal K-types and so if it is nonzero it defines a period integral for the discrete series representations $\Pi \boxtimes \pi$ [18]. If the representation $\Pi \boxtimes \pi$ is not tempered the integral usually does not converge, but nevertheless we can consider periods via symmetry breaking operators.

The next theorem describes for the pair $(G, G') = (SO(n+1, 1), SO(n, 1))$ the θ-stable parameters of the representations in $\mathcal{A}$ which have a nontrivial period $\mathrm{Hom}_{G'}(\Pi \boxtimes \pi, \mathbb{C})$. Recall that the θ-stable parameter of a representation $\Pi \in \mathcal{A}$ of height i is of the form

1. $(\mu_1, \ldots, \mu_i \mid\mid \mu_{i+1}, \ldots, 0)_\delta$ if $G = SO(2m+1, 1)$,
2. $(\mu_1, \ldots, \mu_i \mid\mid \mu_{i+1}, \ldots, \mu_m)_\delta$ if $G' = SO(2m, 1)$.

Theorem V.3. *Suppose that* Π *and* π *are representations of* G*, respectively* G' *in* $\mathcal{A}$ *of height* i *respectively* j*. The following statements are equivalent:*

(i) *The representation* $\Pi \boxtimes \pi$ *has a nontrivial* G'*-period;*

(ii) Π *and* π *have the same signature,* $j = i$ *or* $i - 1$ *and their* θ*-stable parameters satisfy the interlacing conditions of the branching result in Theorem III.6.*

In [11, Props. 10.12 and 10.30] we proved furthermore for $(O(n+1, 1), O(n, 1))$.

Theorem V.4. *If Π and π are representations of O(n+1,1) respectively O(n,1) with trivial infinitesimal character ρ such that $\Pi \otimes \pi$ has a nontrivial G'-period. Then there is a test vector for a nonzero period in the minimal K-type $\Pi \boxtimes \pi$.*

Remark V.5. We expect that Theorem V.4 also holds for unitary representations in $\mathcal{A}$.

Remark V.6. Similar results for cohomologically induced representations of other pairs (G, G') of reductive groups were obtain by B. Sun [16].

V.2 Distinguished representations

Let G be a reductive group, and H a reductive subgroup. We regard H as a subgroup of the direct product group $G \times H$ via the diagonal embedding $H \hookrightarrow G \times H$.

Definition V.7. Let ψ be a one-dimensional representation of H. We say an admissible smooth representation Π of G is (H, ψ)-*distinguished* if

$$\mathrm{Hom}_H(\Pi \boxtimes \psi^\vee, \mathbb{C}) \simeq \mathrm{Hom}_H(\Pi|_H, \psi) \neq \{0\}.$$

If the character ψ is trivial, we say Π is H-*distinguished.*

Repeated application of Theorem III.6 proves

Theorem V.8. *Suppose that $\Pi = A_{\mathfrak{q}}(\lambda) \in \mathcal{A}$ is a representation of $SO(n+1,1)$ of height h. Then Π is $SO(n+1-h,1)$-distinguished.*

Remark V.9. For a different proof and perspective of this theorem see [6].

Since the representations $\Pi_{i,+}$ have height i (see Example II.10), this generalizes the following theorem proved in [11, Thm. 12.4].

Theorem V.10. *Let $0 \leq i \leq \frac{n}{2}$. Then the representations $\Pi_{i,\delta}$ $(\delta \in \{+,-\})$ of $G = SO(n+1,1)$ are $SO(n+1-i,1)$-distinguished.*

In the remainder of the section we recall a formula for the period of representations with trivial infinitesimal character ρ of the pair

$$(G, H) = (SO(n+1,1), SO(m+1,1)) \quad \text{for } m \leq n.$$

We use here the notation of Example II.10 in Section II.4.2.

The period can be computed by applying the composition of the regular symmetry breaking operators that we constructed in [11, Chap. 12, Sect. 1] with respect to the chain of subgroups

$$G = SO(n+1,1) \supset SO(n,1) \supset SO(n-1,1) \supset \cdots \supset SO(m+1,1) = H, \quad (5.18)$$

to test vectors. We write simply Π_i for $\Pi_{i,+}$. We recall from [11, Prop. 14.44] that $\Pi_i \equiv \Pi_{i,+}$ has a minimal K-type $K_{min}(i,+) = \bigwedge^i(\mathbb{C}^{n+1}) \boxtimes \mathbf{1}$.

