F-method for symmetry breaking operators

Dedicated to Professor Michael Eastwood for his 60th birthday

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Abstract

We provide some insights in the study of branching problems of reductive groups, and a method of investigations into symmetry breaking operators. First, we give geometric criteria for finiteness property of linearly independent continuous (respectively, differential) operators that intertwine two induced representations of reductive Lie groups and their reductive subgroups. Second, we extend the 'F-method' known for local operators to non-local operators. We then illustrate the idea by concrete examples in conformal geometry, and explain how the F-method works for detailed analysis of symmetry breaking operators, e.g., finding functional equations and explicit residue formulae of 'regular' symmetry breaking operators with meromorphic parameters.

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Contents

1 Introduction

Let G be a real reductive linear Lie group, and P a parabolic subgroup. Associated to a finite-dimensional representation $\lambda : P \to GL_{\mathbb{C}}(V)$, we define a homogeneous vector bundle $\mathcal{V} := G \times_P V$ over the (generalized) real flag variety $X := G/P$. Then the group G acts continuously on the space $C^\infty(X, \mathcal{V})$ of smooth sections, which is endowed with the natural Fréchet topology of uniform convergence of finite derivatives.

Suppose G' is an (algebraic) reductive subgroup of G, P' is a parabolic subgroup of G', and $Y := G'/P'$. For a finite-dimensional representation $\nu : P' \to GL_{\mathbb{C}}(W)$, we define similarly a homogeneous vector bundle $\mathcal{W} := G' \times_{P'} W$ over Y, and form a continuous representation of G' on $C^\infty(Y, \mathcal{W})$. We denote by $\mathrm{Hom}_{G'}(C^\infty(G/P, \mathcal{V}), C^\infty(G'/P', \mathcal{W}))$ the space of continuous G'-homomorphisms (*symmetry breaking operators*).

Assume that

$$P' \subset P \cap G'. \tag{1.1}$$

Then we have a natural G'-equivariant morphism $\iota : Y \to X$. With this morphism ι, we can define a continuous linear operator $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ to be a *differential operator* in a wider sense than the usual by the following local property:

$$\iota(\mathrm{Supp}(Tf)) \subset \mathrm{Supp}\, f \quad \text{for any } f \in C^\infty(X, \mathcal{V}).$$

In the case where $Y = X$ with ι the identity map, this definition is equivalent to that T is a differential operator in the classical sense by Peetre's theorem. We shall write $\mathrm{Diff}_{G'}(C^\infty(G/P, \mathcal{V}), C^\infty(G'/P', \mathcal{W}))$ for the space of G'-intertwining differential operators.

The object of our study is local and non-local symmetry breaking operators:

$$\mathrm{Diff}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) \subset \mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})). \tag{1.2}$$

We consider the following:

Project A. Construct *explicitly* differential (local)/ continuous (non-local) symmetry breaking operators, and classify them.

In the setting where $X = Y$ and $G = G'$, the Knapp–Stein intertwining operators are a basic example of non-local intertwining operators. On the other hand, the existence condition for non-zero homomorphisms between generalized Verma modules has been studied algebraically by many authors in various cases (see [34] and references therein), which is in turn equivalent to the existence condition for non-zero local G-intertwining operators by the duality between Verma modules and principal series representations (e.g. [12]). However, even in this usual setting, it is already a non-trivial matter to write down explicitly the operators, and a complete classification of such operators is far from being solved for reductive groups G in general.

In the setting that we have in mind, namely, where $G' \subsetneq G$, we face with branching problems of irreducible representations of the group G when we restrict them to the subgroup G' (or branching problems for Lie algebras). The study of the restriction of infinite-dimensional representations is difficult in general even as 'abstract analysis', involving wild features ([16, 21]). On the other hand, Project A asks further 'concrete analysis' of the restriction. Nevertheless, there have been recently some explicit results [5, 14, 22, 28, 30, 31] for specific situations where $X \neq Y$ and $G \neq G'$, in relation to Project A.

In light of the aforementioned state of the art, we first wish to clarify what are reasonably general settings for Project A and what are their limitations, not coming from the existing technical difficulties, but from purely representation theoretic constraints. For this, we remember that the spaces of symmetry breaking operators are not always finite-dimensional if $G \neq G'$, and consequently, the subgroup G' may lose a good control of the irreducible G-module ([16]). Thus we think that Project A should be built on a solid foundation where the spaces of local/non-local symmetry breaking operators are at most finite-dimensional, and preferably, of uniformly bounded dimensions, or even one-dimensional. In short, we pose:

Project B. Single out appropriate settings in which Project A makes sense.

In Section 2, we discuss Project B by applying recent progress on the theory of restrictions of representations [19, 21, 23, 29], and provide concrete geometric conditions that assure the spaces in (1.2) to be finite-dimensional, of uniformly bounded dimensions, or at most one-dimensional (multiplicity-free restrictions).

In Sections 3 to 5 we develop a method of Project A by extending the idea of the 'F-method' studied earlier for local operators ([22, 28, 30]) to non-local operators.

2 Finiteness theorems of restriction maps

This section is devoted to Project B.

We begin with the most general case, namely, the case where P and P' are minimal parabolic subgroups of G and G', respectively. In this case, the assumption (1.1) is automatically satisfied (with P replaced by its conjugate, if necessary). Under the condition, we give finiteness criteria for local and non-local operators:

Theorem 2.1 (local operators)**.** *Assume* $\operatorname{rank}_{\mathbb{R}} G = \operatorname{rank}_{\mathbb{R}} G'$. *Then*

$$\dim \operatorname{Diff}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) < \infty \tag{2.1}$$

for all finite-dimensional representations V *of* P *and* W *of* P'.

Remark 2.2. We give a proof of Theorem 2.1 for more general parabolic subgroups P and P', see Theorem 2.7 below. We note that the real rank assumption of Theorem 2.1 is relaxed in Theorem 2.7.

Theorem 2.3 (non-local operators)**.** 1) (finite multiplicity) *The following two conditions on the pair* (G, G') *of real reductive groups are equivalent:*

(i) *For all finite-dimensional representations* V *of* P *and* W *of* P',

$$\dim \operatorname{Hom}_{G'}(C^\infty(G/P, \mathcal{V}), C^\infty(G'/P', \mathcal{W})) < \infty. \tag{2.2}$$

(ii) *There exists an open* P'*-orbit on the (generalized) real flag variety* G/P.

2) (uniformly bounded multiplicity) *Let* G *be a simple Lie group. We write* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *for the complexifications of the Lie algebras* $\mathfrak{g}_{\mathbb{R}}$ *and* $\mathfrak{g}'_{\mathbb{R}}$ *of* G *and* G', *respectively. Then the following conditions on the pair* (G, G') *are equivalent:*

(i)

$$\sup_V \sup_W \dim \operatorname{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) < \infty, \tag{2.3}$$

where the supremum is taken over all finite-dimensional, irreducible representations V of P and W of P', respectively.

(ii) *There exists an open B'-orbit on the complex flag variety $G_{\mathbb{C}}/B$, where B and B' are Borel subgroups of complexifications $G_{\mathbb{C}}$ and $G'_{\mathbb{C}}$ of G and G' respectively.*

(iii) (strong Gelfand pair) *The pair $(\mathfrak{g}, \mathfrak{g}')$ is one of $(\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C}))$, $(\mathfrak{sl}(n+1,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C}))$ $(n \neq 1)$, $(\mathfrak{so}(n+1,\mathbb{C}), \mathfrak{so}(n,\mathbb{C}))$, or $\mathfrak{g} = \mathfrak{g}'$.*

Theorem 2.3 (1) suggests that the orbit structure $P' \backslash G/P$ is crucial to understand $\operatorname{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$. In fact, the 'regular' symmetry breaking operators are built on the integral transform attached to open P'-orbits on G/P [31], whereas the closed P'-orbit through the origin $o = eP/P \in G/P$ is the support of the distribution kernel (see Proposition 3.1 below) of differential symmetry breaking operators [30]. The whole orbit structure plays a basic role in the classification of all symmetry breaking operators in [31] in the setting that we discuss in Section 5.

