Branching problems of Zuckerman derived functor modules

Dedicated to Gregg Zuckerman on the occasion of his 60th birthday

Toshiyuki Kobayashi*

Abstract

We discuss recent developments on branching problems of irreducible unitary representations π of real reductive groups when restricted to reductive subgroups. Highlighting the case where the underlying $(\mathfrak{g}, K)$-modules of π are isomorphic to Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$, we show various and rich features of branching laws such as infinite multiplicities, irreducible restrictions, multiplicity-free restrictions, and discrete decomposable restrictions. We also formulate a number of conjectures.

Keywords and phrases: branching law, symmetric pair, Zuckerman derived functor module, unitary representation, multiplicity-free representation

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1 Introduction

Zuckerman derived functor is powerful algebraic machinery to construct irreducible unitary representations by cohomological parabolic induction. The $(\mathfrak{g}, K)$-modules $A_{\mathfrak{q}}(\lambda)$, referred to as Zuckerman derived functor modules, give a far reaching generalization of the Borel–Weil–Bott construction of irreducible finite dimensional representations of compact Lie groups. They

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include Harish-Chandra's discrete series representations of real reductive Lie groups as a special case, and may be thought of as a geometric quantization of elliptic orbits (see Fact 6.1).

Branching problems in representation theory ask how irreducible representations π of a group G decompose when restricted to a subgroup G'.

The subject of our study is branching problems with emphasis on the setting when (G, G') is a reductive symmetric pair (Subsection 2.3), and when π is the unitarization of a Zuckerman derived functor module $A_{\mathfrak{q}}(\lambda)$. We see that branching problems in this setting include a wide range of examples: a very special case is equivalent to finding the Plancherel formula for homogeneous spaces (e.g. Proposition 2.4 and Example 4.8) and another special case is of combinatorial nature (e.g. the Blattner formula).

In this article, we give new perspectives on branching problems by revealing the following surprisingly rich and various features:

- The multiplicities may be infinite (Section 2) and may be one (Section 4).
- The restriction may stay irreducible (Section 3).
- The spectrum may be purely continuous and may be discretely decomposable (Section 5).

Finally, we present a number of open problems that might be interesting for further study (see Conjectures 4.2, 4.3, 5.4, and 5.11).

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2 Wild aspects of branching laws

2.1 Analysis and synthesis

One of the most distinguished feature of *unitary* representations is that they are always built up from the smallest objects, namely, irreducible ones. For a locally compact group G, we denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G, endowed with the Fell topology.

Fact 2.1 (Mautner–Teleman)**.** *Every unitary representation π of a locally compact group G is unitarily equivalent to a direct integral of irreducible unitary representations:*

$$\pi \simeq \int_{\widehat{G}}^{\oplus} n_\pi(\sigma)\sigma \, d\mu(\sigma). \tag{2.1}$$

Here, $d\mu$ is a Borel measure on $\widehat{G}$, $n_\pi : \widehat{G} \to \mathbb{N} \cup \{\infty\}$ is a measurable function, and $n_\pi(\sigma)\sigma$ stands for the multiple of an irreducible unitary representation σ with multiplicity $n_\pi(\sigma)$.

The decomposition (2.1) is unique if G is of type I in the sense of von Neumann algebras. Reductive Lie groups are of type I. Then the *multiplicity function* n_π is well-defined up to a measure zero set with respect to $d\mu$. We say that π has a *uniformly bounded multiplicity* if there is $C > 0$ such that $n_\pi(\sigma) \leq C$ almost everywhere; π is *multiplicity-free* if $n_\pi(\sigma) \leq 1$ almost everywhere, or equivalently, if the ring of continuous G-endomorphisms of π is commutative.

2.2 Branching laws and Plancherel formulas

Suppose that G' is a closed subgroup of G. Here are two basic settings where the problem of decomposing unitary representations arises naturally.

1) (Induction $G' \uparrow G$) *Plancherel formula.*

   For simplicity, assume that there exists a G-invariant Borel measure on the homogeneous space G/G'. Then the group G acts unitarily on the Hilbert space $L^2(G/G')$ by translations. The irreducible decomposition of the regular representation of G on $L^2(G/G')$ is called the *Plancherel formula* for G/G'.

2) (Restriction $G \downarrow G'$) *Branching laws.*

   Given an irreducible unitary representation π of G. By the symbol $\pi|_{G'}$, we think of π as a representation of the subgroup G'. The *branching law* of the restriction $\pi|_{G'}$ means the formula of decomposing π into irreducible representations of G'. Special cases of branching laws include the classical Clebsch–Gordan formula, or more generally, the decomposition of the tensor product of two irreducible representations (*fusion rule*), and the Blattner formula, etc.

2.3 Symmetric pairs

We are particularly interested in the branching laws with respect to reductive symmetric pairs. Let us fix some notation.

Suppose σ is an involutive automorphism of a Lie group G. We denote by $G^\sigma := \{g \in G : \sigma g = g\}$, the group of fixed points by σ. We say that (G, G') is a *symmetric pair* if G' is an open subgroup of G^σ. Then the homogeneous space G/G' becomes an affine symmetric space with respect to the canonical G-invariant affine connection. The pair (G, G') is said to be a *reductive symmetric pair* if G is reductive. Further, if G' is compact then G/G' becomes a Riemannian symmetric space.

Example 2.2. 1) (group case) Let $\grave{G}$ be a Lie group, $G := \grave{G} \times \grave{G}$ the direct product group, and $\sigma \in \mathrm{Aut}(G)$ be defined as $\sigma(x, y) := (y, x)$. Then $G^\sigma \equiv \mathrm{diag}(\grave{G}) := \{(x, x) : x \in \grave{G}\}$. Since the homogeneous space G/G^σ is diffeomorphic to $\grave{G}$, we refer to the symmetric pair $(G, G^\sigma) = (\grave{G} \times \grave{G}, \mathrm{diag}(\grave{G}))$ as a *group case*.

2) The followings are chains of reductive symmetric pairs:

$$GL(2n, \mathbb{H}) \supset GL(n, \mathbb{C}) \supset GL(n, \mathbb{R}) \supset GL(p, \mathbb{R}) \times GL(q, \mathbb{R}) \quad (p + q = n),$$
$$O(4p, 4q) \supset U(2p, 2q) \supset Sp(p, q) \supset U(p, q) \supset O(p, q).$$

2.4 Finite multiplicity theorem of van den Ban

Let (G, G') be a reductive symmetric pair.

The irreducible decomposition (2.1) is well-behaved for the induction $G' \uparrow G$, namely, for the Plancherel formula of the symmetric space G/G':

Fact 2.3 (van den Ban [2])**.** *Suppose (G, G') is a reductive symmetric pair. Then the regular representation π on $L^2(G/G')$ has a uniformly bounded multiplicity.*

2.5 Plancherel formulas v.s. branching laws

Fairly many cases of the Plancherel formula for $L^2(G/G')$ treated in Fact 2.3 can be realized as a special example of branching laws of the restriction of irreducible unitary representations of other groups. For example, we recall from [21, Propositions 6.1, 6.2] and [29, Theorem 36]:

Proposition 2.4. *Let G/G' be a reductive symmetric space. Then the regular representation of G on $L^2(G/G')$ is unitarily equivalent to the restriction $\pi|_G$ for some irreducible unitary representation π of a reductive group $\widetilde{G}$ containing G as its subgroup if (G, G') fulfills one of the following conditions:*

(A) *G' is compact and the crown domain D of the Riemannian symmetric space G/G' is a Hermitian symmetric space,*

or

(B) *G'/Z_G has a split center. Here Z_G stands for the center of G.*

Remark 2.5. 1) Most Riemannian symmetric pairs (G, G') satisfy the assumption (A) (see [37] for details).

2) As the proof below shows,

$$\widetilde{G} \supset G \supset G'$$

is a chain of reductive symmetric pairs.

3) We can take π to be the unitarization of some $A_{\mathfrak{q}}(\lambda)$ in (A) and also in (B) when G is a complex reductive Lie group.

