Stable spectrum for pseudo-Riemannian locally symmetric spaces

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Abstract

Let $X = G/H$ be a reductive symmetric space with $\operatorname{rank} G/H = \operatorname{rank} K/K \cap H$, where K (resp. $K \cap H$) is a maximal compact subgroup of G (resp. of H). We investigate the discrete spectrum of certain Clifford–Klein forms $\Gamma\backslash X$, where Γ is a discrete subgroup of G acting properly discontinuously and freely on X: we construct an infinite set of joint eigenvalues for "intrisic" differential operators on $\Gamma\backslash X$, and this set is stable under small deformations of Γ in G. *To cite this article: F. Kassel, T. Kobayashi, C. R. Acad. Sci. Paris, Ser. I 348 (2010).*

Résumé

Spectre stable pour les variétés pseudo-riemanniennes localement symétriques. Soit $X = G/H$ un espace symétrique réductif vérifiant $\operatorname{rang} G/H = \operatorname{rang} K/K \cap H$, où K (resp. $K \cap H$) est un sous-groupe compact maximal de G (resp. de H). Nous étudions le spectre discret de certaines formes de Clifford–Klein $\Gamma\backslash X$, où Γ est un sous-groupe discret de G agissant librement et proprement sur X : nous construisons un ensemble infini de valeurs propres pour les opérateurs différentiels "intrinsèques" sur $\Gamma\backslash X$, et cet ensemble est stable par petites déformations de Γ dans G. *Pour citer cet article : F. Kassel, T. Kobayashi, C. R. Acad. Sci. Paris, Sér. I 348 (2010).*

Version française abrégée

Soit $X = G/H$ un espace symétrique, où G est un groupe de Lie réductif connexe non compact et H la composante neutre du groupe des points fixes de G par un certain automorphisme involutif σ. L'espace X est naturellement muni d'une métrique pseudo-riemannienne G-invariante. Une *forme de Clifford–Klein* de X est un quotient $X_\Gamma = \Gamma\backslash X$ où Γ est un sous-groupe discret de G agissant librement et proprement sur X ; c'est une variété complète localement modelée sur X. Soit $\mathbb{D}(X)$ l'algèbre des opérateurs

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différentiels G-invariants sur X. Tout élément $D \in \mathbb{D}(X)$ (par exemple le laplacien) induit un opérateur différentiel D_Γ sur X_Γ. Le *spectre discret* $\mathrm{Spec}_d(X_\Gamma)$ de X_Γ est l'ensemble des morphismes d'algèbres $\lambda : \mathbb{D}(X) \to \mathbb{C}$ pour lesquels il existe une fonction $f \in L^2(X_\Gamma)$ non nulle vérifiant $D_\Gamma f = \lambda(D) f$ pour tout $D \in \mathbb{D}(X)$ au sens des distributions. Soit $K = G^\theta$ un sous-groupe compact maximal de G, où θ est une involution de Cartan commutant avec σ. Notre résultat principal concerne les formes de Clifford–Klein de X qui sont *standard*, au sens où Γ est inclus dans un sous-groupe réductif de G agissant proprement sur X.

Théorème 0.1 *Supposons* $\mathrm{rang}\, G/H = \mathrm{rang}\, K/K\cap H$. *Le spectre discret* $\mathrm{Spec}_d(X_\Gamma)$ *est infini pour toute forme de Clifford–Klein compacte standard* X_Γ *de* X *; de plus, il existe une partie infinie de* $\mathrm{Spec}_d(X_\Gamma)$ *qui est stable par petites déformations de* Γ *dans* G. *Ceci reste vrai lorsque* Γ *est convexe cocompact dans un sous-groupe réductif de* G *de rang réel* 1.

Dans la situation du théorème 0.1, il existe un voisinage $\mathcal{U} \subset \mathrm{Hom}(\Gamma, G)$ de l'inclusion naturelle tel que pour tout $\varphi \in \mathcal{U}$ le quotient $X_{\varphi(\Gamma)} = \varphi(\Gamma)\backslash X$ soit une forme de Clifford–Klein de X, compacte si X_Γ l'est : cela résulte de [5] (propreté) et [8] (compacité). Le théorème 0.1 affirme que, quitte à réduire le voisinage $\mathcal{U}$, il existe un ensemble infini qui est inclus dans $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ pour tout $\varphi \in \mathcal{U}$. L'étude des petites déformations de formes de Clifford–Klein dans ce cadre général remonte à l'article [10].

