Vector-valued covariant differential operators for the Möbius transformation

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Abstract

We obtain a family of functional identities satisfied by vector-valued functions of two variables and their geometric inversions. For this we introduce particular differential operators of arbitrary order attached to Gegenbauer polynomials. These differential operators are symmetry breaking for the pair of Lie groups $(SL(2,\mathbb{C}), SL(2,\mathbb{R}))$ that arise from conformal geometry.

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1 A family of vector-valued functional identities

Given a pair of functions f, g on $\mathbb{R}^2\setminus\{(0,0)\}$, we consider a $\mathbb{C}^2$-valued function $\vec{F} := \begin{pmatrix} f \\ g \end{pmatrix}$. Define its "twisted inversion" $\vec{F}_\lambda$ with parameter $\lambda \in \mathbb{C}$ by

$$\vec{F}_\lambda^\vee(r\cos\theta, r\sin\theta) := r^{-2\lambda}\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}\vec{F}\left(\frac{-\cos\theta}{r}, \frac{\sin\theta}{r}\right). \tag{1.1}$$

Clearly, $\vec{F} \mapsto \vec{F}_\lambda^\vee$ is involutive, namely, $(\vec{F}_\lambda^\vee)_\lambda^\vee = \vec{F}$.

A pair of differential operators $\mathcal{D}_1$, $\mathcal{D}_2$ on $\mathbb{R}^2$ yields a linear map

$$\mathcal{D} : C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}), \quad (f, g) \mapsto (\mathcal{D}_1 f)(x, 0) + (\mathcal{D}_2 g)(x, 0).$$

We write

$$\mathcal{D} := \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2).$$

Our main concern in this article is the following:

Question A. (1) For which parameters $\lambda, \nu \in \mathbb{C}$, do there exist differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathbb{R}^2$ with the following properties?

- $\mathcal{D}_1$ and $\mathcal{D}_2$ have constant coefficients.
- For any $\vec{F} \in C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2)$, the functional identity

$$(\mathcal{M}_{\lambda,\nu}) \qquad (\mathcal{D}\vec{F}_\lambda^\vee)(x) = |x|^{-2\nu}(\mathcal{D}\vec{F})\left(-\frac{1}{x}\right), \quad \text{for } x \in \mathbb{R}^\times$$

holds, where $\mathcal{D} = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2)$.
(2) Find an explicit formula of such $\mathcal{D} \equiv \mathcal{D}_{\lambda,\nu}$ if exists.

Our motivation will be explained in Section 2 by giving three equivalent formulations of Question A. Here are some examples of the operators $\mathcal{D}_{\lambda,\nu}$ satisfying $(\mathcal{M}_{\lambda,\nu})$.

Example 1.1. (0) $\nu = \lambda$:

$$\mathcal{D}_{\lambda,\nu} := \mathrm{Rest}_{y=0} \circ (\mathrm{id}, 0)\,,$$

namely,

$$\mathcal{D}_{\lambda,\nu}\begin{pmatrix} f \\ g \end{pmatrix}(x) = f(x,0)$$

satisfies $(\mathcal{M}_{\lambda,\nu})$ for $\nu = \lambda$.

(1) $\nu = \lambda + 1$:

$$\mathcal{D}_{\lambda,\nu} := \mathrm{Rest}_{y=0} \circ \left(\frac{\partial}{\partial x}, \lambda\frac{\partial}{\partial y}\right),$$

namely,

$$\mathcal{D}_{\lambda,\nu}\begin{pmatrix} f \\ g \end{pmatrix}(x) = \frac{\partial f}{\partial x}(x,0) + \lambda\frac{\partial g}{\partial y}(x,0)$$

satisfies $(\mathcal{M}_{\lambda,\nu})$ for $\nu = \lambda + 1$.

(2) $\nu = \lambda + 2$:

$$\mathcal{D}_{\lambda,\nu} := \mathrm{Rest}_{y=0} \circ \left(2(2\lambda+1)\frac{\partial^2}{\partial x \partial y}, (\lambda-1)\frac{\partial^2}{\partial x^2} + (\lambda+1)(2\lambda+1)\frac{\partial^2}{\partial y^2}\right),$$

namely,

$$\mathcal{D}_{\lambda,\nu}\begin{pmatrix} f \\ g \end{pmatrix}(x) = 2(2\lambda+1)\frac{\partial^2 f}{\partial x \partial y}(x,0) + (\lambda-1)\frac{\partial^2 g}{\partial x^2}(x,0) + (\lambda+1)(2\lambda+1)\frac{\partial^2 g}{\partial y^2}(x,0)$$

satisfies $(\mathcal{M}_{\lambda,\nu})$ for $\nu = \lambda + 2$.

Given $\mathcal{D} = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2)$, define

$$\mathcal{D}^{\vee} := \mathrm{Rest}_{y=0} \circ (-\mathcal{D}_2, \mathcal{D}_1). \tag{1.2}$$

Clearly, $\mathcal{D}^{\vee}$ is determined only by $\mathcal{D}$, and is independent of the choice of $\mathcal{D}_1$ and $\mathcal{D}_2$. Proposition 1.2 below shows that the map $\mathcal{D} \mapsto \mathcal{D}^{\vee}$ is an automorphism of the set of the operators $\mathcal{D}$ such that $(\mathcal{M}_{\lambda,\nu})$ is satisfied.

Proposition 1.2. *If $\mathcal{D}$ satisfies $(\mathcal{M}_{\lambda,\nu})$ for all $\vec{F}$, so does $\mathcal{D}^{\vee}$.*

Proof. For $\vec{F} = \begin{pmatrix} f \\ g \end{pmatrix}$, we set ${}^{\vee}\vec{F} := \begin{pmatrix} g \\ -f \end{pmatrix}$. Then we have

$${}^{\vee\vee}\vec{F} = -\vec{F}, \quad \mathcal{D}\left({}^{\vee}\vec{F}\right) = (\mathcal{D}^{\vee})\vec{F}, \quad ({}^{\vee}\vec{F})^{\vee}_{\lambda} = {}^{\vee}(\vec{F}^{\vee}_{\lambda}). \tag{1.3}$$

To see this we note that $w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ commutes with $\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$ and that $\mathcal{D}^{\vee}$ and ${}^{\vee}\vec{F}$ are expressed as $\mathcal{D}^{\vee} = \mathcal{D}w^{-1}$ and ${}^{\vee}\vec{F} = w^{-1}\vec{F}$, respectively. Therefore,

$$\begin{aligned}
\left(\mathcal{D}^{\vee}\vec{F}^{\vee}_{\lambda}\right)(x) &= \mathcal{D}\left({}^{\vee}\vec{F}\right)^{\vee}_{\lambda}(x) \\
&= |x|^{-2\nu}\left(\mathcal{D}\,{}^{\vee}\vec{F}\right)\left(-\frac{1}{x}\right) \\
&= |x|^{-2\nu}\left(\mathcal{D}^{\vee}\vec{F}\right)\left(-\frac{1}{x}\right),
\end{aligned}$$

where the passage from the first line to the second one is justified by the fact that ${}^{\vee}\vec{F}$ satisfies $(\mathcal{M}_{\lambda,\nu})$. □

