Intertwining operators and the restriction of representations of rank one orthogonal groups

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Abstract

We give a complete classification of intertwining operators (*breaking symmetry operators*) between spherical principal series representations of $O(n+1,1)$ and $O(n,1)$ together with explicit formulae of the distribution kernels. Further we use this to determine the breaking symmetry operators between their irreducible composition factors.

Résumé

Nous donnons une classification complète des opérateurs d'entrelacement (*opérateurs de brisure de symétrie*) entre les représentations des séries principales sphériques de $O(n+1,1)$ et de $O(n,1)$ ainsi que des formules explicites pour les noyaux de Schwartz de ces opérateurs. Par la suite, nous déterminons les opérateurs de brisure de symétrie entre les facteurs irréductibles des séries de composition correspondantes.

Version française abrégée

Introduction. Toute représentation π d'un groupe G définit, par restriction, une représentation $\pi_{|G'}$ d'un sous-groupe donné $G' \subset G$. En général, ce foncteur ne préserve pas l'irréductibilité. Si G est un groupe compact alors toute représentation irréductible et continue π de G est de dimension finie et la restriction $\pi_{|G'}$ est isomorphe à une somme directe des représentations irréductibles π' de G' avec certaines multiplicités $m(\pi, \pi')$. Ces multiplicités sont étudiées à l'aide des techniques combinatoires. Toutefois, aucun algorithme pour retrouver les $m(\pi, \pi')$ n'est connu au jour d'aujourd'hui lors que G' n'est pas compact. Par ailleurs, si G' est non-compact et la représentation π est de dimension infinie, alors la restriction de π à G' n'est pas, en général, une somme directe de représentations irréductibles [3] et l'on doit considérer une notion alternative de multiplicité.

Pour une représentation continue π d'un groupe de Lie réductif réel G dans un espace de Banach H_π, le sous-espace H_π^∞ des vecteurs C^∞ de H_π est équipé naturellement d'une topologie de Fréchet, et (π, H_π) donne lieu à une représentation continue π^∞ de G dans H_π^∞. Étant donné une autre représentation continue π' d'un sous-groupe réductif G', nous considérons l'espace des opérateurs continus d'entrelacement que l'on appellera *opérateurs de brisure de symétrie* :

$$\mathrm{Hom}_{G'}(\pi^\infty, (\pi')^\infty).$$

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La dimension $m(\pi, \pi')$ de cet espace est déterminée par le $(\mathfrak{g}, K)$-module sous-jacent de π et le $(\mathfrak{g}', K')$-module sous-jacent de π' et cela d'une façon indépendante du choix des globalisations π et π'. Nous utilisons la même notation $m(\pi, \pi')$ pour dsigner cette dimension et l'appelons la *multiplicité* de π' dans la restriction $\pi_{|G'}$. Cette notion fournit une information importante sur la restriction de π à G'. Notons que cette définition reste valable pour des représentations non-unitaires.

En général, $m(\pi, \pi')$ peut être infini. En effet, il a été démontré dans [6] que la multiplicité $m(\pi, \pi')$ est finie pour toutes les représentations irréductibles π de G et π' de G' si et seulement si le sous-groupe parabolique minimal P' de G' possède une orbite ouverte dans la variété de drapeaux rélle G/P, et $m(\pi, \pi')$ est uniformément bornée si et seulement si le sous-groupe de Borel de $G'_{\mathbb{C}}$ possède une orbite ouverte dans la variété de drapeaux complexe de $G_{\mathbb{C}}$. Par exemple, $m(\pi, \pi')$ est uniformément bornée si les algèbres de Lie $(\mathfrak{g}, \mathfrak{g}')$ de (G, G') sont des formes rélles de $(\mathfrak{sl}(n+1, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}))$ ou encore de $(\mathfrak{o}(n+1, \mathbb{C}), \mathfrak{o}(n, \mathbb{C}))$. Par ailleurs, des propriétés plus fines des multiplicités de la restriction d'une représentation admissible et irréductible du groupe $G = O(n)$ sur un corps local à un sous-groupe de la forme $G' = O(n-1)$ sont formulées par la conjecture de Gross et Prasad [1].

