Workshop: Geometric Quantization in the Non-compact Setting

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Abstracts

Geometric quantization, limits, and restrictions– some examples for elliptic and nilpotent orbits

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The Kirillov–Kostant–Duflo orbit philosophy relates the set of equivalence classes of irreducible unitary representations of a Lie group G with the set of coadjoint orbits. Our expectation is that this correspondence is given by a "geometric quantization":

$$Q : \mathfrak{g}^* / \operatorname{Ad}^*(G) \to \widehat{G}, \tag{1}$$

satisfying functorial properties (e.g. $[Q, R] = 0$, $[Q, \mathrm{Limit}] = 0$). This works perfectly for simply connected nilpotent G. However, for reductive G, there is no reasonable bijection between $\widehat{G}$ and $\mathfrak{g}^*/\operatorname{Ad}^*(G)$ (or its subset requiring some integrality conditions). Nevertheless we know more or less what Q should be for semisimple orbits. For example, $Q(\mathcal{O}^G)$ is realized in a certain Dolbeault cohomology group on $\mathcal{O}^G$ for an integral elliptic orbit $\mathcal{O}^G$, and $Q(\mathcal{O}^G)$ is given by a (classical) parabolic induction for a hyperbolic orbit $\mathcal{O}^G$.

Let H be a subgroup of G, $\mathfrak{h} \subset \mathfrak{g}$ their Lie algebras, and $\mathrm{pr} : \mathfrak{g}^* \to \mathfrak{h}^*$ the restriction map. Take any coadjoint orbit $\mathcal{O}^G \subset \mathfrak{g}^*$. Then the natural inclusion $\iota : \mathcal{O}^G \hookrightarrow \mathfrak{g}^*$ gives the momentum map of the Hamiltonian action of G on $\mathcal{O}^G$ endowed with the Kirillov–Kostant–Souriau symplectic form, and the composition $\mu := \mathrm{pr} \cdot \iota : \mathcal{O}^G \to \mathfrak{h}^*$ gives that for H.

For a coadjoint orbit $\mathcal{O}^H \subset \mathfrak{h}^*$, we set

$$n(\mathcal{O}^G, \mathcal{O}^H) := \#(\mu^{-1}(\mathcal{O}^H)/H) = (\mathcal{O}^G \cap \mathrm{pr}^{-1}(\mathcal{O}^H))/H.$$

Our concern is with the case where G and H are non-compact reductive groups. For $\mathcal{O}^G$ such that $Q(\mathcal{O}^G) \in \widehat{G}$ is well-defined, we raise:

Conjecture 1. (1) *The restriction of the unitary representation $Q(\mathcal{O}^G)|_H$ is multiplicity-free, namely, the ring $\operatorname{End}_H(Q(\mathcal{O}^G))$ is commutative if*

$$n(\mathcal{O}^G, \mathcal{O}^H) \le 1 \quad \text{for any } \mathcal{O}^H \in \mathfrak{h}^*/\operatorname{Ad}^*(H). \tag{2}$$

(2) *If $\mathcal{O}^G_\lambda$ is a family of coadjoint orbits with parameter λ such that the restrictions $Q(\mathcal{O}^G_\lambda)|_H$ are multiplicity-free, then* (2) *holds for all $\mathcal{O}^G_\lambda$.*

We present some non-compact settings for Conjecture 1 (2), and show some evidence of the Conjecture. For a simple Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we set

$$\mathcal{C}^*_{\mathfrak{k}} := ([\mathfrak{k}, \mathfrak{k}] + \mathfrak{p})^\perp \subset \mathfrak{g}^*.$$

We note $\mathcal{C}^*_{\mathfrak{k}} \neq 0$ iff G/K is a Hermitian symmetric space. Assume that a coadjoint orbit $\mathcal{O}^G$ satisfies

$$\mathcal{O}^G \cap \mathcal{C}^*_{\mathfrak{k}} \neq \emptyset. \tag{3}$$

Let $\{\nu_1, \dots, \nu_k\}$ be the maximal set of strongly orthogonal set in $\Delta(\mathfrak{p}_+^{-\tau}, \mathfrak{t}^\tau)$ (see [2] for more details). For $\mathbb{A} = \mathbb{Z}$ or $\mathbb{R}$, we define

$$\mathcal{C}_{\mathbb{A}}^+ := \{\sum_{j=1}^k a_j \nu_j : a_1 \geq \cdots \geq a_k \geq 0,\ a_j \in \mathbb{A}\ (1 \leq j \leq k)\}.$$