In order to answer Question A for general (λ, ν), we recall that the Gegenbauer polynomial or ultraspherical polynomial $C^{\alpha}_{\ell}(t)$ is a polynomial in one variable t of degree ℓ given by

$$C^{\alpha}_{\ell}(t) = \sum_{k=0}^{[\frac{\ell}{2}]} (-1)^k \frac{\Gamma(\ell - k + \alpha)}{\Gamma(\alpha)\Gamma(\ell - 2k + 1)k!}(2t)^{\ell-2k},$$

where $[s]$ denotes the greatest integer that does not exceed s. Following [9], we inflate $C^{\alpha}_{\ell}(t)$ to a polynomial of two variables by

$$C^{\alpha}_{\ell}(s,t) := s^{\frac{\ell}{2}} C^{\alpha}_{\ell}\left(\frac{t}{\sqrt{s}}\right). \tag{1.4}$$

By formally substituting $-\frac{\partial^2}{\partial x^2}$ and $\frac{\partial}{\partial y}$ to s and t in $C_\ell^\alpha(s,t)$, respectively, we obtain a homogeneous differential operator $\mathcal{C}_\ell^\alpha := C_\ell^\alpha\left(-\frac{\partial^2}{\partial x^2}, \frac{\partial}{\partial y}\right)$ of order ℓ on $\mathbb{R}^2$. Here are the first four operators:

$$
\begin{aligned}
\mathcal{C}_0^\alpha &= \mathrm{id},\\
\mathcal{C}_1^\alpha &= 2\alpha\tfrac{\partial}{\partial y},\\
\mathcal{C}_2^\alpha &= \alpha\left(-\tfrac{\partial^2}{\partial x^2} + (\alpha+1)\tfrac{\partial^2}{\partial y^2}\right),\\
\mathcal{C}_3^\alpha &= \tfrac{2}{3}\alpha(\alpha+1)\left(3\tfrac{\partial^3}{\partial x^2\partial y} + 2(\alpha+2)\tfrac{\partial^3}{\partial y^3}\right).
\end{aligned}
$$

Theorem A. *Suppose that $a := \nu - \lambda$ is a non-negative integer. For $a > 0$, we define the following pair of homogeneous differential operators of order a on $\mathbb{R}^2$ by*

$$
\begin{aligned}
\mathcal{D}_1 &:= a(2\lambda + a - 1)\frac{\partial}{\partial x}\circ \mathcal{C}_{a-1}^{\lambda+\frac{1}{2}}\\
\mathcal{D}_2 &:= \left(2\lambda^2 + 2(a-1)\lambda + a(a-1)\right)\frac{\partial}{\partial y}\circ \mathcal{C}_{a-1}^{\lambda+\frac{1}{2}}\\
&\qquad\qquad + (\lambda-1)(2\lambda+1)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\circ \mathcal{C}_{a-2}^{\lambda+\frac{3}{2}}.
\end{aligned}
$$

For $a = 0$, we set

$$\mathcal{D}_1 := \mathrm{id}, \quad \mathcal{D}_2 := 0.$$

Then $\mathcal{D} := \mathrm{Rest}_{y=0}\circ(\mathcal{D}_1, \mathcal{D}_2)$ and $\mathcal{D}^\vee := \mathrm{Rest}_{y=0}\circ(\mathcal{D}_2, -\mathcal{D}_1)$ satisfy the functional identity $(\mathcal{M}_{\lambda,\nu})$. Moreover, when $2\lambda \notin \{0, -1, -2, \cdots\}$, there exists a non-trivial solution to $(\mathcal{M}_{\lambda,\nu})$ only if $\nu - \lambda$ is a non-negative integer. and any differential operator satisfying $(\mathcal{M}_{\lambda,\nu})$ is a linear combination of $\mathcal{D}$ and $\mathcal{D}^\vee$.

Notation: $\mathbb{N} := \{0, 1, 2, \ldots\}$
$\mathbb{N}_+ := \{1, 2, \ldots\}$

2 Three equivalent formulations

Question A arises from various disciplines of mathematics. In this section we describe it in three equivalent ways.

2.1 Covariance of $SL(2,\mathbb{R})$ for vector-valued functions

For $\lambda \in \mathbb{C}$, we define a group homomorphism

$$\psi_\lambda : \mathbb{C}^\times \to GL(2,\mathbb{R}), \quad z = re^{i\theta} \mapsto r^\lambda \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.1}$$

For a $\mathbb{C}^2$-valued function $\vec{F}$ on $\mathbb{C} \simeq \mathbb{R}^2$, we set

$$\vec{F}_\lambda^h(z) := \psi_\lambda\left((cz+d)^{-2}\right) \vec{F}\left(\frac{az+b}{cz+d}\right)$$

for $\lambda \in \mathbb{C}$, $h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, and $z \in \mathbb{C}$ such that $cz+d \neq 0$.

Question A′. (1) Determine complex parameters $\lambda, \nu \in \mathbb{C}$ for which there exist differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathbb{R}^2$ with the following property: $\mathcal{D} = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2)$ satisfies

$$(\mathcal{D}\vec{F}_\lambda^h)(x) = |cx+d|^{-2\nu} (\mathcal{D}\vec{F})\left(\frac{ax+b}{cx+d}\right) \tag{2.2}$$

for all $\vec{F} \in C^\infty(\mathbb{C}) \oplus C^\infty(\mathbb{C}), h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, and $x \in \mathbb{R} \setminus \{-\frac{d}{c}\}$.

(2) Find an explicit formula of such $\mathcal{D} \equiv \mathcal{D}_{\lambda,\nu}$.

The equivalence between Questions A and A′ follows from the following three observations:

- The functional identity (2.2) for $h = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ $(t \in \mathbb{R})$ implies that $\mathcal{D} = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2)$ is a translation invariant operator. Therefore, we can take $\mathcal{D}_1$ and $\mathcal{D}_2$ to have constant coefficients.
- $\vec{F}_\lambda^\vee = \vec{F}_\lambda^w$.
- The group $SL(2,\mathbb{R})$ is generated by w and $\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\right\}$.