4) There are some more cases other than (A) or (B) for which the conclusion of Proposition 2.4 holds. For instance, see Example 4.8 for the group case $L^2(GL(n, \mathbb{C}))$ and also for a more general case $L^2(GL(2n, \mathbb{R})/GL(n, \mathbb{C}))$.

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Outline of the proof. The choice of π and $\widetilde{G}$ depends on each case (A) and (B).

(A) We take $\widetilde{G}$ to be the automorphism group of D, and π to be any holomorphic discrete series representation of $\widetilde{G}$ of scalar type. Then π is realized in the Hilbert space consisting of square integrable, holomorphic sections of a G-equivariant holomorphic line bundle over D. Since holomorphic sections are determined uniquely by the restriction to the totally real submanifold G/G', we get a realization of the restriction $\pi|_G$ in a certain Hilbert subspace of $\mathcal{A}(G/G')$, which itself is not $L^2(G/G')$ but is unitarily equivalent to the regular representation on $L^2(G/G')$ (see [12]).

(B) Let P be a maximal parabolic subgroup of G whose Levi part is G'. Take $\widetilde{G}$ to be the direct product $G \times G$, and π to be the outer tensor product representation $\pi_1 \boxtimes \pi_2$ where π_1 is a degenerate unitary principal series representation induced from a unitary character of P and π_2 is the contragredient representation of π_1. Then apply the Mackey theory. $\square$

Example 2.6. 1) The regular representation on $L^2(G/G') = L^2(GL(n,\mathbb{R})/O(n))$ is unitarily equivalent to the restriction of a holomorphic discrete series representation of $\widetilde{G} := Sp(n,\mathbb{R})$ to G.

2) The regular representation on $L^2(GL(n,\mathbb{R})/GL(p,\mathbb{R})\times GL(q,\mathbb{R}))$ with $(p+q=n)$ is unitarily equivalent to the restriction of a degenerate principal representation of $\widetilde{G} := GL(n,\mathbb{R}) \times GL(n,\mathbb{R})$ to G (namely, to the tensor product representation).

2.6 Wild aspects of branching laws

Retain our assumption that (G,G') is a reductive symmetric pair.

Proposition 2.4 suggests that branching problems include a wide range of examples. In fact, while the 'good behavior' in Fact 2.3 for the Plancherel formula of the symmetric space G/G', the branching law of the restriction $\pi|_{G'}$ does not behave well in general. Even when π_K is a Zuckerman derived functor module $A_{\mathfrak{q}}(\lambda)$, we cannot expect:

'False Theorem' 2.7. *Let (G,G') be a reductive symmetric pair, and π an irreducible unitary representation of G. Then the multiplicities of the discrete spectrum in the branching laws $\pi|_{G'}$ are finite.*

Remark 2.8. Such a multiplicity theorem holds for reductive symmetric pairs (G,G') under the assumption that the restriction $\pi|_{G'}$ is infinitesimally discretely decomposable in the sense of Definition 5.3 (cf. [22, 28]). A key to the proof is Theorem 5.6 on a criterion of K'-admissibility and Corollary 5.8 on an estimate of the associated variety. See Remark 5.14 for the case $\pi_K \simeq A_{\mathfrak{q}}(\lambda)$.

Before giving a counterexample to (false) 'Theorem' 2.7 about the discrete spectrum, we discuss an easier case, namely, an example of infinite multiplicities in the continuous spectrum of the branching law:

Proposition 2.9 ($G \times G \downarrow \operatorname{diag} G$)**.** (Gelfand–Graev [8].) *If π_1 and π_2 are two unitary principal series representations of $G = SL(n, \mathbb{C})$ ($n \geq 3$), then the multiplicities in the decomposition of the tensor product $\pi_1 \otimes \pi_2$ are infinite almost everywhere with respect to the measure $d\mu$ in the direct integral* (2.1).

We recall the underlying $(\mathfrak{g}, K)$-modules of unitary principal series representations of a complex reductive Lie group are obtained as a special case of Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$.

Hence we get

Observation 2.10. *The multiplicities of the continuous spectrum in the branching law of the restriction $\pi|_{G'}$ may be infinite even in the setting where $\pi_K \simeq A_{\mathfrak{q}}(\lambda)$ and (G, G') is a reductive symmetric pair.*

Here is a more delicate example, which yields a counterexample to (false) 'Theorem'2.7 about the discrete spectrum.

Proposition 2.11 ($G_{\mathbb{C}} \downarrow G_{\mathbb{R}}$)**.** (see [26]) *There exist an irreducible unitary principal series representation π of $G = SO(5, \mathbb{C})$ and two irreducible unitary representations τ_1 (a holomorphic discrete series representation) and τ_2 (a non-holomorphic discrete series representation) of the subgroup $G' = SO(3, 2)$ such that*

$$0 < \dim \operatorname{Hom}_{G'}(\tau_1, \pi|_{G'}) < \infty \quad \textit{and} \quad \dim \operatorname{Hom}_{G'}(\tau_2, \pi|_{G'}) = \infty.$$

Here, $\operatorname{Hom}_{G'}(\cdot, \cdot)$ *denotes the space of continuous G'-intertwining operators.*

3 Almost irreducible branching laws

Let G be a real reductive Lie group, G' a subgroup, and π an irreducible unitary representation of G.

We have seen some wild aspects of branching laws in the previous section. As its opposite extremal case, this section highlights especially nice cases, namely, where the restriction $\pi|_{G'}$ remains irreducible or almost irreducible in the following (obvious) sense:

Definition 3.1. We say a unitary representation π is *almost irreducible* if π is a finite direct sum of irreducible representations.

It may well happen that the restriction $\pi|_{G'}$ is almost irreducible when G' is a maximal parabolic subgroup of G, but is a rare phenomenon when G' is a reductive subgroup. Nevertheless, we find in Subsections 3.2–3.3 that there exist a small number of examples where the restriction $\pi|_{G'}$ stays irreducible, or is almost irreducible in some cases.

We divide such irreducible unitary representations π of G into three cases, according as π_K are Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$ (see Theorem 3.5), principal series representations (see Theorem 3.8), and minimal representations (see Theorem 3.11). From the view point of the Kostant–Kirillov–Duflo orbit method, they may be thought of as the geometric quantization of elliptic, hyperbolic, and nilpotent orbits, respectively.

3.1 Restriction to compact subgroups

First of all, we observe that almost irreducible restrictions $\pi|_{G'}$ happen only when G' is non-compact if $\dim \pi = \infty$.

Let K be a maximal compact subgroup of a real reductive Lie group G.

Observation 3.2. *For any irreducible infinite dimensional unitary representation π of G, the branching law of the restriction $\pi|_K$ contains infinitely many irreducible representations of K.*

Proof. Clear from Harish-Chandra's admissibility theorem (see Fact 3.4 below). □

For later purpose, we introduce the following terminology:

Definition 3.3. Suppose K' is a compact group and π is a representation of K'. We say π is K'-admissible if $\dim \mathrm{Hom}_{K'}(\tau, \pi) < \infty \quad \text{for any } \tau \in \widehat{K'}$.

With this terminology, we state:

Fact 3.4 (Harish-Chandra's admissibility theorem)**.** *Any irreducible unitary representation π of G is K-admissible.*

We shall apply the notion of K'-admissibility when K' is a subgroup of K, and see that it plays a crucial role in the theory of discretely decomposable restrictions in Section 5.

3.2 Irreducible restriction $\pi|_{G'}$ with $\pi_K = A_{\mathfrak{q}}(\lambda)$

This subsection discusses for which triple (G, G', π) the restriction $\pi|_{G'}$ is (almost) irreducible in the setting that the underlying $(\mathfrak{g}, K)$-module π_K is isomorphic to a Zuckerman derived functor module $A_{\mathfrak{q}}(\lambda)$.

Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, $L := N_G(\mathfrak{q}) \equiv \{g \in G : \mathrm{Ad}(g)\mathfrak{q} = \mathfrak{q}\}$, and $\overline{A_{\mathfrak{q}}(\lambda)}$ the unitary representation of G whose underlying $(\mathfrak{g}, K)$-module is $A_{\mathfrak{q}}(\lambda)$.