Dans la présente note nous décrivons les multiplicités des représentations des séries principales sphériques ainsi que leurs séries de composition pour les groupes $G = O(n+1, 1)$ et $G' = O(n, 1)$.

Soit $I(\lambda)$ l'espace des vecteurs C^∞ d'une représentation de la série principale sphérique de G, et $J(\nu)$ celui de G', où $\lambda, \nu \in \mathbb{C}$. La paramétrisation est choisie de telle sorte que $I(\lambda)$ contient une sous-représentation de dimension finie si et seulement si $-\lambda \in \mathbb{N}$, et $J(\nu)$ en contient une si et seulement si $-\nu \in \mathbb{N}$. Définissons $L_{even} \subset //$ par

$$\begin{aligned} L_{even} :=&\{(-i, -j) : j \leq i \text{ et } i \equiv j \mod 2\}, \\ // :=&\{(\lambda, \nu) \in \mathbb{C}^2 : \lambda - \nu = 0, -2, -4, \cdots\}. \end{aligned}$$

Nous obtenons
Théorème 0.1

$$m(I(\lambda), J(\nu)) = \begin{cases} 1 & si\ (\lambda, \nu) \in \mathbb{C}^2 \setminus L_{even}, \\ 2 & si\ (\lambda, \nu) \in L_{even}, \end{cases}$$

Afin de donner des formules explicites pour ces opérateurs de brisure de symétrie, nous réalisons $I(\lambda)$ dans un sous-espace de $C^\infty(\mathbb{R}^n)$ par restriction à une cellule de Bruhat ouverte, et $J(\nu)$ dans $C^\infty(\mathbb{R}^{n-1})$.
Théorème 0.2 *Il existe une famille d'opérateurs d'entrelacement $\tilde{\mathcal{A}}_{\lambda,\nu} \in \mathrm{Hom}_{G'}(I(\lambda), J(\nu))$ qui dépend holomorphiquement de $(\lambda, \nu) \in \mathbb{C}^2$, qui soit non-nulle pour tout $(\lambda, \nu) \in \mathbb{C}^2 \setminus L_{\mathrm{even}}$ et dont le noyau de Schwartz $\tilde{K}_{\lambda,\nu}(x - y, x_n)$ est donné pour $((x, x_n), y) \in \mathbb{R}^n \oplus \mathbb{R}^{n-1}$ par :*

$$\widetilde{K}_{\lambda,\nu}(x, x_n) := \frac{1}{\Gamma(\frac{\lambda+\nu-n+1}{2})\Gamma(\frac{\lambda-\nu}{2})} |x_n|^{\lambda+\nu-n}(|x|^2 + x_n^2)^{-\nu}.$$

Parmi les propriétés importantes des opérateurs de brisure de symétrie citons l'existence du prolongement analytique en $(\lambda, \nu) \in \mathbb{C}^2$ et le fait qu'ils satisfont des équations fonctionnelles avec les opérateurs d'entrelacement de Knapp–Stein de G et de G', respectivement. Le support du noyau de tout opérateur de brisure de symétrie est un sous-ensemble fermé et invariant par G' de $G/P \times G'/P'$. En calculant les résidus des opérateurs de brisure de symétrie nous obtenons une nouvelle et simple preuve de la formule pour les opérateurs différentiels conformément covariants $\tilde{\mathcal{C}}_{\lambda,\nu}$ pour le plongement $S^{n-1} \hookrightarrow S^n$ qui ont récemment été découverts par A. Juhl [2]. Plus précisément, pour $(\lambda, \nu) \in //$, posons $l := \frac{1}{2}(\nu - \lambda)$ et définissons un opérateur différentiel

$$\tilde{C}_{\lambda,\nu} = \mathrm{rest} \circ \sum_{j=0}^{l} \frac{2^{2l-2j}}{j!(2l-2j)!} \prod_{i=1}^{l-j} \Big(\frac{\lambda+\nu-n-1}{2} + i \Big) \Delta^j_{\mathbb{R}^{n-1}} \Big(\frac{\partial}{\partial x_n} \Big)^{2l-2j}.$$

