INVERSION OF RANKIN–COHEN OPERATORS VIA HOLOGRAPHIC TRANSFORM

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ABSTRACT. The analysis of branching problems for restriction of representations brings the concept of *symmetry breaking transform* and *holographic transform*. Symmetry breaking operators decrease the number of variables in geometric models, whereas holographic operators increase it. Various expansions in classical analysis can be interpreted as particular occurrences of these transforms. From this perspective we investigate two remarkable families of differential operators: the Rankin–Cohen operators and the holomorphic Juhl conformally covariant operators. Then we establish for the corresponding symmetry breaking transforms the Parseval–Plancherel type theorems and find explicit inversion formulæ with integral expression of holographic operators.

The proof uses the F-method which provides a duality between symmetry breaking operators in the holomorphic model and holographic operators in the L^2-model, leading us to deep links between special orthogonal polynomials and branching laws for infinite-dimensional representations of real reductive Lie groups.

Keywords and phrases: *Symmetry breaking, holographic transform, Rankin–Cohen operators, Juhl operators, orthogonal polynomials, branching rules, F-method.*

CONTENTS

1. INTRODUCTION

Let π be an irreducible representation of a group G on a vector space V, and G' a subgroup. The G-module (π, V) may be seen as a G'-module by restriction, for

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which we write $\pi|_{G'}$. For an irreducible representation (ρ, W) of the subgroup G', a *symmetry breaking operator* is a (continuous) linear map $V \to W$ which intertwines $\pi|_{G'}$ and ρ. In recent years *individual* symmetry breaking operators have been studied intensively in different settings ranging from automorphic form theory to conformal geometry, see [2, 3, 6, 9, 16, 19, 20, 25] and references therein.

In this article, we investigate a **collection** of symmetry breaking operators,

$$R_\ell : V \longrightarrow W_\ell, \ \ell \in \Lambda,$$

referred to as a *symmetry breaking transform*, for a family of irreducible representations ρ_ℓ of the subgroup G' on vector spaces W_ℓ with parameter $\ell \in \Lambda$.

Various expansions in classical analysis can be interpreted through this paradigm:

Example 1.1 $(GL_n \downarrow GL_{n-1})$**.** *Arranging homogeneous polynomials of* $x = (x_1, \cdots, x_n)$ *in descending order with respect to the power of* x_n *is an example of symmetry breaking transform for* $(G, G') = (GL_n, GL_{n-1})$*. In fact, taking the ℓ-th component in the expansion*

$$f(x) = \sum_{\ell=0}^{k} f_\ell(x') x_n^{k-\ell}, \quad \text{for } x' = (x_1, \cdots, x_{n-1})$$

defines a G'-homomorphism from $V := \mathrm{Pol}^k[x]$ to $W_\ell := \mathrm{Pol}^\ell[x']$ on which G and G', respectively, act irreducibly.

Traditional representation-theoretic viewpoint tells that the Fourier series expansion or Fourier transform is the irreducible decomposition of the regular representation of the *abelian* group $G' = S^1$ or $\mathbb{R}$, whereas we make use of a *hidden symmetry* of the *noncommutative* group $G = SL(2, \mathbb{R})$ in the sense that G contains G' as a subgroup and that G acts on the space of functions on S^1 or $\mathbb{R}$. The latter viewpoint brings us a new interpretation of the (classical) Fourier series or Fourier transform in the framework of "symmetry breaking" as follows.

Example 1.2. *A spherical principal series representation π_λ of $G = SL(2, \mathbb{R})$ is realized on the vector space of homogeneous functions*

$$V_\lambda := \left\{ f \in C^\infty(\mathbb{R}^2 \setminus \{(0,0)\}) : f(ax, ay) = |a|^\lambda f(x, y) \text{ for all } a \in \mathbb{R}^\times \right\}.$$

This representation is irreducible for all $\lambda \in \mathbb{C} \setminus \mathbb{Z}$.
• (Fourier series) *The representation π_λ of $SL(2, \mathbb{R})$ can be realized in $C^\infty(S^1)$ via the identification $V_\lambda \xrightarrow{\sim} C^\infty(S^1)$, $f(x, y) \mapsto h(\theta) := f(\cos\theta, \sin\theta)$, because any homogeneous function is determined by its restriction to the unit circle S^1. Since S^1 is*

preserved by the subgroup $G' := SO(2)$, the collection of the Fourier coefficients

$$V_\lambda \xrightarrow{\sim} C^\infty(S^1) \longrightarrow \mathrm{Map}(\mathbb{Z}, \mathbb{C}),$$

$$f \mapsto h \quad \mapsto \quad \widehat{h}(\ell) := \frac{1}{2\pi} \int_0^{2\pi} h(\theta) e^{-i\ell\theta} d\theta, \quad \ell \in \mathbb{Z}$$

gives a symmetry breaking transform from the infinite-dimensional representation (π_λ, V_λ) of $G = SL(2, \mathbb{R})$ to the collection of one-dimensional representations χ_ℓ of the abelian subgroup $G' = SO(2) \simeq S^1$ indexed by $\ell \in \mathbb{Z}$.

• (Fourier transform) *Similarly, any function $f(x, y) \in V_\lambda$ is determined by its restriction to the real line $y = 1$, which is preserved by the unipotent subgroup $G'' := \left\{ \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} : \xi \in \mathbb{R} \right\} (\simeq \mathbb{R})$. Thus the Fourier transform*

$$L^1(\mathbb{R}) \longrightarrow C(\mathbb{R}), \quad F \mapsto (\mathcal{F}F)(\xi) := \int_{\mathbb{R}} F(x) e^{-ix\xi} dx,$$

induces another symmetry breaking transform for the pair $(G, G'') = (SL(2, \mathbb{R}), \mathbb{R})$.

Example 1.3 (spherical harmonics). *Expansion of functions on S^n by eigenfunctions of the Laplacian Δ_{S^n} corresponds to a symmetry breaking transform from a spherical principal series representation π of $G = SO(n+1, 1)$ to a collection of irreducible finite-dimensional representations of the compact subgroup $G' = O(n+1)$.*

Reversing the arrows in the definition of a symmetry breaking operator $R_\ell \colon V \to W_\ell$, we consider a G'-homomorphism $\Psi_\ell \colon W_\ell \to V$, going from smaller to larger representation space, and thus referred to as a *holographic operator*. As in the case of symmetry breaking, the collection of holographic operators $\{\Psi_\ell\}$ is said to be a *holographic transform*.

$$G \curvearrowright V \underset{\Psi_\ell}{\overset{R_\ell}{\rightleftarrows}} W_\ell \curvearrowleft G'.$$

To illustrate a holographic transform by an example with both V and W_ℓ being infinite-dimensional, we recall that the classical Poisson integral (see *e.g.* [10, Sec. 0])

$$\mathcal{P}_\nu \colon C_c(\mathbb{R}) \longrightarrow C^\infty(\Pi)$$
$$h(t) \mapsto (\mathcal{P}_\nu h)(x, y) = \int_{-\infty}^\infty \frac{y^\nu}{((x-t)^2 + y^2)^\nu} h(t) dt$$

constructs eigenfunctions of the Laplace–Beltrami differential operator $\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ for the eigenvalue $\nu(\nu - 2)$ on the upper-half plane Π endowed with Poincaré metric. The group $SL(2, \mathbb{R})$ acts isometrically on Π and conformally on its boundary.

Traditionally, the Poisson integral was treated in the context of representations of $SL(2,\mathbb{R})$, however, we highlight the fact that the totality of functions on Π admits a larger symmetry because the group $SL(2,\mathbb{C})$ acts on the conformal compactification of Π. Thus the Poisson integral can be interpreted as a particular occurrence of a holographic operator for the pair $(G, G') = (SL(2,\mathbb{C}), SL(2,\mathbb{R}))$ as below.

Example 1.4 (Poisson integral). *A generic symmetry breaking operator $A_{\lambda,\nu}$ from the spherical principal series representation π_λ of $G = SL(2,\mathbb{C})$ on $C^\infty(S^2)$ to the one ϖ_ν of the subgroup $G' = SL(2,\mathbb{R})$ on $C^\infty(S^1)$ takes the following form (see [20, (7.2)]):*

$$A_{\lambda,\nu} : C_c^\infty(\mathbb{R}^2) \longrightarrow C^\infty(\mathbb{R}),$$

$$f(x,y) \mapsto (A_{\lambda,\nu}f)(x) = \int_{\mathbb{R}^2} f(t,y)K_{\lambda,\nu}(x-t,y)dtdy$$

in the flat coordinates where $K_{\lambda,\nu}$ is a distributional kernel given by

$$(1.1) \qquad\qquad K_{\lambda,\nu}(x,y) = (x^2 + y^2)^{-\nu}|y|^{\lambda+\nu-2}.$$

Then the dual map of $A_{\lambda,\nu}$ yields a holographic operator $\Psi_{\lambda,\nu}$ with the formula

$$g(t) \mapsto (\Psi_{\lambda,\nu}g)(x,y) := \int_{\mathbb{R}} g(t)K_{\lambda,\nu}(x-t,y)dt.$$

Thus the (classical) Poisson integral $\mathcal{P}_\nu$ can be viewed as the restriction of the holographic operator $\Psi_{\lambda,\nu}$ with $\lambda = 2$, namely, defn $\mathcal{P}_\nu = \mathrm{Rest}_\Pi \circ \Psi_{2,\nu}$.

With these interpretations of classical examples in mind, we raise the following two general problems for a symmetry breaking transform $R(v) = \{R_\ell(v)\}_{\ell \in \Lambda}$, where $R_\ell \colon V \longrightarrow W_\ell$ ($\ell \in \Lambda$), are symmetry breaking operators:

Problem A. *Can we recover an element v of V from its symmetry breaking transform $R(v) = \{R_\ell(v)\}_{\ell \in \Lambda}$?*

Problem A includes the following subproblems:

A.0. Tell *a priori* if Λ is sufficiently large for R to be injective.
A.1. Construct a "holographic transform".
A.2. Find an explicit inversion of the symmetry breaking transform R.

When V is a Hilbert space on which G acts unitarily, we also ask for a Parseval–Plancherel type theorem for the symmetry breaking transform:

Problem B. *Find a closed formula for the norm of an element v in V in terms of its symmetry breaking transform $\{R_\ell(v)\}_{\ell \in \Lambda}$.*

In this article, we investigate Problems A and B in the following two cases:
- Rankin–Cohen transform (Section 2);

  • Holomorphic Juhl transform (Section 3).

In both cases, the transform is a collection of holomorphic differential operators between complex manifolds: the first case is associated with the family of the Rankin–Cohen operators that appeared in the theory of holomorphic modular forms [3], whereas the second case originated from Juhl's conformally covariant operators [9].

These transforms can be analyzed in the framework of infinite dimensional representations of Lie groups, namely, the decomposition of the tensor product of two holomorphic discrete series representations of $SL(2, \mathbb{R})$ in the first case, and the branching laws of holomorphic discrete series representations of the conformal Lie group $G = SO_o(2, n)$ when restricted to a subgroup $G' = SO_o(2, n-1)$, in the second case.

The main goal here is to give a solution to Problems A and B for the above two transforms. We provide two types of integral expressions as a solution to Problem A1, see Theorems 2.2 and 3.10. The main results are summarized as below.

$G \supset G'$	$SL_2 \times SL_2 \supset SL_2$	$SO_o(2, n) \supset SO_o(2, n-1)$
Problem A1 construction of holographic transform	Theorem 2.2	Theorem 3.10
Problem A2 inversion of symmetry breaking transform	Theorem 2.5	Theorem 3.2
Problem B L^2-theory for	Theorem 2.7	Theorem 3.2

The key idea of our approach is to introduce "special orthogonal polynomials" $\{P_\ell\}$ associated to symmetry breaking operators. This can be done via the F-method, which we developed in [18, 19], that analyzes the representations through the Fourier transform of their geometric realizations. In this article, we show for the Rankin–Cohen bidifferential operators $\{R_\ell\}$ that the polynomials $\{P_\ell\}$ are the Jacobi polynomials and that the holographic operators are given by the Jacobi transforms along the transversal direction to a codimension-one foliation of the symmetric cone (Section 2); for the holomorphic Juhl operators $\{R_\ell\}$, the holographic operators are associated to the Gegenbauer polynomials $\{P_\ell\}$ (Section 3). Thus Problems A and B for symmetry breaking transforms can be studied as questions on special orthogonal polynomials via the F-method.

The table below shows some new links which the F-method provides between representations and special functions in this setting.

Symmetry breaking operators $\{R_\ell\}$	Special orthogonal polynomials $\{P_\ell\}$
G'-intertwining property	hypergeometric differential equations
operator norm of R_ℓ	L^2-norm of P_ℓ
branching law $\pi\|_{G'}$	L^2-completeness of $\{P_\ell\}$
holographic transform (L^2-model)	integral transform associated to $\{P_\ell\}$

Analogously to the classical Poisson transform (Example 1.4), the holographic transform provides an integral expression of eigenfunctions of certain holomorphic differential operator. We illustrate this idea with the example of the Rankin–Cohen operators, see Theorem 2.30 which is proved as a byproduct of the main results.

In Section 4 we discuss the background of Problems A and B from a viewpoint of the representation theory of real reductive Lie groups.

Notation: $\mathbb{N} = \{0, 1, 2, \cdots\}$, $i = \sqrt{-1}$ (imaginary unit), $(x)_k = x(x + 1)(x + 2)\cdots(x + k - 1)$ for $k \in \mathbb{N}$ (Pochhammer symbol), and $[x]$ is the largest integer that does not exceed $x \in \mathbb{R}$.

2. Rankin–Cohen transform and its holographic transform

The Rankin–Cohen bidifferential operators map functions of two variables to those of one variable, respecting twisted actions of $SL(2, \mathbb{R})$. In this section, we solve Problems A and B stated in Section 1 for the *Rankin–Cohen transform* (Definition 2.4), a collection of such operators.

2.1. **Rankin–Cohen bidifferential operators.** We begin with a quick review of the Rankin–Cohen bidifferential operators.

2.1.1. *Holomorphic discrete series representations of $SL(2,\mathbb{R})^\sim$.* Let $\Pi = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, \ y > 0\}$ be the upper half-plane, and $\mathcal{O}(\Pi)$ the space of holomorphic functions on Π. For $\lambda \in \mathbb{Z}$ we define a representation π_λ of $SL(2,\mathbb{R})$ on $\mathcal{O}(\Pi)$ by

$$\pi_\lambda(g)f(z) = (cz+d)^{-\lambda}f\left(\frac{az+b}{cz+d}\right) \qquad \text{for} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Viewed as a representation of the universal covering group $SL(2,\mathbb{R})^\sim$, the representation π_λ is well-defined for all $\lambda \in \mathbb{C}$. There is a canonical perfect pairing between $(\pi_\lambda, \mathcal{O}(\Pi))$ and the Verma module

$$M_{-\lambda} := U(\mathfrak{g}_\mathbb{C}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{-\lambda},$$

where $U(\mathfrak{g}_\mathbb{C})$ denotes the universal enveloping algebra of $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{b}$ is a Borel subalgebra containing $\mathfrak{k}_\mathbb{C} = \mathfrak{so}(2,\mathbb{C})$. Therefore, $(\pi_\lambda, \mathcal{O}(\Pi))$ is irreducible if and only if $\lambda \in \mathbb{C} \setminus (-\mathbb{N})$ because the $\mathfrak{g}$-module M_ν is reducible if and only if $\nu \in \mathbb{N}$.

Let $p \colon SL(2,\mathbb{R})^\sim \to SL(2,\mathbb{R})$ be the covering homomorphism, and set $SO(2)^\sim = p^{-1}(SO(2))$. For every $\lambda \in \mathbb{C}$, we can form a homogeneous holomorphic line bundle $\mathcal{L}_\lambda$ over $\Pi \simeq SL(2,\mathbb{R})^\sim/SO(2)^\sim$ associated to a character $\mathbb{C}_\lambda$ of $SO(2)^\sim$, and the multiplier representation $(\pi_\lambda, \mathcal{O}(\Pi))$ is equivalent to the natural action of $SL(2,\mathbb{R})^\sim$ on the space $\mathcal{O}(\Pi, \mathcal{L}_\lambda)$ of holomorphic sections of $\mathcal{L}_\lambda$.

2.1.2. *Holomorphic model $\mathcal{H}^2(\Pi)_\lambda$.* For $\lambda > 1$ the weighted Bergman space $\mathcal{H}^2(\Pi)_\lambda := (\mathcal{O} \cap L^2)(\Pi, y^{\lambda-2}dxdy)$ is nonzero, and the Hilbert space $\mathcal{H}^2(\Pi)_\lambda$ admits a reproducing kernel $K_\lambda(z,w) = \frac{\lambda-1}{4\pi}\left(\frac{z-\bar{w}}{2i}\right)^{-\lambda}$, see [5, Prop. XIII.1.2]. The representation $(\pi_\lambda, \mathcal{O}(\Pi))$ yields an irreducible unitary representation of $SL(2,\mathbb{R})^\sim$ on $\mathcal{H}^2(\Pi)_\lambda$, which descends to $SL(2,\mathbb{R})$ when $\lambda \in \mathbb{Z}$. The set of equivalence classes of irreducible unitary representations (*unitary dual*) of $SL(2,\mathbb{R})$ contains a family of those with continuous parameter (*e.g.* principal series representations, complementary series representations), whereas π_λ ($\lambda = 2, 3, \cdots$) form a countable family of irreducible unitary representations realized in the kernel of the Cauchy–Riemann operator. Thus π_λ ($\lambda = 2, 3, \cdots$) is referred to as a *holomorphic discrete series representations* of $SL(2,\mathbb{R})$, and π_λ ($\lambda > 1$) as a relative holomorphic discrete series representation of the covering group $SL(2,\mathbb{R})^\sim$. We call the realization on $\mathcal{H}^2(\Pi)_\lambda$ *holomorphic model* of the representation π_λ. Similarly, the direct product group $SL(2,\mathbb{R})^\sim \times SL(2,\mathbb{R})^\sim$ acts on $\mathcal{H}^2(\Pi \times \Pi)_{(\lambda',\lambda'')} \simeq \mathcal{H}^2(\Pi)_{\lambda'} \widehat{\otimes} \mathcal{H}^2(\Pi)_{\lambda''}$ as an irreducible unitary representation if $\lambda', \lambda'' > 1$, where $\widehat{\otimes}$ stands for the completion of the algebraic tensor product.

We shall deal with another realization (L^2-model) of the same representation π_λ in Section 2.6.2.

2.1.3. *Rankin–Cohen bidifferential operators.*

Consider $\lambda', \lambda'', \lambda''' \in \mathbb{C}$ such that $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$ and define a differential operator $\mathcal{R}_{\lambda',\lambda''}^{\lambda'''} : \mathcal{O}(\Pi \times \Pi) \longrightarrow \mathcal{O}(\Pi \times \Pi)$ by

$$(2.1) \qquad \mathcal{R}_{\lambda',\lambda''}^{\lambda'''}(f)(\zeta_1, \zeta_2) := \sum_{j=0}^{\ell} (-1)^j \frac{(\lambda' + \ell - j)_j \, (\lambda'' + j)_{\ell-j}}{j!(\ell - j)!} \frac{\partial^\ell f}{\partial \zeta_1^{\ell-j} \partial \zeta_2^j}(\zeta_1, \zeta_2).$$

The *Rankin–Cohen bidifferential operator* is a linear map

$$\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''} : \mathcal{O}(\Pi \times \Pi) \longrightarrow \mathcal{O}(\Pi),$$

defined by $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''} := \mathrm{Rest} \circ \mathcal{R}_{\lambda',\lambda''}^{\lambda'''}$, where Rest stands for the restriction map $f(\zeta_1, \zeta_2) \mapsto f(\zeta, \zeta)$ to the diagonal.

The Rankin–Cohen bidifferential operator $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ is a symmetry breaking operator from the tensor product representation $\pi_{\lambda'} \widehat{\otimes} \pi_{\lambda''}$ to $\pi_{\lambda'''}$ with respect to the diagonal embedding $SL(2, \mathbb{R})\widetilde{\ } \hookrightarrow SL(2, \mathbb{R})\widetilde{\ } \times SL(2, \mathbb{R})\widetilde{\ }$, and such a symmetry breaking operator is unique up to scalar multiplication for generic parameters (see [19, Cor. 9.3] for the precise condition). Moreover, $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ induces a continuous map from the weighted Bergman space $\mathcal{H}^2(\Pi \times \Pi)_{(\lambda',\lambda'')}$ to $\mathcal{H}^2(\Pi)_{\lambda'''}$ if $\lambda', \lambda'' > 1$ ([18, Thm. 5.13], see also Proposition 2.27 below for an explicit formula of its operator norm).

2.2. **Notations and two constants $c_\ell(\lambda', \lambda'')$ and $r_\ell(\lambda', \lambda'')$.** The parameter set in Section 2 is $(\lambda', \lambda'', \lambda''') \in \mathbb{C}^3$ with $\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}$. Throughout this section, we use the following notation:

$$(2.2) \qquad \alpha = \lambda' - 1, \quad \beta = \lambda'' - 1, \quad 2\ell = \lambda''' - \lambda' - \lambda''.$$

The main results involve the following two constants

$$
\begin{aligned}
c \equiv c_\ell(\lambda', \lambda'') \quad &:= \quad \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} |P_\ell^{\alpha,\beta}(v)|^2 (1-v)^\alpha (1+v)^\beta dv \\
(2.3) \qquad &= \quad \frac{\Gamma(\lambda' + \ell)\Gamma(\lambda'' + \ell)}{(\lambda' + \lambda'' + 2\ell - 1)\Gamma(\lambda' + \lambda'' + \ell - 1)\ell!}, \\
r \equiv r_\ell(\lambda', \lambda'') \quad &:= \quad \frac{b(\lambda''')}{b(\lambda')b(\lambda'')} \\
(2.4) \qquad &= \quad \frac{\Gamma(\lambda' + \lambda'' + 2\ell - 1)}{2^{2\ell+2}\pi\Gamma(\lambda' - 1)\Gamma(\lambda'' - 1)},
\end{aligned}
$$

where $P_\ell^{\alpha,\beta}(v)$ is the Jacobi polynomial (see (5.4) in Appendix), and $b(\lambda) = 2^{2-\lambda}\pi\Gamma(\lambda - 1)$ is a Plancherel density (see Fact 2.9 below). We note that $c_\ell(\lambda', \lambda'') \neq 0$ if $\mathrm{Re}\,\lambda', \mathrm{Re}\,\lambda'' > 0$ and $\ell \in \mathbb{N}$.