A remarkable feature of Theorem 2.3 (2) is that the equivalent conditions do not depend on the real form $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}'_{\mathbb{R}})$ but depend only on the complexification $(\mathfrak{g}, \mathfrak{g}')$, which are obvious from (ii) or (iii) but are quite non-trivial from (i). On the other hand, the equivalent conditions in Theorem 2.3 (1) depend heavily on the real form $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}'_{\mathbb{R}})$.

Comparing the criteria given in Theorems 2.1 and 2.3, we find that the space of non-local intertwining operators (i.e. the right-hand side of (1.2)) are generally much larger than that of local operators (i.e. the left-hand side of (1.2)). Thus 'appropriate settings' in Project B will be different for local and non-local operators:

Example 2.4. Suppose $(G, G') = (GL(p+q, \mathbb{R}), GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$. Then the finite-dimensionality (2.1) of local operators holds for all p, q, whereas the finite-dimensionality (2.2) of non-local operators fails if $p \geq 2$ or $q \geq 2$. Likewise, for the non-symmetric pair

$$(G, G') = (GL(p+q+r, \mathbb{R}), GL(p, \mathbb{R}) \times GL(q, \mathbb{R}) \times GL(r, \mathbb{R})),$$

the finite-dimensionality (2.1) still holds for all $p, q, r > 0$, whereas (2.2) fails for any $p, q, r > 0$.

Example 2.5 (group case)**.** Let $(G, G') = (H \times H, \operatorname{diag} H)$ with $H = O(p, q)$. Then the condition (ii) of Theorem 2.3 (1) holds if and only if $\min(p, q) \leq 1$. Again, (2.1) always holds but (2.2) fails if $\min(p, q) \geq 2$. Symmetry breaking operators for the pair $(H \times H, \operatorname{diag} H)$ correspond to invariant trilinear forms of representations of H, for which concrete analysis was studied, e.g., in [5] in the case $H = O(p, 1)$.

In a subsequent paper [25], we shall give a complete classification on the level of the Lie algebras of the reductive symmetric pairs (G, G') that satisfy the equivalent conditions of Theorem 2.3 (1). (The classification was given earlier in [17] in the setting where (G, G') is of the form $(H \times H, \operatorname{diag} H)$, and Example 2.5 is essentially the unique example satisfying the equivalence conditions in this case.)

Here are some comments on the proof of Theorem 2.3. The implication (ii) $\Rightarrow$ (i) in both (1) and (2) of Theorem 2.3 is a direct consequence of the main theorems of [29, Theorems A and B] which are stated in a more general setting. The converse implication is proved by using a generalized Poisson transform [29, Theorem 3.1]. The method is based on the theory of system of partial differential equations with regular singularities and hyperfunction boundary values, developed by M. Sato, M. Kashiwara, T. Kawai, and T. Oshima [15] and the compactification of the group manifold with normal-crossing boundaries. Alternatively, the implication (ii) to (i) in Theorem 2.3 (2) could be derived from a recent multiplicity-free theorem [36] and from the classification of real forms of (complex) strongly Gelfand pairs [33]. We note that the proof in [29] does not use any case-by-case argument.

We give a proof of Theorem 2.1 in a more general form below. Let us fix some notation. A semisimple element H of a complex reductive Lie algebra $\mathfrak{g}$ is said to be *hyperbolic* if the eigenvalues of $\operatorname{ad}(H) \in \operatorname{End}_{\mathbb{C}}(\mathfrak{g})$ are all real. Given a hyperbolic element H, we define subalgebras of $\mathfrak{g}$ by

$$\mathfrak{n}^+ \equiv \mathfrak{n}^+(H), \quad \mathfrak{l} \equiv \mathfrak{l}(H), \quad \mathfrak{n}^- \equiv \mathfrak{n}^-(H)$$

to be the sum of the eigenspaces with positive, zero, and negative eigenvalues, respectively. Then $\mathfrak{p}(H) := \mathfrak{l}(H) + \mathfrak{n}^+(H)$ is a parabolic subalgebra of $\mathfrak{g}$. Let

$\mathfrak{g}_{\mathbb{R}}$ be the Lie algebra of a real reductive Lie group G, and $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. If H is a hyperbolic element of $\mathfrak{g}_{\mathbb{R}}$, then

$$P(H) := \{g \in G : \mathrm{Ad}(g)\mathfrak{p}(H) = \mathfrak{p}(H)\} = L(H)\exp(\mathfrak{n}^+(H))$$

is a parabolic subgroup of G, and $\mathfrak{p}_{\mathbb{R}}(H) := \mathfrak{p}(H) \cap \mathfrak{g}_{\mathbb{R}}$ is its Lie algebra.

We define the following subset of $\mathfrak{g}_{\mathbb{R}}$ by

$$\mathfrak{g}_{\mathbb{R}}^{\mathrm{reg,hyp}} := \{H \in \mathfrak{g}_{\mathbb{R}} : H \text{ is hyperbolic, and } L(H) \text{ is amenable}\}. \tag{2.4}$$

For a hyperbolic element H in $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{p}_{\mathbb{R}}(H)$ is a minimal parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$ if and only if $H \in \mathfrak{g}_{\mathbb{R}}^{\mathrm{reg,hyp}}$ by definition.

Definition 2.6 ($\mathfrak{g}'$-compatible parabolic subalgebra)**.** We say a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is $\mathfrak{g}'$-*compatible* if there exists a hyperbolic element H in $\mathfrak{g}'$ such that $\mathfrak{p} \equiv \mathfrak{p}(H)$. We say P is G'-compatible if we can take H in $\mathfrak{g}'_{\mathbb{R}}$.

If P is G'-compatible, then $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{g}'$ becomes a parabolic subalgebra of $\mathfrak{g}'$ with Levi decomposition

$$\mathfrak{p}' = \mathfrak{l}' + (\mathfrak{n}^+)' \equiv (\mathfrak{l} \cap \mathfrak{g}') + (\mathfrak{n}^+ \cap \mathfrak{g}')$$

and $P' := P \cap G'$ becomes a parabolic subgroup of G'. The $\mathfrak{g}'$-compatibility is a sufficient condition for the 'discrete decomposability' of generalized Verma modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F$ when restricted to the reductive subalgebra $\mathfrak{g}'$ (see [21]).

By abuse of notation, we shall write also $\mathcal{L}_\lambda \to X$ for the line bundle associated to (λ, V) instead of the previous notation $\mathcal{V} \to X$ if V is one-dimensional. We write $\lambda \gg 0$ if $\langle d\lambda, \alpha \rangle \gg 0$ for all $\alpha \in \Delta(\mathfrak{n}^+, \mathfrak{j})$ where $\mathfrak{j}$ is a Cartan subalgebra of $\mathfrak{l}$.