Theorem 3.5 ([19])**.** *Suppose that (G, G', L) is one of the following triples:*

G	G'	L
$SU(n,n)$	$Sp(n,\mathbb{R})$	$U(n-1,n)$
$SU(2p,2q)$	$Sp(p,q)$	$U(2p-1,2q)$
$SO_0(2p,2q)$	$SO_0(2p,2q-1)$	$U(p,q)$
$SO_0(4,3)$	$G_2(\mathbb{R})$	$SO_0(4,1) \times SO(2)$
$SO_0(4,3)$	$G_2(\mathbb{R})$	$SO(2) \times SO_0(2,3)$
$SL(2n,\mathbb{C})$	$Sp(n,\mathbb{C})$	$GL(2n-1,\mathbb{C})$
$SO(2n,\mathbb{C})$	$SO(2n-1,\mathbb{C})$	$GL(n,\mathbb{C})$
$SO(7,\mathbb{C})$	$G_2(\mathbb{C})$	$\mathbb{C}^{\times} \times SO(5,\mathbb{C})$
$SU(2n)$	$Sp(n)$	$U(2n-1)$
$SO(2n)$	$SO(2n-1)$	$U(n)$
$SO(7)$	$G_{2,compact}$	$SO(2) \times SO(5)$

Then, the restriction $\overline{A_{\mathfrak{q}}(\lambda)}|_{G'}$ is almost irreducible for any λ satisfying the positivity and integrality condition (see Subsection 6.2). Further, the restriction $\overline{A_{\mathfrak{q}}(\lambda)}|_{G'}$ stays irreducible if the character $\lambda|_{\mathfrak{l}'}$ of $\mathfrak{l}' := \mathfrak{g}' \cap \mathfrak{l}$ is in the good range with respect to $\mathfrak{q}' := \mathfrak{g}'_{\mathbb{C}} \cap \mathfrak{q}$ (see (6.2)). On the level of Harish-Chandra modules, we have an isomorphism

$$A_{\mathfrak{q}}(\lambda) \simeq A_{\mathfrak{q}'}(\lambda|_{\mathfrak{l}'}),$$

as $(\mathfrak{g}', K')$-modules.

Outline of proof. We recall the following well-known representations of spheres:

$$\begin{aligned} Sp(n)/Sp(n-1) &\xrightarrow{\sim} U(2n)/U(2n-1) &&\simeq S^{4n-1}, \\ U(n)/U(n-1) &\xrightarrow{\sim} SO(2n)/SO(2n-1) &&\simeq S^{2n-1}, \\ Spin(5)/Spin(3) &\xrightarrow{\sim} Spin(7)/G_2 &&\simeq S^7. \end{aligned}$$

Then the trick in [20, Lemma 5.1] shows that the natural inclusion map $G'/L' \hookrightarrow G/L$ is in fact surjective for any of the specific triples (G, G', L) in Theorem 3.5, where we set $L' := G' \cap L$. Further, we have $\mathfrak{g}'_{\mathbb{C}} + \mathfrak{q} = \mathfrak{g}_{\mathbb{C}}$ so that the inclusion $\mathfrak{g}'_{\mathbb{C}} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$ induces the bijection $\mathfrak{g}'_{\mathbb{C}}/\mathfrak{q}' \xrightarrow{\sim} \mathfrak{g}_{\mathbb{C}}/\mathfrak{q}$ and L' coincides with $N_{G'}(\mathfrak{q}')$. Thus, the diffeomorphism $G'/L' \xrightarrow{\sim} G/L$ is biholomorphic. In turn, we get an isomorphism of canonical line bundles (see (6.1)):

$$\begin{array}{ccc} G' \times_{L'} \mathbb{C}_{2\rho(\mathfrak{u}')} & \xrightarrow{\sim} & G \times_L \mathbb{C}_{2\rho(\mathfrak{u})} \\ \downarrow & & \downarrow \\ G'/L' & \xrightarrow{\sim} & G/L \end{array}$$

This implies

$$\rho(\mathfrak{u})|_{\mathfrak{l}'} = \rho(\mathfrak{u}')$$

in the setting of Theorem 3.5. Let $\mathcal{L}_\lambda$ be a G-equivariant holomorphic line bundle over G/L for $\lambda \in \sqrt{-1}\mathfrak{l}^*$. Then the pull-back of $\mathcal{L}_{\lambda+2\rho(\mathfrak{u})}$ to G'/L' yields a G'-equivariant holomorphic line bundle $\mathcal{L}_{\lambda|_{\mathfrak{l}'}+2\rho(\mathfrak{u}')}$ over G'/L'. Hence, we have natural isomorphisms

$$H^*_{\bar{\partial}}(G/L, \mathcal{L}_{\lambda+2\rho(\mathfrak{u})}) \xrightarrow{\sim} H^*_{\bar{\partial}}(G'/L', \mathcal{L}_{\lambda|_{\mathfrak{l}'}+2\rho(\mathfrak{u}')})$$

between Dolbeault cohomology groups. Thus, we get Theorem 3.5 in view of the geometric interpretation of Zuckerman derived functor modules (see Section 6). □

Remark 3.6. 1) The pairs (G, G') in Theorem 3.5 are reductive symmetric pairs except for the case $(G, G') = (SO_0(4,3), G_2(\mathbb{R}))$.

2) The pair $(\mathfrak{g}, \mathfrak{l})$ is a reductive symmetric pair in all the cases of Theorem 3.5 ($\mathfrak{q}$ is of symmetric type in the sense of Definition 4.1). Correspondingly there are two choices of θ-stable parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ with $N_G(\mathfrak{q}) \simeq L$. In either case, $\overline{A_{\mathfrak{q}}(\lambda)}|_{G'}$ is almost irreducible.

3) In the compact case (i.e. the last three rows), the restriction is irreducible for all λ.

Example 3.7. 1) In [20] we gave a different proof of Theorem 3.5 for the pair $SO_0(4,3) \downarrow G_2(\mathbb{R})$ based on the Beilinson–Bernstein localization theory, and then applied it to construct (all) discrete series representations for non-symmetric homogeneous spaces $G_2(\mathbb{R})/SL(3,\mathbb{R})$ and $G_2(\mathbb{R})/SU(2,1)$.

2) H. Sekiguchi applied the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to the symmetric pair $U(n,n) \downarrow Sp(n,\mathbb{R})$ for more general $\mathfrak{q}$ to get a range characterization theorem of the Penrose transform (see [41]). Following the notation in

[41, Proposition 1.5], we see that the unitary character $\mathbb{C}_\lambda$ is in the weakly fair range for the θ-stable maximal parabolic subalgebra $\mathfrak{q}$ considered in Theorem 3.5 if and only if $\lambda = \lambda_1 e_1$ with $\lambda_1 \geq -n$. Further, $A_\mathfrak{q}(\lambda)$ is irreducible as a $\mathfrak{u}(n,n)$-module for all $\lambda_1 \geq -n$. Its restriction to $\mathfrak{sp}(n,\mathbb{R})$ stays irreducible for $\lambda_1 > -n$, but splits into two irreducible modules $(W(n,1)_+)_K \oplus (W(n,1)_-)_K$.

3) Dunne and Zierau [6] determined the automorphism groups of elliptic orbits. It follows from their results that our list in Theorem 3.5 exhausts all the cases where $\overline{A_\mathfrak{q}(\lambda)}|_{G'}$ stays irreducible for sufficiently positive λ.

3.3 Irreducible restriction $\pi|_{G'}$ with $\pi = \mathrm{Ind}_P^G(\tau)$

This subsection discusses for which triples (G, G', π) the restriction $\pi|_{G'}$ is (almost) irreducible in the setting that π is a (degenerate) principal series representation $\pi = \mathrm{Ind}_P^G(\tau)$ of G.

Let P be a parabolic subgroup of G with Levi decomposition $P = LN$. For an irreducible unitary representation τ of L, we extend it to P by letting N act trivially, and denote by $\mathrm{Ind}_P^G(\tau)$ the unitarily induced representation of G.