2.2 Conformally covariant differential operators

Let X be a smooth manifold equipped with a Riemannian metric g. Suppose that a group G acts on X by the map $G \times X \to X$, $(h,x) \mapsto h \cdot x$. This action is called *conformal* if there is a positive-valued smooth function (*conformal factor*) Ω on $G \times X$ such that

$$h^*(g_{h\cdot x}) = \Omega(h,x)^2 g_x \quad \text{for any } h \in G \text{ and } x \in X.$$

Given $\lambda \in \mathbb{C}$, we define a G-equivariant line bundle $\mathcal{L}_\lambda \equiv \mathcal{L}_\lambda^{\mathrm{conf}}$ over X by letting G act on the direct product $X \times \mathbb{C}$ by $(x,u) \mapsto (h \cdot x, \Omega(h,x)^{-\lambda} u)$ for $h \in G$. Then we have a natural action of G on the vector space $\mathcal{E}_\lambda(X) := C^\infty(X, \mathcal{L}_\lambda)$ consisting of smooth sections for $\mathcal{L}_\lambda$. Since $\mathcal{L}_\lambda \to X$ is topologically a trivial bundle, we may identify $\mathcal{E}_\lambda(X)$ with $C^\infty(X)$, and corresponding G-action on $C^\infty(X)$ is given as the multiplier representation $\varpi_\lambda \equiv \varpi_\lambda^X$:

$$\left(\varpi_\lambda(h)f\right)(x) = \Omega(h^{-1},x)^\lambda f(h^{-1}\cdot x) \quad \text{for } h \in G \text{ and } f \in C^\infty(X).$$

See [7] for the basic properties of the representation $(\varpi_\lambda, C^\infty(X))$.

Example 2.1. We endow $\mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$ with a Riemannian metric g via the stereographic projection of the unit sphere S^2:

$$\mathbb{R}^3 \supset S^2 \xrightarrow{\sim} \mathbb{C} \cup \{\infty\}, \quad (p,q,r) \mapsto \frac{p + \sqrt{-1}q}{1+r}.$$

Then $g(u,v) = \frac{4}{(1+|z|^2)^2}(u,v)_{\mathbb{R}^2}$ for $u, v \in T_z\mathbb{C} \simeq \mathbb{R}^2$, and the Möbius transformation, defined by

$$\mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C}, \quad z \mapsto g \cdot z = \frac{az+b}{cz+d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}),$$

is conformal with conformal factor

$$\Omega(g,z) = |cz+d|^{-2}. \tag{2.3}$$

Therefore,

$$\left(\varpi_\lambda(h)f\right)(z) = |cz+d|^{-2\lambda} f\left(\frac{az+b}{cz+d}\right) \quad \text{for } h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is a (non-unitary) spherical principal series representation $\mathrm{Ind}_{B_{\mathbb{C}}}^{G_{\mathbb{C}}}(1 \otimes \lambda\alpha \otimes 1)$ of $G_{\mathbb{C}} = SL(2,\mathbb{C})$, where α is the unique positive restricted root which defines a Borel subgroup $B_{\mathbb{C}}$.

Let $\wedge^i T^*X$ be the i-th exterior power of the cotangent bundle T^*X for $0 \leq i \leq n$, where n is the dimension of X. Then sections ω for $\wedge^i T^*X$ are i-th differential forms on X, and G acts on $\mathcal{E}^i(X) = C^\infty(X, \wedge^i T^*X)$ as the pull-back of differential forms:

$$\varpi(h)\omega = (h^{-1})^*\omega \quad \text{for } \omega \in \mathcal{E}^i(X).$$

More generally, the tensor bundle $\mathcal{L}_\lambda \otimes \wedge^i T^*X$ is also a G-equivariant vector bundle over X, and we denote by $\varpi^X_{\lambda,i}$ the regular representation of G on the space of sections

$$\mathcal{E}^i_\lambda(X) := C^\infty(X, \mathcal{L}_\lambda \otimes \wedge^i T^*X).$$

By definition $\mathcal{E}^0_\lambda(X) = \mathcal{E}_\lambda(X)$. In our normalization we have a natural G-isomorphism

$$\mathcal{E}^0_n(X) \simeq \mathcal{E}^n_0(X),$$

if X admits a G-invariant orientation.

Denote by $\mathrm{Conf}(X)$ the full group of conformal transformations of the Riemannian manifold (X, g). Given a submanifold Y of X, we define a subgroup by

$$\mathrm{Conf}(X; Y) := \{\varphi \in \mathrm{Conf}(X) : \varphi(Y) = Y\}.$$

Then the induced action of $\mathrm{Conf}(X; Y)$ on the Riemannian manifold $(Y, g|_Y)$ is again conformal. We then consider the following problem.

Problem 2.2. (1) Given $0 \leq i \leq \dim X$ and $0 \leq j \leq \dim Y$, classify $(\lambda, \nu) \in \mathbb{C}^2$ such that there exists a non-zero local/non-local operator

$$T : \mathcal{E}^i_\lambda(X) \to \mathcal{E}^j_\nu(Y)$$

satisfying

$$\varpi^Y_{\nu,j}(h) \circ T = T \circ \varpi^X_{\lambda,i}(h) \quad \text{for all } h \in \mathrm{Conf}(X; Y).$$

(2) Find explicit formulæ of the operators $T \equiv T^{i,j}_{\lambda,\nu}$.

The case $i = j = 0$ is a question that was raised in [6, Problem 4.2] as a geometric aspect of the branching problem for representations with respect to the pair of groups $\mathrm{Conf}(X) \supset \mathrm{Conf}(X; Y)$.

As a special case, one may ask:

Question A″. Solve Problem 2.2 for covariant differential operators in the setting that $(X, Y) = (S^2, S^1)$ and $(i, j) = (1, 0)$.

We note that, for $(X, Y) = (S^2, S^1)$, there are natural homomorphisms

$$\begin{array}{ccc} G_{\mathbb{C}} := SL(2,\mathbb{C}) & \to & \mathrm{Conf}(X) \\ \cup & & \cup \\ G_{\mathbb{R}} := SL(2,\mathbb{R}) & \to & \mathrm{Conf}(X;Y), \end{array}$$

and the images of $SL(2,\mathbb{C})$ and $SL(2,\mathbb{R})$ coincide with the identity component groups of $\mathrm{Conf}(X) \simeq O(3,1)$ and $\mathrm{Conf}(X;Y)$, respectively. Question A is equivalent to Question A″ with $\mathrm{Conf}(X;Y)$ replaced by its identity component $SO_0(2,1) \simeq SL(2,\mathbb{R})/\{\pm I\}$. In fact, the differential operator $\mathcal{D} = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2)$ in Question A gives a $G_{\mathbb{R}}$-equivariant differential operator

$$\mathcal{E}^1_{\lambda-1}(S^2) \to \mathcal{E}^0_{\nu}(S^1) \equiv \mathcal{E}_{\nu}(S^1)$$

in our normalization, which takes the form

$$\mathcal{E}^1(\mathbb{R}^2) \to C^\infty(\mathbb{R}), \quad f dx + g dy \mapsto (\mathcal{D}_1 f)(x,0) + (\mathcal{D}_2 g)(x,0)$$

in the flat coordinates via the stereographic projection.

