Global analysis by hidden symmetry

Toshiyuki Kobayashi

Dedicated to Roger Howe on the occasion of his 70th birthday with admire on his original contributions to the fields

Abstract Hidden symmetry of a G'-space X is defined by an extension of the G'-action on X to that of a group G containing G' as a subgroup. In this setting, we study the relationship between the three objects:

(A) global analysis on X by using representations of G (hidden symmetry);
(B) global analysis on X by using representations of G';
(C) branching laws of representations of G when restricted to the subgroup G'.

We explain a trick which transfers results for finite-dimensional representations in the compact setting to those for infinite-dimensional representations in the noncompact setting when $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical. Applications to branching problems of unitary representations, and to spectral analysis on pseudo-Riemannian locally symmetric spaces are also discussed.

Key words: reductive group, branching law, hidden symmetry, spherical variety, locally symmetric space, invariant differential operator

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1 Introduction

In the late 80s, I was working on two "new" different topics of study:

(I) (geometry) actions of discrete groups on *pseudo-Riemannian* homogeneous spaces [15],
(II) (representation theory) restriction of unitary representations to *noncompact* subgroups [17],

and trying to find criteria for the setting that will assure the following "best properties":

- properly discontinuous actions (*v.s.* ergodic actions, *etc.*) for (I),
- discretely decomposable restrictions (*v.s.* continuous spectrum) for (II),

respectively. The techniques in solving these problems were quite different.

Roger Howe visited Kyoto in 1994, and raised a question about how these two topics are connected to each other in my mind. Philosophically, there is some similarity in (I) and (II): proper actions are "compact-like" actions on locally compact topological spaces, whereas we could expect that there are also "compact-like" *linear* actions on Hilbert spaces such as discretely decomposable unitary representations [23], see also Margulis [29]. The aim of this paper is to give another answer to his question rigorously from the viewpoint of "analysis with hidden symmetries" by extending the half-formed idea of [24] which is based on the observation as below: our first example of discretely decomposable restrictions of nonholomorphic discrete series representations [16] arose from the geometry of the three-dimensional open complex manifold $X = \mathbb{P}^{1,2}\mathbb{C}$ (see Section 4.2) satisfying the following two properties:

- the de Sitter group $G' = Sp(1,1) \simeq Spin(4,1)$ acts properly on X, and consequently, there exists a three-dimensional compact indefinite Kähler manifold as the quotient of X by a cocompact discrete subgroup of G' [15];
- any irreducible unitary representation of the conformal group $G = U(2,2) (\doteqdot S^1 \times SO(4,2))$ that is realized in the regular representation $L^2(X)$ is discretely decomposable when restricted to the subgroup G' [16, 17].

In this example, we see *hidden symmetry* of X:

$$X \text{ is a homogeneous space of } G'\text{, but also that of an overgroup } G. \tag{1.1}$$

Furthermore, $X = \mathbb{P}^{1,2}\mathbb{C}$ has a quaternionic Hopf fibration

$$F \to X \to Y \tag{1.2}$$

over the four-dimensional ball Y which is the Riemannian symmetric space associated to G' with typical fiber $F \simeq S^2$ (see Section 4.2).

More generally, we shall work with the setting (1.1) of hidden symmetry for a pair of real reductive Lie groups $G \supset G'$. The subject of this article is in three folds:

(A) global analysis on X by using representations of G;
(B) global analysis on X by using representations of G';
(C) branching laws of representations of G when restricted to G'.

A key role is played by a maximal reductive subgroup L' of G' containing H', which induces a G'-equivariant fibration $F \to X \to Y$ (see (2.1)).

In Section 2, we study (B) in two ways:

- (A) + (C),
- analysis of the fiber F and the base space Y,

and deduce a double fibration in Theorem 2.10 for the "discrete part" of the unitary representations of the groups G, G' and L' arising naturally when X is G'-real spherical (Definition 2.7). Assuming a stronger condition that the complexification $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical (Definition-Theorem 3.1), we analyze the ring $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ of $G'_{\mathbb{C}}$-invariant holomorphic differential operators on the complexification $X_{\mathbb{C}}$ by three natural subalgebras $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ defined in (3.10)–(3.12) which commute with $G_{\mathbb{C}}$, $G'_{\mathbb{C}}$, and $L'_{\mathbb{C}}$, respectively (see Theorems 3.5 and 3.6). Turning to real forms $F \to X \to Y$, the relationship between (A), (B), and (C) is formulated utilizing the algebras $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$.

Moreover, an application of the relationship

$$\text{(A)} \quad \text{and} \quad \text{(B)} \implies \text{(C)}$$

will be discussed in Section 5 that includes a new branching law of Zuckerman's derived functor module $A_{\mathfrak{q}}(\lambda)$ with respect to a nonsymmetric pair (Theorem 5.5). An application of the relationship

$$\text{(A)} \quad \text{and} \quad \text{(C)} \implies \text{(B)}$$

was studied in [17, 22] for the existence problem of discrete series representation for *nonsymmetric* homogeneous spaces. Spectral analysis on non-Riemannian locally symmetric spaces is discussed in Section 6 as an application of the relationship

$$\text{(B)} \quad \text{and} \quad \text{(C)} \implies \text{(A)}.$$

This scheme explains the aforementioned example as a special case of the following general results:

H' is compact $\Rightarrow$ (I) X admits compact Clifford-Klein forms (Proposition 6.2)
$\Downarrow$ Remark 5.2
F is compact $\Rightarrow$ (II) discretely decomposable restrictions (Theorem 5.1)

The results of Sections 3–4 and Section 6 will be developed in the forthcoming papers [12] (for compact real forms) and [13] with F. Kassel, respectively.

2 Analysis on homogeneous spaces with hidden symmetry

In this section, we introduce a general framework which relates branching problems of unitary representations and global analysis on homogeneous spaces with hidden symmetries.

2.1 Homogeneous space with hidden symmetry

Suppose that G is a group, and G' its subgroup. If G acts on a set X, then the subgroup G' acts on X by restriction (*broken symmetry*). Conversely, if we regard X as a G'-space, the G-action on X is said to be a *hidden symmetry*, or G is an *overgroup* of (G', X). In the setting that G' acts transitively on X, the action of the overgroup G is automatically transitive. This happens when $X = G/H$ for some subgroup H of G such that $G = G'H$.

We denote by $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}'_{\mathbb{C}}$, and $\mathfrak{h}_{\mathbb{C}}$ the complexified Lie algebras of Lie groups G, G', and H, respectively.

To find the above setting for reductive groups, the following criterion is useful.

Lemma 2.1 ([17, Lemma 5.1]). *Suppose that G, G', and H are real reductive groups. We set $H' := G' \cap H$ and $X = G/H$. Then the following three conditions on the triple (G, G', H) are equivalent:*

(i) *the natural injection $G'/H' \hookrightarrow X$ is bijective;*
(ii) *G' has an open orbit in X;*
(iii) $\mathfrak{g}'_{\mathbb{C}} + \mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$.

The implication (ii) ⇒ (i) does not hold in general if we drop the assumption that H is reductive.

We observe in the condition (iii) of Lemma 2.1:

- the role of G' and H is symmetric;
- the condition is determined only by the complexification of the Lie algebras.

Hence, one example in the compact case yields a number of examples, as is illustrated by the following.

Example 2.2. The unitary group $U(n)$ acts transitively on the unit sphere S^{2n-1} in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, and thus the inclusion $U(n) \hookrightarrow SO(2n)$ induces the isomorphism:

$$U(n)/U(n-1) \overset{\sim}{\to} SO(2n)/SO(2n-1) \simeq S^{2n-1}.$$

From the implication (i) ⇒ (iii) in Lemma 2.1, we have

$$\mathfrak{gl}(n, \mathbb{C}) + \mathfrak{so}(2n-1, \mathbb{C}) = \mathfrak{so}(2n, \mathbb{C}).$$

In turn, taking other real forms or switching G' and H, we get the following isomorphisms:

$$
\begin{aligned}
SO(2n-1)/U(n-1) &\xrightarrow{\sim} SO(2n)/U(n),\\
GL(n,\mathbb{C})/GL(n-1,\mathbb{C}) &\xrightarrow{\sim} SO(2n,\mathbb{C})/SO(2n-1,\mathbb{C}),\\
SO(2n-1,\mathbb{C})/GL(n-1,\mathbb{C}) &\xrightarrow{\sim} SO(2n,\mathbb{C})/GL(n,\mathbb{C}),\\
U(p,q)/U(p-1,q) &\xrightarrow{\sim} SO(2p,2q)/SO(2p-1,2q),\\
SO(2p-1,2q)/U(p,q) &\xrightarrow{\sim} SO(2p,2q)/U(p,q),\\
GL(n,\mathbb{R})/GL(n-1,\mathbb{R}) &\xrightarrow{\sim} SO(n,n)/SO(n-1,n),\\
SO(n-1,n)/GL(n-1,\mathbb{R}) &\xrightarrow{\sim} SO(n,n)/GL(n,\mathbb{R}).
\end{aligned}
$$

See [17, Sect. 5] for further examples. See also Table 4.3 in Section 4.

We shall use the following notation and setting throughout this article.

Setting 2.3 (reductive hidden symmetry) *Suppose a triple (G, G', H) satisfies one of (therefore any of) the equivalent three conditions in Lemma 2.1. We set $H' := G' \cap H$ and $X := G'/H'$. Then X has a hidden symmetry G and we have a natural diffeomorphism $X \simeq G/H$ which respects the action of G'.*

Similarly, we may consider settings for complex Lie groups or compact Lie groups:

Setting 2.4 (complex reductive hidden symmetry) *Let $G'_{\mathbb{C}} \subset G_{\mathbb{C}} \supset H_{\mathbb{C}}$ be a triple of complex reductive groups. Suppose that their Lie algebras satisfy the condition* (iii) *in Lemma 2.1, or equivalently, $G'_{\mathbb{C}}$ acts transitively on $G_{\mathbb{C}}/H_{\mathbb{C}}$. We set $H'_{\mathbb{C}} := G'_{\mathbb{C}} \cap H_{\mathbb{C}}$ and $X_{\mathbb{C}} := G'_{\mathbb{C}}/H'_{\mathbb{C}} \simeq G_{\mathbb{C}}/H_{\mathbb{C}}$.*

Setting 2.5 (compact hidden symmetry) *Let $G'_U \subset G_U \supset H_U$ be a triple compact Lie groups. Suppose that their complexified Lie algebras satisfy the condition* (iii) *in Lemma 2.1, or equivalently, G'_U acts transitively on G_U/H_U. We set $H'_U := G'_U \cap H_U$ and $X_U := G'_U/H'_U \simeq G_U/H_U$.*

By "global analysis with hidden symmetries" we mean analysis of functions and differential equations on $X \simeq G'/H' \simeq G/H$ by using representation theory of two groups G and G'. To perform it, our key idea is based on the following observation.

Observation 2.6. *In Setting 2.3, H' is not necessarily a maximal reductive subgroup of G' even when H is maximal in G.*

We take a reductive subgroup L' of G' which contains H'. (Later, we shall take L' to be a maximal reductive subgroup.) We set

$$F := L'/H' \quad \text{and} \quad Y := G'/L'.$$

Analogous notations will be applied to Settings 2.4 and 2.5. Then we have the following fiber bundle structure of X (also of $X_{\mathbb{C}}$ and X_U):

$$\begin{array}{ccccc} F & \to X & \to Y \\ \cap & \cap & \cap \\ F_{\mathbb{C}} & \to X_{\mathbb{C}} & \to Y_{\mathbb{C}} \\ \cup & \cup & \cup \\ F_U & \to X_U & \to Y_U. \end{array} \tag{2.1}$$

2.2 Preliminaries from representation theory

Let $\widehat{G}$ be the unitary dual of G, *i.e.*, the set of equivalence classes of irreducible unitary representations of a Lie group G. Any unitary representation Π of G is decomposed into a direct integral of irreducible unitary representations:

$$\Pi \simeq \int_{\widehat{G}}^{\oplus} m(\pi)\pi d\mu(\pi), \tag{2.2}$$

where $d\mu$ is a Borel measure of $\widehat{G}$ endowed with the Fell topology, and the measurable function $m : \widehat{G} \to \mathbb{N} \cup \{\infty\}$ stands for the multiplicity. The decomposition (2.2) is unique up to the equivalence of the measure if G is a real reductive Lie group.

Let $\mathcal{H}$ be the Hilbert space on which the unitary representation Π is realized. The *discrete part* Π_d of Π is a subrepresentation defined on the maximal closed G-invariant subspace $\mathcal{H}_d$ of $\mathcal{H}$ that decomposes discretely into irreducible unitary representations. For an irreducible unitary representation π of G, we define the π-*isotypic component* of the unitary representation Π by

$$\Pi[\pi] := \mathrm{Hom}_G(\pi, \mathcal{H}) \otimes \pi, \tag{2.3}$$

where $\mathrm{Hom}_G(\ ,\ )$ stands for the space of continuous G-homomorphisms. This is a unitary representation realized in a closed subspace of $\mathcal{H}_d$, and is a multiple of π. Then we have a unitary equivalence

$$\Pi_d \simeq \sum_{\pi \in \widehat{G}}^{\oplus} \Pi_d[\pi],$$

where $\sum^{\oplus}$ denotes the Hilbert completion of the algebraic direct sum. Let Π_c be the G-submodule ("continuous part") defined on the orthogonal complementary subspace $\mathcal{H}_c$ of $\mathcal{H}_d$ in $\mathcal{H}$.

Given $\pi \in \widehat{G}$ and a subgroup G' of G, we may think of π as a representation of the subgroup G' by restriction, to be denoted by $\pi|_{G'}$. The irreducible decomposition of the restriction $\pi|_{G'}$ is called the *branching law*. We define a subset of $\widehat{G'}$ by

$$\mathrm{Disc}(\pi|_{G'}) := \{\vartheta \in \widehat{G'} : \mathrm{Hom}_{G'}(\vartheta, \pi|_{G'}) \neq \{0\}\}.$$

Such ϑ contributes to the discrete part $(\pi|_{G'})_d$ of the restriction $\pi|_{G'}$.

For a closed unimodular subgroup H, we endow G/H with a G-invariant Radon measure and consider the unitary representation of G on the Hilbert space $L^2(G/H)$. The irreducible decomposition of $L^2(G/H)$ is called the *Plancherel formula*. We define a subset of $\widehat{G}$ by

$$\mathrm{Disc}(G/H) := \{\pi \in \widehat{G} : \mathrm{Hom}_G(\pi, L^2(G/H)) \neq \{0\}\}.$$

Such π is called a *discrete series representation* for G/H. For $H = \{e\}$, $\mathrm{Disc}(G/H)$ consists of Harish-Chandra's discrete series representations. If H is noncompact, elements of $\mathrm{Disc}(G/H)$ are not necessarily tempered representations of G.