Let $v \in \bigwedge^i(\mathbb{C}^{n+1})$ be the image of $1 \in \mathbb{C}$ via the following successive inclusions:

$$\bigwedge^i(\mathbb{C}^{n+1}) \supset \bigwedge^{i-1}(\mathbb{C}^n) \supset \cdots \supset \bigwedge^{i-\ell}(\mathbb{C}^{n+1-\ell}) \supset \cdots \supset \bigwedge^0(\mathbb{C}^{n+1-i}) \simeq \mathbb{C} \ni 1,$$

and we regard v as an element of the minimal K-type $K_{min}(i,+)$ of Π_i.

Theorem V.11 ([11, Thm. 12.5])**.** *Let Π_i be the irreducible representation of $G = SO(n+1,1)$, with infinitesimal character ρ, height i and signature $+$. Let v be the normalized element of its minimal K-type as above. For $0 \leq i \leq n$, the value $F(v)$ of the $SO(n+1-i,1)$-period F on $v \in \Pi_i$ is*

$$\frac{\pi^{\frac{1}{4}i(2n-i-1)}}{((n-i)!)^{i-1}} \times \begin{cases} \frac{1}{(n-2i)!} & \text{if } 2i < n+1, \\ (-1)^{n+1}(2i-n-1)! & \text{if } 2i \geq n+1. \end{cases}$$

VI Bilinear forms on $(\mathfrak{g}, K)$-cohomologies induced by symmetry breaking operators

Consider now the induced map by a symmetry breaking operator

$$T \colon \Pi \to \pi$$

on $(\mathfrak{g}, K)$-cohomologies of a pair of representations Π and π. In what follows, by abuse of notation, we denote an admissible smooth representation and its underlying $(\mathfrak{g}, K)$-module by the same letter when we discuss their $(\mathfrak{g}, K)$-cohomologies.

Recall that a theorem of Vogan–Zuckerman [21] states that every irreducible *unitary* representation Π of $SO(n+1,1)$ with

$$H^*(\mathfrak{g}, K; \Pi \otimes V) \neq \{0\}$$

for a finite-dimensional representation V is of the form $\Pi = A_{\mathfrak{q}}(\lambda)$. If we assume that

$$H^*(\mathfrak{g}, K; \Pi) \neq \{0\}$$

then Π is isomorphic to a unitary irreducible representation with infinitesimal character ρ *i.e.*, it is of the form $A_{\mathfrak{q}}$. See [11, Chap. 14, Sect. 9.4] for $O(n+1,1)$.

Note also that an irreducible representation Π with

$$H^*(\mathfrak{g}, K; \Pi \otimes V) \neq \{0\}$$

for some finite-dimensional representation V is not always unitarizable.

Suppose that Π is a principal series representation of a connected reductive Lie group with nonsingular integral infinitesimal character. If the $(\mathfrak{g}, K)$-cohomology of $\Pi \otimes V$ is nonzero the highest weight of V satisfies the conditions in [1, Chap. III, Thm. 3.3]. For representations of $O(n+1,1)$ the situation is more complicated and the finite-dimensional representation V is also described in [11, Chap. 16, Sect. 4]. Using the results in [11, Chap. 15] about the restriction of representations of $O(n+1,1)$ to $SO(n+1,1)$ we obtain a formula for the representation V for $SO(n+1,1)$.

Let $(G, G') = (SO(n+1,1), SO(n,1))$, and Π, π be representations of G and G', respectively with

$$H^*(\mathfrak{g}, K; \Pi \otimes V) \neq \{0\}$$

and

$$H^*(\mathfrak{g}', K'; \pi \otimes V') \neq \{0\},$$

where V and V' are irreducible finite-dimensional representations of G and G', respectively. Suppose in addition that

1. $\mathrm{Hom}_{G'}(V|_{G'}, V') \neq \{0\}$;
2. Π and π have the same height i;
3. Π and π have the same signature δ.