Theorem 3.8. *Suppose that (G, G', L) is one of the following triples:*

G	G'	L
$SL(2n,\mathbb{C})$	$Sp(n,\mathbb{C})$	$GL(2n-1,\mathbb{C})$
$SO(2n,\mathbb{C})$	$SO(2n-1,\mathbb{C})$	$GL(n,\mathbb{C})$
$SO(7,\mathbb{C})$	$G_2(\mathbb{C})$	$\mathbb{C}^\times \times SO(5,\mathbb{C})$
$SL(2n,\mathbb{R})$	$Sp(n,\mathbb{R})$	$GL(2n-1,\mathbb{R})$
$SO(2n,2n)$	$SO(2n,2n-1)$	$GL(2n,\mathbb{R})$
$SO(4,3)$	$G_2(\mathbb{R})$	$SO(1,1) \times SO(3,2)$

Then, the degenerate unitary principal series representations $\pi = \mathrm{Ind}_P^G(\tau)$ of G are almost irreducible when restricted to the subgroup G' for any one dimensional unitary representation τ of any parabolic subgroup P having L as its Levi part.

Outline of the proof. The subgroup G' acts transitively on the (real) flag variety G/P in the setting of Theorem 3.8, and the isotropy subgroup $P' := G' \cap P$ becomes a parabolic subgroup of G'. Then we get an isomorphism $G'/P' \xrightarrow{\sim} G/P$, and hence the conclusion follows. □

We note that the parabolic subgroup P in Theorem 3.8 is maximal.

Example 3.9. For simplicity, we use $GL(2n, \mathbb{R})$ instead of the semisimple group $SL(2n, \mathbb{R})$ in the fourth row, and consider the reductive symmetric pair $(G, G') = (GL(2n, \mathbb{R}), Sp(n, \mathbb{R}))$. Let P be a maximal parabolic subgroup P of G with Levi subgroup $L = GL(2n-1, \mathbb{R}) \times GL(1, \mathbb{R})$. Then P has an abelian unipotent radical $\mathbb{R}^{2n-1}$ and $P' = G' \cap P$ has a non-abelian unipotent radical which is isomorphic to the Heisenberg group H^{2n-1}. In this case the unitary representation $\pi = \mathrm{Ind}_P^G(\tau)$ is irreducible as a representation of G for any unitary character τ of P. On the other hand, the restriction of π to G' is more delicate. It stays irreducible for generic τ (i.e. $d\tau \neq 0$) and splits into two irreducible representations of G' for singular τ, giving rise to a 'special unipotent representation' of $G' = Sp(n, \mathbb{R})$. See [35] for a detailed analysis in connection with the Weyl operator calculus.

Remark 3.10. For a complex reductive group, the underlying $(\mathfrak{g}, K)$-modules of (degenerate) principal series representations are isomorphic to some $A_{\mathfrak{q}}(\lambda)$. Thus the first three cases in Theorem 3.8 have already appeared in Theorem 3.5 in the context of $A_{\mathfrak{q}}(\lambda)$.

3.4 Irreducible restriction of minimal representation

Thirdly, we present an example of almost irreducible branching laws for representations π which are supposed to be attached to minimal nilpotent coadjoint orbits.

Let ϖ be the irreducible unitary representation of the indefinite orthogonal group $G = O(p, q)$ for $p, q \geq 2$, $(p, q) \neq (2, 2)$ and $p + q$ even, constructed in [3] or [34, Part I]. It is a representation of Gelfand–Kirillov dimension $p + q - 3$, and is *minimal* in the sense that its annihilator in the enveloping algebra $U(\mathfrak{g})$ is the Joseph ideal if $p + q > 6$.

Theorem 3.11 $(O(p, q) \downarrow O(p, q-1))$**.**

$$\varpi|_{O(p,q-1)} \simeq V_+ + V_-$$

where $V_\pm$ are irreducible representations of $O(p, q-1)$.

Proof. See [34, Corollary 7.2.1]. □

Remark 3.12. The irreducible decomposition $V_+ + V_-$ has a geometric meaning in connection to the smallest L^2-eigenvalues of the (ultra-hyperbolic) Laplacian on pseudo-Riemannian space forms.

4 Multiplicity-free conjecture

Irreducible restrictions to reductive subgroups are a somewhat rare phenomenon, as we have seen in the previous section. On the other hand, it happens more often that the restriction is multiplicity-free with respect to reductive symmetric pairs (G, G') (see [29] for examples). In this section, we propose a conjectural sufficient condition for the restriction $\pi|_{G'}$ to be multiplicity-free in the setting where π_K is a Zuckerman derived functor module $A_{\mathfrak{q}}(\lambda)$. Our conjecture is motivated by the propagation theorem of multiplicity-free property under 'visible actions' [31].

Definition 4.1. 1) We say a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is of *symmetric type* if $(\mathfrak{g}, \mathfrak{l})$ forms a symmetric pair.

2) We say that $\mathfrak{q}$ is of *virtually symmetric type* if there exists a θ-stable parabolic subalgebra $\widetilde{\mathfrak{q}}$ of symmetric type such that $\widetilde{L}/L \equiv N_G(\widetilde{\mathfrak{q}})/N_G(\mathfrak{q})$ is compact.

Remark. 1) If $\mathfrak{q}$ is of virtually symmetric type, then we have a fibration $\widetilde{L}/L \to G/L \to G/\widetilde{L}$ with compact fiber $\widetilde{L}/L$.

2) If $\mathfrak{q}$ is of symmetric type, then $\mathfrak{q}$ is obviously of virtually symmetric type.

3) Any parabolic subalgebra is of virtually symmetric type if G is compact.

Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and $\overline{A_{\mathfrak{q}}(\lambda)}$ be the unitarization of $A_{\mathfrak{q}}(\lambda)$. Suppose $(\mathfrak{g}, \mathfrak{g}')$ is a reductive symmetric pair. We then propose the following two conjectures:

Conjecture 4.2. *If a θ-stable parabolic subalgebra $\mathfrak{q}$ is of symmetric type, then the restriction $\overline{A_{\mathfrak{q}}(\lambda)}|_{G'}$ is multiplicity-free for sufficiently regular λ.*

Conjecture 4.3. *If $\mathfrak{q}$ is of virtually symmetric type, then the restriction $\overline{A_{\mathfrak{q}}(\lambda)}|_{G'}$ has a uniformly bounded multiplicity.*

Here are some affirmative cases:

Example 4.4. Suppose G is a non-compact simple Lie group such that G/K is a Hermitian symmetric space. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Cartan decomposition. Then $\mathfrak{p}_{\mathbb{C}} := \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$ decomposes into a direct sum of two irreducible representations of K, say $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$. Then $\mathfrak{q} := \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+$ is a θ-stable parabolic subalgebra of symmetric type. If λ is in the good range, then $A_{\mathfrak{q}}(\lambda)$ is the underlying $(\mathfrak{g}, K)$-module of a holomorphic discrete series representation of scalar type. In this case, we see Conjecture 4.2 holds by the explicit branching law:

$$\begin{aligned} &G' = K && \cdots \text{ Hua [13], Kostant, Schmid [40],} \\ &G' : \text{ non-compact} && \cdots \text{ Kobayashi [29].} \end{aligned}$$

Example 4.5. As a generalization of Example 4.4, we retain that G/K is a Hermitian symmetric space, and assume that $\mathfrak{q}$ is of holomorphic type in the sense that $\mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}} \supset \mathfrak{p}_+$. Then $A_{\mathfrak{q}}(\lambda)$ is at most a finite direct sum of irreducible unitary highest weight modules if λ is in the weakly fair range (see [1]). In this case, Conjecture 4.3 is true for any $A_{\mathfrak{q}}(\lambda)$ (see [30, Theorem B]). Further, it was proved in [30, Theorems A, C] as a special case of the propagation theorem of multiplicity-free property that the restriction $\pi|_{G'}$ is multiplicity-free if π is an irreducible unitary highest weight module of scalar type.

Example 4.6. For $(G, G') = (O(p, q), O(r) \times O(p - r, q))$ and for a θ-stable parabolic subalgebra $\mathfrak{q}$ of maximal dimension, we see from explicit branching laws [18] that Conjecture 4.2 holds in this case. Likewise, Conjecture 4.2 holds for the restriction $O(2p, 2q) \downarrow U(p, q)$ again by explicit branching laws [20].