Theorem 2.7 (local operators)**.** *Let $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$. Suppose $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$ is $\mathfrak{g}'$-compatible.*
1) (finite multiplicity) *For any finite-dimensional representations V and W of the parabolic subgroups P and P', respectively, we have*

$$\dim \mathrm{Diff}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) < \infty.$$

2) (uniformly bounded multiplicity) *If $(\mathfrak{g}, \mathfrak{g}')$ is a reductive symmetric pair and $\mathfrak{n}^+$ is abelian, then*

$$\sup_W \dim \mathrm{Diff}_{G'}(C^\infty(X, \mathcal{L}_\lambda), C^\infty(Y, \mathcal{W})) = 1,$$

for any one-dimensional representation $\mathbb{C}_\lambda$ *of* P *with* $\lambda \gg 0$*. Here* W *runs over all finite-dimensional irreducible representations of* P'*.*

Proof. 1) The classical duality between Verma modules and principal series representations (e.g. [12]) can be extended to the context of the restriction for reductive groups $G \downarrow G'$, and the following bijection holds (see [30, Corollary 2.9]):

$$\mathrm{Hom}_{(\mathfrak{g}',P')}(U(\mathfrak{g}')\otimes_{U(\mathfrak{p}')}W^\vee, U(\mathfrak{g})\otimes_{U(\mathfrak{p})}V^\vee) \simeq \mathrm{Diff}_{G'}(C^\infty(G/P,\mathcal{V}), C^\infty(G'/P',\mathcal{W})).$$

Here $(\lambda^\vee, V^\vee)$ denotes the contragredient representation of (λ, V). Then the proof of Theorem 2.7 is reduced to the next proposition. □

Proposition 2.8. *Let* $\mathfrak{g}'$ *be a reductive subalgebra of* $\mathfrak{g}$*. Suppose that* $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$ *is* $\mathfrak{g}'$*-compatible.*
1) *For any finite-dimensional* $\mathfrak{p}$*-module* F *and* $\mathfrak{p}'$*-module* F'*,*

$$\dim \mathrm{Hom}_{\mathfrak{g}'}(U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} F', U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F) < \infty.$$

2) *If* $(\mathfrak{g}, \mathfrak{g}')$ *is a symmetric pair and* $\mathfrak{n}^+$ *is abelian, then*

$$\sup_{F'} \dim \mathrm{Hom}_{\mathfrak{g}'}(U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} F', U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda) = 1$$

for any one-dimensional representation $\mathbb{C}_\lambda$ *of* $\mathfrak{p}$ *with* $\lambda \ll 0$*. Here the supremum is taken over all finite-dimensional simple* $\mathfrak{p}'$*-modules* F'*.*

Proof. 1) The proof is parallel to [21, Theorem 3.10] which treated the case where F and F' are simple modules of P and P', respectively.
2) See [21, Theorem 5.1]. □

Hence Theorem 2.7 is shown. We refer also to [19, Theorem B] for an analogous statement to Theorem 2.7 (2) which was formulated in the context of unitary representations.

Proof of Theorem 2.1. If H is a generic hyperbolic element H in $\mathfrak{g}'_{\mathbb{R}}$, then $\mathfrak{p}_{\mathbb{R}}(H)$ is a minimal parabolic subalgebra of $\mathfrak{g}_{\mathbb{R}}$ owing to the rank assumption. Then Theorem 2.1 follows from Theorem 2.7. □

Remark 2.9. In most cases where G' is a proper non-compact subgroup of G, for each representation (λ, V) of P, there exist continuously many representations (ν, W) of P' such that

$$\mathrm{Hom}_{G'}(C^\infty(X,\mathcal{V}), C^\infty(Y,\mathcal{W})) \neq \{0\}.$$

However, there are a few exceptional cases where only a finite number of irreducible representations (ν, W) of P' satisfy

$$\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) \neq \{0\}$$

([20, Theorem 3.8]). This happens when $X = Y$ and $G \supsetneqq G'$. Even in this case, Project A brought us a new interaction of the classical analysis (e.g. the Weyl operator calculus) and representation theory (e.g. infinite-dimensional representations of minimal Gelfand–Kirillov dimensions) via symmetry breaking operators, see [4, 27] for $(G, G') = (GL(2n, \mathbb{F}), Sp(n, \mathbb{F}))$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

3 F-method for continuous operators

The 'F-method' is a powerful tool to find singular vectors explicitly in the Verma modules by using the algebraic Fourier transform [22]. We applied the 'F-method' in the previous papers [28, 30] to construct new covariant differential operators including classical Rankin–Cohen's bi-differential operator [6, 7, 35] and Juhl's conformally covariant differential operator [14].

In this section, we generalize the idea of the F-method from local to nonlocal operators. We follow the notation of [30]. In particular, we regard distributions as generalized functions à la Gelfand rather than continuous linear forms on $C_c^\infty(X)$. Concrete examples will be discussed in Section 5.

We retain the setting as before. Let $G_{\mathbb{C}}$ be a complexification of G. According to the Gelfand–Naimark decomposition $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{l} + \mathfrak{n}^+$ of the complex reductive Lie algebra $\mathfrak{g}$, we have a diffeomorphism

$$\mathfrak{n}^- \times L_{\mathbb{C}} \times \mathfrak{n}^+ \to G_{\mathbb{C}}, \quad (X, \ell, Y) \mapsto (\exp X)\ell(\exp Y),$$

onto an open subset $G_{\mathbb{C}}^{\mathrm{reg}}$ containing the identity of the complex Lie group $G_{\mathbb{C}}$. Let

$$p_\pm : G_{\mathbb{C}}^{\mathrm{reg}} \longrightarrow \mathfrak{n}^\pm, \qquad p_o : G_{\mathbb{C}}^{\mathrm{reg}} \to L_{\mathbb{C}},$$

be the projections characterized by the identity

$$\exp(p_-(g))\, p_o(g)\, \exp(p_+(g)) = g.$$

Then the definition of the following maps α and β is independent of the choice of $G_{\mathbb{C}}$:

$$(3.1)\quad (\alpha, \beta) : \mathfrak{g} \times \mathfrak{n}^- \to \mathfrak{l} \oplus \mathfrak{n}^-, \quad (D, C) \mapsto \left.\frac{d}{dt}\right|_{t=0} \left(p_o\left(e^{tD}e^C\right), p_-\left(e^{tD}e^C\right)\right).$$

For example, if $\mathfrak{n}^\pm$ are abelian or equivalently if $(\mathfrak{g}, \mathfrak{l})$ is a symmetric pair, then $\alpha : \mathfrak{g} \times \mathfrak{n}^- \to \mathfrak{l}$ takes the form

$$\alpha(D, C) = \begin{cases} [D, C] & \text{for } C \in \mathfrak{n}^-, D \in \mathfrak{n}^+, \\ 0 & \text{for } C \in \mathfrak{n}^-, D \in \mathfrak{n}^- + \mathfrak{l}. \end{cases}$$

For $D \in \mathfrak{g}$, $\beta(D, \cdot\,)$ induces a linear map $\mathfrak{n}^- \to \mathfrak{n}^-$, and thus we may regard $\beta(D, \cdot)$ as a holomorphic vector field on $\mathfrak{n}^-$ via the identification $\mathfrak{n}^- \ni C \mapsto \beta(D, C) \in \mathfrak{n}^- \simeq T_C\mathfrak{n}^-$.

We recall $\mathfrak{n}^\pm_\mathbb{R}$ are real forms of the complex Lie algebras $\mathfrak{n}^\pm$. Let $N^\pm := \exp(\mathfrak{n}^\pm_\mathbb{R})$. Then

$$\iota : \mathfrak{n}^-_\mathbb{R} \to G/P, \quad Z \mapsto (\exp Z)P \tag{3.2}$$

defines an open Bruhat cell N^-P/P in the real flag variety G/P.

We denote by $\mathbb{C}_{2\rho}$ the one-dimensional representation of P on $|\Lambda^{\dim \mathfrak{n}}(\mathfrak{n})|$. The infinitesimal representation will be denoted by 2ρ.