Theorem $\mathbf{B}_{\mathrm{hol}}$ and $\mathbf{B}_{\mathrm{hol}}^Q$ ([2, 4])**.** *Suppose (G,H) is a symmetric pair of holomorphic type. For any $\mathcal{O}_\lambda^G$ satisfying the condition (3), we have:*

(1) *$\mu : \mathcal{O}_\lambda^G \to \mathfrak{h}^*$ is proper, and $n(\mathcal{O}_\lambda^G, \mathcal{O}^H) \leq 1$ for any H-coadjoint orbit $\mathcal{O}^H$ in $\mathfrak{h}^*$. Further, $n(\mathcal{O}_\lambda^G, \mathcal{O}^H) \neq 0$ only if O^H is elliptic. More precisely,*

$$\mu(\mathcal{O}_\lambda^G) = \coprod_{\mu \in \lambda + \mathcal{C}_{\mathbb{R}}^+} \mathcal{O}_\mu^H.$$

(2) *The restriction of the unitary representation $Q(\mathcal{O}_\lambda^G)|_H$ is discretely decomposable and multiplicity-free. More precisely,*

$$Q(\mathcal{O}_\lambda^G)|_H \simeq \sum_{\mu \in \lambda|_{\mathfrak{t}^\tau} + \rho(\mathfrak{p}_+^{-\tau}) + \mathcal{C}_{\mathbb{Z}}^+}^{\oplus} Q(\mathcal{O}_\mu^H) \qquad \textit{(discrete direct sum).}$$

Theorem $\mathbf{B}_{\mathrm{anti}}$ and $\mathbf{B}_{\mathrm{anti}}^Q$ ([2, 4])**.** *Suppose (G,H) is a symmetric pair of anti-holomorphic type. For any $\mathcal{O}_\lambda^G$ satisfying the condition (3), we have:*

(1) *The momentum map $\mu : \mathcal{O}^G \to \mathfrak{h}^*$ is not proper. Further, $n(\mathcal{O}_\lambda^G, \mathcal{O}^H) \leq 1$ for any H-coadjoint orbit $\mathcal{O}^H$ in $\mathfrak{h}^*$. More precisely, $n(\mathcal{O}_\lambda^G, \mathcal{O}^H) \neq 0$ if and only if $\mathcal{O}^H$ is hyperbolic. Hence,*

$$\mu(\mathcal{O}_\lambda^G) = \coprod_{\mu \in (\mathfrak{a}_{\mathfrak{h}})_+^*} \mathcal{O}_\mu^H.$$

(2) *The restriction $Q(\mathcal{O}_\lambda^G)|_H$ is decomposed only by continuous spectrum:*

$$Q(\mathcal{O}_\lambda^G)|_H \simeq \int_{(\mathfrak{a}_{\mathfrak{h}})_+^*} Q(\mathcal{O}_\mu^H) d\mu \qquad \textit{(direct integral).}$$

A remarkable feature of Theorem $\mathrm{B}_{\mathrm{anti}}$ is that the image $\mu(\mathcal{O}_\lambda^G)$ is independent of λ in contrast to Theorem $\mathrm{B}_{\mathrm{hol}}$.

The geometric quantization of nilpotent orbits is non-trivial. Observing that any nilpotent orbit $\mathcal{O}_{\mathrm{nilp}}$ can be approximated by semisimple orbits $\mathcal{O}_\nu$, we propose:

Problem 1. *Construct a representation $Q(\mathcal{O}_{\mathrm{nilp}})$ from the knowledge of geometric quantizations $Q(\mathcal{O}_\nu)$ for semisimple orbits that approach to $\mathcal{O}_{\mathrm{nilp}}$.*