A nontrivial symmetry breaking operator $T \colon \Pi \otimes V \to \pi \otimes V'$ induces a canonical homomorphism

$$T^* \colon H^j(\mathfrak{g}, K; \Pi \otimes V) \to H^j(\mathfrak{g}', K'; \pi \otimes V') \tag{6.19}$$

and a bilinear form

$$B_T \colon H^j(\mathfrak{g}, K; \Pi \otimes V) \times H^{n-j}(\mathfrak{g}', K'; (\pi \otimes V')^\vee \otimes \chi_{(-1)^n}) \to \mathbb{C} \quad \text{for all } j.$$

where $(\pi \otimes V')^\vee$ denotes the contragredient representation of $\pi \otimes V'$.

The formulas in [11, Chap. 16, Sect. 3] and [1, Chap. III, Thm. 3.3] imply

Theorem VI.1. *Suppose that* Π *and* π *are principal series representations and that*

1. $H^*(\mathfrak{g}, K, \Pi \otimes V) \neq \{0\}$ *and* $H^*(\mathfrak{g}', K', \pi \otimes V') \neq \{0\}$*;*
2. $\mathrm{Hom}_{G'}(V|_{G'}, V') \neq \{0\}$*;*
3. Π *and* π *have the same height* i*;*
4. Π *and* π *have the same signature* δ*;*
5. *there exists a nontrivial symmetry breaking operator* $T\colon \Pi \to \pi$*.*

The symmetry breaking operator T *induces a nontrivial homomorphism*

$$T^i\colon H^i(\mathfrak{g}, K; \Pi \otimes V) \to H^i(\mathfrak{g}', K'; \pi \otimes V')$$

and hence a nontrivial bilinear form

$$B_T\colon H^i(\mathfrak{g}, K; \Pi \otimes V) \times H^{n-i}(\mathfrak{g}', K'; (\pi \otimes V')^\vee \chi_{(-1)^n}) \to \mathbb{C}.$$

The following theorem provides a criterium for the nonvanishing of this bilinear form on the $(\mathfrak{g}, K)$-cohomology of representations with trivial infinitesimal character.

Theorem VI.2 ([11, Thm. 12.11])**.** *Let* $T\colon X \to Y$ *be a* $(\mathfrak{g}', K')$*-homomorphism, where* X *is a* $(\mathfrak{g}, K)$*-module* $A_{\mathfrak{q}}$ *and* Y *is a* $(\mathfrak{g}', K')$*-module* $A_{\mathfrak{q}'}$*. Let* U *be the representation space of the minimal* K*-type* μ *in* X*, and* U' *that of the minimal* K'*-type* μ' *in* Y*. We define a* K'*-homomorphism by*

$$\varphi_T := \mathrm{pr} \circ T|_U \colon U \to U'. \tag{6.20}$$

(1) *If* φ_T *is zero, then the homomorphisms* $T_*\colon H^j(\mathfrak{g}, K; X) \to H^j(\mathfrak{g}', K'; Y)$ *and the bilinear form* B_T *vanish for all degrees* $j \in \mathbb{N}$*.*

(2) *If* φ_T *is* $\mathfrak{p}$*-nonvanishing at degree* j*, then* T_* *and the bilinear forms* B_T *are nonzero for this degree* j*.*

This theorem together with our results [11] implies

Theorem VI.3 (cf. [11, Thm. 12.13])**.** *Let* $(G, G') = (SO(n+1, 1), SO(n, 1))$*,* $0 \leq i \leq n$*, and* $\delta \in \{+, -\}$*. Let* T *be the symmetry breaking operator* $\Pi_{i,\delta} \to \pi_{i,\delta}$ *given in Theorem III.4*

(1) *T induces bilinear forms*

$$B_T\colon H^j(\mathfrak{g},K;\Pi_{i,\delta})\times H^{n-j}(\mathfrak{g}',K';\pi_{n-i,(-1)^n\delta})\to\mathbb{C}\quad \textit{for all } j.$$

(2) *The bilinear form B_T is nonzero if and only if $j=i$ and $\delta=(-1)^i$.*

Remark VI.4. A theorem similar to Theorem VI.3 was proved by B. Sun [15] for the $(\mathfrak{g},K)$-cohomology with nontrivial coefficients of irreducible tempered representations of the pair

$$(GL(n,\mathbb{R}),GL(n-1,\mathbb{R})).$$

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