These sets $\mathrm{Disc}(\pi|_{G'})$ and $\mathrm{Disc}(G/H)$ may be empty. We shall denote by $\underline{\mathrm{Disc}}(\pi|_{G'})$ and $\underline{\mathrm{Disc}}(G/H)$ the multisets counted with multiplicities. Then the discrete part of the unitary representations $\pi|_{G'}$ of G' and $L^2(G/H)$ of G are given as

$$(\pi|_{G'})_d \simeq {\sum}^{\oplus}_{\vartheta \in \underline{\mathrm{Disc}}(\pi|_{G'})} \vartheta,$$
$$L^2(G/H)_d \simeq {\sum}^{\oplus}_{\pi \in \underline{\mathrm{Disc}}(G/H)} \pi,$$

respectively.

Given a unitary representation (τ, W) of H, we form a G-equivariant Hilbert vector bundle $\mathcal{W} := G \times_H W$ over G/H. Then we have a natural unitary representation $\mathrm{Ind}_H^G \tau$ on the Hilbert space $L^2(G/H, \mathcal{W})$ of L^2-sections, and define a subset $\mathrm{Disc}(G/H, \tau)$ of $\widehat{G}$ and a multiset $\underline{\mathrm{Disc}}(G/H, \tau)$, similarly. They are reduced to $\mathrm{Disc}(G/H)$ and $\underline{\mathrm{Disc}}(G/H)$, respectively, if (τ, W) is the trivial one-dimensional representation of H.

2.3 Double fibration for $\underline{\mathrm{Disc}}(G'/H')$

Suppose we are in Setting 2.3, namely, we have a bijection

$$X = G'/H' \xrightarrow{\sim} G/H$$

induced by the inclusion $G' \hookrightarrow G$. Then we may compare the three objects (A), (B), and (C) in Introduction. We wish to obtain new information of the one from the other two. A general framework that provides a relationship between (A), (B), and (C) will be formulated by using the notion of real spherical homogeneous spaces, which we recall from [18].

Definition 2.7. A homogeneous space X of a real reductive Lie group G is said to be *real spherical* if X admits an open orbit of a minimal parabolic subgroup of G.

Example 2.8. (1) Any homogeneous space of a compact Lie group G_U is real spherical because a minimal parabolic subgroup of G_U is the whole group G_U.

(2) Any reductive symmetric space is real spherical.
(3) Any real form $X = G/H$ of a $G_{\mathbb{C}}$-spherical homogeneous space $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ (see Definition-Theorem 3.1 in Section 3) is real spherical [26, Lemma 4.2].
(4) Let N be a maximal unipotent subgroup of G. Then G/N is real spherical, as is seen from the Bruhat decomposition.

The notion of "real sphericity" gives a geometric criterion for X on which the function space is under control by representations of G in the following sense. Let $\widehat{G}_{\mathrm{smooth}}$ be the set of equivalence classes of irreducible admissible representations of G of moderate growth [37, Ch. 11. Sect. 5.1].

Fact 2.9 ([26]). *Let X be an algebraic homogeneous space of a real reductive Lie group G. Then the following two conditions on X are equivalent:*

(i) $\mathrm{Hom}_G(\pi, C^\infty(X, \mathcal{W}))$ *is finite-dimensional for any* $\pi \in \widehat{G}_{\mathrm{smooth}}$ *and for any G-equivariant vector bundle $\mathcal{W} \to X$ of finite rank.*
(ii) X *is real spherical.*

The condition (i) in Fact 2.9 remains the same if we replace C^∞ by $\mathcal{D}'$ (distribution) or if we replace $\mathrm{Hom}_G(\pi, C^\infty(X, \mathcal{W}))$ by $\mathrm{Hom}_{\mathfrak{g},K}(\pi_K, C^\infty(X, \mathcal{W}))$, where π_K stands for the underlying $(\mathfrak{g}, K)$-module of π.

Highlighting the "discrete part" of the unitary representations that are involved, we obtain a basic theorem on analysis with hidden symmetries that relates (A), (B), and (C):

Theorem 2.10. *Assume that $X = G'/H'$ is real spherical in Setting 2.3.*

(1) *The multiplicity of any element in* $\underline{\mathrm{Disc}}(G'/H')$ *is finite.*
(2) *Let L' be a subgroup of G' which contains H' (see Observation 2.6). Then, the discrete part of the unitary representation $L^2(G'/H')$ has the following two expressions:*

$$\begin{aligned} & L^2(G'/H')_d \\ \simeq & \sum\nolimits^{\oplus}_{\pi \in \underline{\mathrm{Disc}}(G/H)} (\pi|_{G'})_d && = \sum\nolimits^{\oplus}_{\pi \in \underline{\mathrm{Disc}}(G/H)} \sum\nolimits^{\oplus}_{\vartheta \in \underline{\mathrm{Disc}}(\pi|_{G'})} \vartheta \\ \simeq & \sum\nolimits^{\oplus}_{\tau \in \underline{\mathrm{Disc}}(L'/H')} L^2(G'/H', \tau)_d && = \sum\nolimits^{\oplus}_{\tau \in \underline{\mathrm{Disc}}(L'/H')} \sum\nolimits^{\oplus}_{\vartheta \in \underline{\mathrm{Disc}}(G'/L', \tau)} \vartheta. \end{aligned}$$

(3) *Assume further that* $\underline{\mathrm{Disc}}(G'/H')$ *is multiplicity-free, i.e.* $\dim \mathrm{Hom}_{G'}(\vartheta, L^2(X)) \leq 1$ *for any* $\vartheta \in \widehat{G'}$. *Then, there is a natural double fibration*

$$\begin{array}{ccc} & \mathrm{Disc}(G'/H') & \\ \mathcal{K}_1 \swarrow & & \searrow \mathcal{K}_2 \\ \mathrm{Disc}(G/H) & & \mathrm{Disc}(L'/H') \end{array}$$

such that the fibers are given by

$$\begin{aligned} \mathcal{K}_1^{-1}(\pi) &= \mathrm{Disc}(\pi|_{G'}) && \text{for } \pi \in \mathrm{Disc}(G/H), \\ \mathcal{K}_2^{-1}(\tau) &= \mathrm{Disc}(G'/L', \tau) && \text{for } \tau \in \mathrm{Disc}(L'/H'). \end{aligned}$$

Remark 2.11. (1) In the case where $F = L'/H'$ is compact, the idea of Theorem 2.10 was implicitly used in [16] to find the branching law of some Zuckerman $A_{\mathfrak{q}}(\lambda)$-modules with respect to reductive symmetric pairs (see [34, 36] for the definition of $A_{\mathfrak{q}}(\lambda)$). In the same spirit, we shall give a new example of branching laws of $A_{\mathfrak{q}}(\lambda)$ with respect to *nonsymmetric* pairs in Theorem 5.5.

(2) We shall see in Section 6 that Theorem 2.10 serves also as a new method for spectral analysis on non-Riemannian locally symmetric spaces in the setting where H' is compact and H is noncompact.

(3) In [12], we shall give a proof of Theorems 3.5 and 3.6 below by using Theorem 2.10 in the special setting where G is compact.

As a direct consequence of Theorem 2.10 (2), we obtain the following:

Corollary 2.12. *Suppose G'/H' is real spherical in Setting 2.3 and let L' be a reductive subgroup of G' containing H'. Then the following three subsets of $\widehat{G'}$ are the same:*

$$\mathrm{Disc}(G'/H') = \bigcup_{\pi\in\mathrm{Disc}(G/H)} \mathrm{Disc}(\pi|_{G'}) = \bigcup_{\tau\in\mathrm{Disc}(L'/H')} \mathrm{Disc}(G'/L',\tau).$$

2.4 Proof of Theorem 2.10

The first assertion of Theorem 2.10 follows from the finite-multiplicity theorem for real spherical homogeneous spaces (see Fact 2.9).

In order to prove Theorem 2.10 (2), we begin with the relation between the multisets $\underline{\mathrm{Disc}}(G'/H')$ and $\underline{\mathrm{Disc}}(L'/H')$. Suppose that L' is a reductive subgroup of G' containing H'. Then $F = L'/H'$ carries an L'-invariant Radon measure. We decompose the unitary representation of L' on $L^2(F)$ into the "discrete" and "continuous part":

$$L^2(F) \simeq L^2(F)_d \oplus L^2(F)_c,$$

where their irreducible decompositions are given by

$$L^2(F)_d = {\sum}^{\oplus}_{\tau\in\underline{\mathrm{Disc}}(L'/H')} \tau \qquad \text{(Hilbert direct sum)},$$

$$L^2(F)_c \simeq \int^{\oplus}_{\widehat{L'}} m(\tau)\tau d\mu(\tau) \qquad \text{(direct integral)}.$$

The inclusive relation $H' \subset L' \subset G'$ induces a G'-equivariant map:

$$X = G'/H' \to Y := G'/L',$$

with typical fiber $F = L'/H'$. Accordingly, the induction by stages gives a decomposition of the regular representation of G':

$$L^2(X) \simeq L^2(G'/L', L^2(F)_d) \oplus L^2(G'/L', L^2(F)_c). \tag{2.4}$$

We shall show that the space $\mathrm{Hom}_{G'}(\vartheta, L^2(G'/L', L^2(F)_c))$ is either zero or infinite-dimensional for any $\vartheta \in \widehat{G'}$.

For a measurable set S in $\widehat{L'}$, we define a subrepresentation of L' on the following closed subspace of $L^2(F)_c$:

$$\mathcal{H}(S) := \int_S^{\oplus} m(\tau)\tau d\mu(\tau).$$

In turn, we obtain a unitary representation of G' defined on the closed subspace $L^2(G'/L', \mathcal{H}(S))$ of $L^2(G'/L', L^2(F)_c)$.

Suppose $\mathrm{Hom}_{G'}(\vartheta, L^2(G'/L', L^2(F)_c)) \neq \{0\}$ for some $\vartheta \in \widehat{G'}$. We claim that there exist measurable subsets $S^{(1)}$ and $S^{(2)}$ of $\widehat{L'}$ with $\mu(S^{(1)} \cap S^{(2)}) = 0$ such that

$$\mathrm{Hom}_{G'}(\vartheta, L^2(G'/L', \mathcal{H}(S^{(j)}))) \neq \{0\} \quad \text{for } j = 1, 2.$$

Indeed, if not, we would have a countable family of measurable sets $S_1 \supset S_2 \supset \cdots$ in $\widehat{L'}$ such that

$$\mathrm{Hom}_{G'}(\vartheta, L^2(G'/L', \mathcal{H}(S_j^c))) = \{0\} \quad \text{for all } j,$$
$$\lim_{j\to\infty} \mu(S_j) = 0,$$

where $S_j^c := \widehat{L'} \setminus S_j$ stands for the complement of S_j in the unitary dual $\widehat{L'}$. But this were impossible because the discrete part of the unitary representation $L^2(F)_c$ is zero.

Therefore, the second factor of (2.4) does not contribute to discrete series representations for G'/H'. Hence

$$L^2(G'/H')_d = \sum_{\pi \in \underline{\mathrm{Disc}}(G'/H')}^{\oplus} \pi \subset \sum_{\tau \in \underline{\mathrm{Disc}}(L'/H')}^{\oplus} L^2(G'/L', \tau).$$

Thus we have proved:

Proposition 2.13. *Assume G'/H' is real spherical and L' is a reductive subgroup of G' containing H'. Then $L^2(G'/H')_d \simeq \sum_{\tau\in\underline{\mathrm{Disc}}(L'/H')}^{\oplus} L^2(G'/L', \tau)_d$, and we have a natural bijection*

$$\underline{\mathrm{Disc}}(G'/H') \simeq \bigcup_{\tau \in \underline{\mathrm{Disc}}(L'/H')} \underline{\mathrm{Disc}}(G'/L', \tau).$$

Similarly to Proposition 2.13 for the fibration $\mathcal{K}_2$, one can prove the following results for the fibration $\mathcal{K}_1 : \underline{\mathrm{Disc}}(G'/H') \to \underline{\mathrm{Disc}}(G/H)$:

Fact 2.14 ([16, Theorem 2.1]). *Suppose we are in Setting 2.3. If $\pi \in \widehat{G}$ is realized as a discrete series representation for $L^2(G/H)$ and if $\vartheta \in \widehat{G'}$ satisfies $\mathrm{Hom}_{G'}(\vartheta, \pi|_{G'}) \neq \{0\}$, then ϑ can be realized in a closed subspace of $L^2(G/H) = L^2(G'/H')$ and thus $\vartheta \in \mathrm{Disc}(G'/H')$. Moreover if G'/H' is real* *spherical, then this correspondence induces a bijection between multisets:*

$$\underline{\mathrm{Disc}}(G'/H') \simeq \bigcup_{\pi \in \underline{\mathrm{Disc}}(G/H)} \underline{\mathrm{Disc}}(\pi|_{G'}), \tag{2.5}$$

and in particular, a bijection between sets:

$$\mathrm{Disc}(G'/H') = \bigcup_{\pi \in \mathrm{Disc}(G/H)} \mathrm{Disc}(\pi|_{G'}).$$

Remark 2.15. A weaker form of Fact 2.14 holds in a more general setting where G' does not act transitively on G/H. See [22, Theorems 5.1 and 8.6] for instance.

Combining Fact 2.14 with Proposition 2.13, we have completed the proof of the second statement of Theorem 2.10.

To see the third statement of Theorem 2.10, assume that $\vartheta \in \widehat{G'}$ satisfies

$$\dim \mathrm{Hom}_{G'}(\vartheta, L^2(X)) = 1.$$

By Theorem 2.10 (2), there exists a unique $\pi \in \widehat{G}$ such that

$$\dim \mathrm{Hom}_G(\pi, L^2(X)) \cdot \dim \mathrm{Hom}_{G'}(\vartheta, \pi|_{G'}) = 1,$$

and there exists a unique $\tau \in \widehat{L}'$ such that

$$\dim \mathrm{Hom}_{L'}(\tau, L^2(F)) \cdot \dim \mathrm{Hom}_{G'}(\vartheta, L^2(G'/L', \tau)) = 1,$$

where we recall $F = L'/H'$. Then all the multiplicities involved are one, and we have

$$L^2(X)[\vartheta] \simeq (\pi|_{G'})[\vartheta] \simeq L^2(G'/L', \tau)[\vartheta] \simeq \vartheta. \tag{2.6}$$

Hence the correspondence $\vartheta \mapsto (\mathcal{K}_1(\vartheta), \mathcal{K}_2(\vartheta)) = (\pi, \tau)$ defines the desired double fibration. Thus Theorem 2.10 (3) is proved. □

2.5 Perspectives of Theorem 2.10

We shall enrich the double fibration in Theorem 2.10 by two general results:

- relations among the infinitesimal characters (or joint eigenvalues of invariant differential operators) of three representations ϑ, $\pi = \mathcal{K}_1(\vartheta)$, and $\tau = \mathcal{K}_2(\vartheta)$ (Theorems 3.6 and 3.8),
- discretely decomposability of the restriction of a unitary representation of G to the subgroup G' under the assumption that the fiber $F = L'/H'$ is compact (Theorem 5.1).