Example 4.7. For any compact group G, the restriction $\pi_\lambda|_{G'}$ is always multiplicity-free if $\mathfrak{q}$ is of symmetric type ([30, Theorems E, F]) and hence, Conjecture 4.2 is true.

Example 4.8. Let $(G, G') = (GL(2n, \mathbb{C}), GL(n, \mathbb{C}) \times GL(n, \mathbb{C}))$, and $\mathfrak{q}$ a θ-stable parabolic subalgebra such that $N_G(\mathfrak{q}) \simeq G'$. Then $\mathfrak{q}$ is of symmetric type. Further, we have the following unitary equivalence:

$$\overline{A_{\mathfrak{q}}(\lambda)}|_{GL(n,\mathbb{C}) \times GL(n,\mathbb{C})} \simeq L^2(GL(n, \mathbb{C})).$$

Thanks to the Plancherel formula of the group $GL(n, \mathbb{C})$ due to the Gelfand school and Harish-Chandra, we see that Conjecture 4.2 holds also in this case.

Let us retain the same θ-stable parabolic subalgebra $\mathfrak{q}$ and consider another reductive symmetric pair $(G, G'') = (GL(2n, \mathbb{C}), GL(2n, \mathbb{R}))$. Then, we get the following unitary isomorphism:

$$\overline{A_{\mathfrak{q}}(\lambda)}|_{GL(2n,\mathbb{R})} \simeq L^2(GL(2n, \mathbb{R})/GL(n, \mathbb{C})).$$

Again, the right-hand side is multiplicity-free by the Plancherel formula for reductive symmetric space due to Oshima, van den Ban, Schlichtkrull, and Delorme [4] among others. (It should be noted that the Plancherel formula for a reductive symmetric space is not multiplicity-free in general.)

Remark 4.9. As we have seen in Example 4.8, Conjectures 4.2 and 4.3 refer to the multiplicities in both discrete and continuous spectrum in the branching law $\overline{A_{\mathfrak{q}}(\lambda)}|_{G'}$.

5 Discretely decomposable branching laws

This section highlights another nice class of branching problems, namely, when the restriction $\pi|_{G'}$ splits discretely without continuous spectrum.

An obvious case is when $\dim \pi < \infty$ or when G' is compact. One of the advantages of discretely decomposable restrictions is that we can expect a combinatorial and detailed study of branching laws by purely algebraic methods because we do not have analytic difficulties arising from continuous spectrum.

Prior to [18], discretely decomposable restrictions $\pi|_{G'}$ were known in some specific settings, e.g. the θ-correspondence for the Weil representation with respect to compact dual pair [11], or when π is a holomorphic discrete series representation and G' is a Hermitian Lie group [14]. A systematic study in the general case including Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$ was initiated by the author in a series of papers [20, 21, 22, 25, 27]. See [9, 18, 20, 34, 39] for a number of concrete examples of branching laws $\pi|_{G'}$ in this framework, [33] for some application to modular symbols, [24] for the construction of new discrete series representations on non-symmetric spaces. See also the lecture notes [28] for a survey on representation theoretic aspects, and [25, 27] for some applications.

In this section, we give a brief overview of discretely decomposable restrictions including some recent developments and open problems.

5.1 Infinitesimally discretely decomposable restrictions

Let us begin with an algebraic formulation. Suppose $\mathfrak{g}'$ is a Lie algebra.

Definition 5.1. A $\mathfrak{g}'$-module V is said to be *discretely decomposable* if there exists an increasing filtration $\{V_n\}$ such that $V = \bigcup_{n=0}^{\infty} V_n$ and each V_n is of finite length as a $\mathfrak{g}'$-module.

In the setting where G' is a real reductive Lie group with maximal compact subgroup K', the terminology 'discretely decomposable' fits well if V is a unitarizable $(\mathfrak{g}', K')$-module, namely, if V is the underlying $(\mathfrak{g}', K')$-module of a unitary representation of G':

Remark 5.2 ([22, Lemma 1.3]). Suppose V is a unitarizable $(\mathfrak{g}', K')$-module. Then V is discretely decomposable as a $\mathfrak{g}'$-module if and only if V is decomposed into an algebraic direct sum of irreducible $(\mathfrak{g}', K')$-modules.

We apply Definition 5.1 to branching problems. Let G be a real reductive Lie group, and G' a reductive subgroup of G. We may and do assume that K is a maximal compact subgroup of G and $K' := K \cap G'$ is that of G'.

Definition 5.3. Let π be a unitary representation of G of finite length. We say the restriction $\pi|_{G'}$ is *infinitesimally discretely decomposable* if the underlying $(\mathfrak{g}, K)$-module π_K is discretely decomposable as a $\mathfrak{g}'$-module.

Here is a comparison between the category of unitary representations and that of $(\mathfrak{g}, K)$-modules:

Conjecture 5.4. *Let π be an irreducible unitary representation of G, and G' a reductive subgroup of G. Then the following two conditions on (G, G', π) are equivalent:*

(i) *The restriction $\pi|_{G'}$ is infinitesimally discretely decomposable.*

(ii) *The unitary representation π decomposes discretely into a direct sum of irreducible unitary representations of G'.*

In general, the implication (i) $\Rightarrow$ (ii) holds. Moreover, the branching law for the restriction of the unitary representation π to G' and that for the restriction of the $(\mathfrak{g}, K)$-module π_K to $(\mathfrak{g}', K')$ are essentially the same under the assumption (i) (see [26, Theorem 2.7]). The converse statement (ii) $\Rightarrow$ (i) remains open; affirmative results have been partially obtained by Duflo and Vargas [5] for discrete series representations π, see also [26, Conjecture D] and [45].

For the study of discretely decomposable restrictions, the concept of K'-admissible restrictions is useful:

Proposition 5.5. *If the restriction $\pi|_{K'}$ is K'-admissible then both the conditions (i) and (ii) in Conjecture 5.4 hold.*

Proof. See [22, Proposition 1.6] and [20, Theorem 1.2], respectively. □

5.2 Analytic approach

We now consider a criterion for the K'-admissibility of a representation π.

Let K' be a closed subgroup of K. Associated to the Hamiltonian K-action on the cotangent bundle $T^*(K/K')$, we consider the momentum map

$$\mu : T^*(K/K') \to \sqrt{-1}\mathfrak{k}^*.$$

Then its image equals $\sqrt{-1}\,\mathrm{Ad}^*(K)(\mathfrak{k}')^\perp$, where $(\mathfrak{k}')^\perp$ is the kernel of the projection $\mathrm{pr}_{\mathfrak{k}\to\mathfrak{k}'} : \mathfrak{k}^* \to (\mathfrak{k}')^*$, the dual to the inclusion $\mathfrak{k}' \subset \mathfrak{k}$ of Lie algebras. The momentum set $C_K(K')$ is defined as the intersection of $\mathrm{Image}\,\mu$ with a dominant Weyl chamber C_+ ($\subset \sqrt{-1}\mathfrak{t}^*$) with respect to a fixed positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ and a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$:

$$C_K(K') := C_+ \cap \sqrt{-1}\,\mathrm{Ad}^*(K)(\mathfrak{k}')^\perp. \tag{5.1}$$

Here we regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{k}^*$ via a K-invariant non-degenerate bilinear form on $\mathfrak{k}$.

Next, let π be a K-module. We write $\mathrm{AS}_K(\pi)$ for the asymptotic K-support introduced by Kashiwara and Vergne [15], that is, the limit cone of the set of highest weights of K-types in π. $\mathrm{AS}_K(\pi)$ is a closed cone in C_+.

We are ready to state a criterion for admissible restrictions.

Theorem 5.6. *Let $G \supset G'$ be a pair of reductive Lie groups, and take maximal compact subgroups $K \supset K'$, respectively. Suppose π is an irreducible unitary representation of G.*

1) *Then the following two conditions are equivalent:*

   (i) $C_K(K') \cap \mathrm{AS}_K(\pi) = \{0\}$.