Let (λ, V) be a finite-dimensional representation of P with trivial action of N^+. Then the dualizing bundle $\mathcal{V}^* := \mathcal{V}^\vee \otimes \Omega_{G/P}$ is given by

$$\mathcal{V}^* \simeq G \times_P (V^\vee \otimes \mathbb{C}_{2\rho})$$

as a homogeneous vector bundle. The pull-back of $\mathcal{V}^* \to G/P$ to $\mathfrak{n}^-_\mathbb{R}$ via (3.2) defines the trivial vector bundle $\mathfrak{n}^-_\mathbb{R} \times V^\vee \to \mathfrak{n}^-_\mathbb{R}$, and induces an injective morphism $\iota^* : C^\infty(G/P, \mathcal{V}^*) \to C^\infty(\mathfrak{n}^-_\mathbb{R}) \otimes V^*$. The infinitesimal representation of the regular representation $C^\infty(G/P, \mathcal{V}^*)$ is given as an operator on $C^\infty(\mathfrak{n}^-_\mathbb{R}) \otimes V^\vee$ by

$$d\pi^*_\lambda(D) := \langle d\lambda^\vee + 2\rho \operatorname{id}_{V^\vee}, \alpha(D, \cdot)\rangle - \beta(D, \cdot) \otimes \operatorname{id}_{V^\vee} \quad \text{for } D \in \mathfrak{g}. \tag{3.3}$$

We may regard $d\pi^*_\lambda(D)$ as an $\operatorname{End}(V^\vee)$-valued holomorphic differential operator on $\mathfrak{n}^- = \mathfrak{n}^-_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$.

Any continuous operator $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ is given by a distribution kernel $K_T \in \mathcal{D}'(X \times Y, \mathcal{V}^* \boxtimes \mathcal{W})$ by the Schwartz kernel theorem. We write

$$m : G \times G' \to G, \quad (g, g') \mapsto (g')^{-1}g,$$

for the multiplication map. Then the pull-back m^*K_T is regarded as an element of $\mathcal{D}'(X, \mathcal{V}^*) \otimes W$. We have from [31] the following two propositions:

Proposition 3.1. *The correspondence $T \mapsto m^* K_T$ induces a bijection:*

$$\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) \xrightarrow{\sim} (\mathcal{D}'(X, \mathcal{V}^*) \otimes W)^{\Delta(P')}.$$

Proposition 3.2. *Assume that the natural multiplication map $P' \times N^- \times P \to G$ is surjective, namely,*

$$P' N^- P = G. \tag{3.4}$$

Then $\iota^* m^* K_T$ is a W-valued tempered distribution on $\mathfrak{n}_{\mathbb{R}}^-$, and the correspondence $T \mapsto \iota^* m^* K_T$ is injective:

$$\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) \hookrightarrow \mathcal{S}'(\mathfrak{n}_{\mathbb{R}}^-) \otimes W. \tag{3.5}$$

The idea of the F-method for non-local operators is to characterize the image of (3.5) by the Fourier transform $\mathcal{S}'(\mathfrak{n}_{\mathbb{R}}^-) \xrightarrow{\sim} \mathcal{S}'(\mathfrak{n}_{\mathbb{R}}^+)$.

For this, we recall the algebraic Fourier transform of the Weyl algebra. Let E be a complex vector space, and denote by $\mathcal{D}(E)$ the ring of holomorphic differential operators on E with polynomial coefficients.

Definition 3.3 (algebraic Fourier transform)**.** We define the *algebraic Fourier transform* as an isomorphism of the Weyl algebras on E and its dual space $E^\vee$:

$$\mathcal{D}(E) \to \mathcal{D}(E^\vee), \qquad S \mapsto \widehat{S},$$

induced by

$$\widehat{\frac{\partial}{\partial z_j}} := -\zeta_j, \quad \widehat{z_j} := \frac{\partial}{\partial \zeta_j}, \quad 1 \leq j \leq n = \dim E,$$

where $(z_1, \ldots, z_n)$ are coordinates on E and $(\zeta_1, \ldots, \zeta_n)$ are the dual coordinates on $E^\vee$. The definition does not depend on the choice of coordinates.

Suppose P is a G'-compatible parabolic subgroup of G (Definition 2.6), and we take P' to be a parabolic subgroup of G' defined by $P \cap G' = L' \exp(\mathfrak{n}_{\mathbb{R}}^+)'$. We define a subspace of $\mathrm{Hom}(V, W)$-valued tempered distributions on $\mathfrak{n}_{\mathbb{R}}^+$ by

$$\mathcal{S}ol(V, W)^\wedge := \{F(\xi) \in \mathcal{S}'(\mathfrak{n}_{\mathbb{R}}^+) \otimes \mathrm{Hom}(V, W) : F \text{ satisfies (3.6) and (3.7) on } \mathfrak{n}_{\mathbb{R}}^+\},$$

where

$$\nu(l) \circ F(\mathrm{Ad}(l^{-1}) \cdot) \circ \lambda(l^{-1}) = F(\cdot) \quad \text{for all } l \in L', \tag{3.6}$$

$$(3.7) \qquad (\widehat{d\pi^*_\lambda(C)} \otimes \mathrm{id}_W + \mathrm{id}_{V^\vee} \otimes d\nu(C))|_{\zeta=-i\xi}\ F = 0 \quad \text{for all } C \in (\mathfrak{n}^+)'.$$

Combining [30] and Proposition 3.2, we obtain:

Theorem 3.4 (F-method for continuous operators)**.** *Let G be a reductive linear Lie group, and G' a reductive subgroup. Suppose P is a G'-compatible parabolic subgroup of G, and $P' = P \cap G'$. We assume* (3.4)*.*

Then the Fourier transform $\mathcal{F}_{\mathbb{R}}$ (see Remark 3.6) of the distribution kernel induces the following bijection:

$$(3.8) \qquad \mathrm{Hom}_{G'}(C^\infty(G/P, \mathcal{V}), C^\infty(G'/P', \mathcal{W})) \xrightarrow{\sim} \mathcal{S}ol(V, W)^\wedge.$$

Remark 3.5. Theorem 3.4 extends the following bijection:

$$(3.9) \qquad \mathrm{Diff}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) \simeq \mathcal{S}ol(V, W)^\wedge \cap \mathrm{Pol}(\mathfrak{n}^+),$$

which was proved in [30] (cf. [28]) without the assumption (3.4).

Remark 3.6. Suppose $E_{\mathbb{R}}$ is a real form of E. If f is a compactly supported distribution on $E_{\mathbb{R}}$, the Fourier transform $\mathcal{F}_{\mathbb{R}} f$ extends holomorphically on the entire complex vector space $E^\vee$. In this case, we may compare two conventions of 'Fourier transforms'

$$\mathcal{F}_c : \mathcal{E}'(E_{\mathbb{R}}) \to \mathcal{O}(E^\vee), \quad f \mapsto \int_{E_{\mathbb{R}}} f(x) e^{\langle x, \zeta \rangle} dx,$$
$$\mathcal{F}_{\mathbb{R}} : \mathcal{E}'(E_{\mathbb{R}}) \to \mathcal{O}(E^\vee), \quad f \mapsto \int_{E_{\mathbb{R}}} f(x) e^{-i\langle x, \xi \rangle} dx,$$

where $\mathcal{E}'(E_{\mathbb{R}})$ stands for the space of compactly supported distributions. We note

$$(3.10) \qquad (\mathcal{F}_{\mathbb{R}} f)(\xi) = (\mathcal{F}_c f)(\zeta) \qquad \text{with } \zeta = -i\xi.$$

In [30] we have adopted $\mathcal{F}_c$ instead of $\mathcal{F}_{\mathbb{R}}$. An advantage of $\mathcal{F}_c$ is that the algebraic Fourier transform defined in Definition 3.3 satisfies

$$(3.11) \qquad \widehat{T} = \mathcal{F}_c \circ T \circ \mathcal{F}_c^{-1} \quad \text{for all } T \in \mathcal{D}(E),$$

which simplifies actual computations in the F-method. In the case where $\mathfrak{n}^+$ is abelian, the bijection (3.9) is compatible with the symbol map

$$\mathrm{Diff}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) \to \mathrm{Pol}(\mathfrak{n}^+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W),$$

if we use $\mathcal{F}_c$ instead of $\mathcal{F}_{\mathbb{R}}$ (see [22, Theorem 3.5] or [30]).