We note that the latter depends heavily on the real forms, whereas the former depends only on the complexifications. This observation allows us to get useful results on infinite-dimensional representations from computation of finite-dimensional representations. We shall illustrate this idea by finding the branching rule of unitary representations for $SO(8,8) \downarrow Spin(1,8)$ from finite-dimensional branching rules for compact groups $SO(16) \downarrow Spin(9)$.

Remark 2.16. One may observe that there is some similarity between Howe's theory of dual pair [6, 7, 8] and Theorem 2.10 in the fibration $F \to X \to Y$ (see (2.1)) even though neither the fiber $F = L'/H'$ nor the base space $Y = G'/L'$ is a group. When $F \to X \to Y$ is a Hopf bundle corresponding to the cases in Table4.1 (i), (iii), and (v) or their noncompact real forms in Table 4.3) a part of Theorem 2.10 may be understood from this viewpoint.

3 Invariant differential operators with hidden symmetry

3.1 Spherical homogeneous spaces—revisited

We give a quick review on known results about spherical homogeneous spaces from the three points of view—geometry, invariant differential operators, and representation theory.

Let $G_{\mathbb{C}}$ be a complex reductive group, $H_{\mathbb{C}}$ an algebraic reductive subgroup, and $X_{\mathbb{C}} := G_{\mathbb{C}}/H_{\mathbb{C}}$. Let $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ be the $\mathbb{C}$-algebra of $G_{\mathbb{C}}$-invariant holomorphic differential operators on $X_{\mathbb{C}}$.

An algebraic subgroup G of $G_{\mathbb{C}}$ is a *real form* if $\mathrm{Lie}(G_{\mathbb{C}}) \simeq \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathrm{Lie}(\ )$ denotes the functor from Lie groups to their Lie algebras. We say that (G, H) is a *real form* of the pair $(G_{\mathbb{C}}, H_{\mathbb{C}})$ if G and its subgroup H are real forms of $G_{\mathbb{C}}$ and $H_{\mathbb{C}}$, respectively. A real form (G, H) is said to be a *compact real form* if G is compact. In this case, we shall use the letter (G_U, H_U) instead of (G, H).

Definition-Theorem 3.1 *The following seven conditions on the pair $(G_{\mathbb{C}}, H_{\mathbb{C}})$ are equivalent. In this case, $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ is called $G_{\mathbb{C}}$-spherical.*
(Geometry)
(i) *$X_{\mathbb{C}}$ admits an open orbit of a Borel subgroup of $G_{\mathbb{C}}$.*
(ii) *$H_{\mathbb{C}}$ has an open orbit in the flag variety of $G_{\mathbb{C}}$.*

(Ring structure of $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$)
(iii) *$\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ is commutative.*
(iv) *$\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ is a polynomial ring.*

(Representation theory)
(v) *If (G_U, H_U) is a compact real form of $(G_{\mathbb{C}}, H_{\mathbb{C}})$, then*

$$\dim \mathrm{Hom}_{G_U}(\pi, C^\infty(G_U/H_U)) \le 1 \quad \textit{for all } \pi \in \widehat{G_U}.$$

(vi) *There exist a real form (G, H) of $(G_{\mathbb{C}}, H_{\mathbb{C}})$ and a constant $C > 0$ such that*

$$\dim \operatorname{Hom}_G(\pi, C^\infty(G/H)) \leq C \qquad \textit{for all } \pi \in \widehat{(G)}_{\mathrm{adm}}.$$

(vii) *There exists a constant $C > 0$ such that*

$$\dim \operatorname{Hom}_G(\pi, \mathcal{D}'(G/H)) \leq C \qquad \textit{for all } \pi \in \widehat{(G)}_{\mathrm{adm}},$$

for all real form (G, H) of $(G_{\mathbb{C}}, H_{\mathbb{C}})$.

Proof. The equivalence (i) ⇔ (ii), and the implications (iv) ⇒ (iii), (v) ⇒ (vi), and (vii) ⇒ (vi) are obvious. For the equivalence (i) ⇔ (iii), see [32]. The equivalence (i) ⇔ (iv) was proved by Knop [14]. For a compact real form G_U of $G_{\mathbb{C}}$, the equivalence (i) ⇔ (v) was proved in Vinberg–Kimelfeld [33]. For noncompact real forms (G, H), we need to take infinite-dimensional representations of G into account, and the equivalence (i) ⇔ (vi) ⇔ (vii) is due to [26].

Remark 3.2. We have confined ourselves to reductive pairs $(G_{\mathbb{C}}, H_{\mathbb{C}})$ in this article, however, the above equivalence extends to a more general setting where $H_{\mathbb{C}}$ is not reductive. See [26] and the references therein for a precise statement.

Example 3.3. Any complex reductive symmetric space $G_{\mathbb{C}}/H_{\mathbb{C}}$ is $G_{\mathbb{C}}$-spherical. Their real forms G/H were classified infinitesimally by Berger [1]. Typical examples are real forms $G/H = SL(n,\mathbb{R})/SO(p,q)$ and $SU(p,q)/SO(p,q)$ $(p + q = n)$ of the complex reductive symmetric spaces $G_{\mathbb{C}}/H_{\mathbb{C}} = SL(n,\mathbb{C})/SO(n,\mathbb{C})$.

There are also nonsymmetric spherical homogeneous spaces $G_{\mathbb{C}}/H_{\mathbb{C}}$ such as

$$GL(2n+1,\mathbb{C})/(\mathbb{C}^\times \times Sp(n,\mathbb{C})) \text{ or } SO(2n+1,\mathbb{C})/GL(n,\mathbb{C}).$$

The homogeneous spaces $G'_{\mathbb{C}}/H'_{\mathbb{C}}$ in Table 4.2 are also nonsymmetric spherical homogeneous spaces $G'_{\mathbb{C}}/H'_{\mathbb{C}}$. See also Krämer [28], Brion [2], and Mikityuk [30] for the classification of spherical homogeneous spaces.

3.2 Preliminaries on invariant differential operators

This section summarizes classical results on the algebra of invariant differential operators on homogeneous spaces of reductive groups. We let the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ act as holomorphic vector fields on $G_{\mathbb{C}}$ in two ways:

$$\text{a right } G_{\mathbb{C}}\text{-invariant vector field given by} \quad x \mapsto dl(Z)_x := \left.\frac{d}{dt}\right|_{t=0} e^{-tZ}x,$$
$$\text{a left } G_{\mathbb{C}}\text{-invariant vector field given by} \quad x \mapsto dr(Z)_x := \left.\frac{d}{dt}\right|_{t=0} xe^{tZ},$$

for $Z \in \mathfrak{g}_{\mathbb{C}}$. Let $U(\mathfrak{g}_{\mathbb{C}})$ be the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Then the Lie algebra homomorphisms $dl : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{X}(G_{\mathbb{C}})$ and $dr : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{X}(G_{\mathbb{C}})$ extend to injective $\mathbb{C}$-algebra homomorphisms from $U(\mathfrak{g}_{\mathbb{C}})$ into the ring $\mathbb{D}(G_{\mathbb{C}})$ of holomorphic differential operators on $G_{\mathbb{C}}$, and we get a $\mathbb{C}$-algebra homomorphism:

$$dl \otimes dr : U(\mathfrak{g}_{\mathbb{C}}) \otimes U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{D}(G_{\mathbb{C}}). \tag{3.1}$$

Let $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$ be the center of $U(\mathfrak{g}_{\mathbb{C}})$. Then we have

$$dl(\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})) = dr(\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})) = dl(U(\mathfrak{g}_{\mathbb{C}})) \cap dr(U(\mathfrak{g}_{\mathbb{C}})).$$

Suppose that $H_{\mathbb{C}}$ is a reductive subgroup of $G_{\mathbb{C}}$, and we set $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ as before. Let $\mathbb{D}(X_{\mathbb{C}})$ be the ring of holomorphic differential operators on $X_{\mathbb{C}}$. We write $U(\mathfrak{g}_{\mathbb{C}})^{H_{\mathbb{C}}}$ for the subalgebra of $U(\mathfrak{g}_{\mathbb{C}})$ consisting of $H_{\mathbb{C}}$-invariant elements under the adjoint action.

Then the homomorphism (3.1) induces the following diagram.

$$\begin{array}{rcl} \mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}) & \otimes \mathbb{C} & \to \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}) \qquad (3.2)\\ & \cap & \cap \\ dl \otimes dr : U(\mathfrak{g}_{\mathbb{C}}) & \otimes U(\mathfrak{g}_{\mathbb{C}})^{H_{\mathbb{C}}} & \to \mathbb{D}(X_{\mathbb{C}}) \\ & \cup & \cup \\ & \mathbb{C} \otimes U(\mathfrak{g}_{\mathbb{C}})^{H_{\mathbb{C}}} & \to \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}) \qquad (3.3) \end{array}$$

These homomorphisms (3.2) and (3.4) map into $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$, however, none of them is very useful for the description of the ring $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ when $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ is a nonsymmetric spherical homogeneous space:

Remark 3.4. (1) $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$ is a polynomial algebra that is well-understood by the Harish-Chandra isomorphism (3.6) below, but the homomorphism (3.2) is rarely surjective when $G_{\mathbb{C}}/H_{\mathbb{C}}$ is nonsymmetric (*i.e.*, the "abstract Capelli problem" à la Howe–Umeda [9] has a negative answer).

(2) (3.3) is always surjective [5], but the ring $U(\mathfrak{g}_{\mathbb{C}})^{H_{\mathbb{C}}}$ is noncommutative and is hard to treat in general.

In Section 3.3, we shall consider simultaneously three rings $\mathbb{D}_{L_{\mathbb{C}}}(F_{\mathbb{C}})$, $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$, and $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ with the notation therein, and Remark 3.4 will be applied to the third one, $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$.

We review briefly the well-known structural results on $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ when $X_{\mathbb{C}}$ is a symmetric space.

Suppose that $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ is a complex reductive symmetric space, *i.e.* $H_{\mathbb{C}}$ is an open subgroup of the group $G_{\mathbb{C}}^{\sigma}$ of fixed points of $G_{\mathbb{C}}$ for some holomorphic involutive automorphism σ. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \mathfrak{q}_{\mathbb{C}}$ be the decomposition of $\mathfrak{g}_{\mathbb{C}}$ into eigenspaces of $d\sigma$, with eigenvalues $+1$, -1, respectively. Fix a maximal semisimple abelian subspace $\mathfrak{a}_{\mathbb{C}}$ of $\mathfrak{q}_{\mathbb{C}}$. Let W be the Weyl group of the restricted root system $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ of $\mathfrak{a}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Then there is a natural isomorphism of $\mathbb{C}$-algebras:

$$\Psi : \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}) \xrightarrow{\sim} S(\mathfrak{a}_{\mathbb{C}})^W, \tag{3.4}$$

known as the Harish-Chandra isomorphism. In turn, any $\nu \in \mathfrak{a}_{\mathbb{C}}^*/W$ gives rise to a $\mathbb{C}$-algebra homomorphism

$$\chi_\nu^{X_{\mathbb{C}}} : \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}) \to \mathbb{C}, \quad D \mapsto \langle \Psi(D), \nu \rangle.$$

Conversely, any $\mathbb{C}$-algebra homomorphism $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}) \to \mathbb{C}$ is written uniquely in this form, and thus we have a natural bijection:

$$\mathfrak{a}_{\mathbb{C}}^*/W \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \mathbb{C}), \quad \nu \mapsto \chi_\nu^{X_{\mathbb{C}}}. \tag{3.5}$$

In the special case that $X_{\mathbb{C}}$ is a group manifold $G_{\mathbb{C}} \simeq (G_{\mathbb{C}} \times G_{\mathbb{C}})/\Delta(G_{\mathbb{C}})$ regarded as a symmetric space by the involution $\sigma(x, y) = (y, x)$, the Harish-Chandra isomorphism (3.4) amounts to the isomorphism

$$\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}) \simeq \mathbb{D}_{G_{\mathbb{C}} \times G_{\mathbb{C}}}(G_{\mathbb{C}}) \simeq S(\mathfrak{j}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}})}, \tag{3.6}$$

where $\mathfrak{j}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and $W(\mathfrak{g}_{\mathbb{C}})$ denotes the Weyl group of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$. Then any $\lambda \in \mathfrak{j}_{\mathbb{C}}^*/W(\mathfrak{g}_{\mathbb{C}})$ induces a $\mathbb{C}$-algebra homomorphism $\chi_\lambda^{G_{\mathbb{C}}} : \mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$, and the bijection (3.5) reduces to:

$$\mathfrak{j}_{\mathbb{C}}^*/W(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}), \mathbb{C}), \quad \lambda \mapsto \chi_\lambda^{G_{\mathbb{C}}}. \tag{3.7}$$

When $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ is $G_{\mathbb{C}}$-spherical, then by work of Knop [14], there is an isomorphism analogous to the Harish-Chandra homomorphism, but it is less explicit.

3.3 Three subalgebras in $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$

Suppose that the triple $(G_{\mathbb{C}}, G'_{\mathbb{C}}, H_{\mathbb{C}})$ of complex Lie groups are in Setting 2.4. It turns out that the subgroup $H'_{\mathbb{C}} := G'_{\mathbb{C}} \cap H_{\mathbb{C}}$ is not necessarily a maximal reductive subgroup of $G'_{\mathbb{C}}$ even when $H_{\mathbb{C}}$ is maximal in $G_{\mathbb{C}}$. We take a complex reductive subgroup $L'_{\mathbb{C}}$ of $G'_{\mathbb{C}}$ containing $H'_{\mathbb{C}}$, and set $F_{\mathbb{C}} := L'_{\mathbb{C}}/H'_{\mathbb{C}}$ and $Y_{\mathbb{C}} := G'_{\mathbb{C}}/L'_{\mathbb{C}}$. Then we have a natural holomorphic fibration

$$F_{\mathbb{C}} \hookrightarrow X_{\mathbb{C}} \twoheadrightarrow Y_{\mathbb{C}}. \tag{3.8}$$

By using the geometry (3.8), we shall give a detailed description of the $\mathbb{C}$-algebra $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ that will enrich the double fibration for representations of the three groups $G_{\mathbb{C}}$, $L'_{\mathbb{C}}$, and $G'_{\mathbb{C}}$ in Theorem 2.10. For this, we introduce the three subalgebras $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ in $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ as below.