Ici, rest dénote la restriction à l'hyperplan $x_n = 0$. Ceci donne un opérateur différentiel d'entrelacement $\tilde{C}_{\lambda,\nu} : I(\lambda) \to J(\nu)$, d'ordre $2l$.
Théorème 0.3 (see Theorem 3.5)

$$\mathrm{Hom}_{G'}(I(\lambda), J(\nu)) = \begin{cases} \mathbb{C}\tilde{A}_{\lambda,\nu} & si\ (\lambda, \nu) \in \mathbb{C}^2 \setminus L_{even}, \\ \mathbb{C}\Gamma\left(\frac{\lambda-\nu}{2}\right) \tilde{A}_{\lambda,\nu} \oplus \mathbb{C}\tilde{C}_{\lambda,\nu} & si\ (\lambda, \nu) \in L_{even}. \end{cases}$$

1. Introduction

Given a pair of Lie groups $G' \subset G$, the restriction to the subgroup G' associates representations of G' to representations π of G. If G is compact, then any irreducible continuous representation π of G is finite-dimensional and the restriction $\pi|_{G'}$ is isomorphic to a direct sum of irreducible representations π' of G' with multiplicities $m(\pi, \pi')$. These multiplicities are studied by using combinatorial techniques. If G' is not compact and the representation π is infinite dimensional, then generically the restriction $\pi|_{G'}$ is not a direct sum of irreducible representations [3] and we have to consider another notion of multiplicity.

For a continuous representation π of a real reductive Lie group G on a Banach space H_π, the space H_π^∞ of C^∞-vectors of H_π is naturally endowed with Fréchet topology, and (π, H_π) gives rise to a continuous representation π^∞ of G on H_π^∞. Given another continuous representation π' of a reductive subgroup G', we consider the space of continuous G'-intertwining operators (*symmetry breaking operators*)

$$\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, (\pi')^\infty).$$

The dimension of this space is determined by the underlying $(\mathfrak{g}, K)$-module of π and the $(\mathfrak{g}', K')$-module of π', and is independent of the choice of the globalizations π and π'. We use the same symbol $m(\pi, \pi')$ to denote this dimension, and call it the *multiplicity* of π' occurring in the restriction $\pi|_{G'}$. This yields important information of the restriction of π to G'. Notice that the above definition of the multiplicity $m(\pi, \pi')$ makes sense for non-unitary representations π and π', too. In general, $m(\pi, \pi')$ may be infinite. It was proved in [6] that the multiplicity $m(\pi, \pi')$ is finite for all irreducible representations π of G and all irreducible representations π' of G' if and only if the minimal parabolic subgroup P' of G' has an open orbit on the real flag variety G/P, and is uniformly bounded if and only if a Borel subgroup of $G'_{\mathbb{C}}$ has an open orbit on the complex flag variety of $G_{\mathbb{C}}$. For example, the multiplicity $m(\pi, \pi')$ is uniformly bounded if the Lie algebras $(\mathfrak{g}, \mathfrak{g}')$ of (G, G') are real forms of $(\mathfrak{sl}(n+1, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}))$ or $(\mathfrak{o}(n+1, \mathbb{C}), \mathfrak{o}(n, \mathbb{C}))$.