2.3. **Integral formula for holographic operators.**

In this section, we introduce integral transforms $\Psi^{\lambda'''}_{\lambda',\lambda''}$ (*holographic operator*) that realize irreducible summands in the tensor product representations $\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''}$.

2.3.1. *Construction of holographic operators for the tensor product.*

Definition 2.1 (holographic operators)**.** For $\lambda', \lambda'', \lambda''' \in \mathbb{C}$ we set $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'')$. Assume that

$$(2.5) \qquad \mathrm{Re}(\lambda' + \ell) > 0, \quad \mathrm{Re}(\lambda'' + \ell) > 0, \quad \text{and} \quad \ell \in \mathbb{N}.$$

For a holomorphic function g on the upper half plane Π, we define a holomorphic function on $\Pi \times \Pi$ by the line integral:

$$(2.6) \qquad \left(\Psi^{\lambda'''}_{\lambda',\lambda''}g\right)(\zeta_1, \zeta_2) := \frac{(\zeta_1 - \zeta_2)^\ell}{2^{\lambda'+\lambda''+2\ell-1}\ell!} \times$$
$$\int_{-1}^{1} g\left(\frac{(\zeta_2 - \zeta_1)v + (\zeta_1 + \zeta_2)}{2}\right)(1 - v)^{\lambda'+\ell-1}(1 + v)^{\lambda''+\ell-1}dv.$$

We note that the set $\left\{\frac{(\zeta_2-\zeta_1)v+(\zeta_1+\zeta_2)}{2} : -1 \leq v \leq 1\right\}$ is the line segment connecting the two points ζ_1 and ζ_2 in Π.

2.3.2. *Basic properties of $\Psi^{\lambda'''}_{\lambda',\lambda''}$.*

The integral transform $\Psi^{\lambda'''}_{\lambda',\lambda''}$ in (2.6) provides a holographic operator in the following sense:

Theorem 2.2 (holographic operator in the upper half plane)**.** *Suppose $\lambda', \lambda'', \lambda''' \in \mathbb{C}$ satisfy* (2.5)*.*

(1) *The map $\Psi^{\lambda'''}_{\lambda',\lambda''} : \mathcal{O}(\Pi) \longrightarrow \mathcal{O}(\Pi \times \Pi)$ intertwines the action of $SL(2,\mathbb{R})^\sim$ from $\pi_{\lambda'''}$ to the tensor product representation $\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''}$.*

(2) *Moreover, if both λ' and λ'' are real and greater than 1, then the linear map $\Psi^{\lambda'''}_{\lambda',\lambda''}$ induces an isometric embedding (up to rescaling) of the weighted Bergman space:*

$$\mathcal{H}^2(\Pi)_{\lambda'''} \longrightarrow \mathcal{H}^2(\Pi \times \Pi)_{(\lambda',\lambda'')}.$$

The image of the holographic operator $\Psi^{\lambda'''}_{\lambda',\lambda''}$ is characterized by a differential equation of second order on $\Pi \times \Pi$ associated to the Casimir element under the diagonal action, see Theorem 2.30. For $\lambda', \lambda'' > 1$, the operator $\Psi^{\lambda'''}_{\lambda',\lambda''}$ is a scalar multiple of the adjoint $\left(\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}\right)^*$ of the Rankin–Cohen bidifferential operator $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$ (see Proposition 2.22), and its operator norm is given in Theorem 2.7 (2).

Theorem 2.2 will be proved in Section 2.7.6.

Remark 2.3. In Section 3, we introduce *relative* reproducing kernels to construct irreducible summands in the holomorphic model. The integral formula given there (see

Theorem 3.10) is different from the one introduced in Definition 2.1. The advantage of the definition (2.6) is that the holographic operator $\Psi_{\lambda',\lambda''}^{\lambda'''}$ is defined also for the nonunitary case, see Theorem 2.2 (1).

2.4. **The Rankin–Cohen transform and its inversion.** In this section we introduce the Rankin–Cohen transform $\mathcal{RC}_{\lambda',\lambda''}$ as the collection of individual operators $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ for fixed λ' and λ''. Its inversion formula is proved in Theorem 2.5 by using the holographic operators, giving a solution to Problem A in Section 1.

Definition 2.4 (Rankin–Cohen transform). For $\lambda', \lambda'' \in \mathbb{C}$, the *Rankin–Cohen transform* $\mathcal{RC}_{\lambda',\lambda''}$ is a linear map

$$(2.7) \qquad \mathcal{RC}_{\lambda',\lambda''} \colon \mathcal{O}(\Pi \times \Pi) \longrightarrow \mathrm{Map}(\mathbb{N}, \mathcal{O}(\Pi)), \quad f \mapsto (\ell \mapsto \mathcal{RC}_{\lambda',\lambda''}(f)_\ell)$$

defined by $(\mathcal{RC}_{\lambda',\lambda''}(f))_\ell := \mathcal{RC}_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell} f \qquad$ for $\ell \in \mathbb{N}$.

The Rankin–Cohen transform $\mathcal{RC}_{\lambda',\lambda''}$ intertwines $(\pi_{\lambda'} \widehat{\otimes} \pi_{\lambda''}, \mathcal{O}(\Pi \times \Pi))$ with the formal direct sum $\widehat{\bigoplus}_{\ell \in \mathbb{N}}(\pi_{\lambda'+\lambda''+2\ell}, \mathcal{O}(\Pi))$, and can be inverted by using the integral operators $\Psi_{\lambda',\lambda''}^{\lambda'''}$ as follows.

Theorem 2.5 (inversion of the Rankin–Cohen transform). *Suppose* $\lambda', \lambda'' > 1$. *Then for any* $f \in \mathcal{H}^2(\Pi)_{\lambda'} \widehat{\otimes} \mathcal{H}^2(\Pi)_{\lambda''}$ *one has*

$$f = \sum_{\ell=0}^{\infty} \frac{1}{c_\ell(\lambda',\lambda'')} \, \Psi_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell} \left(\mathcal{RC}_{\lambda',\lambda''}(f) \right)_\ell .$$

Theorem 2.5 will be proved in Section 2.8.6.

2.5. **Parseval–Plancherel type theorem for the Rankin–Cohen transform and its holographic transform.** In this section we develop an L^2-theory for the Rankin–Cohen transform (Definition 2.4) and for the holographic transform (Theorem 2.7 (2)), thus providing an answer to Problem B for these two transforms.

2.5.1. *Weighted Hilbert sums.* In order to formulate the Parseval–Plancherel type theorem, we fix some notations for the Hilbert direct sum.

Definition 2.6 (weighted Hilbert sum). Let $\{V_\ell\}_{\ell \in \mathbb{N}}$ be a family of Hilbert spaces and $\{a_\ell\}_{\ell \in \mathbb{N}}$ a sequence of positive numbers. The Hilbert sum $\sum_{\ell \in \mathbb{N}}^{\oplus} V_\ell$ associated to the *weights* $\{a_\ell\}_{\ell \in \mathbb{N}}$ is the Hilbert completion of the algebraic direct sum $\bigoplus_{\ell \in \mathbb{N}} V_\ell$ equipped with the pre-Hilbert structure given by

$$(v, v') := \sum_{\ell=0}^{\infty} a_\ell (v_\ell, v'_\ell)_{V_\ell} \quad \text{for } v = (v_\ell)_{\ell \in \mathbb{N}} \text{ and } v' = (v'_\ell)_{\ell \in \mathbb{N}}.$$

2.5.2. *Parseval–Plancherel type theorem.* For $\lambda', \lambda'' > 1$ the bidifferential operators $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ extend to a continuous map between Hilbert spaces. Now, we formulate a Parseval–Plancherel type theorem for the Rankin–Cohen transform as well as the "holographic transform", hence answer Problem B for these transforms.

Theorem 2.7 (Parseval–Plancherel theorem). *Suppose $\lambda', \lambda'' > 1$.*

(1) *The Rankin–Cohen transform $\mathcal{RC}_{\lambda',\lambda''}$ (Definition 2.4) induces an $SL(2,\mathbb{R})^{\sim}$-equivariant unitary operator*

$$\mathcal{H}^2(\Pi)_{\lambda'} \widehat{\otimes} \mathcal{H}^2(\Pi)_{\lambda''} \xrightarrow{\sim} \sum_{\ell \in \mathbb{N}}^{\oplus} \mathcal{H}^2(\Pi)_{\lambda'+\lambda''+2\ell}$$

to the Hilbert sum associated to weights $\left\{ \frac{1}{r_\ell(\lambda',\lambda'')c_\ell(\lambda',\lambda'')} \right\}_{\ell \in \mathbb{N}}$. Thus, for every $f \in \mathcal{H}^2(\Pi)_{\lambda'} \widehat{\otimes} \mathcal{H}^2(\Pi)_{\lambda''}$,

$$\|f\|^2_{\mathcal{H}^2(\Pi)_{\lambda'} \widehat{\otimes} \mathcal{H}^2(\Pi)_{\lambda''}}$$
$$= \sum_{\ell=0}^{\infty} \frac{1}{r_\ell(\lambda',\lambda'')c_\ell(\lambda',\lambda'')} \| (\mathcal{RC}_{\lambda',\lambda''}(f))_\ell \|^2_{\mathcal{H}^2(\Pi)_{\lambda'+\lambda''+2\ell}}.$$

(2) *Collecting the holographic operators $\Psi_{\lambda',\lambda''}^{\lambda'''}$, we define the holographic transform*

$$\Psi_{\lambda',\lambda''} : \bigoplus_{\ell \in \mathbb{N}} \mathcal{H}^2(\Pi)_{\lambda'+\lambda''+2\ell} \longrightarrow \mathcal{H}^2(\Pi)_{\lambda'} \widehat{\otimes} \mathcal{H}^2(\Pi)_{\lambda''}$$

by

$$\Psi_{\lambda',\lambda''} := \bigoplus_{\ell=0}^{\infty} \Psi_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell}.$$

Then $\Psi_{\lambda',\lambda''}$ induces an $SL(2,\mathbb{R})^{\sim}$-equivariant unitary operator

$$\sum_{\ell=0}^{\infty}{}^{\oplus} \mathcal{H}^2(\Pi)_{\lambda'+\lambda''+2\ell} \xrightarrow{\sim} \mathcal{H}^2(\Pi)_{\lambda'} \widehat{\otimes} \mathcal{H}^2(\Pi)_{\lambda''}$$

from the Hilbert sum associated to the weights $\left\{ \frac{c_\ell(\lambda',\lambda'')}{r_\ell(\lambda',\lambda'')} \right\}_{\ell \in \mathbb{N}}$. Thus,

$$\|\Psi_{\lambda',\lambda''}g\|^2_{\mathcal{H}^2(\Pi)_{\lambda'} \widehat{\otimes} \mathcal{H}^2(\Pi)_{\lambda''}} = \sum_{\ell=0}^{\infty} \frac{c_\ell(\lambda',\lambda'')}{r_\ell(\lambda',\lambda'')} \|g_\ell\|^2_{\mathcal{H}^2(\Pi)_{\lambda'+\lambda''+2\ell}}$$

for $g = (g_\ell)_{\ell \in \mathbb{N}}$.

Theorem 2.7 will be proved in Section 2.8.5. It gives quantitative information on the classical branching law (*fusion rule*) of the tensor product of two holomorphic

discrete series representations $\pi_{\lambda'}$ and $\pi_{\lambda''}$ that decomposes into a multiplicity-free direct Hilbert sum of irreducible unitary representations when $\lambda', \lambda'' > 1$ [22, 23]:

$$(2.8) \qquad\qquad \pi_{\lambda'} \widehat{\otimes} \pi_{\lambda''} \simeq \sum_{\ell \in \mathbb{N}}^{\oplus} \pi_{\lambda'+\lambda''+2\ell}.$$

The projection to each irreducible summand in the decomposition (2.8) is given as the composition of the corresponding Rankin–Cohen operator and the holographic operator in the holomorphic model. Thus Theorem 2.5 (and Proposition 2.22 below) shows the following corollary.

Corollary 2.8 (projection operator). *Suppose $\lambda', \lambda'', \lambda''' > 1$ and $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Then*

$$\frac{1}{c_\ell(\lambda', \lambda'')} \Psi_{\lambda',\lambda''}^{\lambda'''} \circ \mathcal{RC}_{\lambda',\lambda''}^{\lambda'''} = \frac{1}{r_\ell(\lambda', \lambda'') c_\ell(\lambda', \lambda'')} (\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''})^* \circ \mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$$

is the projection operator of the Hilbert space $\mathcal{H}^2(\Pi)_{\lambda'} \widehat{\otimes} \mathcal{H}^2(\Pi)_{\lambda''}$ onto the irreducible summand which is isomorphic to $\mathcal{H}^2(\Pi)_{\lambda'''}$, see (2.8).

2.6. **Holographic transform in the L^2-model.** By the Fourier–Laplace transform, the weighted Bergman space $\mathcal{H}^2(\Pi)_\lambda$ realized in the space of holomorphic functions on the upper half plane Π is mapped into the space of functions supported on the positive axis $\mathbb{R}_+$, more precisely, onto the Hilbert space $L^2(\mathbb{R}_+, x^{1-\lambda}dx)$, giving thus rise to an L^2-*model* of the same representation of $SL(2,\mathbb{R})^\sim$(Fact 2.9). We shall find closed formulæ for the symmetry breaking transform and the holographic transform also in this model and give an answer to Problems A and B, see Theorems 2.11, 2.14 and 2.16. The results in the L^2-model give a new interpretation of the classical theory of the Jacobi transform, and also play a key role in proving the theorems for the holomorphic model, see Section 2.7.

2.6.1. *L^2-model of holomorphic discrete series.* For $\lambda > 1$ we consider the Hilbert space $L^2(\mathbb{R}_+)_\lambda := L^2(\mathbb{R}_+, x^{1-\lambda}dx)$.

Fact 2.9. *Suppose $\lambda > 1$. The Fourier–Laplace transform*

$$\mathcal{F} \colon F \mapsto \mathcal{F}F(\zeta) := \int_0^\infty F(z) e^{i\zeta z} dz,$$

is an isometry from $L^2(\mathbb{R}_+)_\lambda$ onto the weighted Bergman space $\mathcal{H}^2(\Pi)_\lambda$ up to scalar multiplication. To be precise, we have

$$\|\mathcal{F}F\|_{\mathcal{H}^2(\Pi)_\lambda}^2 = b(\lambda) \|F\|_{L^2(\mathbb{R}_+)_\lambda}^2$$

for all $F \in L^2(\mathbb{R}_+)_\lambda$ (see e.g. [5, Thm. XIII.1.1]), where

$$b(\lambda) := 2^{2-\lambda} \pi \Gamma(\lambda - 1).$$

For $\lambda > 1$, via the unitary (up to scaling) map $\mathcal{F}\colon L^2(\mathbb{R}_+)_\lambda \xrightarrow{\sim} \mathcal{H}^2(\Pi)_\lambda$, we define an irreducible unitary representation of $SL(2,\mathbb{R})^{\sim}$ on $L^2(\mathbb{R}_+)_\lambda$, which is referred to as the L^2-*model* of the holomorphic discrete series representation π_λ.

We shall write
$$\mathcal{F}_1 \equiv \mathcal{F} \quad \text{and} \quad \mathcal{F}_2 := \mathcal{F} \otimes \mathcal{F}$$
in order to distinguish the framework of functions of one or two variables, respectively, and we write, by abuse of notations,

$$(2.9) \qquad L^2(\mathbb{R}_+^2)_{\lambda',\lambda''} := L^2(\mathbb{R}_+ \times \mathbb{R}_+, x^{1-\lambda'}y^{1-\lambda''}dx\,dy) \simeq L^2(\mathbb{R}_+)_{\lambda'}\widehat{\otimes}L^2(\mathbb{R}_+)_{\lambda''}.$$

2.6.2. *Construction of discrete summands in the L^2-model.* Via the Fourier–Laplace transform, we can define the counterpart for the L^2-model of the Rankin–Cohen bidifferential operator $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ and the holographic integral operator $\Psi_{\lambda',\lambda''}^{\lambda'''}$ (2.6) by

$$(2.10) \qquad\qquad\qquad \widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'''} := \mathcal{F}_1^{-1} \circ \mathcal{RC}_{\lambda',\lambda''}^{\lambda'''} \circ \mathcal{F}_2.$$

$$(2.11) \qquad\qquad\qquad \widehat{\Psi}_{\lambda',\lambda''}^{\lambda'''} := \mathcal{F}_2^{-1} \circ \Psi_{\lambda',\lambda''}^{\lambda'''} \circ \mathcal{F}_1.$$

We know from [18] that $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ is continuous between the weighted Bergman spaces, and so is $\widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'''}$. In turn, $\widehat{\Psi}_{\lambda',\lambda''}^{\lambda'''}$ is continuous between the Hilbert spaces by (2.12) below, hence so is $\Psi_{\lambda',\lambda''}^{\lambda'''}$. Alternatively, the continuity of $\Psi_{\lambda',\lambda''}^{\lambda'''}$ is also given by that of another holographic operator $\Phi_{\lambda',\lambda''}^{\lambda'''}$ introduced in Definition 2.10 (Proposition 2.25). The following commutative diagrams summarize these definitions:
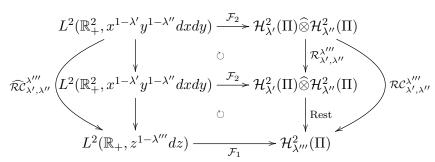
$$
\begin{array}{ccc}
L^2(\mathbb{R}_+^2, x^{1-\lambda'}y^{1-\lambda''}dxdy) & \xrightarrow{\ \mathcal{F}_2\ } & \mathcal{H}_{\lambda'}^2(\Pi)\widehat{\otimes}\mathcal{H}_{\lambda''}^2(\Pi) \\
\downarrow & \circlearrowleft & \downarrow \mathcal{R}_{\lambda',\lambda''}^{\lambda'''} \\
L^2(\mathbb{R}_+^2, x^{1-\lambda'}y^{1-\lambda''}dxdy) & \xrightarrow{\ \mathcal{F}_2\ } & \mathcal{H}_{\lambda'}^2(\Pi)\widehat{\otimes}\mathcal{H}_{\lambda''}^2(\Pi) \\
\downarrow & \circlearrowleft & \downarrow \text{Rest} \\
L^2(\mathbb{R}_+, z^{1-\lambda'''}dz) & \xrightarrow{\ \mathcal{F}_1\ } & \mathcal{H}_{\lambda'''}^2(\Pi)
\end{array}
$$

with outer arrows $\widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'''}$ (left) and $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ (right).

Diagram 2.1. Symmetry breaking operators $\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''} \twoheadrightarrow \pi_{\lambda'''}$
for holomorphic and L^2-models.

$$
\begin{array}{ccc}
L^2(\mathbb{R}_+^2, x^{1-\lambda'}y^{1-\lambda''}dxdy) & \xrightarrow{\ \mathcal{F}_2\ } & \mathcal{H}_{\lambda'}^2(\Pi)\widehat{\otimes}\mathcal{H}_{\lambda''}^2(\Pi) \\
\uparrow{\scriptstyle\widehat{\Psi}_{\lambda',\lambda''}^{\lambda'''}=(c'\Phi_{\lambda',\lambda''}^{\lambda'''})} & \circlearrowleft & \uparrow{\scriptstyle\Psi_{\lambda',\lambda''}^{\lambda'''}(=c''(\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''})^*)} \\
L^2(\mathbb{R}_+, z^{1-\lambda'''}dz) & \xrightarrow{\ \mathcal{F}_1\ } & \mathcal{H}_{\lambda'''}^2(\Pi)
\end{array}
$$

Diagram 2.2. Holographic operators $\pi_{\lambda'''} \hookrightarrow \pi_{\lambda'} \widehat{\otimes} \pi_{\lambda''}$
for holomorphic and L^2-models.

We shall give an explicit integral formula of the symmetry breaking operator $\widehat{\mathcal{RC}}^{\lambda'''}_{\lambda',\lambda''}$ in the L^2-model in Proposition 2.13. On the other hand, we observe the holographic operator in the L^2-model has the following three important characteristics:

(1) the Fourier transform $\widehat{\Psi}^{\lambda'''}_{\lambda',\lambda''}$ of the holographic operator $\Psi^{\lambda'''}_{\lambda',\lambda''}$, see (2.11);
(2) the adjoint of $\widehat{\mathcal{RC}}^{\lambda'''}_{\lambda',\lambda''}$ (Proposition 2.19);
(3) the multiplication operator $\Phi^{\lambda'''}_{\lambda',\lambda''}$, see (2.13) below for definition.

These three approaches may be summarized as the following identities:

$$(2.12) \qquad \widehat{\Psi}^{\lambda'''}_{\lambda',\lambda''} = \left(\widehat{\mathcal{RC}}^{\lambda'''}_{\lambda',\lambda''} \right)^* = i^\ell \Phi^{\lambda'''}_{\lambda',\lambda''},$$

see Propositions 2.19 and 2.20. The third characteristic is remarkable as it does not involve any integration or differentiation. For this reason, we adopt it as our definition of holographic transform in the L^2-model, see Definition 2.10 below.

For $\alpha, \beta \in \mathbb{C}$ and $\ell \in \mathbb{N}$, let $P_\ell^{\alpha,\beta}(x)$ be the Jacobi polynomial of degree ℓ, see (5.3) in Appendix.

Definition 2.10. Retain the setting that $\lambda', \lambda'', \lambda''' \in \mathbb{C}$ with $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. For a function $h(z)$ of one variable z $(z > 0)$, we define a function of two variables x, y $(x, y > 0)$ by

$$(2.13) \qquad \left(\Phi^{\lambda'''}_{\lambda',\lambda''} h \right)(x,y) := \frac{x^{\lambda'-1} y^{\lambda''-1}}{(x+y)^{\lambda'+\lambda''+\ell-1}} P_\ell^{\lambda'-1,\lambda''-1} \left(\frac{y-x}{x+y} \right) \cdot h(x+y).$$

Theorem 2.11 (holographic operator in the L^2-model). *Suppose $\lambda', \lambda'', \lambda''' > 1$ such that $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Then, $\Phi^{\lambda'''}_{\lambda',\lambda''}$ induces an $SL(2,\mathbb{R})\widetilde{}$-equivariant continuous homomorphism between the Hilbert spaces:*

$$\Phi^{\lambda'''}_{\lambda',\lambda''} \colon L^2(\mathbb{R}_+, z^{1-\lambda'''} dz) \longrightarrow L^2(\mathbb{R}_+, x^{1-\lambda'} dx) \widehat{\otimes} L^2(\mathbb{R}_+, y^{1-\lambda''} dy).$$

Theorem 2.11 will be proved in Section 2.7.5.