First we extend $L'_{\mathbb{C}}$-invariant differential operators on the fiber $F_{\mathbb{C}}$ can be extended to $G'_{\mathbb{C}}$-invariant ones on $X_{\mathbb{C}}$, as follows: for any $D \in \mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}})$, for any holomorphic function f defined in an open set V of $X_{\mathbb{C}}$, and for any $g \in G'_{\mathbb{C}}$, we set

$$(\iota(D)f)|_{gF_{\mathbb{C}}} := ((l_g^*)^{-1} \circ D \circ l_g^*)(f|_{gF_{\mathbb{C}}}), \tag{3.9}$$

where $l_g : X_{\mathbb{C}} \to X_{\mathbb{C}}$ is the left translation by g, and $l_g^* : \mathcal{O}(gV) \to \mathcal{O}(V)$ is the pull-back by l_g. Then the right-hand side of (3.9) is independent of the representative g in $gF_{\mathbb{C}}$ since D is $L'_{\mathbb{C}}$-invariant, and thus $\iota(D)$ gives rise to a $G'_{\mathbb{C}}$-invariant holomorphic differential operator on $X_{\mathbb{C}}$. Clearly, $D = 0$ if $\iota(D) = 0$. Thus we have obtained a natural injective $\mathbb{C}$-algebra homomorphism

$$\iota : \mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}}) \to \mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}}).$$

We thus have the following three algebras in $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$:

$$\mathcal{P} := \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \tag{3.10}$$

$$\mathcal{Q} := \iota(\mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}})), \tag{3.11}$$

$$\mathcal{R} := dl(\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}})). \tag{3.12}$$

The subalgebra $\mathcal{P}$ reflects the hidden symmetry of $X_{\mathbb{C}} = G'_{\mathbb{C}}/H'_{\mathbb{C}}$ by the overgroup $G_{\mathbb{C}}$. The subalgebra $\mathcal{Q}$ depends on the choice of $L'_{\mathbb{C}}$, and is interesting if the fiber $F_{\mathbb{C}}$ is nontrivial, equivalently, if $L'_{\mathbb{C}}$ satisfies $H'_{\mathbb{C}} \subsetneqq L'_{\mathbb{C}} \subsetneqq G'_{\mathbb{C}}$. We shall take $L'_{\mathbb{C}}$ to be a maximal reductive subgroup of $G'_{\mathbb{C}}$ containing $H'_{\mathbb{C}}$.

Here is a description of $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ by choosing any two of the three subalgebras $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$:

Theorem 3.5 ([12]). *Assume that $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical in Setting 2.4.*

(1) *The polynomial algebra $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ is generated by $\mathcal{P}$ and $\mathcal{R}$.*
From now, we take a maximal reductive subgroup $L'_{\mathbb{C}}$ of $G'_{\mathbb{C}}$ containing $H'_{\mathbb{C}}$.
(2) *$\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ is generated by $\mathcal{P}$ and $\mathcal{Q}$.*
(3) *$\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ is generated by $\mathcal{Q}$ and $\mathcal{R}$ if $G_{\mathbb{C}}$ is simple.*

It turns out from the classification (see Table 4.2 below) that $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ and $F_{\mathbb{C}} = L'_{\mathbb{C}}/H'_{\mathbb{C}}$ are reductive symmetric spaces in most of the cases in Theorem 3.5. In [12], we find explicitly generators P_k, $\iota(Q_k)$, and $dl(R_k)$ of $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$, respectively, in the following theorem together with their relations.

Theorem 3.6. *Assume that $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical in Setting 2.4.*

(1) *There exist elements P_k of $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$, and elements R_k of $\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}})$ such that*

$$\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}}) = \mathbb{C}[P_1, \cdots, P_m, dl(R_1), \cdots, dl(R_n)]$$

is a polynomial ring in the P_k and $dl(R_k)$.
(2) *Assume further that $G_{\mathbb{C}}$ is simple. We take $L'_{\mathbb{C}}$ to be a maximal reductive subgroup of $G'_{\mathbb{C}}$ containing $H'_{\mathbb{C}}$. Then there exist elements Q_k of $\mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}})$, and integers $s, t \in \mathbb{N}$ with $m + n = s + t$ such that*

$$\begin{aligned}\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}}) &= \mathbb{C}[P_1, \cdots, P_m, \iota(Q_1), \cdots, \iota(Q_n)]\\ &= \mathbb{C}[\iota(Q_1), \cdots, \iota(Q_s), dl(R_1), \cdots, dl(R_t)]\end{aligned}$$

is a polynomial ring in the P_k *and* $\iota(Q_l)$*, and in the* $\iota(Q_k)$ *and* $dl(R_l)$*, respectively.*

The key ingredient of the proof for Theorems 3.5 and 3.6 is to provide explicitly the map

$$\widehat{G'_U} \supset \mathrm{Disc}(G'_U/H'_U) \xrightarrow{\mathcal{K}_1 \times \mathcal{K}_2} \widehat{G_U} \times \widehat{L'_U}, \tag{3.13}$$

which is a special case of Theorem 2.10 for the *compact* real form $X_U = G'_U/H'_U \simeq G_U/H_U$ as below.

Lemma 3.7. *Suppose that we are in Setting 2.5. We take a subgroup* L'_U *of* G'_U *containing* H'_U*. Assume that* $X_{\mathbb{C}} := G'_{\mathbb{C}}/H'_{\mathbb{C}}$ *is* $G'_{\mathbb{C}}$*-spherical.*

(1) $X_{\mathbb{C}}$ *is* $G_{\mathbb{C}}$*-spherical.*
(2) $F_{\mathbb{C}} := L'_{\mathbb{C}}/H'_{\mathbb{C}}$ *is* $L'_{\mathbb{C}}$*-spherical.*
(3) *There is a canonical map*

$$\begin{aligned} \mathcal{K}_1 \times \mathcal{K}_2 : \mathrm{Disc}(G'_U/H'_U) &\to \mathrm{Disc}(G_U/H_U) \times \mathrm{Disc}(L'_U/H'_U), \\ \vartheta \quad &\mapsto \quad (\mathcal{K}_1(\vartheta), \mathcal{K}_2(\vartheta)) \end{aligned}$$

characterized by

$$[\mathcal{K}_1(\vartheta)|_{G'_U} : \vartheta] = 1 \quad \textit{and} \quad [\vartheta|_{L_U} : \mathcal{K}_2(\vartheta)] = 1.$$

Theorems 3.5 and 3.6 apply to analysis of real forms $X = G/H = G'/H'$ in Table 4.3. For this, we set up some notation. Suppose that G is a real form of $G_{\mathbb{C}}$ which leaves a real form X of $X_{\mathbb{C}}$ invariant. Then $G_{\mathbb{C}}$-invariant holomorphic differential operators on $X_{\mathbb{C}}$ induce G-invariant real analytic differential operators on X by restriction. Let $\chi_\lambda^{X_{\mathbb{C}}} \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \mathbb{C})$. For $\mathcal{F} = C^\infty$, L^2, or $\mathcal{D}'$, we define the space of joint eigenfunctions by

$$\mathcal{F}(X; \mathcal{M}_\lambda) := \{f \in \mathcal{F}(X) : Df = \chi_\lambda^{X_{\mathbb{C}}}(D) f \quad \text{for any } D \in \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})\},$$

where solutions are understood in the weak sense for $\mathcal{F} = L^2$ or $\mathcal{D}'$. Then $\mathcal{F}(X; \mathcal{M}_\lambda)$ are G-submodules of regular representations of G on $\mathcal{F}(X)$.

Let $\widehat{(L'_{\mathbb{C}})}_{H'_{\mathbb{C}}}$ be the set of equivalence classes of irreducible finite-dimensional holomorphic representations of $L'_{\mathbb{C}}$ with nonzero $H'_{\mathbb{C}}$-fixed vectors. By Weyl's unitary trick, there is a natural bijection

$$\widehat{(L'_{\mathbb{C}})}_{H'_{\mathbb{C}}} \xrightarrow{\sim} \mathrm{Disc}(L'_U/H'_U) \tag{3.14}$$

if $F_U = L'_U/H'_U$ is a real form of $F'_{\mathbb{C}} = L'_{\mathbb{C}}/H'_{\mathbb{C}}$ and if both $L'_{\mathbb{C}}$ and $H'_{\mathbb{C}}$ are connected.

Theorem 3.8. *Suppose that we are in Setting 2.3 with* $G_{\mathbb{C}}$ *simple and with* $G'_{\mathbb{C}}$ *and* $H_{\mathbb{C}}$ *maximal reductive subgroups. Assume that the complexification* $X_{\mathbb{C}}$ *of* X *is* $G'_{\mathbb{C}}$*-spherical. Let* $L'_{\mathbb{C}}$ *be a maximal complex reductive subgroup of* $G'_{\mathbb{C}}$ *containing* $H'_{\mathbb{C}}$*.*

(1) $\mathfrak{Z}(\mathfrak{l}'_{\mathbb{C}}) \to \mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}})$ *is surjective.*

(2) *There exists a natural map for every* $\tau \in \widehat{(L'_{\mathbb{C}})}_{H'_{\mathbb{C}}}$,

$$\nu_\tau : \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}}), \mathbb{C}) \tag{3.15}$$

with the following property: in (3.13) *if* $\pi \in \mathrm{Disc}(G_U/H_U)$ *is realized in* $L^2(G_U/H_U;\mathcal{M}_\lambda)$, *then the infinitesimal character of any* $\vartheta \in \widehat{G'_U}$ *belonging to* $\mathcal{K}_1^{-1}(\pi) \cap \mathcal{K}_2^{-1}(\tau)$ *is given by* $\nu_\tau(\lambda)$.

(3) *Suppose that a quadruple* $H \subset G \supset G' \supset H'$ *is given as real forms of* $H_{\mathbb{C}} \subset G_{\mathbb{C}} \supset G'_{\mathbb{C}} \supset H'_{\mathbb{C}}$ *such that* $F = L'/H'$ *is compact and that* $\underline{\mathrm{Disc}}(G'/H')$ *is multiplicity-free. Then, for any* $\tau \in \mathrm{Disc}(L'/H') \simeq \widehat{(L'_{\mathbb{C}})}_{H'_{\mathbb{C}}}$ *and for* $\pi \in \mathrm{Disc}(G/H)$ *realized in* $L^2(G/H;\mathcal{M}_\lambda)$, *the infinitesimal character of any* ϑ *belonging to* $\mathcal{K}_1^{-1}(\pi) \cap \mathcal{K}_2^{-1}(\tau)$ *is given by* $\nu_\tau(\lambda)$.

Remark 3.9. In Section 4, we illustrate Theorem 3.8 by examples, and give explicitly of the map ν_τ (see (4.7) and (4.9)), in the case where the rank of the nonsymmetric homogeneous space $X_{\mathbb{C}} = G'_{\mathbb{C}}/H'_{\mathbb{C}}$ is 1 and $2n+1$, respectively.

Proof (Sketch of proof of Theorem 3.8). By Lemma 2.1, $G'_{\mathbb{C}}$ acts transitively on $X_{\mathbb{C}}$. Then the first statement follows from the classification of the quadruple $(H_{\mathbb{C}}, G_{\mathbb{C}}, G'_{\mathbb{C}}, H'_{\mathbb{C}})$ (see Table 4.2), and the second one from Theorem 3.5 (see [12] for details). To see the third statement, let $W_\tau := L^2(L'/H')[\tau]$ be the τ-component (see (2.3)) of the unitary representation of L' on $L^2(L'/H')$. Since $\mathfrak{Z}(\mathfrak{l}'_{\mathbb{C}})$ surjects onto $\mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}})$, every element of $\mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}})$ acts on W_τ as scalar. In turn, $\iota(\mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}}))$ acts as scalars on $L^2(G'/L', \mathcal{W}_\tau)$, which is a G'-invariant closed subspace of $L^2(X)$. By (2.6), and by the assumption that $\underline{\mathrm{Disc}}(G'/H')$ is multiplicity-free, ϑ in $\mathcal{K}_1^{-1}(\pi) \cap \mathcal{K}_2^{-1}(\tau)$ is realized on $L^2(X)[\vartheta] \simeq L^2(G'/H', \mathcal{W}_\tau)[\vartheta]$. Hence the third statement is deduced from (2).

4 Examples of relations among invariant differential operators

In this section, we illustrate Theorems 3.5, 3.6, and 3.8 on invariant differential operators with hidden symmetries by some few examples. We shall carry out in [12] computations thoroughly for all the cases based on the classification (see Table 4.2) of the fibration

$$F_{\mathbb{C}} = L'_{\mathbb{C}}/H'_{\mathbb{C}} \to X_{\mathbb{C}} = G'_{\mathbb{C}}/H'_{\mathbb{C}} \to Y_{\mathbb{C}} = G'_{\mathbb{C}}/L'_{\mathbb{C}}$$

of $G'_{\mathbb{C}}$-spherical $X_{\mathbb{C}}$ with hidden symmetry $G_{\mathbb{C}}$, where $H'_{\mathbb{C}} \subset L'_{\mathbb{C}} \subset G'_{\mathbb{C}} \subset G_{\mathbb{C}}$ and $G_{\mathbb{C}}$ is a complex simple Lie group containing $G'_{\mathbb{C}}$.

The following convention will be used in Sections 4.1 and 4.4.

For $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$, we define $R_k \in \mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$ such that

$$\chi_\nu(R_k) = \sum_{j=1}^{n} \nu_j^k$$

for $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{C}^n/\mathfrak{S}_n$, or equivalently R_k acts on the finite-dimensional representation $F(\mathfrak{g}_\mathbb{C}, \lambda)$ with highest weight $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ ($\lambda_1 \geq \cdots \geq \lambda_n$) as the scalar $\sum_{j=1}^n (\lambda_j + \frac{1}{2}(n+1-2j))^k$.

4.1 Hopf bundle: $X_U = S^1$ bundle over $\mathbb{P}^n\mathbb{C}$

We begin with the underlying geometry of an example in [16] to find an explicit branching law of the unitarization [35] of certain Zuckerman's derived functor modules $A_\mathfrak{q}(\lambda)$ (see Vogan [34] or Vogan–Zuckerman [36] for the definition of $A_\mathfrak{q}(\lambda)$) with respect to a reductive symmetric pair

$$(G, G') = (O(2p, 2q), U(p, q)).$$

The holomorphic setting is given by $X_\mathbb{C} = G_\mathbb{C}/H_\mathbb{C} \simeq G'_\mathbb{C}/H'_\mathbb{C}$ and $F_\mathbb{C} = L'_\mathbb{C}/H'_\mathbb{C}$ with

$$\begin{pmatrix} G_\mathbb{C} & \supset & H_\mathbb{C} \\ \cup & & \cup \\ G'_\mathbb{C} \supset L'_\mathbb{C} & \supset & H'_\mathbb{C} \end{pmatrix} := \begin{pmatrix} SO(2n+2,\mathbb{C}) & \supset & SO(2n+1,\mathbb{C}) \\ \cup & & \cup \\ GL(n+1,\mathbb{C}) \supset GL(n,\mathbb{C}) \times GL(1,\mathbb{C}) & \supset & GL(n,\mathbb{C}) \end{pmatrix}.$$

In the compact form, the fibration $F_U \to X_U \to Y_U$ amounts to the Hopf fibration

$$S^1 \to S^{2n+1} \to \mathbb{P}^n\mathbb{C}.$$

In order to explain a noncompact form of the Hopf fibration, we set

$$S^{p,q} := \{x \in \mathbb{R}^{p+q+1} : \sum_{j=1}^{p+1} x_j^2 - \sum_{k=1}^{q} y_k^2 = 1\} \simeq O(p+1,q)/O(p,q). \tag{4.1}$$

The hypersurface $S^{p,q}$ becomes a pseudo-Riemannian manifold of signature (p, q) as a submanifold of $\mathbb{R}^{p+q+1}$ endowed with the flat pseudo-Riemannian metric

$$ds^2 = dx_1^2 + \cdots + dx_{p+1}^2 - dy_1^2 - \cdots - dy_q^2 \quad \text{on } \mathbb{R}^{p+1,q}.$$

Then $S^{p,q}$ carries a constant sectional curvature $+1$. By switching signature of the metric, $S^{p,q}$ may be regarded also as a pseudo-Riemannian manifold of signature (q, p), having a constant sectional curvature -1 (see [38, Chapter 11]). We note that

$S^{1,q}$ is the anti-de Sitter space, and $S^{p,1}$ is the de Sitter space. Then the noncompact form of the Hopf fibration $F \to X \to Y$ with $G = O(2p+2, 2q)$ amounts to

$$S^1 \to S^{2p+1,2q} \to \mathbb{P}^{p,q}\mathbb{C}, \tag{4.2}$$

where we define an open set of $\mathbb{P}^{p+q}\mathbb{C}$ by

$$\mathbb{P}^{p,q}\mathbb{C} := \{[z : w] \in ((\mathbb{C}^{p+1} \oplus \mathbb{C}^q) \setminus \{0\})/\mathbb{C}^\times : |z|^2 > |w|^2\}. \tag{4.3}$$

Then $\mathbb{P}^{p,q}\mathbb{C}$ carries an indefinite-Kähler structure which is invariant by the natural action of $U(p+1, q)$. If $p = 0$, then $S^{2p+1,2q} = S^{1,2q}$ is the anti-de Sitter space, and $\mathbb{P}^{p,q}\mathbb{C} = \mathbb{P}^{0,q}\mathbb{C}$ is the Hermitian unit ball.