   (ii) *The restriction $\pi|_{K'}$ is K'-admissible.*

2) *If one of the equivalent conditions (i) or (ii) is fulfilled, then the restriction $\pi|_{G'}$ is infinitesimally discretely decomposable (see Definition 5.3), and the restriction $\pi|_{G'}$ is unitarily equivalent to the Hilbert direct sum:*

$$\pi|_{G'} \simeq \sum_{\tau\in G'}^{\oplus} n_\pi(\tau)\tau \qquad \text{with } n_\pi(\tau) < \infty \quad \text{for any} \quad \tau \in \widehat{G}'.$$

Outline of Proof. The proof of the implication (i) $\Rightarrow$ (ii) was proved first by the author [21, Theorem 2.8] by using the singularity spectrum of hyperfunction characters in a more general setting where π is just a K-module such that the multiplicity

$$m_\pi(\tau) := \dim \mathrm{Hom}_K(\tau, \pi)$$

is of infra-exponential growth. In the same spirit, Hansen, Hilgert, and Keliny [10] gave an alternative proof by using the wave front set of distribution characters under the assumption that $m_\pi(\tau)$ is at most of polynomial growth. The last statement was proved in [21, Theorem 2.9] as a consequence of Proposition 5.5. See also [28]. □

The condition (i) in Theorem 5.6 is obviously fulfilled if $C_K(K') = \{0\}$ or if $\mathrm{AS}_K(\pi) = \{0\}$. We pin down the meanings of these extremal cases:

1) $C_K(K') = \{0\} \Leftrightarrow K' = K$. Then the conclusion in Theorem 5.6 2) is nothing but Harish-Chandra's admissibility theorem (see Fact 3.4).

2) $\mathrm{AS}_K(\pi) = \{0\} \Leftrightarrow \dim \pi < \infty$.

5.3 Algebraic approach

For a finitely generated $\mathfrak{g}$-module X, the associated variety $\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X)$ is a subvariety in the nilpotent cone $\mathcal{N}_{\mathfrak{g}_\mathbb{C}}$ of $\mathfrak{g}_\mathbb{C}^*$ (see [42]). In what follows, let X be the underlying $(\mathfrak{g}, K)$-module of $\pi \in \widehat{G}$ and Y the underlying $(\mathfrak{g}', K')$-module of $\tau (\in \widehat{G'})$.

We write $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'} : \mathfrak{g}_\mathbb{C}^* \to (\mathfrak{g}'_\mathbb{C})^*$ for the natural projection dual to $\mathfrak{g}'_\mathbb{C} \hookrightarrow \mathfrak{g}_\mathbb{C}$.

Theorem 5.7 (see [22, Theorem 3.1])**.** *If* $\mathrm{Hom}_{\mathfrak{g}'}(Y, X) \neq \{0\}$*, then*

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X)) \subset \mathcal{V}_{\mathfrak{g}'_\mathbb{C}}(Y). \tag{5.2}$$

Theorem 5.7 leads us to a useful criterion for discrete decomposability by means of associated varieties:

Corollary 5.8. *If the restriction X is infinitesimally discretely decomposable as a $\mathfrak{g}'$-module, then $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(X))$ is contained in the nilpotent cone of $\mathfrak{g}'_\mathbb{C}$.*

Remark 5.9. An analogous statement to Theorem 5.7 fails if we replace $\mathrm{Hom}_{\mathfrak{g}'}(Y, X) \neq \{0\}$ by $\mathrm{Hom}_{G'}(\tau, \pi|_{G'}) \neq \{0\}$.

Remark 5.10. Analogous results to Theorem 5.7 and Corollary 5.8 hold in the category $\mathcal{O}$. See [32].

It is plausible that the following holds:

Conjecture 5.11. *The inclusion* (5.2) *in Theorem 5.7 is equality.*

Here are some affirmative results to Conjecture 5.11.

Proposition 5.12.

1) *X is the Segal–Shale–Weil representation, and $\mathfrak{g}' = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2$ is the compact dual pair in $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$.*

2) *X is the underlying $(\mathfrak{g}, K)$-module of the minimal representation of $O(p, q)$ ($p + q$ even), and $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair.*

3) *X is a (generalized) Verma module, and $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair.*

4) *$X = A_{\mathfrak{q}}(\lambda)$ and $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair.*

Proof. The first statement could be read off from the results in [7, 38] by case-by-case argument though they were not formulated by means of Theorem 5.7. See [34] for the proof of the second, and [32] for that of the third statement, respectively. The fourth statement is proved recently by Y. Oshima by using a $\mathcal{D}$-module argument. □

5.4 Restriction of $A_{\mathfrak{q}}(\lambda)$ to symmetric pair

For the restriction of $A_{\mathfrak{q}}(\lambda)$ to a reductive symmetric pair, our criterion is computable. Let us have a closer look.

Suppose that (G, G') is a symmetric pair defined by an involutive automorphism σ of G. As usual, the differential of σ will be denoted by the same letter. By taking a conjugation by G if necessary, we may and do assume that σ stabilizes K and that $\mathfrak{t}$ and $\Delta^+(\mathfrak{k}, \mathfrak{t})$ are chosen so that

1) $\mathfrak{t}^{-\sigma} := \mathfrak{t} \cap \mathfrak{k}^{-\sigma}$ is a maximal abelian subspace of $\mathfrak{k}^{-\sigma}$,

2) $\sum^+(\mathfrak{k}, \mathfrak{t}^{-\sigma}) := \{\lambda|_{\mathfrak{t}^{-\sigma}} : \lambda \in \Delta^+(\mathfrak{k}, \mathfrak{t})\} \setminus \{0\}$ is a positive system of the restricted root system $\sum(\mathfrak{k}, \mathfrak{t}^{-\sigma})$.

Then the momentum set $C_K(K')$ coincides with the dominant Weyl chamber ($\subset \sqrt{-1}(\mathfrak{t}^{-\sigma})^*$) with respect to $\Sigma^+(\mathfrak{k}, \mathfrak{t}^{-\sigma})$.

Let $\Delta(\mathfrak{u} \cap \mathfrak{p}) \subset \sqrt{-1}\mathfrak{t}^*$ be the set of weights in $\mathfrak{u} \cap \mathfrak{p}$, and $\mathbb{R}_+\Delta(\mathfrak{u} \cap \mathfrak{p})$ the closed cone spanned by $\Delta(\mathfrak{u} \cap \mathfrak{p})$. Then the asymptotic support $\mathrm{AS}_K(A_\mathfrak{q}(\lambda))$ is contained in $\mathbb{R}_+\Delta(\mathfrak{u} \cap \mathfrak{p})$.

Theorem 5.13. *The following six conditions on* $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$ *are equivalent:*

(i) $A_\mathfrak{q}(\lambda)$ *is non-zero and discretely decomposable as a* $\mathfrak{g}'$*-module for some* λ *in the weakly fair range.*

(i)′ $A_\mathfrak{q}(\lambda)$ *is discretely decomposable as a* $\mathfrak{g}'$*-module for any* λ *in the weakly fair range.*

(ii) $\mathbb{R}_+\Delta(\mathfrak{u} \cap \mathfrak{p}) \cap \sqrt{-1}\mathfrak{t}^{-\sigma} = \{0\}$.

(iii) $A_\mathfrak{q}(\lambda)$ *is non-zero and* K'*-admissible for some* λ *in the weakly fair range.*

(iii)′ $A_\mathfrak{q}(\lambda)$ *is* K'*-admissible for any* λ *in the weakly fair range.*

(iv) $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}_\mathbb{C}}(A_\mathfrak{q}(\lambda)))$ *is contained in the nilpotent cone of* $\mathfrak{g}'_\mathbb{C}$.

Proof. The equivalences (i) ⇔ (i)′ and (iii) ⇔ (iii)′ are easy. The implication (ii) ⇒ (iii) was first proved in [20]. Alternatively, we can use Theorem 5.6 and the inclusive relation $\mathrm{AS}_K(A_\mathfrak{q}(\lambda)) \subset \mathbb{R}_+\Delta(\mathfrak{u}\cap\mathfrak{p})$. This was the approach taken in [21]. Other implications are proved in [23] based on Theorem 5.7. □

See [36] for the list of all such triples $(\mathfrak{g}, \mathfrak{g}^\sigma, \mathfrak{q})$.