2.3 Branching laws of Verma modules

Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, and $\mathfrak{b}$ a Borel subalgebra consisting of lower triangular matrices in $\mathfrak{g}$. For $\lambda \in \mathbb{C}$, we define a character of $\mathfrak{b}$, to be denoted by $\mathbb{C}_\lambda$, as

$$\mathfrak{b} \to \mathbb{C}, \quad \begin{pmatrix} -x & 0 \\ y & x \end{pmatrix} \mapsto \lambda x.$$

If $\lambda \in \mathbb{Z}$ then $\mathbb{C}_\lambda$ is the differential of the holomorphic character $\chi_{\lambda,\lambda}$ of the Borel subgroup $B_{\mathbb{C}}$, which will be defined in (3.1) in Section 3.1.

We consider a $\mathfrak{g}$-module, referred to as a Verma module, defined by

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

Then $\mathbf{1}_\lambda := 1 \otimes 1 \in M(\lambda)$ is a highest weight vector with weight $\lambda \in \mathbb{C}$, and it generates $M(\lambda)$ as a $\mathfrak{g}$-module. The $\mathfrak{g}$-module $M(\lambda)$ is irreducible if and only if $\lambda \notin \mathbb{N}$.

We consider the following algebraic question:

Question A‴. (1) Classify $(\mu, \lambda_1, \lambda_2) \in \mathbb{C}^3$ such that

$$\operatorname{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda_1) \otimes M(\lambda_2)) \neq \{0\}.$$

(2) Find an explicit expression of $\varphi(\mathbf{1}_\mu)$ in $M(\lambda_1) \otimes M(\lambda_2)$ for any $\varphi \in \operatorname{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda_1) \otimes M(\lambda_2))$.

An answer to Question A‴ is given as follows:

Proposition 2.3. *If $\lambda_1 + \lambda_2 \notin \mathbb{N}$ then the tensor product $M(\lambda_1) \otimes M(\lambda_2)$ decomposes into the direct sum of Verma modules as follows:*

$$M(\lambda_1) \otimes M(\lambda_2) \simeq \bigoplus_{a=0}^{\infty} M(\lambda_1 + \lambda_2 - 2a).$$

For the proof, consult [4] for instance. In fact, in [4], one finds the (abstract) branching laws of (parabolic) Verma modules in the general setting of the restriction with respect to symmetric pairs. By the duality theorem ([8], [9, Theorem 2.7]) between differential *symmetry breaking operators* (covariant differential operators to submanifolds) and (discretely decomposable) branching laws of Verma modules, we have the following one-to-one correspondence

$$\begin{aligned} &\{\text{The differential operators } \mathcal{D} \text{ yielding the functional identity } (\mathcal{M}_{\lambda,\nu})\} \\ &\leftrightarrow \operatorname{Hom}_{\mathfrak{g}}(M(-2\nu), M(-\lambda-1) \otimes M(-\lambda+1)) \\ &\qquad\qquad \oplus \operatorname{Hom}_{\mathfrak{g}}(M(-2\nu), M(-\lambda+1) \otimes M(-\lambda-1)), \end{aligned} \tag{2.4}$$

because $T_o(G_{\mathbb{C}}/B_{\mathbb{C}}) \otimes \mathbb{C} \simeq \mathbb{C}_{-2} \boxtimes \mathbb{C} + \mathbb{C} \boxtimes \mathbb{C}_{-2}$ as $\mathfrak{b} \otimes \mathbb{C} \simeq \mathfrak{b} \oplus \mathfrak{b}$-modules. Combining this with Proposition 2.3, we obtain

Proposition 2.4. *If $2\lambda \notin -\mathbb{N}$ then a non-zero differential operator $\mathcal{D}$ satisfying $(\mathcal{M}_{\lambda,\nu})$ exists if and only if $\nu - \lambda \in \mathbb{N}$, and the set of such differential operators forms a two-dimensional vector space.*

Owing to Proposition 1.2, we get the two-dimensional solution space as the linear span of $\mathcal{D}$ and $\mathcal{D}^\vee$, once we find a generic solution $\mathcal{D}$.

3 Rankin–Cohen brackets

As a preparation for the proof of Theorem A, we briefly review the Rankin–Cohen brackets, which originated in number theory [1, 2, 11].

3.1 Homogeneous line bundles over $\mathbb{P}^1\mathbb{C}$

First, we shall fix a normalization of three homogeneous line bundles over $X = \mathbb{P}^1\mathbb{C}$, namely, $\mathcal{L}_\lambda^{\mathrm{conf}}$ (Section 2), $\mathcal{L}_\lambda^{\mathrm{hol}}$, and $\mathcal{L}_{n,\lambda}$.

We define a Borel subgroup of $G_\mathbb{C} = SL(2,\mathbb{C})$ by

$$B_\mathbb{C} := \left\{ \begin{pmatrix} a & 0 \\ c & \frac{1}{a} \end{pmatrix} : a \in \mathbb{C}^\times, c \in \mathbb{C} \right\},$$

and identify $G_\mathbb{C}/B_\mathbb{C}$ with $X = \mathbb{P}^1\mathbb{C}$ by $hB_\mathbb{C} \mapsto h \cdot 0$.

Given $n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$, we define a one-dimensional representation of $B_\mathbb{C}$ by

$$\chi_{n,\lambda} : B_\mathbb{C} \to \mathbb{C}^\times, \quad \begin{pmatrix} \frac{1}{re^{i\theta}} & 0 \\ c & re^{i\theta} \end{pmatrix} \mapsto e^{in\theta} r^\lambda, \tag{3.1}$$

and a $G_\mathbb{C}$-equivariant line bundle $\mathcal{L}_{n,\lambda} = G_\mathbb{C} \times_{B_\mathbb{C}} \chi_{n,\lambda}$ as the set of equivalence classes of $G_\mathbb{C} \times \mathbb{C}$ given by

$$(g,u) \sim (gb^{-1}, \chi_{n,\lambda}(b)u) \quad \text{for some } b \in B_\mathbb{C}.$$

The conformal line bundle $\mathcal{L}_\lambda^{\mathrm{conf}}$ defined in Section 2.2 amounts to $\mathcal{L}_{0,2\lambda}$ by the formula (2.3).