Remark 2.12. Using the notation (2.18) below, we may also write:

$$(\Phi^{\lambda'''}_{\lambda',\lambda''} h)(x,y) = (-1)^\ell \frac{x^{\lambda'-1} y^{\lambda''-1}}{(x+y)^{\lambda'''-1}} \widetilde{P}_\ell^{\lambda'-1,\lambda''-1}(x,y) \cdot h(x+y).$$

2.6.3. *Symmetry breaking transform in the L^2-model and its inversion.* In this sub-section, we give an inversion formula of the symmetry breaking operator $\widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'''}$ in the L^2-model by using the holographic operators $\Phi_{\lambda',\lambda''}^{\lambda'''}$ (Definition 2.10). The symmetry breaking operator $\widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'''}$ was defined originally as the Fourier transform of the Rankin–Cohen bidifferential operator $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ (see (2.10)) but we give a simpler expression as an integral operator (*Jacobi transform*).

Proposition 2.13. *Suppose $\lambda', \lambda'' > 1$ and $\ell \in \mathbb{N}$. Then for any $F \in C_c(\mathbb{R}_+ \times \mathbb{R}_+)$, the following identity holds:*

$$\left(\widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell} F\right)(z) = \frac{z^{\ell+1}}{2i^\ell} \int_{-1}^1 P_\ell^{\lambda'-1,\lambda''-1}(v) F\left(\frac{z}{2}(1-v), \frac{z}{2}(1+v)\right) dv.$$

See Section 2.7.3 for a proof.

Collecting the operators $\widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'''}$, we define a symmetry breaking transform

(2.14) $$\widehat{\mathcal{RC}}_{\lambda',\lambda''} \colon L^2(\mathbb{R}_+)_{\lambda'} \widehat{\otimes} L^2(\mathbb{R}_+)_{\lambda''} \longrightarrow \sum_{\ell \in \mathbb{N}}^{\oplus} L^2(\mathbb{R}_+)_{\lambda'+\lambda''+2\ell}$$

by $\left(\widehat{\mathcal{RC}}_{\lambda',\lambda''}(F)\right)_\ell := \widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell}(F)$ for $\ell \in \mathbb{N}$. Then it can be inverted by using the holographic operators $\Phi_{\lambda',\lambda''}^{\lambda'''}$ (Definition 2.10) as follows.

Theorem 2.14. *Suppose $\lambda', \lambda'' > 1$. Then for any element $F \in L^2(\mathbb{R}_+)_{\lambda'} \widehat{\otimes} L^2(\mathbb{R}_+)_{\lambda''}$, one has*

$$F = \sum_{\ell=0}^\infty \frac{i^\ell}{c_\ell(\lambda',\lambda'')} \Phi_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell} \left(\widehat{\mathcal{RC}}_{\lambda',\lambda''}(F)\right)_\ell,$$

and

$$\|F\|_{L^2(\mathbb{R}_+)_{\lambda'} \widehat{\otimes} L^2(\mathbb{R}_+)_{\lambda''}}^2 = \sum_{\ell=0}^\infty \frac{1}{c_\ell(\lambda',\lambda'')} \left\|\left(\widehat{\mathcal{RC}}_{\lambda',\lambda''}(F)\right)_\ell\right\|_{L^2(\mathbb{R}_+)_{\lambda'+\lambda''+2\ell}}^2.$$

Theorem 2.14 will be proved in Section 2.8.7. It gives an answer to Problem A.2 and Problem B in the L^2-model.

Remark 2.15. The Jacobi transform (see *e.g.* [4, Chap. 15]) defined by

$$H(v) \mapsto \left(J^{\alpha,\beta}(H)\right)_\ell := \int_{-1}^1 H(v) P_\ell^{\alpha,\beta}(v)(1-v)^\alpha(1+v)^\beta dv$$

is inverted by the following formula:

(2.15) $$H(v) = \sum_{\ell=0}^\infty d_\ell(\alpha,\beta) \left(J^{\alpha,\beta}(H)\right)_\ell P_\ell^{\alpha,\beta}(v),$$

where we set

$$d_\ell(\alpha, \beta) \quad := \quad \frac{\ell!(\alpha + \beta + 2\ell + 1)\Gamma(\alpha + \beta + \ell + 1)}{2^{\alpha+\beta+1}\Gamma(\alpha + \ell + 1)\Gamma(\beta + \ell + 1)}$$

$$= \quad \frac{1}{2^{\alpha+\beta+1}c_\ell(\alpha + 1, \beta + 1)}.$$

By change of variables, we can see that Theorem 2.14 is equivalent to (2.15) applied to

$$H(v) = (1 - v)^{-\alpha}(1 + v)^{-\beta}F\left(\frac{z}{2}(1 - v), \frac{z}{2}(1 + v)\right)$$

with $\alpha = \lambda' - 1$ and $\beta = \lambda'' - 1$.

2.6.4. *Parseval–Plancherel type theorem for the holographic transform in the L^2-model.* Collecting the holographic operators $\Phi_{\lambda',\lambda''}^{\lambda'''}$, we define the holographic transform

$$(2.16) \qquad \Phi_{\lambda',\lambda''} : \bigoplus_{\ell \in \mathbb{N}} L^2(\mathbb{R}_+)_{\lambda'+\lambda''+2\ell} \longrightarrow L^2(\mathbb{R}_+)_{\lambda'} \widehat{\otimes} L^2(\mathbb{R}_+)_{\lambda''}$$

by

$$\Phi_{\lambda',\lambda''} := \bigoplus_{\ell=0}^{\infty} \Phi_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell}.$$

This transform is the counterpart in the L^2-model of the holographic transform $\Psi_{\lambda',\lambda''}$ (Theorem 2.7 (2)) defined in the holomorphic model.

Theorem 2.16. *Suppose $\lambda', \lambda'' > 1$ and $\ell \in \mathbb{N}$. Then, the holographic transform $\Phi_{\lambda',\lambda''}$ induces an $SL(2,\mathbb{R})^\sim$-equivariant unitary operator*

$$\sum_{\ell=0}^{\infty} {}^{\oplus} L^2(\mathbb{R}_+)_{\lambda'+\lambda''+2\ell} \xrightarrow{\sim} L^2(\mathbb{R}_+)_{\lambda'} \widehat{\otimes} L^2(\mathbb{R}_+)_{\lambda''}$$

subject to the following Parseval–Plancherel type formula:

$$\|\Phi_{\lambda',\lambda''}h\|_{L^2(\mathbb{R}_+^2)_{\lambda',\lambda''}}^2 = \sum_{\ell=0}^{\infty} c_\ell(\lambda', \lambda'')\|h_\ell\|_{L^2(\mathbb{R}_+)_{\lambda'+\lambda''+2\ell}}^2,$$

for $h = (h_\ell)_{\ell \in \mathbb{N}}$ with $h_\ell \in L^2(\mathbb{R}_+)_{\lambda'+\lambda''+2\ell}$.

Theorem 2.16 will be proved in Section 2.8.2.

2.6.5. *Representation theoretic interpretation of the Plancherel density.* The weights $c_\ell(\lambda', \lambda'')$ in the Plancherel formula (Theorem 2.16) are obviously positive when $\lambda', \lambda'' > 1$. We discuss the zeros of the meromorphic continuation of $c_\ell(\lambda', \lambda'')$ when we allow λ' and λ'' to wander outside the region $\lambda', \lambda'' > 1$, so that $\pi_{\lambda'}$ and $\pi_{\lambda''}$ may not be (relative) holomorphic discrete series representations.

Assume furthermore that $\lambda', \lambda'', \lambda''' \in \mathbb{Z}$ such that $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Then the following four conditions on $(\lambda', \lambda'', \lambda''')$ are equivalent (see [19, Thm. 9.1]):

 (i) $c_\ell(\lambda', \lambda'') = 0$;
 (ii) $2 \geq \lambda' + \lambda'' + \lambda'''$ and $\lambda''' \geq |\lambda' - \lambda''| + 2$;
 (iii) the Rankin–Cohen bilinear operator $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$ vanishes;
 (iv) $\dim \mathrm{Hom}_{SL(2,\mathbb{R})^{\sim}}(\mathcal{O}(\Pi \times \Pi, \mathcal{L}_{\lambda'} \boxtimes \mathcal{L}_{\lambda''}), \mathcal{O}(\Pi, \mathcal{L}_{\lambda'''})) = 2$.

2.7. **Proof of Theorems 2.2 and 2.11.**

In this section, we derive from the Rankin–Cohen bidifferential operators $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$ the integral intertwining operators that embed irreducible representations of $SL(2,\mathbb{R})^{\sim}$ into the tensor product representations, and give a proof of Theorems 2.2 and 2.11.

The key idea is to use symmetry breaking operators $\widehat{\mathcal{RC}}^{\lambda'''}_{\lambda',\lambda''}$ in the L^2-model, which fits well into the F-method connecting the Rankin–Cohen operators with the Jacobi polynomials. The scheme of the proof is summarized in the following diagram:

$$(2.17) \qquad \begin{array}{ccc} \mathcal{RC}^{\lambda'''}_{\lambda',\lambda''} & \xleftarrow{\text{Proposition 2.22}} & \Psi^{\lambda'''}_{\lambda',\lambda''} \qquad (\text{Theorem 2.2}) \\ \scriptstyle\text{Proposition 2.13} \updownarrow & & \updownarrow \scriptstyle\text{Proposition 2.20} \\ \widehat{\mathcal{RC}}^{\lambda'''}_{\lambda',\lambda''} & \xrightarrow[\text{Proposition 2.19}]{} & \Phi^{\lambda'''}_{\lambda',\lambda''} \qquad (\text{Theorem 2.11}) \end{array}$$

2.7.1. *Jacobi polynomials and Rankin–Cohen bidifferential operators.* We retain the notation and assumption that $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. lem

The nature of the bidifferential symmetry breaking operator $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$ is explained in [19, Thm. 8.1] by the F-method, which we recall now. We inflate the Jacobi polynomial $P^{\alpha,\beta}_\ell(t)$ (see (5.3)) into a homogeneous polynomial of degree ℓ by

$$\widetilde{P}^{\alpha,\beta}_\ell(x,y) := (-1)^\ell (x+y)^\ell P^{\alpha,\beta}_\ell \left(\frac{y-x}{x+y} \right)$$

$$(2.18) \qquad = \sum_{j=0}^{\ell} \frac{(-1)^{\ell-j}(\alpha+\beta+\ell+1)_j(\alpha+j+1)_{\ell-j}}{(\ell-j)!j!}(x+y)^{\ell-j}x^j.$$

Then we have the following

Proposition 2.17. *Suppose $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Then the Rankin–Cohen bidifferential operator $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ (see (2.1)) is given by $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''} = \mathrm{Rest} \circ \mathcal{R}_{\lambda',\lambda''}^{\lambda'''}$ with*

$$(2.19) \qquad \mathcal{R}_{\lambda',\lambda''}^{\lambda'''} = \widetilde{P}_{\ell}^{\lambda'-1,\lambda''-1}\left(\frac{\partial}{\partial\zeta_1}, \frac{\partial}{\partial\zeta_2}\right).$$

Remark 2.18. In [19, (9.9)], we gave a similar formula

$$(2.20) \qquad \mathcal{R}_{\lambda',\lambda''}^{\lambda'''} = P_{\ell}^{\lambda'-1,1-\lambda'''}\left(\frac{\partial}{\partial\zeta_1}, \frac{\partial}{\partial\zeta_2}\right)$$

by using another two-variable function $P_{\ell}^{\alpha,\beta}(x,y) := y^{\ell}P_{\ell}^{\alpha,\beta}\left(1 + \frac{2x}{y}\right)$. Our expression (2.19) is symmetric with respect to the first and second variables.

Proof of Proposition 2.17. According to the first Kummer's relation for the hypergeometric function we get (see for instance [7, 8.962]):

$$P_{\ell}^{\alpha,\beta}(x) = \left(\frac{1+x}{2}\right)^{\ell} P_{\ell}^{\alpha,-\alpha-\beta-2\ell-1}\left(\frac{3-x}{1+x}\right),$$

and therefore

$$P_{\ell}^{\lambda'-1,1-\lambda'''}(1-2s) = (1-s)^{\ell}P_{\ell}^{\lambda'-1,\lambda''-1}\left(\frac{1+s}{1-s}\right).$$

Hence the right-hand sides of (2.19) and (2.20) are equal to each other. $\qquad\square$

2.7.2. *Coordinate change in the L^2-model.* For the study of symmetry breaking in the L^2-model, we introduce the following coordinates:

$$(2.21) \qquad \iota: \mathbb{R}_+ \times (-1,1) \xrightarrow{\sim} \mathbb{R}_+^2, \quad (z,v) \mapsto (x,y) := \left(\frac{z}{2}(1-v), \frac{z}{2}(1+v)\right).$$

Then, ι is a diffeomorphism with $dxdy = \frac{z}{2}dzdv$. With the convention (2.2) in Section 2.2, we set

$$(2.22) \qquad M(z,v) \equiv M_{\lambda',\lambda'',\lambda'''}(z,v) := 2^{\alpha+\beta}z^{\ell+1}(1-v)^{-\alpha}(1+v)^{-\beta}.$$

If $(x,y) = \iota(z,v)$, then we have

$$(2.23) \qquad \frac{z^{1-\lambda'''}}{x^{1-\lambda'}y^{1-\lambda''}}M(z,v) = z^{-\ell},$$

$$(2.24) \qquad x^{1-\lambda'}y^{1-\lambda''}dxdy = M(z,v)^2z^{-\alpha-\beta-\lambda'''}(1-v)^{\alpha}(1+v)^{\beta}dzdv,$$

whereas the holographic operator $\Phi_{\lambda',\lambda''}^{\lambda'''}$ (see (2.13)) takes the form

$$(2.25) \qquad \left(\Phi_{\lambda',\lambda''}^{\lambda'''}h\right) \circ \iota(z,v) = M(z,v)^{-1}P_{\ell}^{\alpha,\beta}(v)h(z).$$

2.7.3. *Fourier transform of the Rankin–Cohen bidifferential operators.* We are ready to prove Proposition 2.13 for an integral expression of the symmetry breaking operator $\widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'''}$ (see (2.10)).

Proof of Proposition 2.13. For a function $F \in L^2(\mathbb{R}_+^2)_{\lambda',\lambda''}$ we set

$$G(x,y) := \widetilde{P}_\ell^{\lambda'-1,\lambda''-1}(x,y)F(x,y).$$

By Proposition 2.17 the F-method shows that the Rankin–Cohen bidifferential operator is induced from the multiplication by the polynomial $\widetilde{P}_\ell^{\lambda'-1,\lambda''-1}(x,y)$, namely,

(2.26) $$(\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}\mathcal{F}_2 F)(\zeta) = i^\ell(\mathrm{Rest} \circ \mathcal{F}_2 G)(\zeta).$$

The left-hand side of (2.26) equals $\left(\mathcal{F}_1\widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'''}F \right)(\zeta)$ by the definition (2.10). We compute the right-hand side of (2.26). Via the diffeomorphism (2.21), we have $\widetilde{P}_\ell^{\alpha,\beta} \circ \iota(z,v) = (-1)^\ell z^\ell P_\ell^{\alpha,\beta}(v)$. Thus we get

$$
\begin{aligned}
(\mathrm{Rest} \circ \mathcal{F}_2)\, G(\zeta) &= \int_0^\infty \int_0^\infty G(x,y)e^{i(x+y)\zeta}dxdy \\
&= \frac{1}{2}\int_0^\infty \int_{-1}^1 G \circ \iota(z,v)e^{iz\zeta}zdzdv \\
&= \frac{1}{2}\mathcal{F}_1(\mathcal{J}F)(\zeta),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{J}F(z) &:= z\int_{-1}^1 G \circ \iota(z,v)dv \\
&= (-1)^\ell z^{\ell+1}\int_{-1}^1 P_\ell^{\lambda'-1,\lambda''-1}(v)F \circ \iota(z,v)dv.
\end{aligned}
$$

Hence Proposition 2.13 is proved. □

2.7.4. *Three characteristics of holographic operators in the L^2-model.* In Section 2.6.2, we discussed the three characteristics (1), (2), and (3) of holographic operators in the L^2-model. These three characteristics play a key role in the proof of main theorems. In this subsection we explicate the relationship between

(2) and (3) in Proposition 2.19, and
(1) and (3) in Proposition 2.20,

and prove the formula (2.12).

Proposition 2.19. *The adjoint of the holographic operator* $\Phi^{\lambda'''}_{\lambda',\lambda''}$ *(Definition 2.10) is proportional to the Fourier transform of the Rankin–Cohen operator* $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$:

$$\left(\Phi^{\lambda'''}_{\lambda',\lambda''}\right)^* = i^\ell \widehat{\mathcal{RC}}^{\lambda'''}_{\lambda',\lambda''}.$$

Proof. We have already seen in Section 2.6.2 that $\widehat{\mathcal{RC}}^{\lambda'''}_{\lambda',\lambda''}$ is a continuous map between the Hilbert spaces. Hence we shall work with dense subspaces $C_c(\mathbb{R}_+)$ and $C_c(\mathbb{R}^2_+)$ in $L^2(\mathbb{R}_+)_{\lambda'''}$ and $L^2(\mathbb{R}^2_+)_{\lambda',\lambda''}$, respectively. Take $h \in C_c(\mathbb{R}_+)$ and $F \in C_c(\mathbb{R}^2_+)$. By the integral expression of $\widehat{\mathcal{RC}}^{\lambda'''}_{\lambda',\lambda''}$ given in Proposition 2.13, we have

$$
\begin{aligned}
&(h, \widehat{\mathcal{RC}}^{\lambda'''}_{\lambda',\lambda''} F)_{L^2(\mathbb{R}_+, z^{1-\lambda'''} dz)} \\
&= \frac{i^\ell}{2} \int_0^\infty h(z) z^{\ell+1} \overline{\int_{-1}^1 P_\ell^{\lambda'-1,\lambda''-1}(v) F \circ \iota(z,v) dv} z^{1-\lambda'''} dz \\
&= i^\ell \int_0^\infty \int_0^\infty (\Phi^{\lambda'''}_{\lambda',\lambda''} h)(x,y) \overline{F(x,y)} x^{1-\lambda'} y^{1-\lambda''} dx dy \\
&= i^\ell (\Phi^{\lambda'''}_{\lambda',\lambda''} h, F)_{L^2(\mathbb{R}^2_+, x^{1-\lambda'} y^{1-\lambda''} dx dy)}.
\end{aligned}
$$

Here, in the second equality we have used (2.24) and (2.25). Thus Proposition 2.19 is proved. □

Proposition 2.20. *With the notation* (2.11), *we have*

$$\widehat{\Psi}^{\lambda'''}_{\lambda',\lambda''} = i^\ell \Phi^{\lambda'''}_{\lambda',\lambda''}.$$

Before giving a proof of Proposition 2.20, we need the following.

Lemma 2.21. *For any* $g \in \mathcal{H}^2(\Pi)_\lambda$, *we have*

$$\left(\frac{d}{dt}\right)^\ell \int_0^\infty z^{-\ell} (\mathcal{F}_1^{-1} g)(z) e^{izt} dz = i^\ell g(t).$$

Proof. The statement follows from the (classical) Fourier inversion formula and the Paley–Wiener theorem for g. □

Proof of Proposition 2.20. It suffices to show

$$\mathcal{F}_2 \circ \Phi^{\lambda'''}_{\lambda',\lambda''} \circ \mathcal{F}_1^{-1} = (-i)^\ell \Psi^{\lambda'''}_{\lambda',\lambda''}.$$

We set

$$t(v) := \frac{1}{2}\left((\zeta_2 - \zeta_1)v + (\zeta_1 + \zeta_2)\right).$$

By the definitions (2.22) and (2.25) of M and $\Phi^{\lambda'''}_{\lambda',\lambda''}$, we have,

$$2^{\alpha+\beta+1}(\mathcal{F}_2 \circ \Phi_{\lambda',\lambda''}^{\lambda'''}h)(\zeta_1,\zeta_2)$$

$$= \int_0^\infty \int_{-1}^1 z^{-\ell}(1-v)^\alpha(1+v)^\beta P_\ell^{\alpha,\beta}(v)h(z)e^{izt(v)}dvdz$$

$$= \frac{(-1)^\ell}{2^\ell \ell!}\int_0^\infty \int_{-1}^1 z^{-\ell}h(z)e^{izt(v)}\left(\frac{d}{dv}\right)^\ell \left((1-v)^{\alpha+\ell}(1+v)^{\beta+\ell}\right)dvdz$$

$$= \frac{i^\ell}{2^\ell \ell!}\int_{-1}^1 (1-v)^{\alpha+\ell}(1+v)^{\beta+\ell}\left(\frac{dt(v)}{dv}\right)^\ell (\mathcal{F}_1 h)(t(v))dv,$$

where the second equality follows from the Rodrigues formula (5.2) for the Jacobi polynomials, and the third one from integration by parts and Lemma 2.21. Putting $g = \mathcal{F}_1 h$, we obtain

$$(\mathcal{F}_2 \circ \Phi_{\lambda',\lambda''}^{\lambda'''} \circ \mathcal{F}_1^{-1}g)(\zeta_1,\zeta_2)$$

$$= \frac{(\zeta_1-\zeta_2)^\ell(-i)^\ell}{2^{\alpha+\beta+2\ell+1}\ell!}\int_{-1}^1 g(t(v))(1-v)^{\alpha+\ell}(1+v)^{\beta+\ell}dv$$

$$= (-i)^\ell \left(\Psi_{\lambda',\lambda''}^{\lambda'''}g\right)(\zeta_1,\zeta_2).$$

Hence Proposition 2.20 is proved. □

2.7.5. *Proof of Theorem 2.11.* In this subsection we give a proof of Theorem 2.11.