Here is an example for which the idea works. Let $G = O(p,q)$, and set

$$f := E_{12} - E_{21},\ h := E_{1,p+q} + E_{p+q,1} \in \mathfrak{g}.$$

For a parameter $\nu > 0$, we introduce a family of minimal elliptic and hyperbolic orbits

$$\mathcal{O}_\nu^{\mathrm{ell}} := \mathrm{Ad}^*(G)(\nu f), \quad \mathcal{O}_\nu^{\mathrm{hyp}} := \mathrm{Ad}^*(G)(\nu h).$$

Theorem C ([5])**.**

$$\lim_{\nu \downarrow 0} \mathcal{O}_\nu^{\mathrm{hyp}} = \lim_{\nu \downarrow 0} \mathcal{O}_\nu^{\mathrm{ell}} = \mathcal{O}_0^{\mathrm{nilp}} \cup \mathcal{O}_{\mathrm{min}} \cup \{0\}.$$

Here $\mathcal{O}_\nu^{\mathrm{hyp}}$, $\mathcal{O}_\nu^{\mathrm{ell}}$, and $\mathcal{O}_0^{\mathrm{nilp}}$ are hyperbolic, elliptic, and nilpotent orbits of dimension $2(p+q-2)$, and $\mathcal{O}_{\mathrm{min}}$ is the minimal nilpotent orbit. Then, we can construct $Q(\mathcal{O}_{\mathrm{min}})$ from the knowledge of $Q(\mathcal{O}_\nu^{\mathrm{hyp}})$ or $Q(\mathcal{O}_\nu^{\mathrm{ell}})$ as follows:

Theorem C^Q ([5, 6])**.** *For $p+q$ even and $p, q \geq 2$, there exists the following two non-splitting exact sequences of G-modules:*

$$0 \to \varpi_{\mathrm{min}} \to Q(\mathcal{O}_{-1}^{\mathrm{hyp}}) \xrightarrow{\widetilde{\Delta}} Q(\mathcal{O}_1^{\mathrm{hyp}}) \to 0,$$

$$0 \to \varpi_{\mathrm{min}} \to Q(\mathcal{O}_{-1}^{\mathrm{ell}}) \to Q(\mathcal{O}_1^{\mathrm{ell}}) \to 0.$$

Remark. (1) The same representation ϖ_{min} appears as a subrepresentation of the two completely different representations $Q(\mathcal{O}_{-1}^{\mathrm{hyp}})$ and $Q(\mathcal{O}_1^{\mathrm{ell}})$.

(2) We have used Q by a little abuse of notation, namely, as an "analytic continuation" of Q. We note that neither $Q(\mathcal{O}_{\pm 1}^{\mathrm{hyp}})$ nor $Q(\mathcal{O}_{-1}^{\mathrm{ell}})$ is unitarizable.

(3) The intertwining operator $\widetilde{\Delta}$ is given by the Yamabe operator in the conformal geometry (see [6]) for the pseudo-Riemannian manifold $\mathcal{O}_1^{\mathrm{hyp}} \simeq (S^{p-1} \times S^{q-1})/\mathbb{Z}_2$.

Finally we discuss a direct approach to get a quantization $Q(\mathcal{O}_{\mathrm{min}}^G)$, namely, to construct an irreducible unitary representation from a real minimal nilpotent orbit $\mathcal{O}_{\mathrm{min}}^G$. Here is an optimistic approach:

Approach. *Find an appropriate Lagrangian submanifold C of $\mathcal{O}_{\mathrm{min}}^G$, and construct an irreducible unitary representation $Q(\mathcal{O}_{\mathrm{min}}^G)$ of G on $L^2(C)$.*

We list some difficulties:

- The group G cannot act geometrically on any such C.
- There does not exist any invariant polarization on $\mathcal{O}_{\mathrm{min}}^G$.
- For some group G, there is no candidate for $Q(\mathcal{O}_{\mathrm{min}}^G)$.

However, we can give some affirmative results in the following setting:

Theorem D and D^Q ([1, 3])**.** *Suppose G is the conformal group of any real simple Jordan algebra V. Then $C := \mathcal{O}_{\mathrm{min}}^G \cap V$ is Lagrangian in $\mathcal{O}_{\mathrm{min}}^G$, and the above approach works for an appropriate covering of G except for $\mathfrak{g} \simeq \mathfrak{so}(p,q)$ ($p+q$ odd).*

A generalized Fourier transform is studied in details in [3].

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