In this note we describe the multiplicities for spherical principal series representations and their composition factors of the groups $G = O(n+1, 1)$ and $G' = O(n, 1)$. Furthermore, we give a classification of symmetry breaking operators $I(\lambda)^\infty \to J(\nu)^\infty$ for any spherical principal series representations $I(\lambda)$ and $J(\nu)$, and find explicit formulae of distribution kernels of its basis for every $(\lambda, \nu) \in \mathbb{C}^2$. The important property of these symmetry breaking operators $\tilde{\mathcal{A}}_{\lambda,\nu}$ is the existence of the analytic continuation to $(\lambda, \nu) \in \mathbb{C}^2$, and the functional equations that they satisfy with the Knapp–Stein intertwining operators of G and G'. The residue calculus of $\tilde{A}_{\lambda,\nu}$ provides a third method to obtain Juhl's conformally covariant operators for the embedding $S^{n-1} \hookrightarrow S^n$ (see [2], [4] for the existing two proofs; see also [5]).

2. The main results

Let $G = O(n+1, 1)$ be the automorphism group of a quadratic form

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 - x_{n+2}^2$$

and the subgroup $G' = O(n, 1)$ embedded as the stabilizer of the basis vector e_{n+1}.

A spherical principal series representation $I(\lambda)$ of G is induced from a character χ_λ of a minimal parabolic subgroup P for $\lambda \in \mathbb{C}$. By a little abuse of notation, we take the representation space of $I(\lambda)$ to be the space of C^∞-sections of the G-equivariant line bundle $G \times_P (\chi_\lambda, \mathbb{C}) \to G/P$, so that $I(\lambda)$ itself is the smooth Fréchet globalization of moderate growth in the sense of Casselman–Wallach [9]. The parameterization is chosen so that $I(\lambda)$ is reducible if and only if $-\lambda \in \mathbb{N}$ or $\lambda - n \in \mathbb{N}$, and that $I(-i)$ $(i \in \mathbb{N})$ contains a finite dimensional representation $F(i)$ as the unique subrepresentation. The irreducible Fréchet representation $I(-i)/F(i)$ of G is denoted by $T(i)$. Spherical principal series representations of G' are denoted by $J(\nu)$ and are parameterized so that the finite dimensional representations $F(j)$ is a subrepresentation of $J(-j)$. The irreducible Fréchet representation $J(-j)/F(j)$ of G' is denoted by $T(j)$.

Consider pairs of nonpositive integers and define

$$L_{even} = \{(-i, -j) : j \le i \text{ and } i \equiv j \mod 2\},$$
$$L_{odd} = \{(-i, -j) : j \le i \text{ and } i \equiv j + 1 \mod 2\}.$$

Theorem 2.1 (multiplicities for spherical principal series) *We have*

$$m(I(\lambda), J(\nu)) = \begin{cases} 1 & \text{if } (\lambda,\nu) \in \mathbb{C}^2 \setminus L_{even}, \\ 2 & \text{if } (\lambda,\nu) \in L_{even}. \end{cases}$$

About 20 years ago Gross and Prasad [1] formulated a conjecture determining the restriction of an irreducible admissible representation of the group $G = O(n)$ over a local field to a subgroup of the form $G' = O(n-1)$. The conjecture in [1] relates the existence of nontrivial homomorphisms to the value of an L-function at 1/2. It is also conjectured that for a given pair of generic L-packets of G and G', there is a unique non-trivial pairing, up to scalars, between precisely one member of each packet, where G and G' are allowed to vary among inner forms. Recently, it was proved in Sun and Zhu [8] that $m(\pi, \pi') \leq 1$ for all irreducible admissible representations π of G and π' of G'. However, it is more involved to tell whether $m(\pi, \pi') = 0$ or 1.