Proof of Theorem 2.11. As the Rankin–Cohen bidifferential operator $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ intertwines the tensor product $\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''}$ and $\pi_{\lambda'''}$, so does its Fourier transform $\widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'''}$ (see (2.10)), and in turn its adjoint operator $\left(\widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'''}\right)^*$ because $\pi_{\lambda'}, \pi_{\lambda''}$, and $\pi_{\lambda'''}$ are unitary representations. Hence Theorem 2.11 follows from Proposition 2.19. □

2.7.6. *Proof of Theorem 2.2.* Theorem 2.11 together with an argument of holomorphic continuation on parameters completes the proof of Theorem 2.2 as follows.

Proof of Theorem 2.2. The second assertion follows from Proposition 2.20 because $\Phi_{\lambda',\lambda''}^{\lambda'''}$ is an intertwining operator as it was shown in Theorem 2.11.

Let $\ell \in \mathbb{N}$ and $\lambda''' = \lambda' + \lambda'' + 2\ell$. If $(\lambda',\lambda'') \in \mathbb{C}^2$ satisfies (2.5) then the integral (2.6) converges for all $g \in \mathcal{O}(\Pi)$, and $\Psi_{\lambda',\lambda''}^{\lambda'''}$ is continuous viewed as a map from the Montel space $\mathcal{O}(\Pi)$ to the one $\mathcal{O}(\Pi \times \Pi)$.

On the other hand, if furthermore λ', λ'' are real and $\lambda', \lambda'' > 1$, then $\Psi_{\lambda',\lambda''}^{\lambda'''}$ is a G-homomorphism on $\mathcal{H}^2(\Pi)_{\lambda'''}$ by the second statement. Since $\mathcal{H}^2(\Pi)_{\lambda'''}$ is dense in the Montel space $\mathcal{O}(\Pi)$ as its subspace of K-finite functions is already dense in

$\mathcal{O}(\Pi)$, the continuous map $\Psi^{\lambda'''}_{\lambda',\lambda''}\colon \mathcal{O}(\Pi) \longrightarrow \mathcal{O}(\Pi \times \Pi)$ intertwines $\pi_{\lambda'''}$ and the tensor product representation $\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''}$ if $\lambda',\lambda'' > 1$. Since $\Psi^{\lambda'''}_{\lambda',\lambda''}g \in \mathcal{O}(\Pi \times \Pi)$ depends holomorphically on $(\lambda',\lambda'') \in \mathbb{C}^2$ subject to (2.5) and since the actions $\pi_{\lambda'},\pi_{\lambda''}$ and $\pi_{\lambda'+\lambda''+2\ell}$ of $SL(2,\mathbb{R})\widetilde{\ }$also depend holomorphically on $(\lambda',\lambda'') \in \mathbb{C}^2$, the first statement is shown. $\qquad\square$

2.7.7. *Adjoint of the Rankin–Cohen operator.* As the last part of the diagram (2.17), we show that $\Psi^{\lambda'''}_{\lambda',\lambda''}$ is the adjoint of the Rankin–Cohen operator $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$ up to scalar multiplication.

Suppose $\lambda',\lambda'',\lambda''' > 1$ and $\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}$. We regard the Rankin–Cohen operator $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$ as a continuous map between Hilbert spaces

$$\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}\colon \mathcal{H}^2(\Pi)_{\lambda'}\widehat{\otimes}\mathcal{H}^2(\Pi)_{\lambda''} \longrightarrow \mathcal{H}^2(\Pi)_{\lambda'''}.$$

By (2.12) and Lemma 2.24 below, we obtain

Proposition 2.22. *Let $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. The adjoint of $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$ is given by*

$$\left(\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}\right)^* = r_\ell(\lambda',\lambda'')\Psi^{\lambda'''}_{\lambda',\lambda''}.$$

2.8. **Proof of the Parseval–Plancherel type theorem for the symmetry breaking transform and the holographic transform.** In this section we complete the proof of Theorems 2.5 and 2.7 in the holomorphic model and Theorems 2.14 and 2.16 in the L^2-model for the Parseval–Plancherel type results for the symmetry breaking and holographic transforms. Our strategy consists in applying the F-method, and then in reducing the proof of these theorems to the fact that the Jacobi polynomials $\left\{P^{\alpha,\beta}_\ell\right\}_{\ell\in\mathbb{N}}$ form an orthogonal basis of the Hilbert space $L^2([-1,1],(1-v)^\alpha(1+v)^\beta dv)$.

2.8.1. *Some properties of operators on Hilbert spaces.* We review a general fact on operators on Hilbert spaces. Suppose a Hilbert space V is decomposed into a Hilbert direct sum of closed subspaces $\{V_\ell\}_{\ell\in\mathbb{N}}$, that is, $V \simeq \sum^{\oplus}_{\ell\in\mathbb{N}} V_\ell$, where the inner product on V_ℓ is induced from that of V. Let $\mathrm{pr}_{V\to V_\ell}\colon V \longrightarrow V_\ell$ be the projection operator. Let $\{W_\ell\}_{\ell\in\mathbb{N}}$ be another family of Hilbert spaces. Suppose that we are given a continuous map $R_\ell\colon V \longrightarrow W_\ell$ such that the restriction $R_\ell|_{V_\ell} \colon V_\ell \longrightarrow W_\ell$ is a unitary operator up to scalar multiplication and $R_\ell|_{V_\ell^\perp} \equiv 0$ for every $\ell \in \mathbb{N}$. Then the adjoint operator $R_\ell^*\colon W_\ell \longrightarrow V$ is an isometry (up to scalar) onto V_ℓ. We write $\|R_\ell\|_{\mathrm{op}}$ for the operator norm of R_ℓ and set

$$C_\ell := \|R_\ell\|^2_{\mathrm{op}}.$$

The following two lemmas are elementary.

Lemma 2.23.   (1) *The linear map* $R := \bigoplus_{\ell \in \mathbb{N}} R_\ell \colon V \to \bigoplus_{\ell \in \mathbb{N}} W_\ell$ *satisfies*

$$(2.27) \qquad R_\ell^* R_\ell = C_\ell \mathrm{pr}_{V \to V_\ell},$$

$$(2.28) \qquad \|F\|_V^2 = \sum_{\ell \in \mathbb{N}} \frac{1}{C_\ell} \|R_\ell F\|_{W_\ell}^2 \quad \text{for all} \quad F \in V.$$

In particular, we have the following inversion formula and the unitarity of the map R:

(inversion) $F = \sum_{\ell \in \mathbb{N}} \frac{1}{C_\ell} R_\ell^*(R_\ell F),$

(unitarity) R *extends to a unitary operator* $V \xrightarrow{\sim} \sum_{\ell \in \mathbb{N}}^{\oplus} W_\ell$, *where* $\sum_{\ell \in \mathbb{N}}^{\oplus} W_\ell$ *is*

the Hilbert sum associated to the weights $\{C_\ell^{-1}\}_{\ell \in \mathbb{N}}$ *(see Definition 2.6).*

(2) *The linear map* $R^* := \bigoplus_{\ell \in \mathbb{N}} R_\ell^* \colon \bigoplus_{\ell \in \mathbb{N}} W_\ell \to V$ *satisfies*

$$(2.29) \qquad R_\ell R_\ell^* = C_\ell \mathrm{id}_{W_\ell},$$

$$\|R_\ell^* w_\ell\|_V^2 = C_\ell \|w_\ell\|_{W_\ell}^2 \quad \text{for all} \quad w_\ell \in W_\ell.$$

In particular R^* extends to a unitary operator $\sum_{\ell \in \mathbb{N}}^{\oplus} W_\ell \xrightarrow{\sim} V$, *where* $\sum_{\ell \in \mathbb{N}}^{\oplus} W_\ell$ *is the Hilbert sum associated to the weights* $\{C_\ell\}_{\ell \in \mathbb{N}}$.

Lemma 2.24. *Suppose that H_j and L_j ($j = 1, 2$) are Hilbert spaces and that $\mathcal{F}_j \colon L_j \longrightarrow H_j$ are unitary operators up to scalar multiple. Let b_j be positive numbers such that*

$$\|\mathcal{F}_j(F)\|_{H_j}^2 = b_j \|F\|_{L_j}^2 \qquad \text{for all} \quad F \in L_j.$$

Let $\Psi \colon H_1 \longrightarrow H_2$ and $D \colon H_2 \longrightarrow H_1$ be continuous linear maps, and we define $\widehat{\Psi} \colon L_1 \longrightarrow L_2$ and $\widehat{D} \colon L_2 \longrightarrow L_1$ by

$$\widehat{\Psi} := \mathcal{F}_2^{-1} \circ \Psi \circ \mathcal{F}_1, \qquad \widehat{D} := \mathcal{F}_1^{-1} \circ D \circ \mathcal{F}_2.$$

We set $r := \frac{b_1}{b_2}$. Then,

(1) *the operator norms of these operators satisfy*

$$\|\widehat{D}\|_{\mathrm{op}}^2 = \frac{1}{r} \|D\|_{\mathrm{op}}^2, \qquad \|\widehat{\Psi}\|_{\mathrm{op}}^2 = r \|\Psi\|_{\mathrm{op}}^2;$$

(2) *the adjoint operators of D and $\widehat{D}$ are related as*

$$\widehat{D^*} = r \left(\widehat{D}\right)^*.$$

2.8.2. *Parseval–Plancherel type theorem for $\Phi_{\lambda',\lambda''}^{\lambda'''}$.* In this subsection, we prove Theorem 2.16. By the (abstract) branching law (2.8), Theorem 2.16 is deduced from the following proposition.

Proposition 2.25. *Suppose $\lambda', \lambda'', \lambda''' > 1$ satisfy $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Then,*

$$\|\Phi_{\lambda',\lambda''}^{\lambda'''}h\|_{L^2(\mathbb{R}_+^2)_{\lambda',\lambda''}}^2 = c_\ell(\lambda',\lambda'')\|h\|_{L^2(\mathbb{R}_+)_{\lambda'''}}^2,$$

for all $h \in L^2(\mathbb{R}_+)_{\lambda'''}$. Here we recall (2.3) for the definition of $c_\ell(\lambda',\lambda'')$.

We reduce Proposition 2.25 to the fact that the Jacobi polynomials are orthogonal polynomials, see (5.4) in Appendix.

Proof of Proposition 2.25. Via the diffeomorphism (2.21), we get from the formulæ (2.24) for the measure and (2.25) for the holographic operator $\Phi_{\lambda',\lambda''}^{\lambda'''}$:

$$\|\Phi_{\lambda',\lambda''}^{\lambda'''}h\|_{L^2(\mathbb{R}_+^2)_{\lambda',\lambda''}}^2 = \frac{1}{2^{\lambda'+\lambda''-1}} \times$$

$$\int_0^\infty \int_{-1}^1 |h(z)|^2 \left|P_\ell^{\lambda'-1,\lambda''-1}(v)\right|^2 z^{1-\lambda'''}(1-v)^{\lambda'-1}(1+v)^{\lambda''-1} \, dv dz.$$

By the L^2-norm (5.4) of the Jacobi polynomials, we conclude the proposition. $\quad\square$

2.8.3. *Operator norm of the holographic operator $\Psi_{\lambda',\lambda''}^{\lambda'''}$ in the holomorphic model.*

Proposition 2.26. *Suppose $\lambda', \lambda'', \lambda''' > 1$ and $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Then*

$$\|\Psi_{\lambda',\lambda''}^{\lambda'''}g\|_{\mathcal{H}^2(\Pi)_{\lambda'}\widehat{\otimes}\mathcal{H}^2(\Pi)_{\lambda''}}^2 = \frac{c_\ell(\lambda',\lambda'')}{r_\ell(\lambda',\lambda'')}\|g\|_{\mathcal{H}^2(\Pi)_{\lambda'''}}^2 \text{ for all } g \in \mathcal{H}^2(\Pi)_{\lambda'''}.$$

Proof. By Proposition 2.20 and Fact 2.9, we have

$$\|\Psi_{\lambda',\lambda''}^{\lambda'''}g\|_{\mathcal{H}^2(\Pi)_{\lambda'}\widehat{\otimes}\mathcal{H}^2(\Pi)_{\lambda''}}^2 = b(\lambda')b(\lambda'')\|\Phi_{\lambda',\lambda''}^{\lambda'''}\mathcal{F}_1^{-1}g\|_{L^2(\mathbb{R}_+^2)_{\lambda',\lambda''}}^2.$$

By Proposition 2.25 and Fact 2.9 again, the right-hand side of the above equality amounts to

$$\frac{b(\lambda')b(\lambda'')}{b(\lambda''')}c_\ell(\lambda',\lambda'')\|g\|_{\mathcal{H}^2(\Pi)_{\lambda'''}}^2.$$

Now the proposition follows from the definition (2.4) of $r_\ell(\lambda',\lambda'')$. $\quad\square$

2.8.4. *Norm of the Rankin–Cohen bidifferential operators.* We find the operator norm of $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ as below.

Proposition 2.27. *Suppose that $\lambda', \lambda'' > 1$ and $\lambda''' = \lambda' + \lambda'' + 2\ell$ ($\ell \in \mathbb{N}$). Then the operator norm of the Rankin–Cohen bidifferential operator $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ seen as a map from the weighted Bergman space $\mathcal{H}^2(\Pi)_{\lambda'}\widehat{\otimes}\mathcal{H}^2(\Pi)_{\lambda''}$ to $\mathcal{H}^2(\Pi)_{\lambda'''}$ is given by*

$$\|\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}\|_{\mathrm{op}}^2 = r_\ell(\lambda',\lambda'')c_\ell(\lambda',\lambda'').$$

Proof. By Lemma 2.24 (1), we have

$$\|\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}\|_{\mathrm{op}}^2 = r_\ell(\lambda',\lambda'')\|\widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'''}\|_{\mathrm{op}}^2,$$

which equals $r_\ell(\lambda',\lambda'')c_\ell(\lambda',\lambda'')$ by Propositions 2.19 and 2.25. $\square$

2.8.5. *Proof of Theorem 2.7.* Let us complete the proof of the Parseval–Plancherel type theorem for the Rankin–Cohen transform $\mathcal{RC}_{\lambda',\lambda''}$ and the holographic transform $\Psi_{\lambda',\lambda''}$.

Proof of Theorem 2.7. (1) We apply Lemma 2.23 with $R_\ell = \mathcal{RC}_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell}$. By Proposition 2.27, we have

$$\|R_\ell\|_{\mathrm{op}}^2 = r_\ell(\lambda',\lambda'')c_\ell(\lambda',\lambda''),$$

hence the first statement follows from Lemma 2.23 (1).

(2) We apply Lemma 2.23 with $R_\ell = \frac{1}{r_\ell(\lambda',\lambda'')}\mathcal{RC}_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell}$. By Proposition 2.27, we have

$$\|R_\ell\|_{\mathrm{op}}^2 = \frac{c_\ell(\lambda',\lambda'')}{r_\ell(\lambda',\lambda'')}.$$

Since $\Psi_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell} = R_\ell^*$ (see Proposition 2.22), we get the second statement by Lemma 2.23 (2). $\square$

2.8.6. *Proof of Theorem 2.5.* We are ready to complete the proof of Theorem 2.5.

Proof of Theorem 2.5. By Lemma 2.23 (1) applied to $R_\ell = \mathcal{RC}_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell}$, the above proof of Theorem 2.7 (1) implies

$$f = \sum_{\ell=0}^{\infty} \frac{1}{r_\ell(\lambda',\lambda'')c_\ell(\lambda',\lambda'')} R_\ell^* R_\ell f$$

for any $f \in \mathcal{H}^2(\Pi)_{\lambda'} \widehat{\otimes} \mathcal{H}^2(\Pi)_{\lambda''}$. Now Theorem 2.5 follows from the equation $R_\ell^* = r_\ell(\lambda',\lambda'')\Psi_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell}$ (see Proposition 2.22). $\square$

2.8.7. *Proof of Theorem 2.14.* Finally, we show Theorem 2.14.

Proof of Theorem 2.14. We apply Lemma 2.23 (1) with $R_\ell = \widehat{\mathcal{RC}}_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell}$. By Lemma 2.24 and Proposition 2.27, we obtain

$$\|R_\ell\|_{\mathrm{op}}^2 = \frac{1}{r_\ell(\lambda',\lambda'')}\left\|\mathcal{RC}_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell}\right\|_{\mathrm{op}}^2 = c_\ell(\lambda',\lambda'').$$

Since $R_\ell^* = i^\ell\, \Phi_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell}$ by Proposition 2.19, Theorem 2.14 follows from Lemma 2.23 (1). $\square$

2.9. **Some applications of symmetry breaking and holographic transforms.**
We point out two applications of the symmetry breaking and holographic transforms
introduced in the previous section. First, we provide explicit description of the
minimal K-types of the $SL(2,\mathbb{R})\widetilde{}$-module $(\pi_\lambda, \mathcal{O}(\Pi))$ in both holomorphic model
$\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''}$ and L^2-model $L^2(\mathbb{R}^2_+)_{\lambda',\lambda''}$ (see Propositions 2.28 and 2.29). Second, we find
in Theorem 2.30 an integral expression of any eigenfunction for a specific second-
order holomorphic partial differential operator arising from the diagonal action of
the Casimir in the enveloping algebra.

2.9.1. *Minimal K-types.* The minimal K-type of the $SL(2,\mathbb{R})\widetilde{}$-module $(\pi_\lambda, \mathcal{O}(\Pi))$ is
given by $\mathbb{C}(\zeta + i)^{-\lambda}$, see (2.32) for the whole set of K-types. As an application of
the integral formula (2.6) we find an explicit expression for the minimal K-types of
submodules in the tensor product $\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''}$ as follows.

Proposition 2.28. *Suppose* $\operatorname{Re}\lambda', \operatorname{Re}\lambda'' > 0$ *and* $\lambda''' = \lambda' + \lambda'' + 2\ell$ $(\ell \in \mathbb{N})$. *Then
the holomorphic function*

$$(\zeta_1, \zeta_2) \mapsto (\zeta_1 - \zeta_2)^\ell(\zeta_1 + i)^{-\lambda'-\ell}(\zeta_2 + i)^{-\lambda''-\ell}$$

is a minimal K-type in the submodule $\Psi^{\lambda'''}_{\lambda',\lambda''}(\mathcal{O}(\Pi))$ *in* $\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''}$.

Proof. We set $g(\zeta) := (\zeta + i)^{-\lambda'''}$. By the change of variables $t = \frac{1}{2}(1 + v)$, the
definition (2.6) shows

$$\left(\Psi^{\lambda'''}_{\lambda',\lambda''}g\right)(\zeta_1, \zeta_2) = \frac{1}{\ell!}(\zeta_1 - \zeta_2)^\ell(\zeta_1 + i)^{-\lambda'''}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tz)^{-b}dt,$$

where $a = \lambda'' + \ell, b = c = \lambda'''$, and $z = \frac{\zeta_1-\zeta_2}{\zeta_1+i}$.

By the Euler integral representation of the hypergeometric function ${}_2F_1$, and by
the fact that ${}_2F_1(a, b; b; z) = (1 - z)^{-a}$, we obtain

$$(2.30)\quad \left(\Psi^{\lambda'''}_{\lambda',\lambda''}g\right)(\zeta_1, \zeta_2) = \frac{1}{\ell!}B(\lambda' + \ell, \lambda'' + \ell)(\zeta_1 - \zeta_2)^\ell(\zeta_1 + i)^{-\lambda'-\ell}(\zeta_2 + i)^{-\lambda''-\ell},$$

where $B(\cdot, \cdot)$ stands for the Euler beta function. $\qquad\square$

Proposition 2.29. *Suppose* $\lambda', \lambda'' > 1$ *and* $\ell \in \mathbb{N}$. *Then the function*

$$(x, y) \mapsto (x^{\lambda'-1}e^{-x})(y^{\lambda''-1}e^{-y})(x + y)^\ell P_\ell^{\lambda'-1,\lambda''-1}\left(\frac{y - x}{x + y}\right)$$

belongs to $L^2(\mathbb{R}^2_+)_{\lambda',\lambda''}$, *and gives a minimal K-type in the image of the holographic
operator* $\Phi^{\lambda'+\lambda''+2\ell}_{\lambda',\lambda''}$.

Proof. Since $z^{\lambda'''-1}e^{-z}$ belongs to the minimal K-type in the irreducible representation
$L^2(\mathbb{R}_+)_{\lambda'''}$, so does $\Phi^{\lambda'''}_{\lambda',\lambda''}(z^{\lambda'''-1}e^{-z})$ in the irreducible representation $\Phi^{\lambda'''}_{\lambda',\lambda''}(L^2(\mathbb{R}^2_+)_{\lambda',\lambda''})$.

Then the formula (2.13) of the holographic operator $\Phi_{\lambda',\lambda''}^{\lambda'''}$ with $\lambda''' = \lambda' + \lambda'' + 2\ell$ shows Proposition 2.29. $\qquad\square$

2.9.2. *An application of the integral formula.* Fix $\lambda', \lambda'' \in \mathbb{C}$ and consider eigenfunctions of the following holomorphic differential operator on $\Pi \times \Pi$:

$$(2.31) \quad P_{\lambda',\lambda''} := (\zeta_1 - \zeta_2)^2 \frac{\partial^2}{\partial \zeta_1 \partial \zeta_2} + (\lambda'' \zeta_2 + \lambda' - \lambda'')\zeta_1 \frac{\partial}{\partial \zeta_1} + (\lambda' \zeta_1 - \lambda' + \lambda'')\zeta_2 \frac{\partial}{\partial \zeta_2},$$

and define for $\mu \in \mathbb{C}$

$$\mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',\,\mu}) := \{f \in \mathcal{O}(\Pi \times \Pi) \colon P_{\lambda',\lambda''} f = \mu f\}.$$

The integral transform (2.6) constructs all eigenfunctions of $P_{\lambda',\lambda''}$ as follows.