Remark 5.14. The implication (i) ⇒ (iii)′ and Theorem 5.6 show that

$$\dim \mathrm{Hom}_{G'}(\tau, \pi|_{G'}) < \infty \quad \text{for any } \tau \in \widehat{G'}$$

if the restriction $\pi|_{G'}$ is infinitesimally discretely decomposable for any $\pi_K \simeq A_\mathfrak{q}(\lambda)$.

6 Appendix – basic properties of $A_{\mathfrak{q}}(\lambda)$

This section gives a quick summary of basic properties on Zuckeman's derived functor modules and the "geometric quantization" of elliptic coadjoint orbits $\mathcal{O}_\lambda$ in the following scheme:

$$\begin{array}{cl} \lambda \in \sqrt{-1}\mathfrak{g}^* & \text{an elliptic and integral element} \\ \downarrow & \\ \mathcal{L}_{\lambda+\rho_\lambda} \to \mathcal{O}_\lambda & \text{a } G\text{-equivariant holomorphic line bundle} \\ \downarrow & \\ H^*_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_{\lambda+\rho_\lambda}) & \text{a Fréchet representation of } G \\ \downarrow & \\ \pi_\lambda & \text{a unitary representation of } G \end{array}$$

There is no new result in this section, and the normalization of the parameters and formulation follows the expository notes [23, 28]. See [16] for a more complete treatment and references therein.

6.1 Zuckerman derived functor modules

Let G be a connected real reductive Lie group, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of the Lie algebra of $\mathfrak{g}$, and θ the corresponding Cartan involution.

Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Then the normalizer $L = N_G(\mathfrak{q})$ is a connected reductive subgroup of G, and the homogeneous space G/L carries a G-invariant complex structure such that the holomorphic tangent bundle $T(G/L)$ is given as a homogeneous bundle $G \times_L (\mathfrak{g}_{\mathbb{C}}/\mathfrak{q})$. Let $\mathfrak{l}_{\mathbb{C}}$ be the complexification of the Lie algebra $\mathfrak{l}$ of L, and $\mathfrak{u}$ the unipotent radical of $\mathfrak{q}$. Then we have a Levi decomposition $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$. We set $\rho(\mathfrak{u})(X) := \frac{1}{2}\,\mathrm{Trace}(\mathrm{ad}(X) : \mathfrak{u} \to \mathfrak{u})$ for $X \in \mathfrak{l}$.

We say a Lie algebra homomorphism $\lambda : \mathfrak{l} \to \mathbb{C}$ is *integral* if λ lifts to a character of the connected group L, denoted by $\mathbb{C}_\lambda$. Then $\mathcal{L}_\lambda := G \times_{G_\lambda} \mathbb{C}_\lambda$ is a G-equivariant holomorphic line bundle over G/L. For example, $2\rho(\mathfrak{u})$ is integral, and the canonical bundle $\Omega(G/L) := \Lambda^{top}(T^*(G/L))$ is isomorphic to

$$\Omega(G/L) \simeq \mathcal{L}_{2\rho(\mathfrak{u})} \tag{6.1}$$

as a G-equivariant holomorphic line bundle. The Zuckerman derived functor $W \mapsto \mathcal{R}^j_{\mathfrak{q}}(W \otimes \mathbb{C}_{\rho(\mathfrak{u})})$ is a covariant functor from the category of $(\mathfrak{l}, L \cap K)$-modules to the category of $(\mathfrak{g}, K)$-modules. We note that L is not necessarily compact. In this generality, H. Wong proved in [44] that the Dolbeault cohomology groups

$$H^j_{\bar\partial}(G/L, \mathcal{L}_\lambda \otimes \Omega(G/L)) \simeq H^j_{\bar\partial}(G/L, \mathcal{L}_{\lambda+2\rho(\mathfrak{u})})$$

carry a Fréchet topology on which G acts continuously and that $\mathcal{R}^j_{\mathfrak{q}}(\mathbb{C}_{\lambda+\rho(\mathfrak{u})})$ are isomorphic to their underlying $(\mathfrak{g}, K)$-modules. We set $S := \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}})$, and

$$A_{\mathfrak{q}}(\lambda) := \mathcal{R}^S_{\mathfrak{q}}(\mathbb{C}_{\lambda+\rho(\mathfrak{u})}).$$

In our normalization, $A_{\mathfrak{q}}(0)$ is an irreducible and unitarizable $(\mathfrak{g}, K)$-module with non-zero $(\mathfrak{g}, K)$-cohomology [43], and in particular, has the same infinitesimal character with that of the trivial one dimensional representation $\mathbb{C}$ of G.

6.2 Geometric quantization of elliptic coadjoint orbit

Let $\lambda \in \sqrt{-1}\mathfrak{g}^*$. We say that the coadjoint orbit $\mathcal{O}_\lambda := \mathrm{Ad}^*(G)\cdot\lambda$ is *elliptic* if $\lambda|_{\mathfrak{p}} \equiv 0$. We identify $\mathfrak{g}$ with the dual space $\mathfrak{g}^*$ by a non-degenerate G-invariant bilinear form, and write $X_\lambda \in \sqrt{-1}\mathfrak{g}$ for the corresponding element to λ. Then $\mathrm{ad}(X_\lambda)$ is semisimple and all the eigenvalues are pure imaginary. The sum of the eigenspaces for non-negative eigenvalues of $-\sqrt{-1}\,\mathrm{ad}(X_\lambda)$ defines a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$, and consequently, the elliptic orbit $\mathcal{O}_\lambda$ carries a G-invariant complex structure such that the holomorphic tangent bundle is given by $G \times_L (\mathfrak{g}_{\mathbb{C}}/\mathfrak{q})$.

We set $\rho_\lambda := \rho(\mathfrak{u})$. If $\lambda + \rho_\lambda$ is integral, namely, if $\lambda + \rho_\lambda$ lifts to a character of L, then we can define a G-equivariant holomorphic line bundle $\mathcal{L}_{\lambda+\rho_\lambda} := G \times_L \mathbb{C}_{\lambda+\rho_\lambda}$ over $\mathcal{O}_\lambda$.

Here is a brief summary of the important achievements on unitary representation theory in 1980s and 1990s on the geometric quantization of elliptic orbits due to Parthasarathy, Zuckerman, Vogan and Wallach (algebraic construction, unitarizability of Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$), and Schmid and Wong (realization in Dolbeault cohomology, in particular, the closed range property of the $\bar\partial$-operator) among others. See [16, 28] for the original references therein.

Fact 6.1. *Let $\lambda \in \sqrt{-1}\mathfrak{g}^*$ be elliptic such that $\lambda + \rho_\lambda$ is integral.*

1) *(vanishing theorem) $H^j_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_{\lambda+\rho_\lambda}) = 0$ if $j \neq S$.*

2) *The Dolbeault cohomology group* $H^S_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_{\lambda+\rho_\lambda})$ *carries a Fréchet topology, on which* G *acts continuously. It is the maximal globalization of* $\mathcal{R}^S_{\mathfrak{q}}(\mathbb{C}_\lambda) = A_{\mathfrak{q}}(\lambda - \rho_\lambda)$ *in the sense of Schmid.*

3) *(unitarizability) There is a dense subspace* $\mathcal{H}$ *in* $H^S_{\bar\partial}(\mathcal{O}_\lambda, \mathcal{L}_{\lambda+\rho_\lambda})$ *on which a* G*-invariant Hilbert structure exists. We denote by* π_λ *the resulting unitary representation on* $\mathcal{H}$*.*

4) *If* λ *is in the good range in the sense of Vogan, then the unitary representation of* G *on* $\mathcal{H}$ *is irreducible and non-zero.*

Here, by 'good range', we mean that λ satisfies

$$\langle \lambda + \rho_{\mathfrak{l}}, \alpha \rangle > 0 \quad \text{for any} \quad \alpha \in \Delta(\mathfrak{u}, \mathfrak{h}_{\mathbb{C}}), \tag{6.2}$$

where $\mathfrak{h}$ is a fundamental Cartan subalgebra containing X_λ and $\rho_{\mathfrak{l}}$ is half the sum of positive roots for $\Delta(\mathfrak{l}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. (This condition is independent of the choice of $\mathfrak{h}$ and $\Delta^+(\mathfrak{l}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.)