Theorem 2.1 implies that for irreducible spherical principal series representations $I(\lambda)$

$$m(I(\lambda), F(0)) = 1.$$

Furthermore we obtain

Theorem 2.2 (multiplicities for composition factors) (1) *Suppose that* $(-i, -j) \in L_{even}$. *Then*

$$m(T(i), T(j)) = 1, \qquad m(T(i), F(j)) = 0, \qquad m(F(i), F(j)) = 1.$$

(2) *Suppose that* $(-i, -j) \in L_{odd}$. *Then*

$$m(T(i), T(j)) = 0, \qquad m(T(i), F(j)) = 1, \qquad m(F(i), F(j)) = 0.$$

This generalizes the results by Loke [7] for $G = GL(2, \mathbb{C})$ and $G' = GL(2, \mathbb{R})$.

3. A brief outline of the proof.

3.1. *An analytic family of symmetry breaking operators*

We first construct an analytic family of intertwining restriction operators.

Theorem 3.1 (generic symmetry breaking operators) *There exists a family of symmetry breaking operators* $\tilde{\mathcal{A}}_{\lambda,\nu} \in \mathrm{Hom}_{G'}(I(\lambda), J(\nu))$ *that depends holomorphically on the entire* $(\lambda, \nu) \in \mathbb{C}^2$ *with the distribution kernel*

$$\widetilde{K}_{\lambda,\nu}(x, x_n) := \frac{1}{\Gamma(\frac{\lambda+\nu-n+1}{2})\Gamma(\frac{\lambda-\nu}{2})} |x_n|^{\lambda+\nu-n} (|x|^2 + x_n^2)^{-\nu}.$$

Further, $\tilde{\mathcal{A}}_{\lambda,\nu}$ *is nonzero if* $(\lambda, \nu) \in \mathbb{C}^2 \setminus L_{even}$.

We recall that there exist nonzero Knapp–Stein intertwining operators

$$\widetilde{\mathcal{T}}_\nu : J(\nu) \to J(n-1-\nu) \quad \text{and} \quad \widetilde{\mathcal{T}}_\lambda : I(\lambda) \to I(n-\lambda),$$

with holomorphic parameters $\nu \in \mathbb{C}$ and $\lambda \in \mathbb{C}$, respectively. In our normalization

$$\widetilde{\mathcal{T}}_{-\nu+n-1} \circ \widetilde{\mathcal{T}}_\nu = \frac{\pi^{n-1}}{\Gamma(n-1-\nu)\Gamma(\nu)} \mathrm{id} \quad \text{on } J(\nu), \quad \text{and} \quad \widetilde{\mathcal{T}}_{-\lambda+n} \circ \widetilde{\mathcal{T}}_\lambda = \frac{\pi^n}{\Gamma(n-\lambda)\Gamma(\lambda)} \mathrm{id} \quad \text{on } I(\lambda).$$

The following functional identities are crucial in the proof of Theorems 2.2 and 3.7.

Theorem 3.2 (functional identities) *For all* $(\lambda, \nu) \in \mathbb{C}^2$,

$$\widetilde{\mathcal{T}}_{n-1-\nu} \circ \tilde{\mathcal{A}}_{\lambda,n-1-\nu} = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(n-1-\nu)} \tilde{\mathcal{A}}_{\lambda,\nu},$$

$$\tilde{\mathcal{A}}_{n-\lambda,\nu} \circ \widetilde{\mathcal{T}}_\lambda = \frac{\pi^{\frac{n}{2}}}{\Gamma(n-\lambda)} \tilde{\mathcal{A}}_{\lambda,\nu}.$$

Here we regard the left-hand sides as zero if $\nu - n + 1 \in \mathbb{N}$ or $\lambda - n \in \mathbb{N}$, respectively.