We take

$$P_2 := dl(C_{D_n}) \in \mathbb{D}_{G_\mathbb{C}}(X_\mathbb{C}),$$

where $C_{D_n} \in \mathfrak{Z}(\mathfrak{g}_\mathbb{C})$ is the Casimir element of $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(2n+2, \mathbb{C})$. Let E be a generator of the second factor of $\mathfrak{l}'_\mathbb{C} = \mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C})$ such that the eigenvalues of $\mathrm{ad}(E)$ in $\mathfrak{gl}(n+1, \mathbb{C})$ are 0, ± 1. We set

$$Q_1 := dr(E), \quad Q_2 := Q_1^2 \in \mathbb{D}_{L'_\mathbb{C}}(F_\mathbb{C}).$$

We take R_1, $R_2 \in \mathfrak{Z}(\mathfrak{g}'_\mathbb{C})$ for $\mathfrak{g}'_\mathbb{C} = \mathfrak{gl}(n+1, \mathbb{C})$ as

$$R_1 := \sum_{i=1}^{n+1} E_{ii},$$

$R_2 :=$the Casimir element (see the convention at the beginning of this section).

Then the three subalgebras $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ of $\mathbb{D}_{G'_\mathbb{C}}(X_\mathbb{C})$ are polynomial algebras given as

$$\begin{aligned}
\mathcal{P} &= \mathbb{D}_{G_\mathbb{C}}(X_\mathbb{C}) = \mathbb{C}[P_2], \\
\mathcal{Q} &= \mathbb{D}_{L'_\mathbb{C}}(F_\mathbb{C}) = \mathbb{C}[\iota(Q_1)], \\
\mathcal{R} &= dl(\mathfrak{Z}(\mathfrak{g}'_\mathbb{C})) = \mathbb{C}[dl(R_1), dl(R_2)].
\end{aligned}$$

The relations among generators are given by

$$dl(R_1) = -\iota(Q_1), \quad \text{and} \quad P_2 = 2dl(R_2) - dr(Q_2).$$

Then Theorem 3.6 in this case is summarized by the following three descriptions of $\mathbb{D}_{G'_\mathbb{C}}(X_\mathbb{C})$ as polynomial algebras with explicit generators:

$$\mathbb{D}_{G'_\mathbb{C}}(X_\mathbb{C}) = \mathbb{C}[dl(P_2), \iota(Q_1)] = \mathbb{C}[dl(P_2), dr(R_1)] = \mathbb{C}[dl(Q_1), dr(R_2)]. \tag{4.4}$$

4.2 X as an S^2-bundle over the quaternionic unit ball

In this section, we reexamine the example in Introduction from Theorem 3.6. See also [24] for an exposition on this example from a viewpoint of branching laws and spectral analysis.

We begin with the geometric setting. Let X and Y be a three-dimensional complex manifold and a quaternionic unit ball defined by

$$\begin{aligned} X :=& \mathbb{P}^{1,2}\mathbb{C} = \{[z_1 : z_2 : z_3 : z_4] \in \mathbb{P}^3\mathbb{C} : |z_1|^2 + |z_2|^2 > |z_3|^2 + |z_4|^2\}, \\ Y :=& \{\zeta = x + iy + ju + kv \in \mathbb{H} : x^2 + y^2 + u^2 + v^2 < 1\}. \end{aligned}$$

Then X is homotopic to S^2 by the quaternionic Hopf fibration

$$S^2 \to X \to Y,$$

according to the following 5-tuple of real reductive groups:

$$\begin{pmatrix} G & \supset & H \\ \cup & & \cup \\ G' \supset L' \supset H' \end{pmatrix} := \begin{pmatrix} SU(2,2) & \supset & U(1,2) \\ \cup & & \cup \\ Sp(1,1) \supset Sp(1) \times Sp(1) \supset \mathbb{T} \times Sp(1) \end{pmatrix}.$$

There is a unique (up to a positive scalar multiplication) pseudo-Riemannian metric h on X of signature $(++----)$ on which G acts as isometries. The manifold X does not admit a G-invariant Riemannian metric but a G'-invariant one g induced from $-B(\theta\cdot, \cdot)$ where B and θ are the Killing form and a Cartan involution of $\mathfrak{g}' = \mathfrak{sp}(1,1)$, respectively.

Then the ring of G'-invariant differential operators on X is generated by any two of the following three second-order differential operators:

$\Box$:the Laplacian for the G-invariant pseudo-Riemannian metric h on X,
Δ :the Laplacian for the G'-invariant Riemannian metric g on X,
$\iota(\Delta_{S^2})$:the Laplacian on the fiber S^2, extended to X.

Thus

$$\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}}) \simeq \mathbb{D}_{G'}(X) = \mathbb{C}[\Box, \Delta] = \mathbb{C}[\Box, \iota(\Delta_{S^2})] = \mathbb{C}[\Delta, \iota(\Delta_{S^2})].$$

We note that $\Box \in \mathcal{P}$, $\iota(\Delta_{S^2}) \in \mathcal{Q}$, and $\Delta \in \mathcal{R}$ with the notation as in Theorem 3.5 or 3.6. These generators satisfy the following linear relation:

$$\Box = -24\Delta + 12\iota(\Delta_{S^2}),$$

see [24, (6.3)].

4.3 $(G_{\mathbb{C}}, G'_{\mathbb{C}}) = (SO(16, \mathbb{C}), Spin(9, \mathbb{C}))$

The spin representation defines a prehomogeneous vector space $\mathbb{C}^{16}$ with the action of the direct product group $\mathbb{C}^{\times} \times Spin(9, \mathbb{C})$, where the unique open orbit is given as a homogeneous space $(\mathbb{C}^{\times} \times Spin(9, \mathbb{C}))/Spin(7, \mathbb{C})$, see Igusa [10].

In this section, we consider $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}} \simeq G'_{\mathbb{C}}/H'_{\mathbb{C}}$ and $F_{\mathbb{C}} = L'_{\mathbb{C}}/H'_{\mathbb{C}}$ defined by

$$\begin{pmatrix} G_{\mathbb{C}} & \supset & H_{\mathbb{C}} \\ \cup & & \cup \\ G'_{\mathbb{C}} \supset L'_{\mathbb{C}} & \supset & H'_{\mathbb{C}} \end{pmatrix} := \begin{pmatrix} SO(16, \mathbb{C}) & \supset & SO(15, \mathbb{C}) \\ \cup & & \cup \\ Spin(9, \mathbb{C}) \supset Spin(8, \mathbb{C}) & \supset & Spin(7, \mathbb{C}) \end{pmatrix}.$$

In the compact form, the fibration $F_U \to X_U \to Y_U$ amounts to

$$S^7 \to S^{15} \to S^8.$$

In the noncompact form with $G = O(8, 8)$, the fibration $F \to X \to Y$ amounts to

$$S^7 \to S^{8,7} \to \mathbb{H}^8,$$

where $\mathbb{H}^8$ is the simply-connected 8-dimensional hyperbolic space, from which we deduced the existence of compact pseudo-Riemannian manifold of signature $(8, 7)$ of negative constant sectional curvature in [19], see also [27] for a detailed proof by utilizing the Clifford algebra over $\mathbb{R}$.

Similarly to the example in Section 4.1, we shall see below that the ring $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ of invariant differential operators on the nonsymmetric space $X_{\mathbb{C}} = G'_{\mathbb{C}}/H'_{\mathbb{C}} = Spin(9, \mathbb{C})/Spin(7, \mathbb{C})$ is a polynomial ring of two generators, both of which are given by second-order differential operators. Indeed, the three subalgebras $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ in $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ are generated by a single differential operator P_2, Q_2, and $dl(R_2)$, respectively as below and there is a linear relation (4.5) among them.

Let $C_{SO(16)}$, $C_{Spin(8)}$, and $C_{Spin(9)}$ be the Casimir elements of the complex Lie algebras $\mathfrak{so}(16, \mathbb{C})$, $\mathfrak{spin}(8, \mathbb{C})$, and $\mathfrak{spin}(9, \mathbb{C})$, respectively. We set

$$\begin{aligned} P_2 &:= dl(C_{SO(16)}) \in \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \\ Q_2 &:= dr(C_{Spin(8)}) \in \mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}}), \\ R_2 &:= C_{Spin(9)} \in \mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}}). \end{aligned}$$

Proposition 4.1. (1) *We have the following linear relations:*

$$P_2 = 4dl(R_2) - 3\iota(Q_2). \tag{4.5}$$

(2) *The ring of $G'_{\mathbb{C}}$-invariant holomorphic differential operators on $X_{\mathbb{C}}$ is a polynomial algebra of two generators with the following three expressions:*

$$\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}}) = \mathbb{C}[P_2, \iota(Q_2)] = \mathbb{C}[P_2, dl(R_2)] = \mathbb{C}[\iota(Q_2), dl(R_2)].$$

Remark 4.2. Howe and Umeda [9, Sect. 11.11] obtained a weaker from of Proposition 4.1 for the prehomogeneous vector space $(\mathbb{C}^\times \times Spin(9,\mathbb{C}), \mathbb{C}^{16})$. In particular, they proved that the $\mathbb{C}$-algebra homomorphism $dl : \mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}}) \to \mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ is not surjective and that the "abstract Capelli problem" has a negative answer. The novelty here is to introduce the operator $Q_2 \in \mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}})$ coming from the fiber $F_{\mathbb{C}}$ to describe the algebra $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$.

The proof of Proposition 4.1 relies on an explicit computation of the double fibration of Theorem 2.10 (or Lemma 3.7). We briefly state some necessary computations.

We denote by $F(L'_U, \lambda)$ the irreducible finite-dimensional representation of a connected compact Lie group L'_U with extremal weight λ. We set

$$\vartheta_{a,b} := F(Spin(9), \frac{1}{2}(a,b,b,b))$$

for $a \geq b \geq 0$ with $a \equiv b \mod 2$, namely, for $(a,b) \in \Xi(0)$. Then the sets of discrete series representations for G_U/H_U, G'_U/H'_U, and L'_U/H'_U are given as follows:

Lemma 4.3.

$$\begin{aligned}\operatorname{Disc}(SO(16)/SO(15)) &= \{\mathcal{H}^j(\mathbb{R}^{16}) : j \in \mathbb{N}\},\\ \operatorname{Disc}(Spin(9)/Spin(7)) &= \{\vartheta_{j,k} : (j,k) \in \Xi(0)\},\\ \operatorname{Disc}(Spin(8)/Spin(7)) &= \{\mathcal{H}^k(\mathbb{R}^8) : k \in \mathbb{N}\}.\end{aligned}$$

Proof. The first and third equalities follow from the classical theory of spherical harmonics (see *e.g.* [5, Intr. Thm. 3.1]), and the second equality from Krämer [28].

We set

$$\Xi(\mu) := \{(m,n) \in \mathbb{N}^2 : m - n \geq \mu, \quad m - n \equiv \mu \mod 2\}. \tag{4.6}$$

Then the double fibration of Theorem 2.10 (or Lemma 3.7) amounts to

$$\begin{array}{ccc} & \{\vartheta_{j,k} \in \widehat{Spin(9)} : (j,k) \in \Xi(0)\} & \\ \mathcal{K}_1 \swarrow & & \searrow \mathcal{K}_2 \\ \{\mathcal{H}^j(\mathbb{R}^{16}) : j \in \mathbb{N}\} & & \{\mathcal{H}^k(\mathbb{R}^8) : k \in \mathbb{N}\} \end{array}$$

We use the following normalization of Harish-Chandra isomorphisms:

$$\operatorname{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \mathbb{C}) \simeq \mathbb{C}/\mathbb{Z}_2, \qquad \chi^X_\lambda \leftrightarrow \lambda$$

by $\chi^X_\lambda(P_2) = \lambda^2 - 49$.

$$\operatorname{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}}), \mathbb{C}) \simeq \mathbb{C}^4/W(B_4) = \mathbb{C}^4/(\mathfrak{S}_4 \ltimes (\mathbb{Z}_2)^4), \qquad \chi^{G'}_\nu \leftrightarrow \nu$$

such that the $\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}})$-infinitesimal character of the trivial representation of $\mathfrak{g}'_{\mathbb{C}}$ is given by $\chi_\nu^{G'}$ with $\nu = \frac{1}{2}(7,5,3,1)$. Via these identifications, for every $\tau = \mathcal{H}^k(\mathbb{R}^8) \in \mathrm{Disc}(Spin(8)/Spin(7))$, the map

$$\nu_\tau : \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}}), \mathbb{C}), \quad \chi_\lambda^X \mapsto \chi_\nu^{G'} \tag{4.7}$$

in Theorem 3.8 amounts to

$$\mathbb{C}/\mathbb{Z}_2 \to \mathbb{C}^4/W(B_4), \quad \lambda \mapsto \nu = \frac{1}{2}(\lambda, k+5, k+3, k+1). \tag{4.8}$$

4.4 $X_{\mathbb{C}} := GL(2n+1,\mathbb{C})/Sp(n,\mathbb{C})$

The last example treats the case where $X_{\mathbb{C}}$ is of higher rank. We consider $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}} \simeq G'_{\mathbb{C}}/H'_{\mathbb{C}}$ and $F_{\mathbb{C}} = L'_{\mathbb{C}}/H'_{\mathbb{C}}$ defined by

$$\begin{pmatrix} G_{\mathbb{C}} & \supset & H_{\mathbb{C}} \\ \cup & & \cup \\ G'_{\mathbb{C}} \supset & L'_{\mathbb{C}} & \supset H'_{\mathbb{C}} \end{pmatrix} := \begin{pmatrix} GL(2n+2,\mathbb{C}) & \supset & Sp(n+1,\mathbb{C}) \\ \cup & & \cup \\ GL(2n+1,\mathbb{C}) \supset & GL(2n,\mathbb{C}) \times GL(1,\mathbb{C}) & \supset Sp(n,\mathbb{C}) \end{pmatrix}.$$

This is essentially the case in Table 4.2 (iv) except that $G_{\mathbb{C}}$ contains a one-dimensional center. We note that $X_{\mathbb{C}}$ is a nonsymmetric spherical homogeneous space of rank $2n+1$ if we regard $X_{\mathbb{C}} \simeq G'_{\mathbb{C}}/H'_{\mathbb{C}}$, but is a symmetric space of rank $n+1$ if we regard $X_{\mathbb{C}} \simeq G_{\mathbb{C}}/H_{\mathbb{C}}$.