Theorem 2.30. *Suppose $\lambda', \lambda'' \in \mathbb{C}$. Then, the following hold.*

(1) $\mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',\,\mu}) \neq \{0\}$ *if and only if μ is of the form*

$$\mu = -\ell(\lambda' + \lambda'' + \ell - 1) \quad \text{for some } \ell \in \mathbb{N}.$$

(2) *For any $\lambda', \lambda'' \in \mathbb{C}$ and $\ell \in \mathbb{N}$,*

$$(\zeta_1 - \zeta_2)^\ell (\zeta_1 + i)^{-\lambda'-\ell}(\zeta_2 + i)^{-\lambda''-\ell} \in \mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',-\ell(\lambda'+\lambda''+\ell-1)}).$$

(3) *If $\operatorname{Re}\lambda', \operatorname{Re}\lambda'' > 0$ and $\ell \in \mathbb{N}$, then the integral transform (2.6) gives a bijection*

$$\Psi_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell} \colon \mathcal{O}(\Pi) \xrightarrow{\sim} \mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',-\ell(\lambda'+\lambda''+\ell-1)}).$$

The inverse map is proportional to the Rankin–Cohen bidifferential operator, namely,

$$\mathcal{RC}_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell} \circ \Psi_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell} = c_\ell(\lambda', \lambda'')\mathrm{id} \quad \text{on } \mathcal{O}(\Pi),$$

where $c_\ell(\lambda', \lambda'')$ is defined as in (2.3).

2.9.3. *Quick review of representations of the universal covering group $SL(2, \mathbb{R})^{\widetilde{\ }}$.* In order to prove Theorem 2.30 we recall some properties of representations of $SL(2, \mathbb{R})^{\widetilde{\ }}$. The universal covering group $SO(2)^{\widetilde{\ }}$ of the maximal compact subgroup $K = SO(2)$ is isomorphic to $\mathbb{R}$. We parametrize its characters χ_λ by $\lambda \in \mathbb{C}$ as an extension of the following group homomorphisms originally defined for $\lambda \in \mathbb{Z}$:

$$\mathbb{R} \simeq SO(2)^{\widetilde{\ }} \longrightarrow SO(2) \longrightarrow \mathbb{C}^\times, \quad \theta \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto e^{i\lambda\theta}.$$

The representation π_λ on $\mathcal{O}(\Pi)$ given in Section 2.1.1 is a highest weight module with highest weight $-\lambda$ because it has the following K-types:

$$(2.32) \qquad\qquad\qquad -\lambda, -\lambda - 2, -\lambda - 4, \ldots.$$

Choose the standard basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$:

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the Casimir element C is expressed as $C = \frac{1}{8}(H^2 + 2XY + 2YX)$.

The infinitesimal action $d\pi_\lambda$ is given by holomorphic differential operators:

$$(2.33) \qquad d\pi_\lambda(H) = -\lambda - 2z\frac{d}{dz}, \quad d\pi_\lambda(X) = -\frac{d}{dz}, \quad d\pi_\lambda(Y) = \lambda z + z^2\frac{d^2}{dz^2},$$

and the Casimir element C acts on $(d\pi_\lambda, \mathcal{O}(\Pi))$ as $d\pi_\lambda(C) = \frac{1}{8}\lambda(\lambda - 2)\mathrm{id}$. In general, if π is a highest weight module of $SL(2, \mathbb{R})^\sim$ with highest weight ν ($\nu \in \mathbb{C}$), then the Casimir element is given via $d\pi$ as the scalar multiplication $\frac{1}{8}\nu(\nu + 2)\mathrm{id}$.

2.9.4. *Proof of Theorem 2.30.*

Lemma 2.31. *The Casimir element C of $\mathfrak{sl}(2, \mathbb{R})$ acts on $\mathcal{O}(\Pi \times \Pi)$ via $d\pi_{\lambda'} \otimes d\pi_{\lambda''}$ by*

$$(d\pi_{\lambda'} \otimes d\pi_{\lambda''})(\mathrm{diag}(\mathrm{C})) = -\frac{1}{2}P_{\lambda',\lambda''} + \frac{1}{8}(\lambda' + \lambda'')(\lambda' + \lambda'' - 2),$$

where the holomorphic differential operator $P_{\lambda',\lambda''}$ is defined in (2.31).

Proof. By (2.33) and the Leibniz rule, we have

$$\begin{aligned}
(d\pi_{\lambda'} \otimes d\pi_{\lambda''})(\mathrm{diag}(\mathrm{H})) &= -\lambda' - \lambda'' - 2\left(\zeta_1\frac{\partial}{\partial\zeta_1} + \zeta_2\frac{\partial}{\partial\zeta_2}\right), \\
(d\pi_{\lambda'} \otimes d\pi_{\lambda''})(\mathrm{diag}(\mathrm{X})) &= -\frac{\partial}{\partial\zeta_1} - \frac{\partial}{\partial\zeta_2}, \\
(d\pi_{\lambda'} \otimes d\pi_{\lambda''})(\mathrm{diag}(\mathrm{Y})) &= \lambda'\zeta_1 + \lambda''\zeta_2 + \left(\zeta_1^2\frac{\partial^2}{\partial\zeta_1^2} + \zeta_2^2\frac{\partial^2}{\partial\zeta_2^2}\right).
\end{aligned}$$

Now the lemma follows by a direct computation. $\qquad\qquad\square$

The tensor product representation $\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''}$ does not always split into a direct sum of irreducible representations in the nonunitary case when $\lambda', \lambda'' \in \mathbb{C}$, see [19] for instance. We determine the set of possible infinitesimal characters of subrepresentations of the tensor product $\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''}$ in this case.

Lemma 2.32. *Let $\lambda', \lambda'' \in \mathbb{C}$. Suppose π is a subrepresentation of $\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''}$ such that the Casimir element C acts as scalar multiplication via $d\pi$. Then this scalar must be of the form*

$$\frac{1}{8}(\lambda' + \lambda'' + 2\ell)(\lambda' + \lambda'' + 2\ell - 2) \quad \text{for some } \ell \in \mathbb{N}.$$

Proof. We use the general theory of discretely decomposable restrictions of (nonunitary) representations [11]. First we observe from (2.32) that the K-types of the tensor product representation $\pi_{\lambda'} \widehat{\otimes} \pi_{\lambda''}$ are of the form

$$-\lambda' - \lambda'' - 2(\ell' + \ell'') \quad \text{for some } \ell', \ell'' \in \mathbb{N}.$$

Thus the tensor product representation $\pi_{\lambda'} \widehat{\otimes} \pi_{\lambda''}$ on $\mathcal{O}(\Pi \times \Pi)$ contains the direct sum of K-isotypic spaces

$$(2.34) \qquad \bigoplus_{\ell \in \mathbb{N}} (\ell + 1)\chi_{-\lambda'-\lambda''-2\ell}$$

as a dense subset, where $(\ell + 1)$ stands for the multiplicity.

In particular, each K-type occurs in $\pi_{\lambda'} \widehat{\otimes} \pi_{\lambda''}$ with at most finite multiplicities. Hence, any subrepresentation π is admissible, and of highest weight $-\lambda' - \lambda'' - 2\ell$ for some $\ell \in \mathbb{N}$. Therefore, if the Casimir element acts as a scalar via $d\pi$, then this scalar must coincide with $\frac{1}{8}(\lambda' + \lambda'' + 2\ell)(\lambda' + \lambda'' + 2\ell - 2)$. $\qquad\square$

Proof of Theorem 2.30. (1) By Lemma 2.31, $\mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',\mu})$ is characterized as the eigenspace of the Casimir operator C as follows:

$$(2.35) \quad \mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',\mu})$$
$$= \{f \in \mathcal{O}(\Pi \times \Pi) \colon (d\pi_{\lambda'} \otimes d\pi_{\lambda''})(\operatorname{diag}(C)) = e(\lambda', \lambda'', \mu)f\},$$

where we set $e(\lambda', \lambda'', \mu) = -\frac{1}{2}\mu + \frac{1}{8}(\lambda' + \lambda'')(\lambda' + \lambda'' - 2)$.

On the other hand, Lemma 2.32 tells that $e(\lambda', \lambda'', \mu) = \frac{1}{8}(\lambda' + \lambda'' + 2\ell)(\lambda' + \lambda'' + 2\ell - 2)$ for some $\ell \in \mathbb{N}$. This gives the desired formula for μ, showing the "if" part of the first statement. The "only if" part follows from the second statement.

(2) By the assumption $\operatorname{Re}\lambda', \operatorname{Re}\lambda'' > 0$, the integral (2.6) converges for any $\ell \in \mathbb{N}$. Since the Casimir element C acts on $\mathcal{O}(\Pi)$ as the scalar $\frac{1}{8}\lambda'''(\lambda''' - 2)$ via $d\pi_{\lambda'''}$, and since $\Psi_{\lambda',\lambda''}^{\lambda'''}$ is an intertwining operator, the Casimir element C acts on the image of $\Psi_{\lambda',\lambda''}^{\lambda'''}$ by the same scalar. Therefore $\Psi_{\lambda',\lambda''}^{\lambda'+\lambda''+2\ell}(\mathcal{O}(\Pi)) \subset \mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',-\ell(\lambda'+\lambda''+\ell-1)})$ by (2.35). In turn, Proposition 2.28 and Lemma 2.31 imply that

$$(P_{\lambda',\lambda''} + \ell(\lambda' + \lambda'' + \ell - 1))\left((\zeta_1 - \zeta_2)^\ell(\zeta_1 + i)^{-\lambda'-\ell}(\zeta_2 + i)^{-\lambda''-\ell}\right) = 0$$

as far as $\operatorname{Re}\lambda', \operatorname{Re}\lambda'' > 0$. By the analytic continuation, the equation holds for all $\lambda', \lambda'' \in \mathbb{C}$.

(3) We begin with the case where λ' and λ'' are real and $\lambda', \lambda'' > 1$. Then it follows from Proposition 2.27, Lemma 2.23 (2) and Proposition 2.22 that

$$\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''} \circ \Psi_{\lambda',\lambda''}^{\lambda'''} = c_\ell(\lambda', \lambda'')\operatorname{id} \quad \text{on } \mathcal{H}^2(\Pi)_{\lambda'''}.$$

Since $\mathcal{H}^2(\Pi)_{\lambda'''}$ is dense in $\mathcal{O}(\Pi)$, the equality holds on the whole $\mathcal{O}(\Pi)$ by continuity.

Moreover, since the operators $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$ and $\Psi^{\lambda'''}_{\lambda',\lambda''}$ depend holomorphically on $(\lambda',\lambda'') \in \mathbb{C}^2$ satisfying (2.5), we conclude that

$$(2.36) \qquad \mathcal{RC}^{\lambda'''}_{\lambda',\lambda''} \circ \Psi^{\lambda'''}_{\lambda',\lambda''} = c_\ell(\lambda',\lambda'')\mathrm{id} \quad \text{on } \mathcal{O}(\Pi)$$

for any $(\lambda',\lambda'',\lambda''')$ subject to (2.5) by analytic continuation. In particular, $\Psi^{\lambda'''}_{\lambda',\lambda''}$ is injective and $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$ is surjective because $c_\ell(\lambda',\lambda'') \neq 0$ in this case.

Let us prove that $\Psi^{\lambda'''}_{\lambda',\lambda''} \colon \mathcal{O}(\Pi) \longrightarrow \mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',-\ell(\lambda'+\lambda''+\ell-1)})$ is surjective. We let $SL(2,\mathbb{R})^{\sim}$ act on $\mathcal{O}(\Pi \times \Pi)$ via $\pi_{\lambda'}\widehat{\otimes}\pi_{\lambda''}$. Since the map $\mathbb{N} \longrightarrow \mathbb{C}, \ell \mapsto -\ell(\lambda'+\lambda''+\ell-1)$ is injective if $\mathrm{Re}\,\lambda', \mathrm{Re}\,\lambda'' > 0$, we have the following inclusion of $SL(2,\mathbb{R})^{\sim}$-submodules of $\mathcal{O}(\Pi \times \Pi)$:

$$\bigoplus_{\ell\in\mathbb{N}} \Psi^{\lambda'+\lambda''+2\ell}_{\lambda',\lambda''}\left(\mathcal{O}(\Pi)\right) \subset \bigoplus_{\ell\in\mathbb{N}} \mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',-\ell(\lambda'+\lambda''+\ell-1)}).$$

We observe that K-multiplicities coincide by (2.34) because the holographic operator $\Psi^{\lambda'+\lambda''+2\ell}_{\lambda',\lambda''}$ is injective for any $\ell \in \mathbb{N}$. Therefore,

$$\Psi^{\lambda'+\lambda''+2\ell}_{\lambda',\lambda''} \colon \mathcal{O}(\Pi) \longrightarrow \mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',-\ell(\lambda'+\lambda''+\ell-1)})$$

is surjective on the level of $(\mathfrak{g},K)$-modules.

Since $c_\ell(\lambda',\lambda'') \neq 0$, the symmetry breaking operator $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$ is injective on $\Psi^{\lambda'+\lambda''+2\ell}_{\lambda',\lambda''}\left(\mathcal{O}(\Pi)\right)$ by (2.36), and therefore the underlying $(\mathfrak{g},K)$-module of $\mathrm{Ker}\left(\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}\right)\cap \mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',-\ell(\lambda'+\lambda''+\ell-1)})$ must be zero for any $\lambda''' = \lambda'+\lambda''+2\ell$ ($\ell \in \mathbb{N}$). Hence $\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''}$ is injective when restricted to $\mathcal{S}ol(\Pi \times \Pi, \mathcal{M}_{\lambda',\lambda'',-\ell(\lambda'+\lambda''+\ell-1)})$. Now we conclude that $\Psi^{\lambda'''}_{\lambda',\lambda''}$ is surjective. Thus the theorem is proved. $\qquad\square$

3. Holomorphic Juhl transform and its holographic transform

Another remarkable family of differential operators is provided by conformally covariant differential operators for the pair $S^n \supset S^{n-1}$ of Riemannian manifolds, introduced by Juhl [9]. These operators are symmetry breaking operators from spherical principal series representations of the Lorentz group $O(1, n+1)$ to those of the subgroup $O(1, n)$. This setting is intimately related to the holographic or AdS/CFT correspondence in string theory (see *e.g.* [21, 24]).

The *holomorphic Juhl operators* are the holomorphic continuation of Juhl's operators, which map holomorphic functions on the n-dimensional Lie ball to those on the $(n-1)$-dimensional Lie ball, intertwining (relative) discrete series representations of $G = SO_o(2, n)$ with those of the subgroup $G' = SO_o(2, n-1)$, see [19].

In this section we solve Problems A and B stated in Section 1 for the symmetry breaking transform associated with the holomorphic Juhl operators. We assume

$n \geq 3$ throughout this section. The case $n = 2$ can be recovered from the previous section with an appropriate change of parameters.

3.1. **Holomorphic Juhl operators.**

3.1.1. *Holomorphic discrete series of* $SO_o(2, n)$. Let $Q_{p,q}$ be the standard quadratic form of signature (p, q) on $\mathbb{R}^{p+q}$. The indefinite orthogonal group

$$O(p, q) := \left\{ g \in GL(p + q, \mathbb{R}) : Q_{p,q}(gx) = Q_{p,q}(x) \text{ for all } x \in \mathbb{R}^{p+q} \right\}$$

has four connected components when $p, q > 0$. Let $G = SO_o(2, n)$ be the identity component of $O(2, n)$ and K a maximal compact subgroup of G. We write $\mathfrak{c}(\mathfrak{k})$ for the first factor of the Lie algebra $\mathfrak{k} \simeq \mathbb{R} \oplus \mathfrak{so}(n)$, and fix a characteristic element $H_0 \in \mathfrak{c}(\mathfrak{k})$ such that $\mathrm{ad}(H_0)$ gives the eigenspace decomposition of $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$ as

$$(3.1) \qquad\qquad \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{n}_+ + \mathfrak{n}_-$$

for eigenvalues $0, -i$ and i, respectively. The complex structure of the homogeneous space G/K is given by the G-translation of $\exp\left(\mathrm{ad}\left(\frac{\pi}{2}H_0\right)\right) \in GL_{\mathbb{R}}(T_o(G/K))$, or equivalently, it is induced from the Borel embedding $G/K \subset G_{\mathbb{C}}/K_{\mathbb{C}} \exp \mathfrak{n}_+$.

Let $\widetilde{G}$ be the universal covering of $G = SO_o(2, n)$, $p : \widetilde{G} \longrightarrow G$ the covering homomorphism, and set $\widetilde{K} := p^{-1}(K)$. For $\lambda \in \mathbb{C}$, we define a character of $\mathfrak{c}(\mathfrak{k})$ by $tH_0 \mapsto \lambda t$, which lifts to a character $\mathbb{C}_\lambda$ of $\widetilde{K}$.

Then one can define a $\widetilde{G}$-equivariant holomorphic line bundle $\mathcal{L}_\lambda = \widetilde{G} \times_{\widetilde{K}} \mathbb{C}_\lambda$ over $X := \widetilde{G}/\widetilde{K} \simeq G/K$ for all $\lambda \in \mathbb{C}$, and obtain representations $\pi_\lambda^{(n)}$ of $\widetilde{G}$ on the space $\mathcal{O}(X, \mathcal{L}_\lambda)$ of holomorphic sections. The representation $\pi_\lambda^{(n)}$ descends to G if $\lambda \in \mathbb{Z}$.

If $\lambda \in \mathbb{R}$, then the line bundle $\mathcal{L}_\lambda \longrightarrow X$ carries a $\widetilde{G}$-invariant Hermitian metric, and therefore we can define a Hilbert space $\left(\mathcal{O} \cap L^2\right)(X, \mathcal{L}_\lambda)$. This Hilbert space is nonzero if and only if $\lambda > n - 1$, and the resulting unitary representation of $\widetilde{G}$, to be denoted by the same symbol $\pi_\lambda^{(n)}$, is called a *(relative) holomorphic discrete series representation* of $\widetilde{G}$. For actual computations we use its realization in the weighted Bergman space as below.

We define the tube domain

$$T_\Omega \equiv T_{\Omega(n)} := \mathbb{R}^n + i\Omega(n)$$

by taking $\Omega(n)$ to be the time-like cone in the Minkowski space $\mathbb{R}^{1,n-1}$, namely,

$$\Omega(n) := \left\{ \eta \in \mathbb{R}^n \ : \ Q_{1,n-1}(\eta) > 0, \ \eta_1 > 0 \right\}.$$

Then the tube domain T_Ω is biholomorphic to the bounded symmetric domain of type IV_n, sometimes referred to as the Lie ball. From a group-theoretic viewpoint,

T_Ω is isomorphic to the Hermitian symmetric space $X = G/K$, which is realized as an open subset of $\mathfrak{n}_-(\simeq \mathbb{C}^n)$ via the following maps

(3.2)
$$\mathfrak{n}_- \underset{\text{open}}{\hookrightarrow} G_\mathbb{C}/K_\mathbb{C} \exp(\mathfrak{n}_+) \underset{\text{open}}{\supset} G/K.$$

The homogeneous holomorphic line bundle $\mathcal{L}_\lambda \longrightarrow X$ is trivialized via the Bruhat decomposition, and the Hilbert space $(\mathcal{O} \cap L^2)(X, \mathcal{L}_\lambda)$ is then identified with the weighted Bergman space

$$\mathcal{H}^2(T_{\Omega(n)})_\lambda := \mathcal{O}(T_{\Omega(n)}) \cap L^2(T_{\Omega(n)}, Q_{1,n-1}(\eta)^{\lambda-n}d\xi d\eta),$$

on which G acts as a multiplier representation by

$$f(\zeta) \mapsto b_\lambda(g, \zeta)f(g^{-1}.\zeta)$$

for $g \in G$ and $f(\zeta) \in \mathcal{O}(T_{\Omega(n)})$. Here the multiplier

$$b_\lambda : G \times T_{\Omega(n)} \longrightarrow \mathbb{C}^\times$$

is a 1-cocycle defined by $b_\lambda(g, \zeta) := \chi_{-\lambda}(k(g, \zeta))$, where $\chi_\lambda : K_\mathbb{C} \longrightarrow \mathbb{C}^\times$ is the holomorphic extension of the character $\mathbb{C}_\lambda$ of K and $k(g, \zeta)$ is an element of $K_\mathbb{C}$ determined by

$$g^{-1} \exp(\zeta) \in \exp(g^{-1}.\zeta)k(g, \zeta) \exp(\mathfrak{n}_+),$$

see (3.2).

For $\lambda > n - 1$ this Hilbert space admits a reproducing kernel $K_\lambda(\zeta, \tau)$ given by:

(3.3)
$$K_\lambda(\zeta, \tau) = k_{\lambda,n} Q_{1,n-1}(\zeta - \bar{\tau})^{-\lambda},$$

where $Q_{1,n-1}(\zeta)$ stands for the holomorphic extension of $Q_{1,n-1}$, and we set

$$k_{\lambda,n} := \frac{(2i)^{2\lambda}}{(4\pi)^n} \frac{\left(\lambda - \frac{n}{2}\right)\Gamma(\lambda)}{\Gamma(\lambda - n + 1)},$$

see [5, Prop. XIII.1.2]. We note that $k_{\lambda,n} \neq 0$ if $\lambda > n - 1$.

We realize $O(2, n-1)$ as the subgroup of $O(2, n)$ which fixes the $(n+2)$-th coordinate, and set $G' = SO_o(2, n-1)$ as its identity component. By abuse of notation, we write $\widetilde{G}'$ for the connected subgroup of $\widetilde{G} = SO_o(2, n)\tilde{}$ corresponding to $G' \subset G$. The subgroup $\widetilde{G}'$ is simply connected if $n \neq 4$. Similarly, a (relative) holomorphic discrete series representation $\pi_\nu^{(n-1)}$ of $\widetilde{G}'$ is defined for $\nu > n - 2$, as an irreducible unitary representation on the weighted Bergman space $\mathcal{H}^2(T_{\Omega(n-1)})_\nu$. By abuse of notation, the same symbol $\pi_\nu^{(n-1)}$ will be used to denote a (nonunitary) representation on $\mathcal{O}(T_{\Omega(n-1)}, \mathcal{L}_\nu)$ for $\nu \in \mathbb{C}$.