3.2. *Other families of symmetry breaking operators*

We define

$$\begin{aligned} // :=& \{(\lambda,\nu) \in \mathbb{C}^2 : \lambda - \nu = 0, -2, -4, \ldots\}, \\ \backslash\backslash :=& \{(\lambda,\nu) \in \mathbb{C}^2 : \lambda + \nu = n-1, n-3, n-5 \ldots\}, \\ \mathbb{X} =& \backslash\backslash \cap //. \end{aligned}$$

For $(\lambda,\nu) \in //$, we set $l := \frac{1}{2}(\nu - \lambda)$ and define a differential operator

$$\tilde{\mathcal{C}}_{\lambda,\nu} = \operatorname{rest} \circ \sum_{j=0}^{l} \frac{2^{2l-2j}}{j!(2l-2j)!} \prod_{i=1}^{l-j} \Big(\frac{\lambda+\nu-n-1}{2} + i\Big) \Delta_{\mathbb{R}^{n-1}}^{j} \Big(\frac{\partial}{\partial x_n}\Big)^{2l-2j}.$$

Here rest denotes the restriction to the hyperplane $x_n = 0$. It gives a differential G'-intertwining operator $\tilde{\mathcal{C}}_{\lambda,\nu} : I(\lambda) \to J(\nu)$ of order $2l$, and coincides with the conformally covariant differential operator for the embedding $S^{n-1} \hookrightarrow S^n$, which was discovered recently by A. Juhl in [2].

For $(\lambda,\nu) \in \backslash\backslash$, we define another family of G'-intertwining operators $\tilde{\mathcal{B}}_{\lambda,\nu} : I(\lambda) \to J(\nu)$ that depends holomorphically on $\lambda \in \mathbb{C}$ (or on $\nu \in \mathbb{C}$) by the distribution kernel

$$\widetilde{K}^B_{\lambda,\nu}(x, x_n) := \frac{1}{\Gamma(\frac{\lambda-\nu}{2})} (|x|^2 + x_n^2)^{-\nu} \delta^{(2k)}(x_n).$$

For $\nu \in -\mathbb{N}$, $\tilde{\tilde{\mathcal{A}}}_{\lambda,\nu} := \Gamma(\frac{\lambda-\nu}{2})\tilde{\mathcal{A}}_{\lambda,\nu}$ extends to a non-zero G'-intertwining map, that depends holomorphically on $\lambda \in \mathbb{C}$.

We obtain:

Proposition 3.3 *Every operator in $\operatorname{Hom}_{G'}(I(\lambda), J(\nu))$ is in the span of the operators defined in Sections 3.1 and 3.2.*

3.3. *Reduction of symmetry-breaking operators*

Examining the linear independence of symmetry breaking operators constructed above we prove in particular

Theorem 3.4 (residue formulae)

(1) *For $(\lambda,\nu) \in \backslash\backslash \setminus \mathbb{X}$, we define $k := \frac{1}{2}(n - 1 - \lambda - \nu) \in \mathbb{N}$. Then*

$$\tilde{\mathcal{A}}_{\lambda,\nu} = \frac{(-1)^k}{2^k(2k-1)!!} \tilde{\mathcal{B}}_{\lambda,\nu}.$$

(2) *For $(\nu,\lambda) \in //$, we define $l := \frac{1}{2}(\nu - \lambda)$. Then*

$$\tilde{\mathcal{A}}_{\lambda,\nu} = \frac{(-1)^l l! \pi^{\frac{n-1}{2}}}{\Gamma(\nu) 2^{2l}} \tilde{\mathcal{C}}_{\lambda,\nu}.$$

(3) *Suppose $(\lambda,\nu) \in \mathbb{X}$. We define $k, l \in \mathbb{N}$ as above. Then*

$$\tilde{\mathcal{B}}_{\lambda,\nu} = \frac{(-1)^{l-k} 2^{k-2l} \pi^{\frac{n-1}{2}} l! (2k-1)!!}{\Gamma(\nu)} \tilde{\mathcal{C}}_{\lambda,\nu}.$$

Using the support of the operators, we conclude the following refinement of Theorem 2.1.