First, for the symmetric space $X_{\mathbb{C}} = GL(2n+2,\mathbb{C})/Sp(n+1,\mathbb{C})$, the restricted root system $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ is of type A_n. We take the standard basis $\{h_1, \cdots, h_{n+1}\}$ of $\mathfrak{a}_{\mathbb{C}}^*$ such that

$$\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}}) = \{h_j - h_k : 1 \le j < k \le n+1\}.$$

By these coordinates, the Harish-Chandra isomorphism amounts to:

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \mathbb{C}) \simeq \mathfrak{a}_{\mathbb{C}}^*/W(A_n) \simeq \mathbb{C}^{n+1}/\mathfrak{S}_{n+1}, \qquad \chi_\lambda^X \leftrightarrow \lambda.$$

For $k \in \mathbb{N}$, we define $P_k \in \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ by

$$\chi_\lambda^X(P_k) = \sum_{j=1}^{n+1} \lambda_j^k \quad \text{for } \lambda = (\lambda_1, \cdots, \lambda_{n+1}) \in \mathbb{C}^{n+1}/\mathfrak{S}_{n+1}.$$

Second, the fiber $F_{\mathbb{C}}$ of the bundle $X_{\mathbb{C}} = G'_{\mathbb{C}}/H'_{\mathbb{C}} \to G'_{\mathbb{C}}/L'_{\mathbb{C}}$ is also a symmetric space:

$$F_{\mathbb{C}} = L'_{\mathbb{C}}/H'_{\mathbb{C}} \simeq (GL(2n,\mathbb{C})/Sp(n,\mathbb{C})) \times GL(1,\mathbb{C}).$$

We define similarly $Q, Q_k \in \mathbb{D}_{L'_{\mathbb{C}}}(F'_{\mathbb{C}})$ for $k \in \mathbb{N}$ by

$$\chi^F_\mu(Q) = \mu_0,$$
$$\chi^F_\mu(Q_k) = \sum_{j=1}^n \mu_j^k,$$

for $\mu = (\mu_1, \cdots, \mu_n; \mu_0) \in (\mathbb{C}^n \oplus \mathbb{C})/(\mathfrak{S}_n \times \{1\})$. Then the Harish-Chandra isomorphism gives the description of the polynomial algebras $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ and $\mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}})$:

$$\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}) = \mathbb{C}[P_1, \cdots, P_{n+1}],$$
$$\mathbb{D}_{L'_{\mathbb{C}}}(F_{\mathbb{C}}) = \mathbb{C}[Q, Q_1, \cdots, Q_n].$$

In this case, Theorems 3.6 and 3.8 amount to the following:

Proposition 4.4. (1) *The generators P_k, Q, Q_k and R_k are subject to the following relations:*

$$P_k + \iota(Q_k) = 2^k dl(R_k) \quad \textit{for all } k \in \mathbb{N},$$
$$P_1 - \iota(Q) = dl(R_1).$$

(2) *The ring of $G'_{\mathbb{C}}$-invariant holomorphic differential operators on $X_{\mathbb{C}}$ is a polynomial algebra of $(2n+1)$-generators with the following expressions:*

$$\begin{aligned}\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}}) =& \mathbb{C}[P_1, \cdots, P_{n+1}, \iota(Q_1), \cdots, \iota(Q_n)] \\ =& \mathbb{C}[P_2, \cdots, P_{n+1}, \iota(Q), \iota(Q_1), \cdots, \iota(Q_n)] \\ =& \mathbb{C}[\iota(Q_1), \cdots, \iota(Q_n), dl(R_1), \cdots, dl(R_{n+1})] \\ =& \mathbb{C}[P_1, \cdots, P_{n+1}, dl(R_1), \cdots, dl(R_n)] \\ =& \mathbb{C}[P_1, \cdots, P_n, dl(R_1), \cdots, dl(R_{n+1})].\end{aligned}$$

(3) *For $\tau = F(U(2n), (k_1, k_1, k_2, k_2, \cdots, k_n, k_n)) \boxtimes F(U(1), k_0) \in \mathrm{Disc}(L'_U/H'_U)$, the map ν_τ in Theorem 3.8 is given as*

$$\begin{array}{cccc} \nu_\tau : \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \mathbb{C}) & \to & \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}}), \mathbb{C}), & \\ \wr| & & \wr| & \\ \mathbb{C}^{n+1}/\mathfrak{S}_{n+1} & \to & \mathbb{C}^{2n+1}/\mathfrak{S}_{2n+1}, & \lambda \mapsto \nu_\tau(\lambda), \end{array}$$

where

$$\nu_\tau(\lambda) := (\frac{\lambda_1}{2}, \cdots, \frac{\lambda_n}{2}, k_1 + n - 1, k_2 + n - 3, \cdots, k_n - n + 1, k_0). \quad (4.9)$$

4.5 List of examples

We give an exhaustive list of quadruples (G_U, H_U, G'_U, H'_U) in Table 4.1 up to finite coverings of groups, subject to the following four conditions:

· G_U is a compact simple Lie group,
· H_U and G'_U are maximal proper subgroups of G_U,
· $G_U = H_U G'_U$,
· $G'_{\mathbb{C}}/H'_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.

In Table 4.1, we also write a maximal proper subgroup L'_U of G'_U that contains H'_U. The complexifications $(G_{\mathbb{C}}, H_{\mathbb{C}}, G'_{\mathbb{C}}, H'_{\mathbb{C}})$ of the quadruples (G_U, H_U, G'_U, H'_U) in Table 4.1 together with $F_{\mathbb{C}} := L'_{\mathbb{C}}/H'_{\mathbb{C}}$ are given in Table 4.2, and their real forms are in Table 4.3 up to finite coverings and finitely many disconnected components. When two subgroups K_1 and K_2 commute each other and $K_1 \cap K_2$ is a finite group, we write $K_1 \cdot K_2$ for the quotient group $(K_1 \times K_2)/K_1 \cap K_2$ in Tables 4.1-4.3.

Theorems 3.5 and 3.6 apply to Table 4.2. The pair $(\mathcal{K}_1, \mathcal{K}_2)$ of maps in Lemma 3.7 (the compact case of Theorem 2.10) will be computed explicitly in [12] for all the cases in Table 4.1. Theorem 5.1 for discretely decomposable restrictions apply to those in Table 4.3 with $F = L'/H'$ compact. Theorem 6.3 for spectral analysis on non-Riemannian locally symmetric spaces apply to those in Table 4.3 with H' compact.

Table 4.1 compact case

	G_U	H_U	G'_U	H'_U	L'_U
(i)	$SO(2n+2)$	$SO(2n+1)$	$U(n+1)$	$U(n)$	$U(n)\cdot U(1)$
(ii)	$SO(2n+2)$	$U(n+1)$	$SO(2n+1)$	$U(n)$	$SO(2n)$
(iii)	$SU(2n+2)$	$U(2n+1)$	$Sp(n+1)$	$Sp(n)\cdot U(1)$	$Sp(n)\cdot Sp(1)$
(iv)	$SU(2n+2)$	$Sp(n+1)$	$U(2n+1)$	$Sp(n)\cdot U(1)$	$U(2n)\cdot U(1)$
(v)	$SO(4n+4)$	$SO(4n+3)$	$Sp(n+1)\cdot Sp(1)$	$Sp(n)\cdot\Delta(Sp(1))$	$Sp(n)\cdot Sp(1)^2$
(vi)	$SO(16)$	$SO(15)$	$Spin(9)$	$Spin(7)$	$Spin(8)$
(vii)	$SO(8)$	$Spin(7)$	$SO(5)\cdot SO(3)$	$SU(2)\cdot\Delta(SU(2))$	$SO(4)\cdot SO(3)$
(viii)	$SO(7)$	$G_{2(-14)}$	$SO(5)\cdot SO(2)$	$SU(2)\cdot\Delta(SO(2))$	$SO(4)\cdot SO(2)$
(ix)	$SO(7)$	$G_{2(-14)}$	$SO(6)$	$SU(3)$	$U(3)$
(x)	$SO(7)$	$SO(6)$	$G_{2(-14)}$	$SU(3)$	$SU(3)$
(xi)	$SO(8)$	$Spin(7)$	$SO(7)$	$G_{2(-14)}$	$G_{2(-14)}$
(xii)	$SO(8)$	$SO(7)$	$Spin(7)$	$G_{2(-14)}$	$G_{2(-14)}$
(xiii)	$SO(8)$	$Spin(7)$	$SO(6)\cdot SO(2)$	$SU(3)\cdot\Delta(SO(2))$	$U(3)\cdot SO(2)$
(xiv)	$SO(8)$	$SO(6)\cdot SO(2)$	$Spin(7)$	$SU(3)\cdot\Delta(SO(2))$	$Spin(6)$

In Table 4.2, we have used the following notation:

$$\begin{aligned} OG_n(\mathbb{C}) :=& O(2n,\mathbb{C})/GL(n,\mathbb{C}), \\ GS_n(\mathbb{C}) :=& GL(2n,\mathbb{C})/Sp(n,\mathbb{C}), \\ S^n_{\mathbb{C}} :=& \{(z_1,\cdots,z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{j=1}^{n+1} z_j^2 = 1\}. \end{aligned}$$

Table 4.2 Complexification of the quadruples (G_U, H_U, G'_U, H'_U) and $F_U = L'_U/H'_U$ in Table 4.1

	$G_{\mathbb{C}}$	$H_{\mathbb{C}}$	$G'_{\mathbb{C}}$	$H'_{\mathbb{C}}$	F_U
(i)$_{\mathbb{C}}$	$SO(2n+2,\mathbb{C})$	$SO(2n+1,\mathbb{C})$	$GL(n+1,\mathbb{C})$	$GL(n,\mathbb{C})$	$\mathbb{C}^\times$
(ii)$_{\mathbb{C}}$	$SO(2n+2,\mathbb{C})$	$GL(n+1,\mathbb{C})$	$SO(2n+1,\mathbb{C})$	$GL(n,\mathbb{C})$	$OG_n(\mathbb{C})$
(iii)$_{\mathbb{C}}$	$SL(2n+2,\mathbb{C})$	$GL(2n+1,\mathbb{C})$	$Sp(n+1,\mathbb{C})$	$Sp(n,\mathbb{C})\cdot\mathbb{C}^\times$	$S^2_{\mathbb{C}}$
(iv)$_{\mathbb{C}}$	$SL(2n+2,\mathbb{C})$	$Sp(n+1,\mathbb{C})$	$GL(2n+1,\mathbb{C})$	$Sp(n,\mathbb{C})\cdot\mathbb{C}^\times$	$GS_n(\mathbb{C})$
(v)$_{\mathbb{C}}$	$SO(4n+4,\mathbb{C})$	$SO(4n+3,\mathbb{C})$	$Sp(1,\mathbb{C})\cdot Sp(n+1,\mathbb{C})$	$Sp(n,\mathbb{C})\cdot\Delta(Sp(1,\mathbb{C}))$	$S^3_{\mathbb{C}}$
(vi)$_{\mathbb{C}}$	$SO(16,\mathbb{C})$	$SO(15,\mathbb{C})$	$Spin(9,\mathbb{C})$	$Spin(7,\mathbb{C})$	$S^7_{\mathbb{C}}$
(vii)$_{\mathbb{C}}$	$SO(8,\mathbb{C})$	$Spin(7,\mathbb{C})$	$SO(5,\mathbb{C})\cdot SO(3,\mathbb{C})$	$SL(2,\mathbb{C})\cdot\Delta(SL(2,\mathbb{C}))$	$S^3_{\mathbb{C}}$
(viii)$_{\mathbb{C}}$	$SO(7,\mathbb{C})$	$G_2(\mathbb{C})$	$SO(5,\mathbb{C})\cdot SO(2,\mathbb{C})$	$SL(2,\mathbb{C})\cdot\Delta(SO(2,\mathbb{C}))$	$S^3_{\mathbb{C}}$
(ix)$_{\mathbb{C}}$	$SO(7,\mathbb{C})$	$G_2(\mathbb{C})$	$SO(6,\mathbb{C})$	$SL(3,\mathbb{C})$	$\mathbb{C}^\times$
(x)$_{\mathbb{C}}$	$SO(7,\mathbb{C})$	$SO(6,\mathbb{C})$	$G_2(\mathbb{C})$	$SL(3,\mathbb{C})$	$\{\mathrm{pt}\}$
(xi)$_{\mathbb{C}}$	$SO(8,\mathbb{C})$	$Spin(7,\mathbb{C})$	$SO(7,\mathbb{C})$	$G_2(\mathbb{C})$	$\{\mathrm{pt}\}$
(xii)$_{\mathbb{C}}$	$SO(8,\mathbb{C})$	$SO(7,\mathbb{C})$	$Spin(7,\mathbb{C})$	$G_2(\mathbb{C})$	$\{\mathrm{pt}\}$
(xiii)$_{\mathbb{C}}$	$SO(8,\mathbb{C})$	$Spin(7,\mathbb{C})$	$SO(6,\mathbb{C})\cdot SO(2,\mathbb{C})$	$SL(3,\mathbb{C})\cdot\Delta(\mathbb{C}^\times)$	$\mathbb{C}^\times$
(xiv)$_{\mathbb{C}}$	$SO(8,\mathbb{C})$	$SO(6,\mathbb{C})\cdot SO(2,\mathbb{C})$	$Spin(7,\mathbb{C})$	$SL(3,\mathbb{C})\cdot\Delta(\mathbb{C}^\times)$	$OG_3(\mathbb{C})$