4 Conformally covariant symmetry breaking

In this section we set up some notation for conformally covariant operators in the setting where the groups $G' \subset G$ act conformally on two pseudo-Riemannian manifolds $Y \subset X$, respectively.

Let X be a smooth manifold equipped with a pseudo-Riemannian structure g. Suppose a group G acts conformally on X. The action will be denoted by $L_h : X \to X,\ x \mapsto L_h x$ for $h \in G$. Then there exists a positive-valued function Ω on $G \times X$ such that

$$L_h^*(g_{L_h x}) = \Omega(h,x)^2 g_x \quad \text{for any } h \in G, \text{ and } x \in X.$$

Fix $\lambda \in \mathbb{C}$, and we define a linear map $\varpi_\lambda(h^{-1}) : C^\infty(X) \to C^\infty(X)$ by

$$(\varpi_\lambda(h^{-1})f)(x) := \Omega(h,x)^\lambda f(L_h x).$$

Since the conformal factor Ω satisfies the cocycle condition:

$$\Omega(h_1 h_2, x) = \Omega(h_1, L_{h_2}x)\Omega(h_2, x) \quad \text{for } h_1, h_2 \in G, x \in X,$$

we have formed a family of representations $\varpi_\lambda \equiv \varpi_\lambda^X$ of G on $C^\infty(X)$ with complex parameter λ (see [26, Part I] for details).

Remark 4.1. 1) If G acts on X as isometries, then $\Omega \equiv 1$ and therefore the representation ϖ_λ does not depend on λ.
2) Let n be the dimension of X. Then in our normalization, $(\varpi_n, C^\infty(X))$ is isomorphic to the representation on $C^\infty(X, \Omega_X)$ where Ω_X denotes the bundle of volume densities.

Let $\mathrm{Conf}(X)$ be the full group of conformal transformations on (X,g). Suppose Y is a submanifold of X such that the restriction $g|_Y$ is non-degenerate. Clearly, this assumption is automatically satisfied if (X,g) is a Riemannian manifold. We define a subgroup of $\mathrm{Conf}(X)$ by

$$\mathrm{Conf}(X;Y) = \{\varphi \in \mathrm{Conf}(X) : \varphi(Y) \subset Y\}.$$

For $\varphi \in \mathrm{Conf}(X;Y)$, φ induces a conformal transformation on $(Y, g|_Y)$, and we get a natural group homomorphism

$$\mathrm{Conf}(X;Y) \to \mathrm{Conf}(Y).$$

We write $\mathrm{Conf}_Y(X)$ for its image.

Thus, for $\lambda, \nu \in \mathbb{C}$, we have the following two representations:

$$\varpi_\lambda^X : \mathrm{Conf}(X) \to GL_{\mathbb{C}}(C^\infty(X)),$$
$$\varpi_\nu^Y : \mathrm{Conf}(X; Y) \to GL_{\mathbb{C}}(C^\infty(Y)).$$

We are ready to state the following problem:

Problem 4.2. 1) Classify $(\lambda, \nu) \in \mathbb{C}^2$ such that there exists a non-zero continuous/differential operator

$$T_{\lambda,\nu} : C^\infty(X) \to C^\infty(Y)$$

satisfying

$$\varpi_\nu^Y(h) \circ T_{\lambda,\nu} = T_{\lambda,\nu} \circ \varpi_\lambda^X(h) \quad \text{for all } h \in \mathrm{Conf}(X; Y).$$

2) Find explicit formulas of the operators $T_{\lambda,\nu}$.

We begin with an obvious example.

Example 4.3. Suppose $\lambda = \nu$. We take $T_{\lambda,\nu}$ to be the restriction of functions from X to Y. Clearly, $T_{\lambda,\nu}$ intertwines ϖ_λ and ϖ_ν.

Problem 4.2 concerns a geometric aspect of the general branching problem for representations with respect to the restriction $G \downarrow G'$ in the case where

$$(G, G') = (\mathrm{Conf}(X), \mathrm{Conf}_Y(X)).$$

We shall see in Section 5 that a special case of Problem 4.2 is a special case of Project A. In Section 5, we shall write ϖ_λ^G and $\varpi_\nu^{G'}$ for $\varpi_\lambda \equiv \varpi_\lambda^X$ and $\varpi_\nu \equiv \varpi_\nu^Y$, respectively, in order to emphasize the groups G and G'.

We continue with some further examples of Problem 4.2.

Example 4.4 (Eastwood–Graham)**.** Let $X = Y$ be the sphere S^n endowed with a standard Riemannian metric, and we take $G = G'$ to be the Lorentz group $SO(n+1, 1)$ that act conformally on S^n by the Möbius transform. In this case, G is a semisimple Lie group of real rank one, and all local/non-local intertwining operators were classified by Gelfand–Graev–Vielenkin [9] for $n = 1, 2$, by Eastwood–Graham [8] for general n when $\mathcal{V}$ and $\mathcal{W}$ are line bundles. In particular, all conformal invariants for densities in this case are given by residues of continuous conformal intertwiners. It is noted that an analogous statement is not always true when $G' \neq G$ (cf. Remark 5.3).

Example 4.5. Suppose $X = Y$. Then $G = G'$. Let n be the dimension of the manifold, and we consider the following specific parameter:

$$\lambda = \frac{1}{2}n - 1, \quad \nu = \frac{1}{2}n + 1.$$

Then the Yamabe operator $\widetilde{\Delta_X}$ satisfies $\varpi_\nu \circ \widetilde{\Delta_X} = \widetilde{\Delta_X} \circ \varpi_\lambda$, where

$$\widetilde{\Delta_X} := \Delta_X - \frac{n-2}{4(n-1)}\kappa.$$

Here Δ_X is the Laplacian for the pseudo-Riemannian manifold (X, g), and κ is the scalar curvature.

In particular, if X is the direct product of two spheres $S^p \times S^q$ endowed with the pseudo-Riemannian structure $g_{S^p} \oplus (-g_{S^q})$ of signature (p, q), then the kernel $\mathrm{Ker}(\widetilde{\Delta_X})$ gives rise to an important irreducible unitary representation of $\mathrm{Conf}(X) \simeq O(p+1, q+1)$, so-called a *minimal representation* for $p+q$ even, $p, q \geq 1$, and $p+q \geq 6$ [18, 26] in the sense that its annihilator in the enveloping algebra $U(\mathfrak{o}(p+q+2, \mathbb{C}))$ is the Joseph ideal [3, 32]. The same representation is known to have different realizations and constructions, e.g. the local theta correspondence [13], and the Schrödinger model [24].

Example 4.6 ([14, 28, 31])**.** Let X be the standard sphere S^n and $Y = S^{n-1}$ a totally geodesic hypersurface ('great circle' when $n = 2$). Then we have covering maps

$$\begin{array}{lcl} G := O(n+1, 1) & \twoheadrightarrow & \mathrm{Conf}(X), \\ \quad\cup & & \quad\cup \\ G' := O(n, 1) & \twoheadrightarrow & \mathrm{Conf}_Y(X). \end{array}$$

Then non-zero G'-equivariant differential operators $T_{\lambda,\nu} : C^\infty(X) \to C^\infty(Y)$ exist if and only if the parameter (λ, ν) satisfies $\nu - \lambda \in \{0, 2, 4, \cdots\}$. (Here, the parity condition arises from the fact that G and G' are disconnected groups, cf. [14, 28].) In this case $\dim \mathrm{Diff}_{G'}(C^\infty(X), C^\infty(Y)) = 1$.