On the other hand, if $\lambda = n \in \mathbb{Z}$ then $\chi_{\lambda,\lambda}$ is a holomorphic character of $B_\mathbb{C}$, and consequently, $\mathcal{L}_{\lambda,\lambda} \to X$ becomes a holomorphic line bundle, which we denote by $\mathcal{L}_\lambda^{\mathrm{hol}}$. The complexified cotangent bundle $(T^*X) \otimes \mathbb{C}$ splits into a Whitney sum of the holomorphic and anti-holomorphic cotangent bundle $(T^*X)^{1,0} \oplus (T^*X)^{0,1}$, which amounts to $\mathcal{L}_{2,2} \oplus \mathcal{L}_{-2,2}$. In summary, we have:

Lemma 3.1. *We have the following isomorphisms of $G_\mathbb{C}$-equivariant line bundles over $X \simeq \mathbb{P}^1\mathbb{C}$.*

$$\begin{aligned} \mathcal{L}_\lambda^{\mathrm{hol}} &\simeq \mathcal{L}_{\lambda,\lambda} \quad && \textit{for } \lambda \in \mathbb{Z}, \\ \mathcal{L}_\lambda^{\mathrm{conf}} &\simeq \mathcal{L}_{0,2\lambda} && \textit{for } \lambda \in \mathbb{C}, \\ (T^*X)^{1,0} &\simeq \mathcal{L}_{2,2}, \\ (T^*X)^{0,1} &\simeq \mathcal{L}_{-2,2}. \end{aligned}$$

The line bundle $\mathcal{L}_{n,\lambda} \to X$ is $G_\mathbb{C}$-equivariant; thus, there is the regular representation $\pi_{n,\lambda}$ of $G_\mathbb{C}$ on $C^\infty(X, \mathcal{L}_{n,\lambda})$. This is called the (unnormalized, non-unitary) principal series representation of $G_\mathbb{C}$. The restriction to the open Bruhat cell $\mathbb{C} \hookrightarrow X = \mathbb{C} \cup \{\infty\}$ yields an injection $C^\infty(X, \mathcal{L}_{n,\lambda}) \hookrightarrow C^\infty(\mathbb{C})$, on which $\pi_{n,\lambda}$ is given as a multiplier representation:

$$(\pi_{n,\lambda}(h)F)(z) = \left(\frac{cz+d}{|cz+d|}\right)^{-n} |cz+d|^{-\lambda} F\left(\frac{az+b}{cz+d}\right) \quad \text{for } h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Comparing this with the conformal construction of the representation ϖ_λ in Example 2.1, we have $\varpi_\lambda \simeq \pi_{0,2\lambda}$.

Similarly to the smooth line bundle $\mathcal{L}_{n,\lambda}$, we consider holomorphic sections for the holomorphic line bundle $\mathcal{L}_\lambda^{\mathrm{hol}}$. For this, let D be a domain of $\mathbb{C}$ and G a subgroup of $G_{\mathbb{C}}$, which leaves D invariant. Then we can define a representation, to be denoted by $\pi_\lambda^{\mathrm{hol}}$, of G on the space $\mathcal{O}(D) \equiv \mathcal{O}(D, \mathcal{L}_\lambda^{\mathrm{hol}})$ of holomorphic sections, which is identified with a multiplier representation

$$\left(\pi_\lambda^{\mathrm{hol}}(h)F\right)(z) = (cz+d)^{-\lambda} F\left(\frac{az+b}{cz+d}\right) \quad \text{for } F \in \mathcal{O}(D).$$

Example 3.2. (1) $D = \{z \in \mathbb{C} : |z| < 1\}$, $G = SU(1,1)$.
(2) $D = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, $G = SL(2, \mathbb{R})$.
(For the application below we shall use the unit disc model.)

3.2 Rankin–Cohen bidifferential operator

Let D be a domain in $\mathbb{C}$. For $a \in \mathbb{N}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, the bidifferential operator $\mathcal{RC}^a_{\lambda_1,\lambda_2} : \mathcal{O}(D) \otimes \mathcal{O}(D) \to \mathcal{O}(D)$, referred to as the *Rankin–Cohen bracket* [1, 11], is defined by

$$\mathcal{RC}^a_{\lambda_1,\lambda_2}(f_1 \otimes f_2)(z) := \sum_{\ell=0}^{a} (-1)^\ell \binom{\lambda_1 + a - 1}{\ell} \binom{\lambda_2 + a - 1}{a - \ell} \frac{\partial^{a-\ell} f_1}{\partial z^{a-\ell}}(z) \frac{\partial^\ell f_2}{\partial z^\ell}(z).$$

In the theory of automorphic forms, $\mathcal{RC}^a_{\lambda_1,\lambda_2}$ yields a new holomorphic modular form of weight $\lambda_1 + \lambda_2 + 2a$ out of two holomorphic modular forms f_1 and f_2 of weights λ_1 and λ_2, respectively.

From the viewpoint of representation theory, $\mathcal{RC}^a_{\lambda_1,\lambda_2}$ is an intertwining operator:

$$\pi^{\mathrm{hol}}_{\lambda_1+\lambda_2+2a}(h) \circ \mathcal{RC}^a_{\lambda_1,\lambda_2} = \mathcal{RC}^a_{\lambda_1,\lambda_2} \circ \left(\pi^{\mathrm{hol}}_{\lambda_1}(h) \otimes \pi^{\mathrm{hol}}_{\lambda_2}(h)\right) \tag{3.2}$$

for all $h \in G$.

The coefficients of the Rankin–Cohen brackets look somewhat complicated. Eicheler–Zagier [2, Chapter 3] found that they are related to those of a classical orthogonal polynomial. A short proof for this fact is given by the *F-method* in [9].

To see the relation, we define a polynomial $\mathrm{RC}^a_{\lambda_1,\lambda_2}(x,y)$ of two variables x and y by

$$\mathrm{RC}^a_{\lambda_1,\lambda_2}(x,y) := \sum_{\ell=0}^{a}(-1)^\ell \binom{\lambda_1+a-1}{\ell}\binom{\lambda_2+a-1}{a-\ell} x^{a-\ell}y^\ell, \tag{3.3}$$

so that the Rankin–Cohen bidifferential operator $\mathcal{RC}^a_{\lambda_1,\lambda_2}$ is given by

$$\mathcal{RC}^a_{\lambda_1,\lambda_2} = \mathrm{Rest}_{z_1=z_2=z} \circ \mathrm{RC}^a_{\lambda_1,\lambda_2}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right).$$

The polynomial $\mathrm{RC}^a_{\lambda_1,\lambda_2}(x,y)$ is of homogeneous degree a. Clearly we have:

Lemma 3.3. $\mathrm{RC}^a_{\lambda_1,\lambda_2}(x,y) = (-1)^a \mathrm{RC}^a_{\lambda_2,\lambda_1}(y,x)$.