Table 4.3 Real forms of the quintuples in Table 4.2

	G	H	G'	H'	F
(i)$_{\mathbb{R}}$	$SO(2p,2q)$	$SO(2p,2q-1)$	$U(p,q)$	$U(p,q-1)$	S^1
(i)$_{\mathbb{R}}$	$SO(n,n)$	$SO(n,n-1)$	$GL(n,\mathbb{R})$	$GL(n-1,\mathbb{R})$	$\mathbb{R}$
(ii)$_{\mathbb{R}}$	$SO(2p,2q)$	$U(p,q)$	$SO(2p,2q-1)$	$U(p,q-1)$	$OU_{p,q-1}$
(ii)$_{\mathbb{R}}$	$SO(n,n)$	$GL(n,\mathbb{R})$	$SO(n,n-1)$	$GL(n-1,\mathbb{R})$	OG_{n-1}
(iii)$_{\mathbb{R}}$	$SU(2p,2q)$	$U(2p,2q-1)$	$Sp(p,q)$	$Sp(p,q-1)\cdot U(1)$	S^2
(iii)$_{\mathbb{R}}$	$SL(2n,\mathbb{R})$	$GL(2n-1,\mathbb{R})$	$Sp(n,\mathbb{R})$	$Sp(n-1,\mathbb{R})\cdot GL(1,\mathbb{R})$	$S^{1,1}$
(iv)$_{\mathbb{R}}$	$SU(2p,2q)$	$Sp(p,q)$	$U(2p,2q-1)$	$Sp(p,q-1)\cdot U(1)$	$US_{p,q-1}$
(iv)$_{\mathbb{R}}$	$SL(2n,\mathbb{R})$	$Sp(n,\mathbb{R})$	$GL(2n-1,\mathbb{R})$	$Sp(n-1,\mathbb{R})\cdot GL(1,\mathbb{R})$	GS_{n-1}
(v)$_{\mathbb{R}}$	$SO(4p,4q)$	$SO(4p,4q-1)$	$Sp(p,q)\cdot Sp(1)$	$Sp(p,q-1)\cdot\Delta(Sp(1))$	S^3
(vi)$_{\mathbb{R}}$	$SO(8,8)$	$SO(8,7)$	$Spin(8,1)$	$Spin(7)$	S^7
(vi)$_{\mathbb{R}}$	$SO(8,8)$	$SO(8,7)$	$Spin(5,4)$	$Spin(3,4)$	$S^{3,4}$
(vii)$_{\mathbb{R}}$	$SO(4,4)$	$Spin(4,3)$	$SO(4,1)\cdot SO(3)$	$SU(2)\cdot\Delta(SU(2))$	S^3
(viii)$_{\mathbb{R}}$	$SO(4,3)$	$G_2(\mathbb{R})$	$SO(4,1)\cdot SO(2)$	$SU(2)\cdot\Delta(SO(2))$	S^3
(viii)$_{\mathbb{R}}$	$SO(4,3)$	$G_2(\mathbb{R})$	$SO(2,3)\cdot SO(2)$	$SL(2,\mathbb{R})\cdot\Delta(SO(2))$	$S^{2,1}$
(viii)$_{\mathbb{R}}$	$SO(4,3)$	$G_2(\mathbb{R})$	$SO(3,2)\cdot SO(1,1)$	$SL(2,\mathbb{R})\cdot\Delta(SO(1,1))$	$S^{2,1}$
(ix)$_{\mathbb{R}}$	$SO(4,3)$	$G_2(\mathbb{R})$	$SO(3,3)$	$SL(3,\mathbb{R})$	$\mathbb{R}$
(ix)$_{\mathbb{R}}$	$SO(4,3)$	$G_2(\mathbb{R})$	$SO(4,2)$	$SU(2,1)$	S^1
(x)$_{\mathbb{R}}$	$SO(4,3)$	$SO(3,3)$	$G_2(\mathbb{R})$	$SL(3,\mathbb{R})$	$\{\mathrm{pt}\}$
(x)$_{\mathbb{R}}$	$SO(4,3)$	$SO(4,2)$	$G_2(\mathbb{R})$	$SU(2,1)$	$\{\mathrm{pt}\}$
(xi)$_{\mathbb{R}}$	$SO(4,4)$	$Spin(4,3)$	$SO(4,3)$	$G_2(\mathbb{R})$	$\{\mathrm{pt}\}$
(xii)$_{\mathbb{R}}$	$SO(4,4)$	$SO(4,3)$	$Spin(4,3)$	$G_2(\mathbb{R})$	$\{\mathrm{pt}\}$
(xiii)$_{\mathbb{R}}$	$SO(4,4)$	$Spin(4,3)$	$SO(4,2)\cdot SO(2)$	$SU(2,1)\cdot\Delta(SO(2))$	S^1
(xiii)$_{\mathbb{R}}$	$SO(4,4)$	$Spin(4,3)$	$SO(3,3)\cdot SO(1,1)$	$SL(3,\mathbb{R})\cdot\Delta(SO(1,1))$	$\mathbb{R}$
(xiv)$_{\mathbb{R}}$	$SO(4,4)$	$SO(4,2)\cdot SO(2)$	$Spin(4,3)$	$SU(2,1)\cdot\Delta(SO(2))$	$OU_{2,1}$
(xiv)$_{\mathbb{R}}$	$SO(4,4)$	$SO(3,3)\cdot SO(1,1)$	$Spin(4,3)$	$SL(3,\mathbb{R})\cdot\Delta(SO(1,1))$	OG_3

In Table 4.3, we have used the following notation:

$$\begin{aligned} OU_{p,q} &:= O(2p,2q)/U(p,q), \\ OG_n &:= O(n,n)/GL(n,\mathbb{R}), \\ US_{p,q} &:= U(2p,2q)/Sp(p,q), \\ GS_n &:= GL(2n,\mathbb{R})/Sp(n,\mathbb{R}). \end{aligned}$$

We note that $OU_{p,q}$ (or $US_{p,q}$) is compact if and only if $p = 0$ or $q = 0$.

5 Applications to branching laws

Branching problems ask how irreducible representations π of a group G behave (*e.g.,* decompose) when restricted to its subgroup G'. In general, branching problems of infinite-dimensional representations of real reductive Lie groups $G \supset G'$ are difficult: for instance, there is no general "algorithm" like the finite-dimensional case. We apply the results on invariant differential operators (Theorems 3.5 and 3.6) to branching problems. We shall see a trick transferring results for finite-dimensional representations in the compact setting to those for infinite-dimensional representations which are realized in the space of functions or distributions on real forms X of $G'_{\mathbb{C}}$-spherical homogeneous spaces $X_{\mathbb{C}}$ in the noncompact setting by the following scheme:

Finite-dimensional representations of compact Lie groups G_U and G'_U

$\rightsquigarrow$

Invariant differential operators on $X_{\mathbb{C}}$ for the complexified groups $G_{\mathbb{C}}$ and $G''_{\mathbb{C}}$
(Theorems 3.5 and 3.6)

$\rightsquigarrow$

Infinite-dimensional representations of noncompact real forms G and G'

5.1 *Discrete decomposability of restriction of unitary representations*

Let G be a real reductive Lie group with maximal compact subgroup K. A $(\mathfrak{g}, K)$-module (π_K, V) is said to be *discretely decomposable* if there exists an increasing filtration $\{V_n\}_{n\in\mathbb{N}}$ such that $V = \cup_{n\in\mathbb{N}} V_n$ and that each V_n is a $(\mathfrak{g}, K)$-module of finite length. If π_K is the underlying $(\mathfrak{g}, K)$-module of a unitary representation π of G, then this condition implies that π decomposes discretely into a Hilbert direct sum of irreducible unitary representations of G ([23]).

In this section, as an application of "global analysis with hidden symmetry"((A) and (B) in Introduction) to branching problems ((C) in Introduction), we give a geometric sufficient condition for the restriction of an irreducible unitary representation of a reductive Lie group G not to have continuous spectrum when restricted to a subgroup G'. In Setting 2.3, we take a maximal reductive subgroup L' of G' containing H', and set $F := L'/H'$ so that we have a fibration $F \to X \to Y$ (see (2.1)).

Theorem 5.1 (discrete decomposability of restriction). *Suppose we are in Setting 2.3. Assume that $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical and that F is compact.*

(1) (($\mathfrak{g}, K$)-modules) *Any irreducible $(\mathfrak{g}, K)$-module π_K occurring as a subquotient of the regular representation of G on the space $\mathcal{D}'(X)$ of distributions on X is discretely decomposable as a $(\mathfrak{g}', K')$-module.*
(2) (unitary representation) *For any irreducible unitary representation π of G realized in $\mathcal{D}'(X)$, the restriction $\pi|_{G'}$ decomposes discretely into a Hilbert direct sum of irreducible unitary representations of G'.*
(3) (discrete series) *In (2), if π is a discrete series representation for G/H, then any irreducible summand of the restriction $\pi|_{G'}$ is a discrete series representation for G'/H'.*

Remark 5.2. (1) In the case where H' is compact, Theorem 5.1 will be discussed in detail in [13] in connection to spectral analysis on non-Riemannian locally symmetric spaces $\Gamma\backslash G/H$, see Section 6. We note that if H' is compact, we can take L' to be a maximal compact subgroup of G' containing H' so that $F = L'/H'$ is compact.
(2) A general criterion for discrete decomposability of the restrictions of irreducible unitary representations was given in [20, 21] in terms of invariants of representations. Representations π treated in Theorem 5.1 are much limited, however, we can tell *a priori* from Theorem 5.1 discrete decomposability of the restriction $\pi|_{G'}$ before knowing what the representations π are.

Proof (Sketch of the proof of Theorem 5.1).

(1) Suppose π_K is realized in a subspace V of $\mathcal{D}'(X)$. (We remark that V is automatically contained in $C^\infty(X)$ by the elliptic regularity theorem.) Since $X_{\mathbb{C}}$ is $G_{\mathbb{C}}$-spherical, $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ is finitely generated as a $dl(\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}}))$-module ([14]). Since $\mathfrak{Z}(\mathfrak{g}_{\mathbb{C}})$ acts on V as scalars, the $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$-module $\widetilde{V} := \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}) \cdot V$ is $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$-finite.
Now we consider the G'-equivariant fibration $F \to X \to Y$. Decomposing $\widetilde{V}$ along the compact fiber $F = L'/H'$, we see that there is an irreducible finite-dimensional representation $\tau \in \mathrm{Disc}(L'/H')$ such that the τ-component $\widetilde{V}_\tau$ of $\widetilde{V}$ from the right is nonzero.
Since $\mathfrak{Z}(\mathfrak{l}'_{\mathbb{C}})$ acts on τ as scalars, the action of the subalgebra generated by $\mathcal{P} = \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ and $\mathcal{Q} = \iota(\mathbb{D}_{L_{\mathbb{C}}}(F_{\mathbb{C}}))$ factors through a finite-dimensional algebra, and so does the action of $\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}})$ by Theorem 3.5 (2). Since the $(\mathfrak{g}, K)$-module

$\widetilde{V}$ contains a $\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}})$-finite $\mathfrak{g}'$-module $\widetilde{V}_\tau$, $\widetilde{V}$ is discretely decomposable as a $(\mathfrak{g}', K')$-module by [21].

(2) The statement follows from (1) and [23, Theorem 2.7].

(3) The third statement follows from (1) and [22, Theorem 8.6].

Example 5.3. Let $G = O(p,q)$, $H = O(p-1,q)$ and

$$X := G/H \simeq S^{p-1,q}.$$

In what follows, π stands for any irreducible subquotient module of G of the regular representation on the space $\mathcal{D}'(X)$ of distributions, and π_K for the underlying $(\mathfrak{g}, K)$-module.

(1) ($O(2p', 2q') \downarrow U(p', q')$) Suppose $p = 2p'$ and $q = 2q'$ with $p', q' \in \mathbb{N}$. Let $G' = U(p', q')$ be a natural subgroup of G. As one can observe from Tables 4.1 and 4.3 $(\mathrm{i})_{\mathbb{R}}$,

$$F = L'/H' = (U(p', q'-1) \times U(1))/U(p', q'-1) \simeq U(1)$$

is compact (see also (4.2)). By Theorem 5.1, any π_K is discretely decomposable as a $(\mathfrak{g}', K')$-module.

(2) ($O(4p'', 4q'') \downarrow Sp(p'', q'')$) Suppose $p = 4p''$ and $q = 4q''$ with $p'', q'' \in \mathbb{N}$. Let $G'' := Sp(p'', q'')$ be a natural subgroup of G. Then by Tables 4.1 and 4.3 $(\mathrm{v})_{\mathbb{R}}$,

$$\begin{aligned} F =& L''/H'' \\ =& (Sp(p'', q''-1) \times Sp(1) \times Sp(1))/Sp(p'', q''-1) \times \Delta(Sp(1)) \simeq Sp(1) \end{aligned}$$

is compact. By Theorem 5.1, any π_K is discretely decomposable as a $(\mathfrak{g}'', K'')$-module.

Remark 5.4. (1) In the setting of Example 5.3, explicit branching laws were given in [17] in terms of Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$ when π is a discrete series representation for X, namely, when π is an irreducible unitary representation of G which can be realized in a closed invariant subspace of the Hilbert space $L^2(X)$.

(2) Any π_K in $\mathcal{D}'(X)$ occurs as a subquotient of the most degenerate principal series representation of G that was the main object of Howe–Tan [8], and *vice versa*. The restrictions $O(2p', 2q') \downarrow U(p', q')$ and $O(4p'', 4q'') \downarrow Sp(p'', q'')$ were discussed also in [8] from the viewpoint of the "see-saw" dual pairs.

5.2 Branching law $SO(8,8) \downarrow Spin(1,8)$

We apply the previous results (*e.g.*, Theorems 2.10, 3.8, and 5.1) to find new branching laws of the restriction of unitary representations with respect to the nonsymmetric pair

$$(G, G') = (SO_0(8,8), Spin(1,8))$$

when G' is realized in G *via* the spin representation. The subscript 0 stands for the identity component.

The main results of this section is Theorem 5.5, which might be interesting on its own since not much is known about the restriction of Zuckerman's derived functor module $A_{\mathfrak{q}}(\lambda)$ with respect to pairs (G, G') of reductive groups except for the case where (G, G') is a symmetric pair or there is a subgroup G'' such that $G \supset G'' \supset G'$ is a chain of symmetric pairs (e.g. $(G, G'', G') = (O(4p, 4q), U(2p, 2q), Sp(p, q))$. (Cf. [4, 16, 17, 25, 31] for branching laws with respect to symmetric pairs).

In order to state Theorem 5.1, we fix some notation. Let π_λ ($\lambda \in \mathbb{N}_+$) be irreducible unitary representations of $G = SO_0(8,8)$ attached to minimal elliptic orbits in the philosophy of orbit method. For the reader's convenience, we collect some properties of π_λ:

- The underlying $(\mathfrak{g}, K)$-module $(\pi_\lambda)_K$ of π_λ is given by Zuckerman derived functor module $A_{\mathfrak{q}}(\lambda - 7)$ where $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$ such that the normalizer of $\mathfrak{q}$ in G is $SO(2) \times SO_0(6,8)$. Concerning the ρ-shift of $A_{\mathfrak{q}}(\lambda)$, we adopt the same normalization as in Vogan–Zuckerman [36].
- The $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character of π_λ is $(\lambda, 6, 5, 4, 3, 2, 1, 0)$.
- The K-type formula of π_λ is given by

$$(\pi_\lambda)_K \simeq \bigoplus_{(m,n) \in \Xi(\lambda+1)} \mathcal{H}^m(\mathbb{R}^8) \boxtimes \mathcal{H}^n(\mathbb{R}^8),$$

  where we recall from (4.6) the definition of the parameter set $\Xi(\mu)$.