3.1.2. *Holomorphic Juhl operators.* Let $\Delta_{\mathbb{C}^{1,n-2}} := \frac{\partial^2}{\partial \zeta_1^2} - \frac{\partial^2}{\partial \zeta_2^2} - \cdots - \frac{\partial^2}{\partial \zeta_{n-1}^2}$ be the holomorphic Laplacian on $\mathbb{C}^{n-1}$ associated to the complexified quadratic form $Q_{1,n-2}$. For $\alpha \in \mathbb{C}$ and $\ell, k \in \mathbb{N}$ with $\ell \geq 2k$, we define a polynomial of α of degree $\ell - k$ by

$$(3.4) \qquad a_k(\ell, \alpha) := \frac{(-1)^k 2^{\ell-2k} \cdot \Gamma(\alpha + \ell - k)}{\Gamma(\alpha) \, k!(\ell - 2k)!}.$$

The coefficients $a_k(\ell, \alpha)$ appear in the definition (5.5) of the Gegenbauer polynomials $C_\ell^\alpha(x)$, see Appendix. We define a holomorphic differential operator $\mathcal{D}_\ell^\alpha$ on $\mathbb{C}^n$ by

$$(3.5) \qquad \mathcal{D}_\ell^\alpha := \sum_{k=0}^{\left[\frac{\ell}{2}\right]} a_k(\ell, \alpha) \left(\frac{\partial}{\partial \zeta_n}\right)^{\ell-2k} \Delta_{\mathbb{C}^{1,n-2}}^k.$$

For $\lambda, \nu \in \mathbb{C}$ with $\ell := \nu - \lambda \in \mathbb{N}$, the holomorphic Juhl operator $D_{\lambda \to \nu} \colon \mathcal{O}(T_{\Omega(n)}) \longrightarrow \mathcal{O}(T_{\Omega(n-1)})$ is defined as the composition

$$(3.6) \qquad D_{\lambda \to \nu} := \mathrm{Rest}_{\zeta_n=0} \circ \mathcal{D}_\ell^{\lambda - \frac{n-1}{2}}.$$

The operator $D_{\lambda \to \nu}$ may be viewed as the holomorphic extension of the original Juhl operator [9], which is a conformally covariant differential operator $C^\infty(S^n) \to C^\infty(S^{n-1})$. In our setting, the hyperbolic space H^n is realized as a totally real submanifold of the tube domain $T_{\Omega(n)}$, and likewise, H^{n-1} is that of $T_{\Omega(n-1)}$. The restriction of the holomorphic Juhl operator to these real manifolds also yields a conformally covariant operator $C^\infty(H^n) \longrightarrow C^\infty(H^{n-1})$, see [16, Thm. E].

The holomorphic Juhl operator $D_{\lambda \to \nu}$ gives yet another symmetry breaking operator, intertwining the (relative) holomorphic discrete series representation $\pi_\lambda^{(n)}$ of $\widetilde{G}$ and the one $\pi_\nu^{(n-1)}$ of the subgroup $\widetilde{G}'$ [19, Thm. 6.3]. Moreover, the differential operator $D_{\lambda \to \nu}$ induces a *continuous* map between the Bergman spaces by the general theory [18, Thm. 5.13]. Its adjoint is denoted by $D_{\lambda \to \nu}^*$. We determine the operator norm of $D_{\lambda \to \nu}$ in Proposition 3.6.

3.2. **Two constants $c_\ell(\lambda)$ and $r_\ell(\lambda)$.** Throughout Section 3 the parameter set is $(\lambda, \nu) \in \mathbb{C}^2$ with $\nu - \lambda \in \mathbb{N}$ and $n \geq 3$. We use the following notation:

$$(3.7) \qquad \alpha = \lambda - \frac{n-1}{2}, \qquad \ell = \nu - \lambda.$$

As in Section 2.2 devoted to the tensor product case, the main results in this section involve the following two constants:

$$c \equiv c_\ell(\lambda) \quad := \quad \int_{-1}^{1} |C_\ell^\alpha(v)|^2 (1 - v^2)^{\alpha - \frac{1}{2}} dv$$

$$(3.8) \qquad\qquad = \quad \frac{\pi 2^{n-2\lambda} \Gamma(2\lambda + \ell - n + 1)}{\ell! \left(\lambda + \ell - \frac{n-1}{2}\right) \Gamma\left(\lambda - \frac{n-1}{2}\right)^2},$$

$$r \equiv r_\ell(\lambda) \quad := \quad \frac{b_{n-1}(\nu)}{b_n(\lambda)}$$

$$(3.9) \qquad\qquad = \quad \frac{\Gamma\left(\lambda + \ell - \frac{n-1}{2}\right) \Gamma(\lambda + \ell - n + 2)}{(2\pi)^{\frac{3}{2}} 2^{2\ell+1} \Gamma\left(\lambda - \frac{n}{2}\right) \Gamma(\lambda - n + 1)},$$

where $b_n(\lambda)$ is a Plancherel density (see (3.13) below).

3.3. **Holomorphic Juhl transforms.**

Definition 3.1 (holomorphic Juhl transform)**.** For $\lambda \in \mathbb{C}$, the holomorphic Juhl transform D_λ is a collection of the holomorphic Juhl operators

$$D_\lambda \colon \mathcal{O}(T_{\Omega(n)}) \longrightarrow \mathrm{Map}(\mathbb{N}, \mathcal{O}(T_{\Omega(n-1)})), \quad f \mapsto \{(D_\lambda f)_\ell\}_{\ell \in \mathbb{N}},$$

where $(D_\lambda f)_\ell := D_{\lambda \to \lambda + \ell} f$.

The holomorphic Juhl transform D_λ intertwines $(\pi_\lambda^{(n)}, \mathcal{O}(T_{\Omega(n)}))$ with the formal direct sum $\widehat{\bigoplus}_{\ell \in \mathbb{N}} (\pi_{\lambda+\ell}^{(n-1)}, \mathcal{O}(T_{\Omega(n-1)}))$; its inversion formula and the corresponding Parseval–Plancherel type theorems are given as follows.

Theorem 3.2. *Suppose* $\lambda > n - 1$.

    (1) (inversion formula). *Any* $f \in \mathcal{H}^2(T_{\Omega(n)})_\lambda$ *is recovered from* $D_\lambda f$ *by*

$$f = \sum_{\ell=0}^{\infty} \frac{1}{r_\ell(\lambda) c_\ell(\lambda)} D_{\lambda \to \lambda + \ell}^* (D_\lambda f)_\ell.$$

    (2) (Parseval–Plancherel type theorem). *For every* $f \in \mathcal{H}^2(T_{\Omega(n)})_\lambda$, *we have*

$$(3.10) \qquad \|f\|_{\mathcal{H}^2_\lambda(T_{\Omega(n)})}^2 = \sum_{\ell=0}^{\infty} \frac{1}{r_\ell(\lambda) c_\ell(\lambda)} \| (D_\lambda f)_\ell \|_{\mathcal{H}^2_{\lambda+\ell}(T_{\Omega(n-1)})}^2.$$

Theorem 3.2 is proved in Section 3.5.4. It gives an answer to Problems A.2 and B raised in Section 1 for the holomorphic Juhl transform D_λ. An answer to Problem A.1 (explicit integral formula for holographic transform) will be given in Theorem 3.10.

From a representation-theoretic viewpoint, Theorem 3.2 gives quantitative information on the corresponding branching law for the restriction of the (relative) holomorphic discrete series representation $\pi_\lambda^{(n)}$ of $\widetilde{G}$ to the subgroup $\widetilde{G}'$, which decomposes the restriction $\pi_\lambda^{(n)}\big|_{\widetilde{G}'}$ into a multiplicity-free direct Hilbert sum of irreducible representations of the subgroup $\widetilde{G}'$, see [13, Thm. 8.3]:

$$(3.11) \qquad \pi_\lambda^{(n)}\big|_{\widetilde{G}'} \simeq \sum_{\ell \in \mathbb{N}}^{\oplus} \pi_{\lambda+\ell}^{(n-1)}.$$

Corollary 3.3 (projection operator). *Suppose $\lambda > n - 1$ and $\ell := \nu - \lambda \in \mathbb{N}$. Then*

$$\frac{1}{r_\ell(\lambda)c_\ell(\lambda)} D_{\lambda \to \nu}^* D_{\lambda \to \nu}$$

is the projection operator from the Hilbert space $\mathcal{H}^2(T_{\Omega(n)})_\lambda$ onto the summand which is unitarily equivalent to the irreducible representation $(\pi_\nu^{(n-1)}, \mathcal{H}^2(T_{\Omega(n-1)})_\nu)$, see (3.11).

3.4. **Key operators in the proof of Theorem 3.2.**

Analogously to the case of the tensor product representations (Section 2), we introduce the following continuous operators for the proof of Theorem 3.2:

$$D_{\lambda \to \nu}^*\colon \mathcal{H}^2(T_{\Omega(n-1)})_\nu \longrightarrow \mathcal{H}^2(T_{\Omega(n)})_\lambda \qquad \text{adjoint of } D_{\lambda \to \nu},$$
$$\widehat{D}_{\lambda \to \nu}\colon L^2(\Omega(n))_\lambda \longrightarrow L^2(\Omega(n-1))_\nu \qquad \text{Fourier transform of } D_{\lambda \to \nu},$$
$$\Phi_\lambda^\nu\colon L^2(\Omega(n-1))_\nu \longrightarrow L^2(\Omega(n))_\lambda \qquad \text{holographic operator.}$$

See (3.23) and (3.17) below for the definitions of $\widehat{D}_{\lambda \to \nu}$ and Φ_λ^ν, respectively.

The link between these operators may be summarized in the following diagram:

$$(3.12) \qquad \begin{array}{ccc} D_{\lambda \to \nu} & \xrightarrow{\text{Theorem 3.10}} & D_{\lambda \to \nu}^* \\ {\scriptstyle \text{Proposition 3.7}}\Big\updownarrow & & \Big\updownarrow{\scriptstyle \text{Proposition 3.8}} \\ \widehat{D}_{\lambda \to \nu} & \xrightarrow[\text{Proposition 3.5}]{} & \Phi_\lambda^\nu \end{array}$$

Among them, the holographic operator Φ_λ^ν in the L^2-model will play a crucial role in the proof of Theorem 3.2.

3.5. Holographic transform in the L^2-model on the time-like cone $\Omega(n)$.

3.5.1. *L^2-model of holomorphic discrete series.* For $\lambda > n-1$, we denote by $L^2(\Omega)_\lambda \equiv L^2(\Omega, m_\lambda(y)dy)$ the Hilbert space of square integrable functions on the time-like cone $\Omega \equiv \Omega(n)$ with respect to the measure $m_\lambda(y)dy$, where m_λ is a positive-valued function on Ω given by

$$m_\lambda(y) := Q_{1,n-1}(y)^{\frac{n}{2}-\lambda}.$$

Let $\langle y, \zeta \rangle = \sum_{j=1}^{n} y_j \zeta_j$. Since the cone Ω is self-dual in $\mathbb{R}^n$, the Fourier–Laplace transform

$$(\mathcal{F}_n F)(\zeta) := \int_\Omega F(y)e^{i\langle y,\zeta \rangle}dy$$

is a holomorphic function of $\zeta \in T_\Omega$ if $F \in C_c(\Omega)$.

Fact 3.4 (Faraut–Koranyi [5, Thm. XIII.1.1]). *For $\lambda > n - 1$, we set*

(3.13) $$b_n(\lambda) := (2\pi)^{\frac{3n}{2}-1} 2^{-2\lambda+n} \Gamma\left(\lambda - \frac{n}{2}\right) \Gamma(\lambda - n + 1).$$

Then the Fourier–Laplace transform $\mathcal{F}_n \colon C_c(\Omega) \longrightarrow \mathcal{O}(T_\Omega)$ extends to a linear bijection:

(3.14) $$\mathcal{F}_n \colon L^2(\Omega)_\lambda \overset{\sim}{\to} \mathcal{H}_\lambda^2(T_{\Omega(n)}).$$

with

$$\|\mathcal{F}_n F\|_{\mathcal{H}_\lambda^2(T_{\Omega(n)})}^2 = b_n(\lambda)\|F\|_{L^2(\Omega(n))_\lambda}^2 \quad \text{for all } F \in L^2(\Omega)_\lambda.$$

Via the isomorphism (3.14), an irreducible unitary representation of $\widetilde{G}$ is defined on $L^2(\Omega)_\lambda$ for $\lambda > n-1$, to which we refer as the L^2-*model* of the (relative) holomorphic discrete series representation $\pi_\lambda^{(n)}$.

Similarly, we define a positive-valued function

$$m_\nu'(y') := Q_{1,n-2}(y')^{\frac{n-1}{2}-\nu}$$

on the time-like cone $\Omega' \equiv \Omega(n-1)$ in $\mathbb{R}^{1,n-2}$, and we set $L^2(\Omega')_\nu := L^2(\Omega', m_\nu'(y')dy')$ on which the L^2-model of the (relative) holomorphic discrete series representation $\pi_\nu^{(n-1)}$ of $\widetilde{G}'$ is defined via the unitary map

$$\mathcal{F}_{n-1} \colon L^2(\Omega')_\nu \overset{\sim}{\to} \mathcal{H}_\nu^2(T_{\Omega(n-1)}) \qquad \text{for} \quad \nu > n - 2$$

up to rescaling $b_{n-1}(\nu)^{-\frac{1}{2}}$.

3.5.2. *Gegenbauer polynomial and Juhl's conformally covariant operator.* Let $a_k(\ell, \alpha)$ be as in (3.4). We define a polynomial of two variables by

$$(3.15) \qquad (I_\ell C_\ell^\alpha)(u, v) := \sum_{k=0}^{\left[\frac{\ell}{2}\right]} a_k(\ell, \alpha) u^k v^{\ell - 2k}.$$

For instance, we have $(I_0 C_0^\alpha)(u, v) = 1$, $(I_1 C_1^\alpha)(u, v) = 2\alpha v$, and $(I_2 C_2^\alpha)(u, v) = \alpha(2(\alpha + 1)v^2 - u)$. By definition, $(I_\ell C_\ell^\alpha)(w^2, v)$ is a homogeneous polynomial of degree ℓ, and $(I_\ell C_\ell^\alpha)(1, v)$ coincides with the Gegenbauer polynomial $C_\ell^\alpha(v)$, see (5.5) in Appendix. (This is the reason why we adopted the notation $I_\ell C_\ell^\alpha$.)

The F-method [19, Thm. 6.3] shows that the differential operator $\mathcal{D}_\ell^\alpha$ (see (3.5)) is expressed as

$$(3.16) \qquad \mathcal{D}_\ell^\alpha = i^{-\ell} (I_\ell C_\ell^\alpha) \left(-\Delta_{\mathbb{C}^{1, n-2}}, i \frac{\partial}{\partial \zeta_n} \right).$$

3.5.3. *Construction of discrete summands in the L^2-model.* Following the idea of the F-method [15], we introduce the *holographic operator* Φ_λ^ν as a multiplication operator like in the tensor product case (*cf.* Definition 2.10). By the simplicity of its formula, the holographic operator Φ_λ^ν in the L^2-model plays a crucial role in the proof of Theorem 3.2.

Retain the basic setting (3.7) where $\ell = \nu - \lambda \in \mathbb{N}$ and $\alpha = \lambda - \frac{n-1}{2}$. For a function $h(y')$ on $\Omega(n - 1)$, we define $(\Phi_\lambda^\nu h)(y)$ on $\Omega(n)$ by

$$(3.17) \qquad (\Phi_\lambda^\nu h)(y) := Q_{1, n-2}(y')^{-\left(\ell + \frac{1}{2}\right)} (1 - y_n^2)^{\lambda - \frac{n}{2}} (I_\ell C_\ell^\alpha)(Q_{1, n-2}(y'), -y_n) h(y').$$

Then Φ_λ^ν gives rise to a holographic operator in the L^2-model in the following sense:

Proposition 3.5. *Suppose that $\lambda > n - 1$ and $\nu = \lambda + \ell$ for some $\ell \in \mathbb{N}$.*

(1) *The linear map $\Phi_\lambda^\nu \colon L^2(\Omega(n-1))_\nu \longrightarrow L^2(\Omega(n))_\lambda$ is an isometry up to scalar:*

$$\|\Phi_\lambda^\nu(h)\|_{L^2(\Omega(n))_\lambda}^2 = c_\ell(\lambda) \|h\|_{L^2(\Omega(n-1))_\nu}^2 \quad \text{for all } h \in L^2(\Omega(n-1))_\nu,$$

where we recall that the constant $c_\ell(\lambda)$ is given in (3.8).

(2) *Φ_λ^ν intertwines the irreducible unitary representation $\pi_\nu^{(n-1)}$ of the subgroup $\widetilde{G}'$ with the restriction $\pi_\lambda^{(n)}\big|_{\widetilde{G}'}$.*

We also discuss some further basic properties of the holographic operators Φ_λ^ν in Proposition 3.8.

3.5.4. *Proof of Theorem 3.2.* Postponing the proof of Proposition 3.5 until Section 3.5.7 and Proposition 3.8 below until Section 3.5.8 we complete the proof of Theorem 3.2.

Proposition 3.6. *Suppose $\lambda > n - 1$ and $\nu = \lambda + \ell$ with $\ell \in \mathbb{N}$. Then the differential operator $D_{\lambda \to \nu}$ extends to a continuous linear map $D_{\lambda \to \nu} \colon \mathcal{H}^2(T_{\Omega(n)})_\lambda \longrightarrow \mathcal{H}^2(T_{\Omega(n-1)})_\nu$ with the following operator norm:*

$$\|D_{\lambda \to \nu}\|_{\mathrm{op}}^2 = r_\ell(\lambda) c_\ell(\lambda).$$

Proof. It follows from Lemma 2.24 (1) and Propositions 3.5 and 3.8 that

$$\|D_{\lambda \to \nu}\|_{\mathrm{op}}^2 = r_\ell(\lambda) \|\widehat{D}_{\lambda \to \nu}\|_{\mathrm{op}}^2 = r_\ell(\lambda) \|\Phi_\lambda^\nu\|_{\mathrm{op}}^2 = r_\ell(\lambda) c_\ell(\lambda).$$

$\square$

Proof of Theorem 3.2. By Lemma 2.23 and by the (abstract) branching law (3.11) for the restriction $\widetilde{G} \downarrow \widetilde{G}'$, the theorem follows from the expression of the operator norm of the differential operator $D_{\lambda \to \nu}$ given in Proposition 3.6. $\square$

The rest of this section is devoted to the proof of Proposition 3.5.

3.5.5. *Coordinate change in the L^2-model.* As in Section 2.7.2 for the tensor product case, we introduce the following coordinates of the time-like cone $\Omega(n)$ in $\mathbb{R}^{1,n-1}$:

$$(3.18) \qquad \iota \colon \Omega(n-1) \times (-1, 1) \xrightarrow{\sim} \Omega(n), \quad (y', v) \mapsto (y', -\sqrt{Q_{1,n-2}(y')}v),$$

which is a bijection because $(y', y_n) \in \Omega(n)$ if and only if $y' \in \Omega(n-1)$ and $y_n^2 < Q_{1,n-2}(y')$.

We define a function on $\Omega(n-1) \times (-1, 1)$ by

$$(3.19) \qquad M(y', v) \equiv M_{\lambda, \nu}(y', v) := Q_{1,n-2}(y')^{\frac{\ell+1}{2}} (1 - v^2)^{\frac{n}{2} - \lambda}.$$

Via the isomorphism ι, the ratio of the densities $m_\lambda(y)$ and $m_\nu(y')$, and the holographic operator Φ_λ^ν are expressed as follows:

$$(3.20) \qquad \frac{m_\nu'(y')}{m_\lambda(y)} = M(y', v)^{-1} Q_{1,n-2}(y')^{-\frac{\ell}{2}},$$

$$(3.21) \qquad m_\lambda(y) dy = M(y', v)^2 m_\nu'(y') dy' (1 - v^2)^{\lambda - \frac{n}{2}} dv,$$

$$(3.22) \qquad (\Phi_\lambda^\nu h) \circ \iota(y', v) = M(y', v)^{-1} C_\ell^\alpha(v) h(y').$$

3.5.6. *Fourier transform of the holomorphic Juhl operators.* For $\lambda > n - 1$ and $\nu = \lambda + \ell$ ($\ell \in \mathbb{N}$), the holomorphic Juhl operator $D_{\lambda \to \nu}$ gives rise to a continuous operator $\mathcal{H}^2(T_{\Omega(n)})_\lambda \longrightarrow \mathcal{H}^2(T_{\Omega(n-1)})_\nu$ between the weighted Bergman spaces [18, Thm. 5.13].

We define a linear map $\widehat{D}_{\lambda \to \nu} : L^2(\Omega)_\lambda \longrightarrow L^2(\Omega')_\nu$ by

$$(3.23) \qquad \widehat{D}_{\lambda \to \nu} := \mathcal{F}_{n-1}^{-1} \circ D_{\lambda \to \nu} \circ \mathcal{F}_n.$$

Then the idea of the F-method [18] implies that the "Fourier transform" $\widehat{D}_{\lambda \to \nu}$ of the holomorphic Juhl operator $D_{\lambda \to \nu}$ is given by a Gegenbauer transform (*cf.* [4, Chap. 15]) along the trajectory in (3.18) where the parameter v moves:

Proposition 3.7. *The operator $\widehat{D}_{\lambda \to \nu}$ is given by the following integral transform:*

$$\left(\widehat{D}_{\lambda \to \nu} F\right)(y') = i^{-\ell} Q_{1,n-2}(y')^{\frac{\ell+1}{2}} \int_{-1}^1 F \circ \iota(y', v) C_\ell^\alpha(v) dv \quad \text{for} \quad y' \in \Omega',$$

where we set $\alpha = \lambda - \frac{n-1}{2}$.