In order to describe this differential operator $T_{\lambda,\nu}$ explicitly, we use the stereographic projection

$$S^n \to \mathbb{R}^n \cup \{\infty\}, \quad (s, \sqrt{1-s^2}\omega) \mapsto \sqrt{\frac{1-s}{1+s}}\omega, \tag{4.1}$$

and the corresponding twisted pull-back (see [26, Part I]) for the conformal map (4.1),

$$\iota_\lambda^* : C^\infty(S^n) \hookrightarrow C^\infty(\mathbb{R}^n), \quad f \mapsto F \tag{4.2}$$

is given by

$$F(r\omega) := (1+r^2)^{-\lambda} f\Big(\frac{1-r^2}{1+r^2}, \frac{2r}{1+r^2}\omega\Big) \quad \text{for } r > 0 \text{ and } \omega \in S^{n-1}.$$

In the coordinates, we realize the submanifold Y correspondingly to the hyperplane $x_n = 0$ via (4.1), namely, we have a commutative diagram of the stereographic projections:

$$\begin{array}{ccl} X = S^n & \longrightarrow & \mathbb{R}^n \cup \{\infty\} \\ \cup & & \cup \\ Y = S^{n-1} & \longrightarrow & \mathbb{R}^{n-1} \cup \{\infty\} = \{(x_1, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n : x_n = 0\} \cup \{\infty\}. \end{array}$$

Accordingly, the subgroup G' is defined as the isotropy subgroup of G at $e_n = {}^t(0, \cdots, 0, 1, 0) \in \mathbb{R}^{n+1}$.

Then for $\nu - \lambda = 2l$ ($l \in \mathbb{N}$), the equivariant differential operator $T_{\lambda,\nu}$ is a scalar multiple of the following differential operator:

$$\widetilde{\mathbb{C}}_{\lambda,\nu} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n-1}),$$

$$F \mapsto \sum_{j=0}^{l} \frac{2^{2l-2j} \prod_{i=1}^{l-j} \left(\frac{\lambda+\nu-n-1}{2} + i\right)}{j!(2l-2j)!} \Delta_{\mathbb{R}^{n-1}}^j \Big(\frac{\partial}{\partial x_n}\Big)^{2l-2j} F|_{x_n=0}. \tag{4.3}$$

The differential operator $\widetilde{\mathbb{C}}_{\lambda,\nu}$ can be written by using the Gegenbauer polynomial as follows. Let $C_N^\mu(t)$ be the Gegenbauer polynomial of degree N, and we inflate $C_N^\mu(t)$ to a polynomial of two variables v, t by

$$\begin{aligned} \widetilde{C}_{2l}^\mu(v,t) :=& \frac{\Gamma(\mu)}{\Gamma(\mu+l)} v^l C_{2l}^\mu\Big(\frac{t}{\sqrt{v}}\Big) \\ =& \sum_{j=0}^{l} \frac{(-1)^j 2^{2l-2j}}{j!(2l-2j)!} \prod_{i=1}^{l-j} (\mu+l+i-1) v^j t^{2l-2j}. \end{aligned} \tag{4.4}$$

We note that the definition (4.4) makes sense if v, t are elements in any commutative algebra R. In particular, taking $R = \mathbb{C}[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}]$, we have the following expression:

$$\widetilde{\mathbb{C}}_{\lambda,\nu} F = \widetilde{C}_{2l}^{\lambda - \frac{n-1}{2}}\Big(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}\Big) F|_{x_n=0}.$$

A. Juhl [14] proved the formula (4.3) by a considerably long computation based on recurrence relations. In [28] we have provided a new and simple proof by introducing another differential equation (see (5.5) below) which controls the operators $\widetilde{\mathbb{C}}_{\lambda,\nu}$ (*F-method*). In the next section we give yet another proof of the formula (4.3) from the residue calculations of non-local symmetry breaking operators with meromorphic parameter (see Theorem 5.2).

5 Conformally covariant non-local operators

In this section we analyze both non-local and local, conformally covariant operators in the setting of Example 4.6 by using the F-method.

From the viewpoint of representation theory, Example 4.6 deals with symmetry breaking differential operators between spherical principal series representations ϖ_λ^G and those $\varpi_\nu^{G'}$ when $(G, G') = (O(n+1,1), O(n,1))$. In this case all the assumptions for Theorems 2.1 and 2.3 are fulfilled, and therefore we tell a priori that both of the sides in (1.2) are of uniformly bounded dimensions (actually, at most two-dimensional). In the joint work [31] with B. Speh, we give a complete classification of such symmetry breaking operators between line bundles for both non-local and local ones with explicit generators, which seems to be the first complete example of Project A in a setting where $G' \subsetneq G$.

However, the techniques employed in [31] are not the F-method. Therefore it might be of interest to illuminate some of the key results of [31] from the scope of a generalized F-method (Section 3). To achieve this aim for the current section, we focus on the functional equations (Theorem 5.6) among non-local symmetry breaking operators $\widetilde{\mathbb{A}}_{\lambda,\nu}$ (see (5.2) below) and the relation between $\widetilde{\mathbb{A}}_{\lambda,\nu}$ and the differential operators $\widetilde{\mathbb{C}}_{\lambda,\nu}$ (Theorem 5.2).

The novelty here is the following correspondence:

Symmetry breaking operators	F-method
Functional equations (Theorem 5.6)	Kummer's relation
Residue formulae of $\widetilde{\mathbb{A}}_{\lambda,\nu}$(Theorem 5.2)	${}_2F_1(a,b;c;z)$ reduces to a polynomial if $a \in -\mathbb{N}$

First of all we review quickly some notation and results from [31]. Then the rest of this subsection will be devoted to provide some perspectives from the F-method.

We set $|x| := (x_1^2 + \cdots + x_{n-1}^2)^{\frac{1}{2}}$ for $x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$. If $(\lambda, \nu) \in \mathbb{C}^2$ satisfies

$$\operatorname{Re}(\lambda - \nu) > 0 \quad \text{and} \quad \operatorname{Re}(\lambda + \nu) > n - 1, \tag{5.1}$$

then

$$K^{\mathbb{A}}_{\lambda,\nu}(x, x_n) := |x_n|^{\lambda+\nu-n}(|x|^2 + x_n^2)^{-\nu}$$

is locally integrable on $\mathbb{R}^n$, and the integral operator

$$C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{n-1}), \quad F \mapsto \int_{\mathbb{R}^n} F(y, y_n) K^{\mathbb{A}}_{\lambda,\nu}(x - y, -y_n) dy dy_n$$

extends to a G'-intertwining operator via (4.2)

$$\mathbb{A}_{\lambda,\nu} : C^\infty(S^n) \to C^\infty(S^{n-1}),$$

namely,

$$\mathbb{A}_{\lambda,\nu} \circ \varpi^G_\lambda(h) = \varpi^{G'}_\nu(h) \circ \mathbb{A}_{\lambda,\nu} \quad \text{for all } h \in G'.$$

The important property of our symmetry breaking operators $\mathbb{A}_{\lambda,\nu}$ is the existence of the meromorphic continuation to $(\lambda, \nu) \in \mathbb{C}^2$ (see Theorem 5.1) and the functional equations satisfied by $\mathbb{A}_{\lambda,\nu}$ and the Knapp–Stein intertwining operators (see Theorem 5.6). We note that the celebrated theorem [1, 2] on meromorphic continuation of distributions does not apply immediately to our distribution $K^{\mathbb{A}}_{\lambda,\nu}$ because the two singularities $x_n = 0$ and $|x|^2 + x_n^2 = 0$ (i.e. the origin) have an inclusive relation and are not transversal. Further, it is more involved to find the location of the poles and their residues. In [31] we have found all the poles and their residues explicitly, and in particular, we have the following theorems:

Theorem 5.1. *We normalize*

$$\widetilde{\mathbb{A}}_{\lambda,\nu} := \frac{1}{\Gamma(\frac{\lambda+\nu-n+1}{2})\Gamma(\frac{\lambda-\nu}{2})} \mathbb{A}_{\lambda,\nu}. \tag{5.2}$$

Then $\widetilde{\mathbb{A}}_{\lambda,\nu} : C^\infty(S^n) \to C^\infty(S^{n-1})$, initially holomorphic in the domain given by (5.1)*, extends to a continuous operator for all $(\lambda, \nu) \in \mathbb{C}^2$ and satisfies*

$$\widetilde{\mathbb{A}}_{\lambda,\nu} \circ \varpi^G_\lambda(h) = \varpi^{G'}_\nu(h) \circ \widetilde{\mathbb{A}}_{\lambda,\nu} \quad \textit{for all } h \in G'.$$

Further, $\widetilde{\mathbb{A}}_{\lambda,\nu} f$ *is a holomorphic function of* (λ, ν) *on the entire space* $\mathbb{C}^2$ *for all* $f \in C^\infty(S^n)$*.*

Theorem 5.2 (Residue formula)**.** *If* $\nu - \lambda = 2l$ *with* $l \in \mathbb{N}$ *then*

$$\widetilde{\mathbb{A}}_{\lambda,\nu} = \frac{(-1)^l l! \pi^{\frac{n-1}{2}}}{2^{2l}\Gamma(\nu)} \widetilde{\mathbb{C}}_{\lambda,\nu}.$$

Remark 5.3. For (λ, ν) belonging to

$$L_{\text{even}} := \{(\lambda, \nu) \in \mathbb{Z}^2 : \lambda \le \nu \le 0, \lambda - \nu \equiv 0 \mod 2\},$$

the conformally covariant differential operator $\widetilde{\mathbb{C}}_{\lambda,\nu}$ cannot be obtained as the residue of $\widetilde{\mathbb{A}}_{\lambda,\nu}$. This discrete set L_{even} is exactly the zero-set of the symmetry breaking operators $\widetilde{\mathbb{A}}_{\lambda,\nu}$ and is the most interesting place of symmetry breaking [31]. (We note that L_{even} is of codimension two in $\mathbb{C}^2$!)

The proof of Theorem 5.2 in [31] is to use explicit formulae of the action of $\widetilde{\mathbb{A}}_{\lambda,\nu}$ and $\widetilde{\mathbb{C}}_{\lambda,\nu}$ on K-fixed vectors (*spherical vectors*). Instead, we apply here the generalized F-method and give an alternative proof of Theorem 5.2, which is of more analytic nature and without using computations for specific K-types.

We write $\widetilde{K}^{\mathbb{A}}_{\lambda,\nu}$ and $\widetilde{K}^{\mathbb{C}}_{\lambda,\nu}$ for the distribution kernels of the normalized symmetry breaking operator $\widetilde{\mathbb{A}}_{\lambda,\nu}$ and the conformally covariant differential operator $\widetilde{\mathbb{C}}_{\lambda,\nu}$, respectively. For (λ, ν) belonging to the open domain (5.1), we have

$$\begin{aligned} \widetilde{K}^{\mathbb{A}}_{\lambda,\nu}(x, x_n) =& \frac{1}{\Gamma(\frac{\lambda+\nu-n+1}{2})\Gamma(\frac{\lambda-\nu}{2})} K^{\mathbb{A}}_{\lambda,\nu}(x, x_n) \\ =& \frac{1}{\Gamma(\frac{\lambda+\nu-n+1}{2})\Gamma(\frac{\lambda-\nu}{2})} |x_n|^{\lambda+\nu-n}(|x|^2 + x_n^2)^{-\nu}. \end{aligned}$$

For $(\lambda, \nu) \in \mathbb{C}$ such that $\nu - \lambda = 2l$ $(l \in \mathbb{N})$, we have from (4.3):

$$\begin{aligned} \widetilde{K}^{\mathbb{C}}_{\lambda,\nu} =& \sum_{j=0}^{l} \frac{2^{2l-2j} \prod_{i=1}^{l-j}(\frac{\lambda+\nu-n-1}{2} + i)}{j!(2l-2j)!} (\Delta^j_{\mathbb{R}^{n-1}} \delta(x_1, \cdots, x_{n-1}))\delta^{(2l-2j)}(x_n) \\ =& \widetilde{C}^{\lambda-\frac{n-1}{2}}_{2l}(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n})\, \delta(x_1, \cdots, x_{n-1})\delta(x_n). \end{aligned} \tag{5.3}$$

Proposition 5.4. 1) *The tempered distribution* $\mathcal{F}_{\mathbb{R}}\widetilde{K}^{\mathbb{A}}_{\lambda,\nu} \in \mathcal{S}'(\mathbb{R}^n)$ *is a real analytic function (in particular, locally integrable) in the open subset*

$$\{(\xi, \xi_n) \in \mathbb{R}^{n-1} \oplus \mathbb{R} : |\xi| > |\xi_n|\},$$

where $|\xi| = (\xi_1^2 + \cdots + \xi_{n-1}^2)^{\frac{1}{2}}$. *By the analytic continuation (cf.* (3.10)*), we have for* $|\zeta| > |\zeta_n|$
(5.4)

$$(\mathcal{F}_c\widetilde{K}^{\mathbb{A}}_{\lambda,\nu})(\zeta, \zeta_n) = \frac{\pi^{\frac{n-1}{2}} e^{\frac{\pi i}{2}(\nu-\lambda)}|\zeta|^{\nu-\lambda}}{\Gamma(\nu)2^{\nu-\lambda}} {}_2F_1(\frac{\lambda-\nu}{2}, \frac{\lambda+\nu+1-n}{2}; \frac{1}{2}; -\frac{\zeta_n^2}{|\zeta|^2}).$$

2) *Suppose* $\nu - \lambda = 2l$ $(l \in \mathbb{N})$*. Then*

$$(\mathcal{F}_c\widetilde{K}^{\mathbb{C}}_{\lambda,\nu})(\zeta, \zeta_n) = \widetilde{C}^{\lambda-\frac{n-1}{2}}_{2l}(-|\zeta|^2, \zeta_n).$$

Proof. 1) We use the integration formula

$$\mathcal{F}_{\mathbb{R}^n}K^{\mathbb{A}}_{\lambda,\nu}(\xi, \xi_n) = \frac{2^{-\nu+\frac{n+1}{2}}\pi^{\frac{n-1}{2}}}{\Gamma(\nu)|\xi|^{-\nu+\frac{n-1}{2}}} \int_{\mathbb{R}} |t|^{\lambda-\frac{n+1}{2}} K_{-\nu+\frac{n-1}{2}}(|t\xi|)e^{-it\xi_n}dt,$$

where $K_\mu(t)$ denotes the K-Bessel function. We then apply the following integration formula (see [10, 6.699.4]):

$$\int_0^\infty x^\gamma K_\mu(ax)\cos(bx)dx$$
$$= 2^{\gamma-1}a^{-\gamma-1}\Gamma(\frac{\mu+\gamma+1}{2})\Gamma(\frac{1+\gamma-\mu}{2}){}_2F_1(\frac{\mu+\gamma+1}{2}, \frac{1+\gamma-\mu}{2}; \frac{1}{2}; -\frac{b^2}{a^2})$$

for $\mathrm{Re}(-\gamma \pm \mu) < 1$, $\mathrm{Re}\, a > 0$, $b > 0$.
2) Clear from (5.3) and the definition of $\mathcal{F}_c$. ☐

Proof of Theorem 5.2. By using the following formula of the Gegenbauer polynomial of even degree

$$C^\mu_{2l}(x) = \frac{(-1)^l\Gamma(l+\mu)}{l!\Gamma(\mu)} {}_2F_1(-l, l+\mu; \frac{1}{2}; x^2),$$

we get

$$\mathcal{F}_c\widetilde{K}^{\mathbb{A}}_{\lambda,\nu} = \frac{(-1)^l l! \pi^{\frac{n-1}{2}}}{2^{2l}\Gamma(\nu)} \mathcal{F}_c\widetilde{K}^{\mathbb{C}}_{\lambda,\nu}$$

for $|\zeta| > |\zeta_n|$.