Let us recall the classification of the Harish-Chandra discrete series representation for $G' = Spin(1,8)$. For $\varepsilon = \pm$ and $b = (b_1, b_2, b_3, b_4) \in \mathbb{Z}^4$ or $\mathbb{Z}^4 + \frac{1}{2}(1,1,1,1)$ such that $b_1 \geq b_2 \geq b_3 \geq b_4 \geq 1$, we write ϑ_b^ε for the discrete series representation of G' with

Harish-Chandra parameter: $(b_1 + \frac{5}{2}, b_2 + \frac{3}{2}, b_3 + \frac{1}{2}, b_4 - \frac{1}{2})$,

Blattner parameter: $(b_1, b_2, b_3, \varepsilon b_4)$.

Then any discrete series representation of G' is of this form. For $k \geq l \geq 2$ with $k \equiv l \mod 2$, we set

$$\vartheta_{k,l} := \vartheta^+_{\frac{1}{2}(k,k,k,l)}.$$

We are ready to state a branching law of the unitary representation on π_λ with respect to the nonsymmetric pair $(G, G') = (SO_0(8,8), Spin(1,8))$.

Theorem 5.5 ($SO_0(8,8) \downarrow Spin(1,8)$). *For any $\lambda \in \mathbb{N}_+$, the irreducible unitary representation π_λ of $G = SO_0(8,8)$ decomposes discretely as a representation of $G' = Spin(1,8)$ in accordance with the following branching rule.*

$$\pi_\lambda|_{G'} \simeq \sum_{l=0}^{\infty}{}^{\oplus} \vartheta_{\lambda+2l+1,\lambda+1}.$$

Remark 5.6. In general, if π is a Harish-Chandra discrete series representation of a real reductive Lie group G, then any irreducible summand of the restriction $\pi|_{G'}$ to a reductive subgroup G' is a Harish-Chandra discrete series representation of G' ([22]). Theorem 5.5 shows that the converse statement is not always true because π_λ is a nontempered representation of G whereas any $\vartheta_{k,l}$ is a Harish-Chandra discrete series of G'.

For the proof of Theorem 5.5, we compute explicitly the double fibration in Theorem 2.10. We begin with an explicit K-type formula of ϑ_b^ε:

Lemma 5.7. *Suppose ϑ_b^ε is the (Harish-Chandra) discrete series representation of $G' = Spin(1,8)$ with Blattner parameter $(b_1, b_2, b_3, \varepsilon b_4)$. Then the restriction of ϑ_b^ε to a maximal compact subgroup $L' = Spin(8)$ of G' decomposes as*

$$\vartheta_b^\varepsilon|_{Spin(8)} \simeq \sum_{\mu\in Z(b)}{}^{\oplus} F(Spin(8), (\mu_1, \mu_2, \mu_3, \varepsilon\mu_4))$$

where, for $b = (b_1, b_2, b_3, b_4)$, we set

$$Z(b) := \{\mu \in \mathbb{Z}^4 + b : \mu_1 \geq b_1 \geq \mu_2 \geq b_2 \geq \mu_3 \geq b_3 \geq \mu_4 \geq b_4\}.$$

For $k \in \mathbb{N}$, we set $\tau_k := F(Spin(8), \frac{1}{2}(k,k,k,k))$. The unitary representation of L' on $L^2(L'/H') = L^2(Spin(8)/Spin(7))$ is multiplicity-free, and we have

$$\underline{\mathrm{Disc}}(L'/H') = \mathrm{Disc}(L'/H') = \{\tau_k : k \in \mathbb{N}\}.$$

Let $\mathcal{W}_{\tau_k} = G' \times_{L'} \tau_k$ be the homogeneous vector bundle over the 8-dimensional hyperbolic space $Y := G'/L' = Spin(1,8)/Spin(8)$.

Proposition 5.8. *Let $k \in \mathbb{N}$. There are at most finitely many discrete series representations for $L^2(Y, \mathcal{W}_{\tau_k})$, and they are given as follows, where the sum is multiplicity-free:*

$$L_d^2(Y, \mathcal{W}_{\tau_k}) \simeq \bigoplus_{2\leq l\leq k,\, l\equiv k \mod 2} \vartheta_{k,l}.$$

Proof. By Lemma 5.7, τ_k occurs in ϑ_b^ε as a K-type if and only if $\varepsilon = +$ and $b = \frac{1}{2}(k,k,k,l)$ for some $l \in 2\mathbb{Z} + k$ with $2 \leq l \leq k$, namely, $\vartheta_b^\varepsilon = \vartheta_{k,l}$. Thus the proposition follows from the Frobenius reciprocity.

We have thus shown

$$\mathcal{K}_2^{-1}(\tau_k) = \{\vartheta_{k,l} : 2 \leq l \leq k, l \equiv k \mod 2\}, \tag{5.1}$$
$$\underline{\mathrm{Disc}}(G'/H') = \bigcup_{k\in\mathbb{N}} \mathcal{K}_2^{-1}(\tau_k) = \{\vartheta_{k,l} : (k,l) \in \Xi(2)\},$$

where we recall from (4.6) for the definition of $\Xi(\mu)$. In particular, discrete series for G'/H' is multiplicity-free, *i.e.*, $\underline{\mathrm{Disc}}(G'/H') = \mathrm{Disc}(G/H)$.

On the other hand, we recall the geometry $X = G/H$ where $H = SO_0(7,8)$ and a realization π_λ in the regular representation $L^2(X)$:

- $\mathrm{Disc}(G/H) = \{\pi_\lambda : \lambda \in \mathbb{N}_+\}$.
- $X := G/H \simeq S^{8,7}$ carries a pseudo-Riemannian metric of signature $(8,7)$, normalized so that the sectional curvature is constant equal to -1 (see (4.1)). Then G acts isometrically on the pseudo-Riemannian space form $X \simeq S^{8,7}$, and the Laplacian $\Box_X$ acts as the scalar $\lambda^2 - 49$ on the representation space of π_λ in $L^2(X)$.

Therefore, the double fibration of Theorem 2.10 amounts to

$$
\begin{array}{ccc}
 & \{\vartheta_{k,l} \in \widehat{Spin(1,8)} : (k,l) \in \Xi(2)\} & \\
\mathcal{K}_1 \swarrow & & \searrow \mathcal{K}_2 \\
\{\pi_\lambda \in \widehat{SO_0(8,8)} : \lambda \in \mathbb{N}_+\} & & \{\tau_k \in \widehat{Spin(8)} : k \in \mathbb{N}\}.
\end{array}
$$

We already know the map $\mathcal{K}_2$ explicitly by (5.1). Let us find the map $\mathcal{K}_1$ explicitly by using Theorem 3.8. We recall that the branching law of the restriction $\pi_\lambda|_{G'}$ is nothing but to determine the fiber of the projection $\mathcal{K}_1$.

Suppose $\vartheta_{k,l} \in \mathcal{K}_1^{-1}(\pi_\lambda)$. Since $\mathcal{K}_2(\vartheta_{k,l}) = \tau_k$, the $\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}})$-infinitesimal character of $\vartheta_{k,l}$ is subject to Theorem 3.8. By (4.8), we have

$$\frac{1}{2}(k+5, k+3, k+1, l-1) \equiv \frac{1}{2}(\lambda, k+5, k+3, k+1) \mod W(B_4) \simeq \mathfrak{S}_4 \ltimes (\mathbb{Z}_2)^4.$$

Hence $\lambda = l - 1$, and $\mathcal{K}_1(\vartheta_{k,\lambda+1}) = \pi_\lambda$. Thus the fiber of $\mathcal{K}_1$ is given by

$$\mathcal{K}_1^{-1}(\pi_\lambda) = \{\vartheta_{k,\lambda+1} : \lambda + 1 \leq k, k \equiv \lambda + 1 \mod 2\}.$$

Now Theorem 5.5 is proved.

6 Application to spectral analysis on non-Riemannian locally symmetric spaces $\Gamma\backslash G/H$

In this section we discuss briefly an application of Theorem 3.8 to the analysis on *non-Riemannian* locally symmetric spaces $\Gamma\backslash G/H = \Gamma\backslash X$, for which we initiated a new line of investigation in [11] by a different approach.

We begin with a brief review on the geometry. Suppose that a discrete group Γ acts continuously on X. We recall that the action is said to be *properly discontinuous* if any compact subset of X meets only finitely many of its Γ-translates. If Γ acts properly discontinuously and freely, the quotient $\Gamma\backslash X$ is of Hausdorff topology and carries a natural C^∞-manifold structure such that

$$X \to \Gamma\backslash X$$

is a covering map. The quotient $\Gamma\backslash X = \Gamma\backslash G/H$ is said to be a *Clifford–Klein form of* $X = G/H$.

Suppose $X = G/H$ with H noncompact. Then not all discrete subgroups of G act properly discontinuously: for instance, de Sitter space $S^{n,1} = O(n+1,1)/O(n,1)$ does not admit any infinite properly discontinuous action of isometries (Calabi–Markus phenomenon [3]). Also, infinite subgroups of H never act properly discontinuously on X, because the origin $o := eH \in X$ is a fixed point. In fact, determining which subgroups act properly discontinuously is a delicate question, which was first considered in full generality in [15] in the late 1980s; we refer to [27] for a survey.

A large and important class of examples is constructed as follows (see [15]):

Definition 6.1 (standard Clifford–Klein form). The quotient $\Gamma\backslash X$ of X by a discrete subgroup Γ of G is said to be *standard* if Γ is contained in some reductive subgroup G' of G acting properly on X.

Any $G_{\mathbb{C}}$-invariant holomorphic differential operator on $X_{\mathbb{C}}$ defines a G-invariant (in particular, Γ-invariant) differential operator on X by restriction, and hence induces a differential operator, to be denoted by D_Γ on $\Gamma\backslash X$.

Given $\lambda \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \mathbb{C})$, we set the space of joint eigenfunctions

$$C^\infty(\Gamma\backslash X; \mathcal{M}_\lambda) := \{f \in C^\infty(\Gamma\backslash X) : D_\Gamma f = \lambda(D) f \quad \text{for all } D \in \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})\}.$$

There has been an extensive study on spectral analysis $\Gamma\backslash X$ when X is a reductive symmetric space G/H under additional assumptions:

- $\Gamma = \{e\}$, or
- H is a maximal compact subgroup.

However, not much is known about $C^\infty(\Gamma\backslash X; \mathcal{M}_\lambda)$ when $X = G/H$ with H noncompact. In fact, if we try to attack a problem of spectral analysis on $\Gamma\backslash G/H$ in the general case where H is noncompact and Γ is infinite, then new difficulties may arise from several points of view:

(1) Geometry. The G-invariant pseudo-Riemannian structure on $X = G/H$ is not Riemannian anymore, and discrete groups of isometries of X do not always act properly discontinuously on such X as we discussed above.
(2) Analysis. The Laplacian Δ_X on $\Gamma\backslash X$ is not an elliptic differential operator. Furthermore, it is not clear if Δ_X has a self-adjoint extension on $L^2(\Gamma\backslash X)$.
(3) Representation theory. If Γ acts properly discontinuously on $X = G/H$ with H noncompact, then the volume of $\Gamma\backslash G$ is infinite, and the regular representation $L^2(\Gamma\backslash G)$ may have infinite multiplicities. In turn, the group G may not have a good control of functions on $\Gamma\backslash G$.

Let us discuss a connection of the spectral analysis on a non-Riemannian locally homogeneous space $\Gamma\backslash X$ with the results in the previous section.

Suppose that we are in Setting 2.3. This means that a reductive subgroup G' of G acts transitively on X and $X \simeq G'/H'$ where $H' = G' \cap H$. Then we have:

Proposition 6.2 ([15]). *If H' is compact, then G' acts properly on X, and consequently, any torsion-free discrete subgroup Γ of G' acts properly discontinuously and freely, yielding a standard Clifford–Klein form $\Gamma\backslash X$. In particular, there exists a compact standard Clifford–Klein form of X by taking a torsion-free cocompact Γ in G'.*

From now, we assume that H' is compact and that the complexification $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical. (We can read from Table 4.3 the list of quadruples (G, H, G', H') satisfying these assumptions.)

Take a maximal compact subgroup K' of G' containing H'. The group K' plays the same role with L' in Section 2.1, and we set $F := K'/H'$. For each $(\tau, W) \in \widehat{K'}$, we form a vector bundle

$$\mathcal{W}_\tau := \Gamma\backslash G' \times_{K'} W$$

over the Riemannian locally symmetric space $\Gamma\backslash Y := \Gamma\backslash G'/K'$.

For a $\mathbb{C}$-algebra homomorphism $\nu : \mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}}) \to \mathbb{C}$, we define a subspace of $C^\infty(\Gamma\backslash Y, \mathcal{W}_\tau)$ by

$$C^\infty(\Gamma\backslash Y, \mathcal{W}_\tau; \mathcal{N}_\nu) := \{f \in C^\infty(\Gamma\backslash Y, \mathcal{W}_\tau) : dl(z)f = \nu(z)f \quad \text{for all } z \in \mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}})\},$$

which may be regarded as a G'-submodule $C^\infty(\Gamma\backslash G'; \mathcal{N}_\nu)$ of the regular representation of G' on $C^\infty(\Gamma\backslash G')$.

Suppose now $(\tau, W) \in \mathrm{Disc}(K'/H')$. By Lemma 3.7, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K'}(\tau, C^\infty(K'/H')) = 1,$$

and therefore there is a natural map

$$i_\tau : C^\infty(\Gamma\backslash Y, \mathcal{W}_\tau) \to C^\infty(\Gamma\backslash X).$$

Here is another application of Theorem 3.8 to the fiber bundle $F \to \Gamma\backslash X \to \Gamma\backslash Y$ (see [13] for details).

Theorem 6.3. *Suppose we are in Setting 2.3. Assume that $G_{\mathbb{C}}$ is simple and $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical. Let Γ be a torsion-free discrete subgroup of G' so that the locally homogeneous space $\Gamma\backslash X$ is standard.*

(1) *Let ν_τ be the map*

$$\nu_\tau : \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}), \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{g}'_{\mathbb{C}}), \mathbb{C}), \quad \lambda \mapsto \nu_\tau(\lambda)$$

given in Theorem 3.8 for $\tau \in \mathrm{Disc}(K'/H')$. Then, the following two conditions on $\varphi \in C^\infty(\Gamma\backslash Y, \mathcal{W}_\tau)$ are equivalent:

(i) $i_\tau(\varphi) \in C^\infty(\Gamma\backslash X; \mathcal{M}_\lambda)$,

(ii) $\varphi \in C^\infty(\Gamma\backslash Y, \mathcal{W}_\tau; \mathcal{N}_{\nu_\tau(\lambda)})$.

(2) *For every* $\lambda \in \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C})$*, the joint eigenspace* $C^\infty(\Gamma\backslash X; \mathcal{M}_\lambda)$ *contains*

$$\bigoplus_{\tau\in\widehat{F}} i_\tau(C^\infty(Y_\Gamma, \mathcal{W}_\tau; \mathcal{N}_{\nu(\tau)}))$$

as a dense subspace.

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