Proof. Let $\alpha = \lambda - \frac{n-1}{2}$. Then it follows from (3.5) and (3.16) that

$$\left(i^\ell D_{\lambda \to \nu} \circ \mathcal{F}_n\right) F = \text{Rest}_{\zeta_n=0} \circ (I_\ell C_\ell^\alpha) \left(-\Delta_{\mathbb{C}^{1,n-2}}, i\frac{\partial}{\partial \zeta_n}\right) \mathcal{F}_n F$$

$$(3.24) \qquad\qquad\qquad = \text{Rest}_{\zeta_n=0} \circ \mathcal{F}_n \left(I_\ell C_\ell^\alpha(Q_{1,n-2}(y'), -y_n)F\right),$$

for $F \in L^2(\Omega(n))_\lambda$. Since $(I_\ell C_\ell^\alpha)(u^2, w) = u^\ell C_\ell^\alpha\left(\frac{w}{u}\right)$, the right-hand side amounts to

$$\int_{\Omega(n-1)} \int_{-1}^1 F \circ \iota(y', v) Q_{1,n-2}(y')^{\frac{\ell+1}{2}} C_\ell^\alpha(v) e^{i\langle y', \zeta'\rangle} dy' dv$$

$$= \mathcal{F}_{n-1} \left(Q_{1,n-1}(y')^{\frac{\ell+1}{2}} \int_{-1}^1 F \circ \iota(y', v) C_\ell^\alpha(v) dv\right)$$

via the diffeomorphism (3.18). Since the left-hand side of (3.24) is equal to $i^\ell \mathcal{F}_{n-1} \circ \widehat{D}_{\lambda \to \nu} F$ by the definition (3.23) of $\widehat{D}_{\lambda \to \nu}$, we proved Proposition 3.7. $\square$

3.5.7. *Proof of Proposition 3.5.*

Proof of Proposition 3.5. Let $\alpha = \lambda - \frac{n-1}{2}$ as before. By (3.21) and (3.22) we have

$$\|\Phi_\lambda^\nu h\|_{L^2(\Omega(n))_\lambda}^2 = \|h\|_{L^2(\Omega(n-1))_{\lambda+\ell}}^2 \int_{-1}^1 |C_\ell^\alpha(v)|^2 (1 - v^2)^{\alpha - \frac{1}{2}} dv.$$

Thus the first assertion holds by the definition (3.8) of $c_\ell(\lambda)$.

The second assertion follows readily from Proposition 3.8 below. $\square$

3.5.8. *Holographic operators and the adjoint of symmetry breaking operators.* We have constructed operators Φ_λ^ν in the L^2-model and $\widehat{\Phi}_\lambda^\nu$ in the holomorphic model using the F-method. On the other hand, the adjoint of symmetry breaking operators between unitary representations yield holographic operators in general (*cf.* Theorem 4.4 (1) below). In our setting, these operators are proportional to each other because the branching law (3.11) is multiplicity-free. We determine the proportionality constants:

Proposition 3.8. *Suppose $\lambda > n - 1$ and $\ell = \nu - \lambda \in \mathbb{N}$. Then we have*

$$\left(\widehat{D}_{\lambda \to \nu}\right)^* = i^\ell \Phi_\lambda^\nu,$$

$$D_{\lambda \to \nu}^* = i^\ell r_\ell(\lambda) \widehat{\Phi}_\lambda^\nu.$$

Proof. Take any $h \in L^2(\Omega(n-1))_\nu$ and $F \in L^2(\Omega(n))_\lambda$ with $\nu - \lambda = \ell \in \mathbb{N}$. Since $\alpha = \lambda - \frac{n-1}{2}$ is real, $C_\ell^\alpha(v)$ is real-valued for $-1 < v < 1$, hence we have from Proposition 3.7

$$\left(h, \widehat{D}_{\lambda \to \nu} F\right)_{L^2(\Omega(n-1))_\nu}$$

$$= i^\ell \int_{\Omega(n-1)} h(y') \left(\int_{-1}^1 Q_{1,n-2}(y')^{\frac{\ell+1}{2}} \overline{F \circ \iota(y', v)} C_\ell^\alpha(v) dv\right) m_\nu'(y') dy'$$

$$= i^\ell \int_{\Omega(n)} h(y') \overline{F(y)} (I_\ell C_\ell^\alpha)(Q_{1,n-2}(y'), -y_n) m_\nu'(y') dy$$

$$= i^\ell \int_{\Omega(n)} (\Phi_\lambda^\nu h)(y) \overline{F(y)} m_\lambda(y) dy$$

$$= i^\ell (\Phi_\lambda^\nu h, F)_{L^2(\Omega(n))_\lambda}.$$

Here we have used (3.20) and (3.22) in the third identity. Hence the first equality is shown. By Lemma 2.24, the second equality follows from the first one. $\square$

3.6. **Explicit integral formula for the holographic operator.** In this section, we give an integral formula for the holographic operator $D_{\lambda \to \nu}^*$ in the holomorphic model, giving thus an answer to Problem A.1 in Section 1, see Theorem 3.10 below.

3.6.1. *Construction of discrete summands in the holomorphic model.* In contrast to the holographic operators $\Psi_{\lambda',\lambda''}^{\lambda'''}$ (Definition 2.1) in the tensor product case in Section 2, we do not have a simple integral expression for a holographic operator like (2.6) in the present setting. Instead, we adopt an alternative approach to construct a holographic operator by introducing a *relative reproducing kernel* $K_{\lambda,\nu}(\zeta, \tau')$ as below.

For $\lambda, \nu \in \mathbb{C}$ with $\ell := \nu - \lambda \in \mathbb{N}$, we set

$$(3.25) \qquad K_{\lambda,\nu}(\zeta, \tau') := \frac{\left((\zeta_1 - \overline{\tau}_1)^2 - (\zeta_2 - \overline{\tau}_2)^2 - \cdots - (\zeta_{n-1} - \overline{\tau}_{n-1})^2 - \zeta_n^2\right)^{-\nu}}{\zeta_n^{\lambda-\nu}},$$

where $\zeta = (\zeta_1, \cdots, \zeta_n) \in T_{\Omega(n)}$ and $\tau' = (\tau_1, \cdots, \tau_{n-1}) \in T_{\Omega(n-1)}$.

Remark 3.9. $K_{\lambda,\nu}(\zeta, \tau')$ may be viewed as the holomorphic counterpart of the distribution kernel of a conformally covariant *integral* symmetry breaking operator for the pair of Riemannian manifolds (S^n, S^{n-1}), see [20]. See also (1.1) for the case $n = 2$.

Let $d\mu_\nu$ be a measure on $T_{\Omega(n-1)}$ given by

$$d\mu_\nu(\tau') := \left(\frac{i}{2}\right)^{n-1} Q_{1,n-2}(\text{Im } \tau')^{\nu-n+1} d\tau' d\overline{\tau}'.$$

Theorem 3.10 (holographic operator). *Let $n \geq 3$. Suppose $\lambda > n - 1$ and $\nu = \lambda + \ell$ with $\ell \in \mathbb{N}$. Then the integral*

$$\int_{T_{\Omega(n-1)}} K_{\lambda,\nu}(\zeta, \tau') g(\tau') d\mu_\nu(\tau')$$

converges for all $g \in \mathcal{H}^2(T_{\Omega(n-1)})_\nu$ and $\zeta \in T_{\Omega(n)}$. Moreover, it gives the adjoint operator $D^*_{\lambda \to \nu}$ up to scalar multiplication:

$$\left(D^*_{\lambda \to \nu} g\right)(\zeta) = C \int_{T_{\Omega(n-1)}} K_{\lambda,\nu}(\zeta, \tau') g(\tau') d\mu_\nu(\tau'),$$

where the constant C is given by

$$(3.26) \qquad C = \frac{2^{2\lambda-2n+\ell-1}(\lambda - n + 1)_{n+\ell-1}(2\lambda - n)_{\ell+1}}{i^{2\lambda+2\ell}\pi^n \ell!}.$$

In particular, it yields an injective continuous $\widetilde{G}'$-intertwining operator between weighted Bergman spaces, $\mathcal{H}^2(T_{\Omega(n-1)})_\nu \hookrightarrow \mathcal{H}^2(T_{\Omega(n)})_\lambda$.

We first show that Theorem 3.10 is derived from the following Bernstein–Sato type identity for the holomorphic Juhl operator.

Theorem 3.11. *Let $\mathcal{D}_\ell^\alpha$ be the differential operator as in (3.16). We set*

$$(3.27) \qquad\qquad q(n, \ell; \lambda) := \frac{2^\ell}{\ell!} (2\lambda - n + 1)_\ell (\lambda)_\ell.$$

Then,

$$\zeta_n^{-\ell} \mathcal{D}_\ell^{\lambda - \frac{n-1}{2}} Q_{1,n-1}(\zeta)^{-\lambda} = q(n, \ell; \lambda) \, Q_{1,n-1}(\zeta)^{-\lambda-\ell}.$$

Remark 3.12. Theorem 3.11 shows that the complex power of the quadratic form $Q_{1,n-1}$ satisfies the Bernstein–Sato type identity not only for the power of the Laplacian (see (3.32) below) but also for another operator closely related to the holomorphic Juhl operator.

Postponing the proof of Theorem 3.11, we complete the proof of Theorem 3.10. For this, we also use the following lemma.

Lemma 3.13. *Let D_j $(j = 1, 2)$ be complex manifolds, and $\mathcal{H}_j$ Hilbert spaces contained in $\mathcal{O}(D_j)$ with reproducing kernels $K^{(j)}(\cdot, \cdot)$. Suppose that $\mathcal{R} \colon \mathcal{H}_1 \to \mathcal{H}_2$ is a continuous linear map, and $\mathcal{R}^* \colon \mathcal{H}_2 \to \mathcal{H}_1$ is its adjoint operator. Then,*

(1) $\overline{\mathcal{R}K^{(1)}(\cdot, \zeta)(\tau')} = \left(\mathcal{R}^* K^{(2)}(\cdot, \tau')\right)(\zeta)$ *for $\zeta \in D_1, \tau' \in D_2$;*

(2) $\left(\mathcal{R}^* g\right)(\zeta) = \left(g, \mathcal{R}K^{(1)}(\cdot, \zeta)\right)_{\mathcal{H}_2}$ *for $g \in \mathcal{H}_2$ and $\zeta \in D_1$.*

Proof. (1) The first assertion results from the reproducing property of $K^{(j)}(\cdot, \cdot)$ applied to the following identity:

$$(\mathcal{R}^* K^{(2)}(\cdot, \tau'), K^{(1)}(\cdot, \zeta))_{\mathcal{H}_1} = (K^{(2)}(\cdot, \tau'), \mathcal{R}K^{(1)}(\cdot, \zeta))_{\mathcal{H}_2}.$$

(2) The statement is immediate from the following:

$$\left(\mathcal{R}^* g\right)(\zeta) = \left(\mathcal{R}^* g, K^{(1)}(\cdot, \zeta)\right)_{\mathcal{H}_1} = (g, \mathcal{R}K^{(1)}(\cdot, \zeta))_{\mathcal{H}_2}.$$

$\square$

Proof of Theorem 3.10. Applying Lemma 3.13 to the triple

$$(\mathcal{R}, \mathcal{H}_1, \mathcal{H}_2) = (D_{\lambda \to \nu}, \mathcal{H}^2(T_{\Omega(n)})_\lambda, \mathcal{H}^2(T_{\Omega(n-1)})_\nu),$$

we obtain the following integral expression of the adjoint operator $D_{\lambda \to \nu}^*$:

$$(3.28) \qquad \left(D_{\lambda \to \nu}^* g\right)(\zeta) = \int_{T_{\Omega(n-1)}} g(\tau') \overline{D_{\lambda \to \nu} K_\lambda(\cdot, \zeta)(\tau')} d\mu_\nu(\tau').$$

Here, we have viewed the reproducing kernel $K_\lambda(\tau, \zeta) = k_{\lambda,n} Q_{1,n-1}(\tau - \bar{\zeta})^{-\lambda}$ defined in (3.3) as a function of $\tau \in T_{\Omega(n)}$ with parameter $\zeta \in T_{\Omega(n)}$ and applied the holomorphic Juhl operator $D_{\lambda \to \nu}$. Writing τ as $\tau = (\tau', \tau_n)$, we get from Theorem 3.11:

$$
\begin{aligned}
&D_{\lambda \to \nu} K_\lambda(\tau, \zeta) \\
={}& k_{\lambda,n}\, q(n, \ell; \lambda) \operatorname{Rest}_{\tau_n = 0} \circ (\tau_n - \bar{\zeta}_n)^\ell Q_{1,n-1}(\tau - \bar{\zeta})^{-\lambda-\ell} \\
={}& (-1)^\ell k_{\lambda,n} q(n, \ell; \lambda) \overline{K_{\lambda,\nu}(\zeta, \tau')}
\end{aligned}
$$

for $\tau' \in T_{\Omega(n-1)}$, by the definition (3.25) of the relative reproducing kernel $K_{\lambda,\nu}(\zeta, \tau')$. Since $q(n, \ell; \lambda) \in \mathbb{R}$ when $\lambda \in \mathbb{R}$, the integral formula of Theorem 3.10 is shown with the constant $C = (-1)^\ell \overline{k_{\lambda,n}} q(n, \ell; \lambda)$. A short computation shows the formula (3.26). $\square$

The rest of this section is devoted to the proof of Theorem 3.11.

3.6.2. *Proof of Theorem 3.11.*

Lemma 3.14. *Suppose $\ell \in \mathbb{N}$. Then there exist $q_j \equiv q_j(n, \ell; \lambda)$ $(0 \leq 2j \leq \ell)$ such that*

$$\mathcal{D}_\ell^\alpha Q_{1,n-1}(\zeta)^{-\lambda} = \sum_{j=0}^{\left[\frac{\ell}{2}\right]} q_j \zeta_n^{\ell-2j} Q_{1,n-1}(\zeta)^{-\lambda-\ell+j}.$$

Proof. It is easy to see that an analogous statement holds for $\left(\frac{\partial}{\partial \zeta_n}\right)^\ell$ instead of $\mathcal{D}_\ell^\alpha$, namely, there exist $q_j' \equiv q_j'(n, \ell; \lambda)$ $(0 \leq 2j \leq \ell)$ such that

$$(3.29) \qquad \left(\frac{\partial}{\partial \zeta_n}\right)^\ell Q_{1,n-1}(\zeta)^{-\lambda} = \sum_{j=0}^{\left[\frac{\ell}{2}\right]} q_j' \zeta_n^{\ell-2j} Q_{1,n-1}(\zeta)^{-\lambda-\ell+j}.$$

We rewrite $\mathcal{D}_\ell^\alpha$ as a polynomial of $\Delta_{\mathbb{C}^{1,n-1}} = \frac{\partial^2}{\partial \zeta_1^2} - \frac{\partial^2}{\partial \zeta_2^2} - \cdots - \frac{\partial^2}{\partial \zeta_n^2}$ and $\frac{\partial}{\partial \zeta_n}$ by substituting $\Delta_{\mathbb{C}^{1,n-2}} = \Delta_{\mathbb{C}^{1,n-1}} + \frac{\partial^2}{\partial \zeta_n^2}$ into (3.5):

$$(3.30) \qquad \mathcal{D}_\ell^\alpha = \sum_{k=0}^{\left[\frac{\ell}{2}\right]} p_k(n, \ell; \alpha) \left(\frac{\partial}{\partial \zeta_n}\right)^{\ell-2k} \left(\Delta_{\mathbb{C}^{1,n-1}}\right)^k,$$

where the first coefficient is given by

$$(3.31) \qquad p_0(n, \ell; \alpha) = \sum_{k=0}^{\left[\frac{\ell}{2}\right]} a_k(\ell, \alpha) = C_\ell^\alpha(1) = \frac{(2\alpha)_\ell}{\ell!}.$$

An iterated use of the formula

$$\Delta_{\mathbb{C}^{1,n-1}} Q_{1,n-1}(\zeta)^{-\lambda} = 2\lambda(2\lambda - n + 2)Q_{1,n-1}(\zeta)^{-\lambda-1},$$

leads us to

$$(3.32) \qquad \left(\Delta_{\mathbb{C}^{1,n-1}}\right)^k Q_{1,n-1}(\zeta)^{-\lambda} = s_k(n, \lambda)Q_{1,n-1}(\zeta)^{-\lambda-k},$$

for some polynomials $s_k(n, \lambda)$ of λ of degree $2k$. We note that $s_0(n, \lambda) = 1$. Now the lemma follows from (3.29). $\qquad\square$

Clearly the coefficients $q_j = q_j(n, \ell; \lambda)$ in Lemma 3.14 are unique. The proof of Theorem 3.11 is reduced to the following proposition on these coefficients $q_j(n, \ell; \lambda)$.

Proposition 3.15.

    (1) (the first term). *Recall that $q(n, \ell; \lambda)$ is defined in (3.27). Then,*

$$q_0(n, \ell; \lambda) = q(n, \ell; \lambda).$$

(2) (vanishing of higher terms).

$$q_j\,(n,\ell;\lambda) = 0 \quad \text{for all } j \geq 1.$$

Proof of Proposition 3.15 (1). In the expression

$$\left(\frac{\partial}{\partial\zeta_n}\right)^{\ell-2k} (\Delta_{\mathbb{C}^{1,n-1}})^k\, Q_{1,n-1}(\zeta)^{-\lambda} = s_k(n,\lambda) \left(\frac{\partial}{\partial\zeta_n}\right)^{\ell-2k} Q_{1,n-1}(\zeta)^{-\lambda-k},$$

the term $\zeta_n^\ell\, Q_{1,n-1}(\zeta)^{-\lambda-\ell}$ occurs only when $k = 0$, and its coefficient is given by $s_0(n,\lambda)2^\ell(\lambda)_\ell = 2^\ell(\lambda)_\ell$. By (3.30), we get

$$q_0(n,\ell;\lambda) = p_0(n,\ell;\alpha) \cdot 2^\ell(\lambda)_\ell.$$

Now the first assertion of Proposition 3.15 follows from (3.31).   □

In order to prove the second assertion of Proposition 3.15, we discuss the kernel of the holomorphic Juhl operator $D_{\lambda\to\nu}\colon \mathcal{O}(T_{\Omega(n)}) \longrightarrow \mathcal{O}(T_{\Omega(n-1)})$.

Proposition 3.16. *Suppose $\lambda - \frac{n-1}{2} \notin \{0, -1, -2, \cdots\}$. Then for any $N \in \mathbb{N}$ we have*

$$\bigcap_{j=0}^{N} \mathrm{Ker}\,(D_{\lambda\to\lambda+j})$$

$$= \left\{ f \in \mathcal{O}(T_{\Omega(n)})\colon \mathrm{Rest}_{\zeta_n=0} \circ \left(\frac{\partial}{\partial\zeta_n}\right)^j f = 0 \text{ for all } 0 \leq j \leq N \right\}.$$

Proof. By the definition (3.6) of $D_{\lambda\to\nu}$, the right-hand side is clearly contained in the left-hand side. To see the opposite inclusion, we recall from the definition (3.6) that the symmetry breaking operator $D_{\lambda\to\lambda+j}$ is of the form

$$D_{\lambda\to\lambda+j} = \mathrm{Rest}_{\zeta_n=0} \circ \left(a_0 \left(\frac{\partial}{\partial\zeta_n}\right)^j + \sum_{k=1}^{\left[\frac{j}{2}\right]} a_k \left(\frac{\partial}{\partial\zeta_n}\right)^{j-2k} \Delta_{\mathbb{C}^{1,n-2}}^k \right),$$

where the first coefficient $a_0 \equiv a_0\,(j,\alpha)$ is given by

$$a_0\,(j,\alpha) = \frac{2^j}{j!}(\alpha)_\ell \qquad \text{with} \quad \alpha = \lambda - \frac{n-1}{2}.$$

We now prove the proposition by induction on N. The statement is clear for $N = 0$ and 1 because $D_{\lambda\to\lambda} = \mathrm{Rest}_{\zeta_n=0}$ and $D_{\lambda\to\lambda+1} = \mathrm{Rest}_{\zeta_n=0} \circ \frac{\partial}{\partial\zeta_n}$.

Suppose that $f \in \mathcal{O}(T_{\Omega(n)})$ satisfies $D_{\lambda\to\lambda+j}f = 0$ for $0 \leq j \leq N+1$. By the inductive assumption, we get $\mathrm{Rest}_{\zeta_n=0} \circ \left(\frac{\partial}{\partial\zeta_n}\right)^j f = 0$ $(0 \leq j \leq N)$. Since

$a_0 \equiv a_0(j, \alpha)$ is nonzero for any j by the assumption on λ, $D_{\lambda \to \lambda + N + 1} f = 0$ implies $\mathrm{Rest}_{\zeta_n = 0} \circ \left(\frac{\partial}{\partial \zeta_n} \right)^{N+1} f = 0$. Thus the proposition is proved by induction. $\qquad \square$

Proof of Proposition 3.15 (2). Let $\nu = \lambda + \ell$. By (3.28) and Lemma 3.14, we obtain

$$(3.33) \qquad \left(D_{\lambda \to \nu}^* g \right) (\zeta)$$
$$= (-1)^\ell \overline{k_{\lambda, n}} \sum_{j=0}^{\left[\frac{\ell}{2} \right]} q_j \zeta_n^{\ell - 2j} \int_{T_{\Omega(n-1)}} g(\tau') Q_{1, n-1} (\zeta - \overline{\tau})^{-\lambda - \ell + j} d\mu_\nu(\tau'),$$

where we write $\tau = (\tau', 0) = (\tau_1', \cdots, \tau_{n-1}', 0)$ by abuse of notations.

On the other hand, the composition map

$$D_{\lambda \to \lambda + j} \circ D_{\lambda \to \lambda + \ell}^* : \mathcal{H}^2(T_{\Omega(n-1)})_{\lambda + \ell} \longrightarrow \mathcal{H}^2(T_{\Omega(n-1)})_{\lambda + j}$$

is a G'-intertwining operator for any j. Since the G'-modules $\mathcal{H}^2(T_{\Omega(n-1)})_{\lambda + j}$ $(j \in \mathbb{N})$ are irreducible and mutually inequivalent if $\lambda > n - 1$, such an intertwining operator must be zero unless $j = \ell$. Therefore

$$\mathrm{Image} \left(D_{\lambda \to \lambda + \ell}^* \right) \subset \bigcap_{j=0}^{\ell - 1} \mathrm{Ker} \left(D_{\lambda \to \lambda + j} \right).$$

We now prove that $q_j \equiv q_j(n, \ell; \lambda)$ vanishes for all $1 \le j \le \left[\frac{\ell}{2} \right]$ by downward induction on j. For simplicity, we treat the case where ℓ is even, say $\ell = 2m$. The case where ℓ is odd can be dealt with similarly.

By Proposition 3.16, we have

$$\mathrm{Rest}_{\zeta_n = 0} \circ D_{\lambda \to \nu}^* g = 0 \quad \text{for all} \quad g \in \mathcal{H}^2(T_{\Omega(n-1)})_\nu.$$

Then it follows from (3.33) that

$$(-1)^\ell \overline{k_{\lambda, n}} q_m \int_{T_{\Omega(n-1)}} g(\tau') Q_{1, n-2} (\zeta' - \overline{\tau})^{-\lambda - m} d\mu_\nu(\tau') = 0.$$

Thus we conclude that $q_m = 0$ because $k_{\lambda, n} \ne 0$.