Second we recall that the Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ is a polynomial of one variable t of degree ℓ given by

$$P_\ell^{\alpha,\beta}(t) = \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+\beta+\ell+1)} \sum_{m=0}^{\ell} \frac{\Gamma(\alpha+\beta+\ell+m+1)}{(\ell-m)!m!\Gamma(\alpha+m+1)}\left(\frac{t-1}{2}\right)^m.$$

We inflate it to a homogeneous polynomial of two variables x and y of degree ℓ by

$$P_\ell^{\alpha,\beta}(x,y) := y^\ell P_\ell^{\alpha,\beta}\left(2\frac{x}{y}+1\right).$$

For instance, $P_0^{\alpha,\beta}(x,y) = 1$ and $P_1^{\alpha,\beta}(x,y) = (2+\alpha+\beta)x + (\alpha+1)y$. It turns out that

$$\mathrm{RC}^a_{\lambda_1,\lambda_2}(x,y) = (-1)^a P_a^{\lambda_1-1,-\lambda_1-\lambda_2-2a+1}(x,y).$$

In particular, the following holds.

Lemma 3.4. *We have*

$$\mathcal{RC}^a_{\lambda_1,\lambda_2} = (-1)^a \mathrm{Rest}_{z_1=z_2=z} \circ P_a^{\lambda_1-1,-\lambda_1-\lambda_2-2a+1}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right).$$

4 Holomorphic trick

In this section we give a proof for Theorem A by using the results of the previous sections

4.1 Restriction to a totally real submanifold

Consider a totally real embedding of $X = \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$ defined by

$$\iota : \mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C} \times \mathbb{P}^1\mathbb{C}, \quad z \mapsto (z, \bar{z}). \tag{4.1}$$

The map ι respects the action of $G_\mathbb{C}$ via the following group homomorphism (we regard $G_\mathbb{C}$ as a real group), denoted by the same letter,

$$\iota : G_\mathbb{C} \to G_\mathbb{C} \times G_\mathbb{C}, \quad g \mapsto (g, \bar{g}).$$

This is because $G_\mathbb{C}/B_\mathbb{C} \simeq \mathbb{P}^1\mathbb{C}$ and because the Borel subgroup $B_\mathbb{C}$ is stable by the complex conjugation $g \mapsto \bar{g}$. Then the following lemma is immediate from Lemma 3.1.

Lemma 4.1. *We have an isomorphism of $G_\mathbb{C}$-equivariant line bundles:*

$$\iota^*\left(\mathcal{L}^{\mathrm{hol}}_{\lambda_1} \boxtimes \mathcal{L}^{\mathrm{hol}}_{\lambda_2}\right) \simeq \mathcal{L}_{\lambda_1-\lambda_2,\lambda_1+\lambda_2}.$$

In particular,

$$\iota^*(\mathcal{L}^{\mathrm{hol}}_{\lambda+1} \boxtimes \mathcal{L}^{\mathrm{hol}}_{\lambda-1}) \simeq \mathcal{L}^{\mathrm{conf}}_{\lambda-1} \otimes (T^*X)^{1,0}, \tag{4.2}$$

$$\iota^*(\mathcal{L}^{\mathrm{hol}}_{\lambda-1} \boxtimes \mathcal{L}^{\mathrm{hol}}_{\lambda+1}) \simeq \mathcal{L}^{\mathrm{conf}}_{\lambda-1} \otimes (T^*X)^{0,1}. \tag{4.3}$$

Proposition 4.2. *The isomorphisms* (4.2) *and* (4.3) *induce injective $G_\mathbb{C}$-equivariant homomorphisms between equivariant sheaves:*

$$(\iota^*)^{1,0} : \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda+1}) \otimes \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda-1}) \to \mathcal{E}^{1,0}_{\lambda-1}, \quad f_1(z_1) \otimes f_2(z_2) \mapsto f_1(z)f_2(\bar{z})dz,$$
$$(\iota^*)^{0,1} : \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda-1}) \otimes \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda+1}) \to \mathcal{E}^{0,1}_{\lambda-1}, \quad f_1(z_1) \otimes f_2(z_2) \mapsto f_1(z)f_2(\bar{z})d\bar{z},$$

that is, $(\iota^*)^{1,0}$ and $(\iota^*)^{0,1}$ are injective on every open set D in $\mathbb{P}^1\mathbb{C}$, and

$$(\iota^*)^{1,0} \circ \left(\pi^{\mathrm{hol}}_{\lambda+1}(g) \otimes \pi^{\mathrm{hol}}_{\lambda-1}(\bar{g})\right) = \varpi^1_{\lambda-1}(g) \circ (\iota^*)^{1,0}$$
$$(\iota^*)^{0,1} \circ \left(\pi^{\mathrm{hol}}_{\lambda-1}(g) \otimes \pi^{\mathrm{hol}}_{\lambda+1}(\bar{g})\right) = \varpi^1_{\lambda-1}(g) \circ (\iota^*)^{0,1}$$

hold for any g whenever they make sense.

Proof. The injectivity follows from the identity theorem of holomorphic functions because $\iota : \mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C} \times \mathbb{P}^1\mathbb{C}$ is a totally real embedding. The covariance property is derived from (4.2) and (4.3). $\square$

Fix $\lambda \in \mathbb{Z}$ and $a \in \mathbb{N}$, and set $\nu = \lambda + a$. We want to relate the Rankin–Cohen brackets $\mathcal{RC}^a_{\lambda\pm1,\lambda\mp1}$ to our differential operator $\mathcal{D}$ (see Question A) in the sense that both of the following diagrams commute:

$$\begin{array}{ccccccc}
\mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda+1}) \otimes \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda-1}) & \xhookrightarrow{(\iota^*)^{1,0}} & \mathcal{E}^{1,0}_{\lambda-1}(\mathbb{C}) \subset \mathcal{E}^1_{\lambda-1}(\mathbb{R}^2) & \simeq & C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2) \\
{\scriptstyle \mathcal{RC}^a_{\lambda+1,\lambda-1}}\downarrow & & & & \downarrow{\scriptstyle \mathcal{D}} \\
\mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{2\lambda+2a}) & \xhookrightarrow{\quad\iota^*\quad} & \mathcal{E}_\nu(\mathbb{R}) & \simeq & C^\infty(\mathbb{R}),
\end{array}$$

and

$$\begin{array}{ccccccc}
\mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda-1}) \otimes \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda+1}) & \xhookrightarrow{(\iota^*)^{0,1}} & \mathcal{E}^{0,1}_{\lambda-1}(\mathbb{C}) \subset \mathcal{E}^1_{\lambda-1}(\mathbb{R}^2) & \simeq & C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2) \\
{\scriptstyle (-1)^a\mathcal{RC}^a_{\lambda-1,\lambda+1}}\downarrow & & & & \downarrow{\scriptstyle \mathcal{D}} \\
\mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{2\lambda+2a}) & \xhookrightarrow{\quad\iota^*\quad} & \mathcal{E}_\nu(\mathbb{R}) & \simeq & C^\infty(\mathbb{R}).
\end{array}$$