In view that $\mathrm{Supp}\, \widetilde{K}^{\mathbb{A}}_{\lambda,\nu} \subset \{0\}$ for $\nu - \lambda \in 2\mathbb{N}$ ([31]), both $\mathcal{F}_c\widetilde{K}^{\mathbb{A}}_{\lambda,\nu}$ and $\mathcal{F}_c\widetilde{K}^{\mathbb{C}}_{\lambda,\nu}$ are holomorphic functions on $\mathbb{C}^n$. Hence Theorem 5.2 follows. ☐

Remark 5.5. The assumption (3.4) of Theorem 3.4 is satisfied, and therefore, $\mathrm{Hom}_{G'}(C^\infty(G/P, \mathcal{L}_\lambda), C^\infty(G'/P', \mathcal{L}_\nu))$ can be identified with the following subspace of the Schwartz distributions:

$$\mathcal{S}ol(\lambda,\nu)^\wedge = \{F \in \mathcal{S}'(\mathbb{R}^n) : F \text{ satisfies the following three equations}\}.$$

$$F(m\cdot) = F(\cdot) \quad \text{for } m \in O(n-1)\times O(1),$$
$$(\sum_{i=1}^n \zeta_i \frac{\partial}{\partial \zeta_i} + \lambda - \nu))F = 0,$$
$$(\nu \frac{\partial}{\partial \zeta_j} - \frac{1}{2}\Delta_{\mathbb{R}^n}\zeta_j)F = 0 \quad (1 \le j \le n-1). \tag{5.5}$$

The differential operators in the last equation are the *fundamental differential operators* on the light cone [24, Chapter 2] (or Bessel operators in the context of Jordan algebras). The heart of the F-method is that this differential equation explains why the Gauss hypergeometric functions (and the Gegenbauer polynomials as special cases) arise in the formula of $\mathcal{F}_c \widetilde{K}^{\mathbb{A}}_{\lambda,\nu}$ and $\mathcal{F}_c \widetilde{K}^{\mathbb{C}}_{\lambda,\nu}$ in Proposition 5.4. (In [14], the relation between $\widetilde{\mathbb{C}}_{\lambda,\nu}$ and the Gegenbauer polynomial was pointed out, but the proof was based on the comparison of coefficients determined by recurrence relations.)

We recall that the Riesz potential

$$\widetilde{K}^{\mathbb{T}}_{n-\lambda,\lambda}(x,x_n) := \frac{1}{\Gamma(\lambda+\frac{n}{2})}(x_1^2+\cdots+x_n^2)^\lambda$$

gives the normalized Knapp–Stein intertwining operator by

$$\widetilde{\mathbb{T}}_{n-\lambda,\lambda} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \quad f \mapsto \widetilde{K}^{\mathbb{T}}_{n-\lambda,\lambda} * f.$$

Then $\widetilde{\mathbb{T}}_{n-\lambda,\lambda}$ depends holomorphically on $\lambda \in \mathbb{C}$ and satisfies

$$\varpi^G_\lambda(h) \circ \widetilde{\mathbb{T}}_{n-\lambda,\lambda} = \widetilde{\mathbb{T}}_{n-\lambda,\lambda} \circ \varpi^G_{n-\lambda}(h) \quad \text{for all } h \in G.$$

Here are the functional equations among the three operators: our operators $\widetilde{\mathbb{A}}_{\lambda,\nu}$, the Knapp–Stein operators $\widetilde{\mathbb{T}}_{\nu,m-\nu}$ for G' and $\widetilde{\mathbb{T}}_{n-\lambda,\lambda}$ for G.

Theorem 5.6 ([31]). *Let* $m = n-1$.

$$\widetilde{\mathbb{T}}_{\nu,m-\nu} \circ \widetilde{\mathbb{A}}_{\lambda,\nu} = \frac{\pi^{\frac{m}{2}}}{\Gamma(\nu)} \widetilde{\mathbb{A}}_{\lambda,m-\nu}. \tag{5.6}$$

$$\widetilde{\mathbb{A}}_{\lambda,\nu} \circ \widetilde{\mathbb{T}}_{n-\lambda,\lambda} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\lambda)} \widetilde{\mathbb{A}}_{n-\lambda,\nu}. \tag{5.7}$$

Heuristic idea of a proof based on the F-method. First, we compute the Fourier transform of the Riesz potential, and obtain

$$\mathcal{F}_c(\widetilde{K}^{\mathbb{T}}_{n-\lambda,\lambda})(\zeta,\zeta_n)=\frac{e^{\frac{\pi i}{2}(2\lambda-n)}\pi^{\frac{n}{2}}}{2^{2\lambda-n}\Gamma(\lambda)}(|\zeta|^2+\zeta_n^2)^{\lambda-\frac{n}{2}}. \tag{5.8}$$

Combining (5.4) and (5.8), we would have the following identity of holomorphic functions on $\{(\zeta,\zeta_n)\in\mathbb{C}^n : |\zeta|>|\zeta_n|\}$ as analytic continuation:

$$\begin{aligned}&\mathcal{F}_c(\widetilde{K}^{\mathbb{A}}_{\lambda,\nu}*\widetilde{K}^{\mathbb{T}}_{n-\lambda,\lambda})\\ =&\mathcal{F}_c(\widetilde{K}^{\mathbb{A}}_{\lambda,\nu})\mathcal{F}_c(\widetilde{K}^{\mathbb{T}}_{n-\lambda,\lambda})\\ =&\frac{\pi^{n-\frac{1}{2}}e^{\frac{\pi i}{2}(\lambda+\nu-n)}(|\zeta|^2+\zeta_n^2)^{\lambda-\frac{n}{2}}|\zeta|^{\nu-\lambda}}{\Gamma(\lambda)\Gamma(\nu)2^{\lambda+\nu-n}}{}_2F_1\left(\frac{\lambda-\nu}{2},\frac{\lambda+\nu+1-n}{2};\frac{1}{2};-\frac{\zeta_n^2}{|\zeta|^2}\right).\end{aligned}$$

Then the desired functional equation (5.6) would be reduced to Kummer's relation on Gauss hypergeometric functions:

$$F(\alpha,\beta;\gamma;z)=(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma;z).$$

The identity (5.7) is similar and simpler. □

An advantage of the F-method indicated as above is that we can discover the functional identities such as (5.6) and (5.7) as a disguise of a very simple and classical identity (i.e. Kummer's relation), and the proof does not depend heavily on representation theory. On the other hand, since the convolution (or the multiplication) of two Schwartz distributions are not well-defined in the usual sense in general, a rigorous proof in this direction requires some careful analysis when we deal with such functional equations for non-local operators. (For local operators, we do not face with these analytic difficulties. In this case, we have used the F-method in [28] to prove functional identities for differential operators, e.g. factorization identities in [14].)

In [31], we take a completely different approach based on the uniqueness of symmetry breaking operators for generic parameters (cf. [11, 36]) and the evaluation of spherical vectors applied by symmetry breaking operators for the proof of Theorem 5.6 and its variants.

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