Suppose that we have shown $q_j = 0$ for $j = m, m - 1, \cdots, m + 1 - s$ for some $s \ge 1$. If $s \le m - 1$, then $2s \le \ell - 1 (= 2m - 1)$ and we can proceed by applying Proposition 3.16 with $N = \ell - 1$, hence

$$\mathrm{Rest}_{\zeta_n = 0} \circ \left(\frac{\partial}{\partial \zeta_n} \right)^{2s} \circ D_{\lambda \to \nu}^* g = 0.$$

By the inductive assumption, we obtain

$$(-1)^\ell \overline{k_{\lambda, n}} q_{m-s} \int_{T_{\Omega(n-1)}} g(\tau') Q_{1, n-2} (\zeta' - \overline{\tau})^{-\lambda - m - s} d\mu_\nu(\tau') = 0$$

for all $g \in \mathcal{H}^2(T_{\Omega(n-1)})_\nu$. Thus we conclude that $q_{m-s} = 0$ as far as $s \leq m-1$. Hence we have shown $q_j = 0$ for all $j \geq 1$. $\square$

Thus the proof of Theorem 3.11 (hence, also the one of Theorem 3.10) is completed.

4. Perspectives of symmetry breaking and holographic transforms

We end this article with discussion on a representation-theoretic background of Problems A and B in a broader framework.

In Section 4.1, we consider these problems from the viewpoint of the branching laws of unitary representations of locally compact groups. In Section 4.2, we investigate Problems A and B for triples (G, G', π) such that

- (G, G') is a reductive symmetric pair of holomorphic type;
- π is a unitary highest weight module of G of scalar type,

generalizing the settings for the main results in Sections 2 and 3. The role of special orthogonal polynomials in these cases is clarified in Section 4.3.

4.1. Branching laws, symmetry breaking transform and holographic transform.
Let $G \supset G'$ be a pair of groups, π an irreducible G-module, and ρ an irreducible G'-module. We recall from Section 1 that an element in $\mathrm{Hom}_{G'}(\pi|_{G'}, \rho)$ (resp. in $\mathrm{Hom}_{G'}(\rho, \pi|_{G'})$) is said to be a symmetry breaking operator (resp. a holographic operator). We also recall that a symmetry breaking transform (resp. a holographic transform) is a collection of symmetry breaking operators (resp. holographic operators) where (ρ, W) runs over a certain set Λ of irreducible representations of the subgroup G'.

If π is a unitary representation of a locally compact group G on a Hilbert space V, then Mautner's theorem guarantees that the restriction $(\pi|_{G'}, V)$ is unitarily equivalent to the direct integral of irreducible unitary representations of the subgroup G':

$$(4.1) \qquad \pi|_{G'} \simeq \int_{\widehat{G'}}^{\oplus} m_\pi(\rho) \rho \, d\mu(\rho),$$

where $\widehat{G'}$ is the set of equivalence classes of irreducible unitary representations of G' (*unitary dual*), μ is a Borel measure on $\widehat{G'}$ endowed with the Fell topology, and $m_\pi \colon \widehat{G'} \longrightarrow \mathbb{N} \cup \{\infty\}$ is a measurable function (*multiplicity*). The irreducible decomposition (4.1) is called branching law of the restriction $\pi|_{G'}$, which is unique up to isomorphism if G' is a type I group, in particular, if G' is a real reductive group by a theorem of Harish-Chandra [8].

The (abstract) branching law (4.1) would be enriched through Problems A and B by geometric realizations of irreducible representations and explicit intertwining

operators:

from (LHS) to (RHS)   *symmetry breaking transform*;

from (RHS) to (LHS)   *holographic transform.*

In the unitary case, it is natural to take Λ to be the support of the measure μ in (4.1). If Λ is a countable set, then the branching law (4.1) is *discretely decomposable* without continuous spectrum. A criterion for the triple (G, G', π) to admit a discretely decomposable restriction $\pi|_{G'}$ was studied in [11, 12] when $G \supset G'$ is a pair of real reductive groups.

On the other hand, the multiplicity $m_\pi(\rho)$ in (4.1) is not always finite when π and ρ are infinite-dimensional. A geometric criterion for the pair (G, G') to assure that $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \rho^\infty)$ is finite-dimensional for all smooth irreducible G-modules π^∞ and G'-modules ρ^∞ was established in [17].

If the branching law (4.1) is discretely decomposable and multiplicity free, then we could expect a simple and detailed study of symmetry breaking transform and holographic transform. In this case, since the vector space $\mathrm{Hom}_{G'}(\pi|_{G'}, \rho)$ is one-dimensional, symmetry breaking operator is unique up to scaling for every ρ, and the symmetry breaking transform is defined as the collection of countably many such operators.

4.2. Symmetric pairs of holomorphic type.

In this section, we provide a geometric condition (see Setting 4.1 below) that assures the branching law $\pi|_{G'}$ to be discretely decomposable and multiplicity free. In this case, we see that every symmetry breaking operator is a differential operator (*e.g.* the Rankin–Cohen transforms studied in Section 2 and the holomorphic Juhl transforms in Section 3), and that our symmetry breaking transform D is injective, hence giving an affirmative answer to Problem A.0 in Section 1. The main results in Sections 2 and 3 are built on special cases of this general setting.

Let us fix some notations. Let G be a connected reductive Lie group, θ a Cartan involution, $K = \{g \in G : \theta g = g\}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition, and $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} + \mathfrak{p}_\mathbb{C}$ its complexification. Assume that there exists a central element Z of $\mathfrak{k}_\mathbb{C}$ such that

$$\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} + \mathfrak{n}_+ + \mathfrak{n}_-$$

is the eigenspace decomposition of $\mathrm{ad}(Z)$ with eigenvalues $0, 1$, and -1, respectively. This assumption is satisfied if and only if G is locally isomorphic to a direct product of compact Lie groups (with $Z = 0$) and noncompact Lie groups of Hermitian type. Then the associated Riemannian symmetric space $X = G/K$ becomes a Hermitian symmetric space with complex structure induced from the Borel embedding $G/K \subset$

$G_{\mathbb{C}}/K_{\mathbb{C}} \exp(\mathfrak{n}_+)$. Take a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$, and write $\rho(\mathfrak{n}_+)$ for half the sum of roots in $\Delta(\mathfrak{n}_+, \mathfrak{t}_{\mathbb{C}})$.

Setting 4.1. *Let (G, G') be a reductive symmetric pair of holomorphic type, that is, $X = G/K$ and $Y = G'/K'$ are both Hermitian symmetric spaces and the natural embedding $\iota\colon Y \hookrightarrow X$ is holomorphic. Let $\mathcal{L} = G \times_K \mathbb{C}_\lambda$ be a G-equivariant holomorphic line bundle over X associated to a unitary character $\mathbb{C}_\lambda$ of K, and we set $\mathcal{H}^2(X, \mathcal{L}) := (\mathcal{O} \cap L^2)(X, \mathcal{L})$. Assume λ satisfies the following condition:*

$$(4.2) \qquad \begin{cases} \langle \lambda, \alpha \rangle = 0 & \forall \alpha \in \Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}), \\ \langle \lambda - \rho(\mathfrak{n}_+), \alpha \rangle > 0 & \forall \alpha \in \Delta(\mathfrak{n}_+, \mathfrak{t}_{\mathbb{C}}). \end{cases}$$

The Hilbert space $\mathcal{H}^2(X, \mathcal{L})$ is naturally identified with a weighted Bergman space, which is nonzero if λ satisfies the condition (4.2). We denote by π the representation of G on the Hilbert space $\mathcal{H}^2(X, \mathcal{L})$, which is irreducible and unitary, and is called a *holomorphic discrete series representation* of G. The list of irreducible symmetric pairs (G, G') of holomorphic type may be found in [13, Table 3.4.1].

Fact 4.2 (see [13, Thm. B]). *In Setting 4.1, the restriction $\pi|_{G'}$ is discretely decomposable and multiplicity free.*

Any irreducible G'-module that occurs in the branching law (4.1) for the unitary representation $(\pi, \mathcal{H}^2(X, \mathcal{L}))$ is of the form $\mathcal{H}^2(Y, \mathcal{W})$ for some G'-equivariant holomorphic vector bundle $\mathcal{W}$ over Y associated to an irreducible finite-dimensional unitary representation W of K', and such bundles $\mathcal{W}$ are classified. Thus Λ is parametrized by a subset of $\widehat{K'}$, or by a subset of dominant integral weights which can be described in terms of the root data (see [13, Thm. 8.3]). We write ρ_ℓ for the irreducible unitary representation of G' corresponding to $\ell \in \Lambda$, and identify Λ as a subset of $\widehat{G'}$ by $\ell \mapsto \rho_\ell$. Here is a summary on general results about symmetry breaking operators in this setting:

Fact 4.3. *In Setting 4.1, let $\mathcal{W}$ be the G'-equivariant holomorphic vector bundle corresponding to $\ell \in \Lambda$.*

(1) *Any continuous G'-homomorphism $\mathcal{O}(X, \mathcal{L}) \longrightarrow \mathcal{O}(Y, \mathcal{W})$ is given as a holomorphic differential operator, and induces a continuous G'-homomorphism between the Hilbert spaces $\mathcal{H}^2(X, \mathcal{L}) \longrightarrow \mathcal{H}^2(Y, \mathcal{W})$.*

(2) *Any continuous G'-homomorphism $\mathcal{H}^2(X, \mathcal{L}) \longrightarrow \mathcal{H}^2(Y, \mathcal{W})$ extends to a continuous G'-homomorphism $\mathcal{O}(X, \mathcal{L}) \longrightarrow \mathcal{O}(Y, \mathcal{W})$.*

Proof. (1) The first statement is proved in [18, Thm. 5.3] (*localness theorem*), and the second one is in [18, Thm. 5.13].

(2) By (1) there is a natural injective map

$$(4.3) \qquad \mathrm{Hom}_{G'}(\mathcal{O}(X, \mathcal{L})\big|_{G'}, \mathcal{O}(Y, \mathcal{W})) \hookrightarrow \mathrm{Hom}_{G'}(\mathcal{H}^2(X, \mathcal{L})\big|_{G'}, \mathcal{H}^2(Y, \mathcal{W})).$$

To prove that (4.3) is surjective, we observe that the left-hand side of (4.3) is understood by the branching law for the generalized Verma module [14, Thm. 5.2] via the duality theorem [18, Thm. A], whereas the right-hand side of (4.3) is given by the branching law of the unitary representation $\mathcal{H}^2(X, \mathcal{L})|_{G'}$ ([13, Thm. 8.3]), and that they coincide under the condition (4.2). Hence (4.3) is bijective. $\qquad\square$

In order to clarify the dependence of the parameter ℓ, we write $\mathcal{W}_\ell$ for the G'-equivariant vector bundle corresponding to $\ell \in \Lambda$ from now. Then Fact 4.3 tells that the one-dimensional vector space

$$\mathrm{Hom}_{G'}\left(\mathcal{O}(X, \mathcal{L})\big|_{G'}, \mathcal{O}(Y, \mathcal{W}_\ell)\right) \simeq \mathrm{Hom}_{G'}\left(\mathcal{H}^2(X, \mathcal{L})\big|_{G'}, \mathcal{H}^2(Y, \mathcal{W}_\ell)\right)$$

is spanned by a *differential* symmetry breaking operator. We fix such a generator D_ℓ for every $\ell \in \Lambda$.

Since $D_\ell \colon \mathcal{H}^2(X, \mathcal{L}) \to \mathcal{H}^2(Y, \mathcal{W}_\ell)$ is a continuous operator between the Hilbert spaces, its operator norm $\|D_\ell\|_{\mathrm{op}}$ is finite and its adjoint D_ℓ^* is a continuous linear operator. Set

$$C_\ell := \|D_\ell\|_{\mathrm{op}}^2, \quad \Psi_\ell := \frac{1}{C_\ell} D_\ell^*.$$

Let $D = (D_\ell)_{\ell \in \Lambda}$ be the symmetry breaking transform. Then we have the following:

Theorem 4.4. *Suppose we are in Setting 4.1.*
   (1) $\Psi_\ell \colon \mathcal{H}^2(Y, \mathcal{W}_\ell) \longrightarrow \mathcal{H}^2(X, \mathcal{L})$ *is a holographic operator. Moreover, it is an isometry up to renormalization.*
   (2) *The symmetry breaking transform D is injective on $\mathcal{H}^2(X, \mathcal{L})$.*
   (3) *Any $f \in \mathcal{H}^2(X, \mathcal{L})$ is recovered from its symmetry breaking transform Df by*

$$f = \sum_{\ell \in \Lambda} \Psi_\ell \left(Df\right)_\ell.$$

   (4) *The norm $\|f\|_{\mathcal{H}^2(X, \mathcal{L})}$ is recovered from the sequence of norms $\|\left(Df\right)_\ell\|_{\mathcal{H}^2(Y, \mathcal{W}_\ell)}$ by*

$$\|f\|_{\mathcal{H}^2(X, \mathcal{L})}^2 = \sum_{\ell \in \Lambda} \frac{1}{C_\ell} \|\left(Df\right)_\ell\|_{\mathcal{H}^2(Y, \mathcal{W}_\ell)}^2.$$

Proof. The unitary representation of G on the Hilbert space $\mathcal{H}^2(X, \mathcal{L})$ is decomposed discretely and multiplicity freely into the Hilbert direct sum:

$$(4.4) \qquad\qquad \mathcal{H}^2(X, \mathcal{L})|_{G'} \simeq \sum_{\ell \in \Lambda}{}^{\oplus} \mathcal{H}^2(Y, \mathcal{W}_\ell)$$

as unitary representations of the subgroup G' by Fact 4.3.
(1) The adjoint operator D_ℓ^* is a G'-homomorphism because $\mathcal{H}^2(X, \mathcal{L})$ and both $\mathcal{H}^2(Y, \mathcal{W}_\ell)$ are unitary representations. The second assertion follows from Schur's lemma because $\mathrm{Hom}_{G'}(\mathcal{H}^2(Y, \mathcal{W}_\ell), \mathcal{H}^2(X, \mathcal{L})\big|_{G'})$ is one-dimensional.

(2) Expand $f \in \mathcal{H}^2(X, \mathcal{L})$ as $f = \sum_{\ell \in \Lambda} f_\ell$ according to the decomposition (4.4). Then $(Df)_\ell$ is a nonzero multiple of f_ℓ by Schur's lemma since the decomposition (4.4) is multiplicity free. Hence, if $Df = 0$, then $f_\ell = 0$ for all $\ell \in \Lambda$, and therefore $f = 0$.

Statements (3) and (4) are direct consequences of Lemma 2.23. $\square$

4.3. **Role of orthogonal polynomials.** In this section we investigate Problems A and B in Setting 4.1, and clarify the role of the F-method and special orthogonal polynomials for the L^2-theory of symmetry breaking transforms consisting of holomorphic differential operators.

Suppose we are in Setting 4.1. As we have seen in Section 4.2, Theorem 4.4 (2) solves Problem A.0, whereas Theorem 4.4 (1), (3), and (4) give a framework for Problems A.1, A.2, and B, respectively, for the symmetry breaking transform $D = (D_\ell)_{\ell \in \Lambda}$. Thus the solution to Problems A and B is reduced to the following four questions of finding explicit description and closed formulæ of

- the support of Λ;
- holomorphic differential operators D_ℓ;
- the operator norm $\|D_\ell\|_{\mathrm{op}}$;
- the adjoint operator D_ℓ^*.

Let us summarize briefly what was known, what has been proved in this article, and what looks promising.

As we mentioned in Section 4.2, the explicit description of Λ, equivalently, the branching law (4.4) for the restriction $\pi|_{G'}$ in Section 4.1 was proved in [13, Thm. 8.3], which gives a generalization of the Hua–Kostant–Schmid formula in the case when $G' = K$. Denote by $\mathrm{rank}_\mathbb{R} G/G'$ the split rank of the reductive symmetric space G/G'. Then it turns out that Λ is a free abelian semigroup generated by $\mathrm{rank}_\mathbb{R} G/G'$ elements, see [13].

It is more involved to construct symmetry breaking operators D_ℓ explicitly than determining Λ, namely, the branching law of the restriction $\pi|_{G'}$. As of now, an explicit construction of D_ℓ for all $\ell \in \Lambda$ with exhaustion theorem is known when $\mathrm{rank}_\mathbb{R} G/G' = 1$, see [19]. There are six families of such symmetric pairs (G, G'), and the resulting symmetry breaking operators include the Rankin–Cohen operators and the holomorphic Juhl operators.

In order to obtain the operator norm $\|D_\ell\|_{\mathrm{op}}$ of such holomorphic differential operators D_ℓ, we have developed the idea of the F-method to connect $\|D_\ell\|_{\mathrm{op}}$ with the L^2-norm of special polynomials P_ℓ in the following two cases in this article.

D_ℓ	P_ℓ
Rankin–Cohen operators	Jacobi polynomials
Juhl operators	Gegenbauer polynomials

The relationship between D_ℓ and P_ℓ follows from the fact that the G'-equivariance condition on the operator D_ℓ is transformed into a certain differential equation (*e.g.* Jacobi differential equation (5.1)) for the polynomial P_ℓ. It is plausible that this idea would work in the full generality of Setting 4.1.

Concerning the adjoint operator D_ℓ^*, this article has provided two kinds of integral formulæ, that is, by the line integral (Definition 2.1), see Proposition 2.22, and by the integral over the tube domain (Theorem 3.10). The former has an advantage that the formula is simple and does not require the unitarity of representations, whilst the latter uses a natural idea of the "relative reproducing kernel" $K_{\lambda,\nu}(\zeta,\tau')$, see (3.25).

5. Appendix: Jacobi polynomials and Gegenbauer polynomials

5.1. **The Jacobi polynomials.** Suppose $\alpha, \beta \in \mathbb{C}$ and $\ell \in \mathbb{N}$. The Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ is a polynomial solution to the Jacobi differential equation

$$(5.1) \qquad \left((1-t^2)\frac{d^2}{dt^2} + (\beta - \alpha - (\alpha + \beta + 2)t)\frac{d}{dt} + \ell(\ell + \alpha + \beta + 1) \right) y = 0,$$

which is normalized by $P_\ell^{\alpha,\beta}(1) = \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+1)\ell!} = \frac{(\alpha+1)_\ell}{\ell!}$. Then it satisfies the Rodrigues formula

$$(5.2) \qquad (1-t)^\alpha(1+t)^\beta P_\ell^{\alpha,\beta}(t) = \frac{(-1)^\ell}{2^\ell \ell!} \left(\frac{d}{dt} \right)^\ell \left((1-t)^{\ell+\alpha}(1+t)^{\ell+\beta} \right).$$

The Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ is nonzero and is a polynomial of degree ℓ for generic parameters (see [19, Thm. 11.2] for the precise condition). Explicitly, one has

$$
\begin{aligned}
P_\ell^{\alpha,\beta}(t) &= \frac{(\alpha+1)_\ell}{\ell!}\, {}_2F_1\left(-\ell, \alpha+\beta+\ell+1; \alpha+1; \frac{1-t}{2} \right) \\
(5.3) \qquad &= \sum_{j=0}^\ell \frac{(\alpha+\beta+\ell+1)_j(\alpha+j+1)_{\ell-j}}{(\ell-j)!j!} \left(\frac{t-1}{2} \right)^j.
\end{aligned}
$$

The first Jacobi polynomials are
- $P_0^{\alpha,\beta}(t) = 1$,
- $P_1^{\alpha,\beta}(t) = \frac{1}{2}(\alpha - \beta + (2 + \alpha + \beta)t)$.

For real α, β with $\alpha, \beta > -1$, the Jacobi polynomials $\left\{ P_\ell^{\alpha,\beta} \right\}_{\ell \in \mathbb{N}}$ form an orthogonal basis in the Hilbert space $L^2 \left((-1, 1), (1 - x)^\alpha (1 + x)^\beta dx \right)$ with the following norm (see [1, page 301] for example):

$$(5.4) \qquad \int_{-1}^1 \left| P_\ell^{\alpha,\beta}(x) \right|^2 (1 - x)^\alpha (1 + x)^\beta dx = \frac{2^{\alpha+\beta+1} \Gamma(\ell + \alpha + 1) \Gamma(\ell + \beta + 1)}{(2\ell + \alpha + \beta + 1) \Gamma(\ell + \alpha + \beta + 1) \ell!}.$$

When $\alpha = \beta$ these polynomials yield Gegenbauer polynomials (see (5.6) below), and they further reduce to Legendre polynomials in the case when $\alpha = \beta = 0$.

5.2. **The Gegenbauer polynomials.** For $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$, the Gegenbauer polynomial (or ultraspherical polynomial) $C_\ell^\alpha(t)$ is defined by

$$(5.5) \qquad \qquad C_\ell^\alpha(t) = \sum_{k=0}^{\left[\frac{\ell}{2} \right]} a_k(\ell, \alpha) t^{\ell - 2k},$$

where $a_k(\ell, \alpha)$ is given in (3.4). The Gegenbauer polynomials are special cases of the Jacobi polynomials by

$$(5.6) \qquad \qquad \frac{(2\alpha + 1)_\ell}{(\alpha + 1)_\ell} P_\ell^{\alpha,\alpha}(x) = C_\ell^{\alpha + \frac{1}{2}}(x),$$

and have the generating function:

$$(1 - 2tr + r^2)^{-\alpha} = \sum_{\ell \in \mathbb{N}} C_\ell^\alpha(t) r^\ell.$$

We note that $C_\ell^\alpha(1) = \frac{(2\alpha)_\ell}{\ell!}$. If $\alpha > -\frac{1}{2}$, then the Gegenbauer polynomials $\{ C_\ell^\alpha(v) \}_{\ell \in \mathbb{N}}$ form an orthonormal basis in the Hilbert space $L^2 \left((-1, 1), (1 - v^2)^{\alpha - \frac{1}{2}} dv \right)$ with the following L^2-norm (see [7, 7.313]):

$$(5.7) \qquad \qquad \int_{-1}^1 |C_\ell^\alpha(v)|^2 (1 - v^2)^{\alpha - \frac{1}{2}} dv = \frac{\pi 2^{1-2\alpha} \Gamma(\ell + 2\alpha)}{\ell!(\ell + \alpha) \Gamma(\alpha)^2}.$$

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