Here we have used the following identification:

$$\mathcal{E}^1_{\lambda-1}(\mathbb{R}^2) \simeq C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2), \quad f dx + g dy \mapsto (f, g).$$

We define homogeneous polynomials D_1, D_2 with real coefficients so that

$$D_1(x,y) + \sqrt{-1}D_2(x,y) = 2^{-a}\mathrm{RC}^a_{\lambda+1,\lambda-1}(x - \sqrt{-1}y, x + \sqrt{-1}y),$$

where $\mathrm{RC}^a_{\lambda_1,\lambda_2}(x,y)$ is a polynomial defined in (3.3). We set

(4.4)
$$\mathcal{D}_1 := D_1\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \quad \mathcal{D}_2 := D_2\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right), \quad \mathcal{D} := \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2).$$

Lemma 4.3. *For any holomorphic functions f_1 and f_2,*

$$\mathcal{D}\left((\iota^*)^{1,0}(f_1 \otimes f_2)\right) = \iota^*\mathcal{RC}^a_{\lambda+1,\lambda-1}(f_1 \otimes f_2),$$
$$\mathcal{D}\left((\iota^*)^{0,1}(f_1 \otimes f_2)\right) = (-1)^a\iota^*\mathcal{RC}^a_{\lambda-1,\lambda+1}(f_1 \otimes f_2).$$

Proof. Let $\omega := (\iota^*)^{1,0}(f_1 \otimes f_2) = f_1(z)f_2(\bar{z})dz$. If we write $\omega = fdx + gdy$ then $f(z) = f_1(z)f_2(\bar{z})$ and $g = \sqrt{-1}f$. Therefore,

$$\mathcal{D}\omega = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2) \begin{pmatrix} f \\ g \end{pmatrix},$$

$$(\mathcal{D}_1, \mathcal{D}_2) \begin{pmatrix} f \\ g \end{pmatrix} = \left(\mathcal{D}_1 + \sqrt{-1}\mathcal{D}_2\right)(f_1(z)f_2(\bar{z}))\right).$$

If we write $\mathrm{RC}^a_{\lambda+1,\lambda-1}(x, y) = \sum_{\ell=0}^{a} r_\ell x^{a-\ell} y^\ell$ then

$$\begin{aligned}(\mathcal{D}_1 + \sqrt{-1}\mathcal{D}_2)\,(f_1(z)f_2(\bar{z})) &= \mathrm{RC}^a_{\lambda+1,\lambda-1}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)(f_1(z)f_2(\bar{z})) \\ &= \sum_{\ell=0}^{a} r_\ell \frac{\partial^{a-\ell} f_1}{\partial z^{a-\ell}}(z) \frac{\partial^\ell f_2}{\partial \bar{z}^\ell}(\bar{z}),\end{aligned}$$

because f_1 and f_2 are holomorphic. Taking the restriction to $y = 0$, we get

$$\mathcal{D}(\omega) = \sum_{\ell=0}^{a} r_\ell \frac{\partial^{a-\ell} f_1}{\partial x^{a-\ell}}(x) \frac{\partial^\ell f_2}{\partial x^\ell}(x) = \iota^* \mathcal{RC}^a_{\lambda+1,\lambda-1}(f_1 \otimes f_2).$$

Hence we have proved the first identity. The second identity follows from Lemma 3.3. $\square$

Remark 4.4. If we multiply the bidifferential operator $\mathcal{RC}^a_{\lambda+1,\lambda-1}$ by $\sqrt{-1}$ then obviously (3.2) holds, where the role of $(\mathcal{D}_1, \mathcal{D}_2)$ is changed into $(-\mathcal{D}_2, \mathcal{D}_1)$ because

$$\sqrt{-1}(\mathcal{D}_1 + \sqrt{-1}\mathcal{D}_2) = -\mathcal{D}_2 + \sqrt{-1}\mathcal{D}_1.$$

This explains Proposition 1.2 from the "holomorphic trick."

4.2 Identities of Jacobi polynomials

For $a \in \mathbb{N}_+$, we define the following three meromorphic functions of λ by

$$\begin{aligned}A_a(\lambda) &:= \frac{2\lambda^2 + 2(a-1)\lambda + a(a-1)}{a(2\lambda + a - 1)}, \\ B_a(\lambda) &:= \frac{(\lambda - 1)(2\lambda + 1)}{a(2\lambda + a - 1)}, \\ U_a(\lambda) &:= \frac{2\left(\lambda + [\frac{a}{2}]\right)_{[\frac{a-1}{2}]}}{\left(\lambda + \frac{1}{2}\right)_{[\frac{a-1}{2}]}},\end{aligned}$$

where $(\mu)_k := \mu(\mu+1)\cdots(\mu+k-1) = \frac{\Gamma(\mu+k)}{\Gamma(\mu)}$ is the Pochhammer symbol.

Proposition 4.5. *For $a \in \mathbb{N}_+$, we have*

$$\begin{aligned}&(1-z)^a P_a^{\lambda,-2\lambda-2a+1}\left(\frac{3+z}{1-z}\right)\\&= (-1)^{a-1}U_a(\lambda)\Big((1-A_a(\lambda)z)C_{a-1}^{\lambda+\frac{1}{2}}(z) + B_a(\lambda)(1-z^2)C_{a-2}^{\lambda+\frac{3}{2}}(z)\Big).\end{aligned}$$

Equivalently,

$$\begin{aligned}&P_a^{\lambda,-2\lambda-2a+1}(x-\sqrt{-1}y, x+\sqrt{-1}y)\\&= (\sqrt{-1})^{a-1}U_a(\lambda)\Big(xC_{a-1}^{\lambda+\frac{1}{2}}(-x^2,y) + \sqrt{-1}\left(A_a(\lambda)yC_{a-1}^{\lambda+\frac{1}{2}}(-x^2,y) + B_a(\lambda)(x^2+y^2)C_{a-2}^{\lambda+\frac{3}{2}}(-x^2,y)\right)\Big)\end{aligned}$$

Proposition 4.5 will be used in the proof of Theorem A in the next subsection. We want to note that we wondered if the first equation of Proposition 4.5 was already known; however, we could not find the identity in the literature.

One might give an alternative proof of Proposition 4.5 by applying the F-method to a vector bundle case. We will discuss this approach in a subsequent paper.