References

[1] J. Adams, Unitary highest weight modules, Adv. in Math. **63** (1987), 113–137.

[2] E. van den Ban, Invariant differential operators on a semisimple symmetric space and finite multiplicities in a Plancherel formula, Arkiv Mat. **25** (1987), 175–187.

[3] B. Binegar and R. Zierau, Unitarization of a singular representation of $SO(p, q)$, Commun. Math. Phys. **138** (1991), 245–258.

[4] P. Delorme, Formule de Plancherel pour les espaces symétriques réductifs, Ann. of Math. (2), **147** (1998), 417–452.

[5] M. Duflo and J. A. Vargas, Branching laws for square integrable representations, Proc. Japan Acad. Ser. A, Math. Sci. **86** (2010), 49–54.

[6] E. Dunne and R. Zierau, The automorphism groups of complex homogeneous spaces. Math. Ann. **307** (1997), 489–503.

[7] T. Enright and J. Willenbring, Hilbert series, Howe duality and branching for classical groups, Ann. of Math. (2) **159** (2004), 337–375.

[8] I. M. Gelfand and M. I. Graev, Geometry of homogeneous spaces, representations of groups in homogeneous spaces and related questions of integral geometry. I, Trudy Moskov. Mat. Obšč. 8 (1959), 321–390.

[9] B. Gross and N. Wallach, Restriction of small discrete series representations to symmetric subgroups, *Proc. Sympos. Pure Math.*, **68** (2000), Amer. Math. Soc., 255–272.

[10] S. Hansen, J. Hilgert and S. Keliny, Asymptotic K-support and restrictions of representations, Represent. Theory **13** (2009), 460–469.

[11] R. Howe, θ-series and invariant theory, Proc. Symp. Pure Math. **33** (1979), Amer. Math. Soc., 275–285.

[12] R. Howe, Reciprocity laws in the theory of dual pairs, Progr. in ath. Birkhäuser, 40 (1983), 159–175

[13] L. K. Hua, Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains, Amer. Math. Soc., 1963.

[14] H. P. Jakobsen and M. Vergne, Restrictions and expansions of holomorphic representations, J. Funct. Anal. **34** (1979), 29–53.

[15] M. Kashiwara and M. Vergne, K-types and singular spectrum, In: Lect. Notes in Math. **728**, 1979, Springer-Verlag, 177–200.

[16] A. W. Knapp and D. Vogan, Jr., Cohomological Induction and Unitary Representations, Princeton U.P., 1995.

[17] T. Kobayashi, Unitary representations realized in L^2-sections of vector bundles over semisimple symmetric spaces, Proceedings at the 27-28th Symp. of Functional Analysis and Real Analysis (1989), Math. Soc. Japan, 39–54.

[18] T. Kobayashi, The restriction of $A_{\mathfrak{q}}(\lambda)$ to reductive subgroups, *Proc. Japan Acad.*, **69** (1993), 262–267.

[19] T. Kobayashi, Irreducible restriction of $A_{\mathfrak{q}}(\lambda)$ to reductive subgroups, Lecture at Summer workshop on representation theory, Polytechnic University, August 24, 1993.

[20] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its application, *Invent. Math.*, **117** (1994), 181–205.

[21] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$, II. —micro-local analysis and asymptotic K-support, *Ann. of Math.*, **147** (1998), 709–729.

[22] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$, III. —restriction of Harish-Chandra modules and associated varieties, *Invent. Math.*, **131** (1998), 229–256.

[23] T. Kobayashi, Harmonic analysis on homogeneous manifolds of reductive type and unitary representation theory, *Sugaku*, **46** (1994), Math. Soc. Japan (in Japanese), 124–143; *Translations, Series II*, Selected Papers on Harmonic Analysis, Groups, and Invariants (K. Nomizu, ed.), **183** (1998), Amer. Math. Soc., 1–31.

[24] T. Kobayashi, Discrete series representations for the orbit spaces arising from two involutions of real reductive Lie groups, *J. Funct. Anal.*, **152** (1998), 100–135.

[25] T. Kobayashi, Theory of discrete decomposable branching laws of unitary representations of semisimple Lie groups and some applications, *Sugaku*, **51** (1999), Math. Soc. Japan (in Japanese), 337–356; English translation, *Sugaku Exposition*, **18** (2005), Amer. Math. Soc. 1–37.

[26] T. Kobayashi, Discretely decomposable restrictions of unitary representations of reductive Lie groups — examples and conjectures, *Advanced Study in Pure Math.*, **26** (2000), 98–126.

[27] T. Kobayashi, Unitary representations and branching laws, *Proceedings of the I.C.M. 2002 at Beijing*, **2** (2002), 615–627.

[28] T. Kobayashi, Restrictions of unitary representations of real reductive groups, *Progr. in Math.* **229**, pages 139–207, Birkhäuser, 2005.

[29] T. Kobayashi, Multiplicity-free representations and visible actions on complex manifolds, Publ. Res. Inst. Math. Sci. **41**(2005), 497–549 (a special issue of Publications of the Research Institute for Mathematical Sciences commemorating the fortieth anniversary of the founding of the Research Institute for Mathematical Sciences).

[30] T. Kobayashi, Multiplicity-free theorems of the restrictions of unitary highest weight modules with respect to reductive symmetric pairs. Progr. Math. **255**, pages 45–109. Birkhäuser, 2007.

[31] T. Kobayashi, Visible actions on symmetric spaces. Transformation Groups, **12** (2007), 671–694.

[32] T. Kobayashi, Restrictions of generalized Verma modules to symmetric pairs, submitted, arXiv:1008.4544

[33] T. Kobayashi and T. Oda, Vanishing theorem of modular symbols on locally symmetric spaces, *Comment. Math. Helvetici*, **73** (1998), 45–70.

[34] T. Kobayashi and B. Ørsted, Analysis on minimal representations of $O(p,q)$, Part II. Branching Laws, *Adv. in Math.*, **180** (2003), 513–550.

[35] T. Kobayashi, B. Ørsted, and M. Pevzner, Geometric analysis on small unitary representations of $GL(n,\mathbb{R})$, *J. Funct. Anal.*, **260** (2011), 1682–1720.

[36] T. Kobayashi and Y. Oshima, Classification of discretely decomposable $A_{\mathfrak{q}}(\lambda)$ with respect to reductive symmetric pairs, submitted, arXiv:1104.4400

[37] B. Krötz and R. J. Stanton, Holomorphic extensions of representations. I. Automorphic functions, Ann. of Math. (2) **159** (2004), 641–724.

[38] K. Nishiyama, H. Ochiai, and K. Taniguchi, Bernstein degree and associated cycles of Harish-Chandra modules — Hermitian symmetric case —, *Astérisque*, **273** (2001), 13–80.

[39] B. Ørsted and B. Speh, Branching laws for some unitary representations of $SL(4,\mathbb{R})$, SIGMA 4 (2008) doi:10.3842/SIGMA.2008.017.

[40] W. Schmid, Die Randwerte holomorphe Funktionen auf hermetisch symmetrischen Raumen. Invent. Math. **9** (1969–70), 61–80.

[41] H. Sekiguchi, The Penrose transform for $Sp(n, \mathbb{R})$ and singular unitary representations, J. Math. Soc. Japan **54** (2002), 215–253.

[42] D. A. Vogan, Jr., Associated varieties and unipotent representations, Harmonic Analysis on Reductive Lie Groups, *Progress in Math.*, **101** (1991), Birkhäuser, 315–388.

[43] D. A. Vogan, Jr. and G. J. Zuckerman, Unitary representations with nonzero cohomology, Compositio Math. **53** (1984), 51–90.

[44] H. Wong, Dolbeault cohomological realization of Zuckerman modules associated with finite rank representations, J. Funct. Anal. **129** (1995), 428–454.

[45] F. Zhu and K. Liang, On a branching law of unitary representations and a conjecture of Kobayashi, C. R. Acad. Sci. Paris, Ser. I, **348** (2010), 959–962.

Graduate School of Mathematical Sciences, IPMU,
the University of Tokyo, Komaba, Meguro, Tokyo, 153-8914 Japan
toshi@ms.u-tokyo.ac.jp