Theorem 3.5 (explicit basis) *For* $(\lambda,\nu)\in\mathbb{C}^2$*, we have*

$$\mathrm{Hom}_{G'}(I(\lambda),J(\nu))=\begin{cases}\mathbb{C}\tilde{\tilde{\mathcal{A}}}_{\lambda,\nu}\oplus\mathbb{C}\tilde{\mathcal{C}}_{\lambda,\nu} & \text{if } (\lambda,\nu)\in L_{even},\\ \mathbb{C}\tilde{\mathcal{A}}_{\lambda,\nu} & \text{if } (\lambda,\nu)\in\mathbb{C}^2\setminus L_{even}.\end{cases}$$

3.4. *Image of symmetry breaking operators*

Denote by $\mathbf{1}_\lambda$ and $\mathbf{1}_\nu$ the unit spherical vectors in $I(\lambda)$ and $J(\nu)$, respectively. The image of spherical vector $\mathbf{1}_\lambda$ under the G'-intertwining operators $\tilde{\mathcal{A}}_{\lambda,\nu}$ and $\tilde{\mathcal{B}}_{\lambda,\nu}$ is nonzero if and only if $\lambda\neq 0,-1,-2,-3\ldots$, whereas it is always nonzero under $\tilde{\mathcal{C}}_{\lambda,\nu}$. More precisely

Theorem 3.6 (image of spherical vectors) (1) *For* $(\lambda,\nu)\in\mathbb{C}^2$, $\tilde{\mathcal{A}}_{\lambda,\nu}(\mathbf{1}_\lambda)=\pi^{\frac{n-1}{2}}/\Gamma(\lambda)\mathbf{1}_\nu$.

(2) *For* $(\lambda,\nu)\in\backslash\backslash$*, we set* $k:=\frac{1}{2}(n-1-\lambda-\nu)$*. Then* $\tilde{\mathcal{B}}_{\lambda,\nu}(\mathbf{1}_\lambda)=(-1)^k2^k\pi^{\frac{n-1}{2}}(2k-1)!!/\Gamma(\lambda)\mathbf{1}_\nu$.

(3) *For* $(\lambda,\nu)\in//$*, we set* $l:=\frac{1}{2}(\nu-\lambda)\in\mathbb{N}$*. Then* $\tilde{\mathcal{C}}_{\lambda,\nu}(\mathbf{1}_\lambda)=(-1)^l2^{2l}(\lambda)_{2l}/l!\mathbf{1}_\nu$.

We also determine the image of the underlying $(\mathfrak{g},K)$-module $I(\lambda)_K$ of $I(\lambda)$ by the symmetry breaking operators for all the parameters $(\lambda,\nu)\in\mathbb{C}^2$. Using the basis in Theorem 3.5, we have:

Theorem 3.7 (image of breaking symmetry operator) (1) *Suppose that* $(\lambda,\nu)\in L_{even}$*. We set* $j:=-\nu\in\mathbb{N}$*.*

$$\mathrm{Image}\,\tilde{\tilde{\mathcal{A}}}_{\lambda,\nu}=F(j)\quad\text{and}\quad\mathrm{Image}\,\tilde{\mathcal{C}}_{\lambda,\nu}=J(\nu)_{K'}.$$

(2) *Suppose that* $(\lambda,\nu)\notin L_{even}$*.*

$$\begin{array}{lll}\text{(2-a)} & \mathrm{Image}\,\tilde{\mathcal{A}}_{\lambda,\nu}=F(-\nu) & \text{if } \nu\in-\mathbb{N},\\ \text{(2-b)} & \mathrm{Image}\,\tilde{\mathcal{A}}_{\lambda,\nu}=T(\nu+1-n)_{K'} & \text{if } (\lambda,\nu)\in\backslash\backslash \text{ and } \nu+1-n\in\mathbb{N},\\ \text{(2-c)} & \mathrm{Image}\,\tilde{\mathcal{A}}_{\lambda,\nu}=J(\nu)_{K'} & \text{otherwise.}\end{array}$$

A detailed proof will be given in another paper.

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