4.3 Proof of Theorem A

The relations in Lemma 4.3 and the covariance property (3.2) of the Rankin–Cohen brackets imply that the differential operator $\mathcal{D}$ defined in (4.4) satisfies the covariance relations (2.2) on the image

$$(\iota^*)^{1,0}\big(\mathcal{O}(\mathcal{L}_{\lambda+1}^{\mathrm{hol}}) \otimes \mathcal{O}(\mathcal{L}_{\lambda-1}^{\mathrm{hol}})\big) + (\iota^*)^{0,1}\big(\mathcal{O}(\mathcal{L}_{\lambda-1}^{\mathrm{hol}}) \otimes \mathcal{O}(\mathcal{L}_{\lambda+1}^{\mathrm{hol}})\big).$$

In order to prove (2.2), we need to show that the image is dense in $C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2)$ topologized by uniform convergence on compact sets. To see this we note that the image contains a linear span of the following 1-forms

$$z^m\bar{z}^n dz, \quad z^m\bar{z}^n d\bar{z}, \quad (m,n \in \mathbb{N}).$$

Since a linear span of $(x+iy)^m(x-iy)^n$ $(m,n \in \mathbb{N})$ is dense in $C^\infty(\mathbb{R}^2)$ by the Stone–Weierstrass theorem, we conclude that $\mathcal{D}$ satisfies (2.2). An explicit formula for the operators $(\mathcal{D}_1, \mathcal{D}_2)$ is derived from the Rankin–Cohen brackets by using Lemma 3.4 and Proposition 4.5 for $\lambda \in \mathbb{Z}$. Then the covariance relations (1.1) are satisfied for all $\lambda \in \mathbb{C}$ because $\mathbb{Z}$ is Zariski dense in $\mathbb{C}$.

If $2\lambda \notin -\mathbb{N}$ then the dimension of solutions is two by Proposition 2.3 and the one-to-one correspondence (2.4). Since $\mathcal{D}$ and $\mathcal{D}^\vee$ are linearly independent for our solution $\mathcal{D}$, the linear span of $\mathcal{D}$ and $\mathcal{D}^\vee$ exhausts all the solutions by Proposition 1.2. Hence Theorem A is proved.

4.4 Scalar-valued case

So far we have discussed a family of vector-valued differential operators that yield functional identities satisfied by vector-valued functions. We close this article with some comments on the scalar-valued case.

Let $\lambda \in \mathbb{C}$. Given $f \in C^\infty(\mathbb{R}^2 \setminus \{(0,0)\}) \simeq C^\infty(\mathbb{C} \setminus \{0\})$, we define its *twisted inversion* $f_\lambda^\vee$ by

$$f_\lambda^\vee(r\cos\theta, r\sin\theta) := r^{-2\lambda} f\left(\frac{-\cos\theta}{r}, \frac{\sin\theta}{r}\right)$$

as in (1.1), and more generally,

$$f_\lambda^h(z) := |cz+d|^{-2\lambda} f\left(\frac{az+b}{cz+d}\right) \quad \text{for } h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$$

as in (2.1).

For a differential operator $\mathcal{D}$ on $\mathbb{R}^2$, we define a linear operator $\tilde{\mathcal{D}}$: $C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R})$ by

$$\tilde{\mathcal{D}} := \mathrm{Rest}_{y=0} \circ \mathcal{D}.$$

Fix $\lambda, \nu \in \mathbb{C}$. As in Questions A, A′, A″, and A‴, we may consider the following equivalent questions:

Question B. Find $\tilde{\mathcal{D}}$ with constant coefficients such that

$$\left(\tilde{\mathcal{D}} f_\lambda^\vee\right)(x) = |x|^{-2\nu} (\tilde{\mathcal{D}} f)\left(-\frac{1}{x}\right) \quad \text{for all } f \in C^\infty(\mathbb{R}^2) \text{ and } x \in \mathbb{R}^\times.$$

Question B′. Find $\tilde{\mathcal{D}}$ such that

$$\left(\tilde{\mathcal{D}} f_\lambda^h\right)(x) = |cx+d|^{-2\nu} (\tilde{\mathcal{D}} f)\left(\frac{ax+b}{cx+d}\right) \quad \text{for all } f \in C^\infty(\mathbb{C}), h \in SL(2,\mathbb{R}), \text{ and } x \in \mathbb{R}^\times.$$

Question B″. Find an explicit formula of conformally covariant differential operator $\mathcal{E}_\lambda(S^2) \to \mathcal{E}_\nu(S^1)$.

Question B‴. Find an explicit expression of the element $\varphi(\mathbf{1}_{-\nu})$ for any $\varphi \in \operatorname{Hom}_{\mathfrak{g}}(M(-\nu), M(-\lambda) \otimes M(-\lambda))$, where $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

An answer to Question B″ (and also in the case $S^{n-1} \subset S^n$ for arbitrary $n \geq 2$) was first given by Juhl [3]. In the flat model (Questions B and B′), if $a := \nu - \lambda \in \mathbb{N}$ then

$$\widetilde{\mathcal{C}_a^{\lambda-\frac{1}{2}}} \equiv \operatorname{Rest}_{y=0} \circ C_a^{\lambda-\frac{1}{2}}\left(-\frac{\partial^2}{\partial x^2}, \frac{\partial}{\partial y}\right) : \mathcal{E}_\lambda(\mathbb{R}^2) \to \mathcal{E}_\nu(\mathbb{R}^1)$$

intertwines the $SL(2,\mathbb{R})$-action. There have been several proofs for this (and also for more general cases) based on:

- Recurrence relations among coefficients of $\mathcal{D}$ ([3]),
- F-method ([5, 8, 9]), and
- Residue formulæ of a meromorphic family of non-local symmetry breaking operators [6, 10].

The holomorphic trick in Section 4 applied to this case gives yet another proof by using the Rankin–Cohen brackets and the following proposition analogous to (and much simpler than) Proposition 4.5.

Proposition 4.6. *For $a \in \mathbb{N}$, we have*

$$(1-z)^a P_a^{\lambda-1,-2\lambda-2a+1}\left(\frac{3+z}{1-z}\right) = (-1)^a \frac{\left(\lambda + [\frac{a}{2}]\right)_{[\frac{a+1}{2}]}}{\left(\lambda - \frac{1}{2}\right)_{[\frac{a+1}{2}]}} C_a^{\lambda-\frac{1}{2}}(z).$$

Equivalently,

$$P_a^{\lambda-1,-2\lambda-2a+1}(x-\sqrt{-1}y, x+\sqrt{-1}y) = (\sqrt{-1})^a \frac{\left(\lambda + [\frac{a}{2}]\right)_{[\frac{a+1}{2}]}}{\left(\lambda - \frac{1}{2}\right)_{[\frac{a+1}{2}]}} C_a^{\lambda-\frac{1}{2}}(-x^2, y).$$

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