Soit $\mathfrak{j}_\mathbb{C}$ un sous-espace abélien semi-simple maximal de l'ensemble des points fixes de $-d\sigma$ dans l'algèbre de Lie complexifiée $\mathfrak{g}_\mathbb{C}$ de G, soit $\mathfrak{j}_\mathbb{C}^*$ son dual, et soit W le groupe de Weyl de $\mathfrak{j}_\mathbb{C}$ dans $\mathfrak{g}_\mathbb{C}$. Le spectre discret de toute forme de Clifford–Klein de X s'identifie naturellement à une partie de $\mathfrak{j}_\mathbb{C}^*/W$. Sous les hypothèses du théorème 0.1 on peut supposer que $\mathfrak{j}_\mathbb{C} = \mathfrak{j} \otimes_\mathbb{R} \mathbb{C}$ pour un certain sous-espace abélien maximal $\mathfrak{j}$ de $\sqrt{-1}\,\mathfrak{k}$, où $\mathfrak{k}$ est l'algèbre de Lie de K. Fixons un système $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ de racines positives de $\mathfrak{j}_\mathbb{C}$ dans $\mathfrak{g}_\mathbb{C}$, ce qui définit une chambre de Weyl positive $\mathfrak{j}_+^*$ de $\mathfrak{j}^*$. Soient $\rho \in \mathfrak{j}^*$ et $\rho_c \in \mathfrak{j}^*$ les demi-sommes respectives des racines de $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ et $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \cap \Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, et soit Λ_+ l'intersection de $\mathfrak{j}_+^*$ avec le réseau de $\mathfrak{j}$ engendré par les plus hauts poids des représentations irréductibles de K ayant des vecteurs $(K \cap H)$-invariants non nuls. Pour tout $\lambda \in \mathfrak{j}_+^*$ nous notons $d(\lambda)$ la "distance pondérée" naturelle de λ aux murs de $\mathfrak{j}_+^*$ (voir paragraphe 2). Avec ces notations, voici une version plus précise du théorème 0.1.

Théorème 0.2 *Sous les hypothèses du théorème 0.1, il existe une constante* $R > 0$ *et un voisinage* $\mathcal{U} \subset \mathrm{Hom}(\Gamma, G)$ *de l'inclusion naturelle tels que* $\{\lambda \in 2\rho_c - \rho + \Lambda_+ :\ d(\lambda) \geq R\} \subset \mathrm{Spec}_d(X_{\varphi(\Gamma)})$ *pour tout* $\varphi \in \mathcal{U}$.

Nous donnons une liste d'espaces symétriques X auxquels nos théorèmes s'appliquent, et décrivons explicitement une partie infinie du *spectre discret stable* des formes de Clifford–Klein compactes standard de $X = \mathrm{SO}(2,4)/\mathrm{U}(1,2)$ (en utilisant [11]) et de l'*espace anti-de Sitter* $X = \mathrm{AdS}^3 = \mathrm{SO}(2,2)/\mathrm{SO}(1,2)$ (voir (2)). Rappelons que les variétés anti-de Sitter (c'est-à-dire lorentziennes de courbure constante < 0) compactes de dimension 3 sont les formes de Clifford–Klein compactes de AdS^3, à revêtement fini, isométrie et renormalisation près [7], [13]. Nous démontrons un résultat analogue aux théorèmes 0.1 et 0.2 pour *toutes* ces formes de Clifford–Klein compactes, même celles qui ne sont pas standard.

Théorème 0.3 *Le spectre discret de toute variété anti-de Sitter compacte de dimension 3 est infini, et contient une partie infinie qui est stable par petites déformations de la structure anti-de Sitter.*

Pour démontrer nos résultats, nous construisons des fonctions propres sur les formes de Clifford–Klein de X à partir de fonctions propres sur X construites par Flensted-Jensen [2]. Nous donnons des estimées asymptotiques uniformes de ces dernières, en fonction de la projection $\nu : G \to \overline{\mathfrak{b}_+}$ associée à une décomposition $G = KBH$ (voir paragraphe 3). Nous relions la projection ν à la projection $\mu : G \to \overline{\mathfrak{a}_+}$ associée à une décomposition de Cartan $G = KAK$ où $A \supset B$ (voir paragraphe 3), et utilisons les estimées de [5] et [6] sur la restriction de μ à Γ et à ses déformés. Les détails seront publiés ultérieurement.

1. A general program

Let $X = G/H$ be a reductive symmetric space, where G is a connected noncompact reductive linear Lie group and $H = (G^\sigma)_0$ the identity component of the set of fixed points of G under some involutive automorphism σ. We note that X naturally carries a G-invariant pseudo-Riemannian metric, which is induced by the Killing form of the Lie algebra $\mathfrak{g}$ of G if G is semisimple. A *Clifford–Klein form* of X is a quotient $X_\Gamma = \Gamma\backslash X$ where Γ is a discrete subgroup of G acting properly discontinuously and freely on X; it is a complete manifold locally modelled on X. Any G-invariant differential operator D on X (such as the Laplacian) induces a differential operator D_Γ on X_Γ, and the map $D \mapsto D_\Gamma$ is an injective $\mathbb{C}$-algebra homomorphism from the ring $\mathbb{D}(X)$ of G-invariant differential operators on X into the ring of differential operators on X_Γ. We may think of its image as the set of "intrisic" differential operators on X_Γ. The *discrete spectrum* $\mathrm{Spec}_d(X_\Gamma)$ of X_Γ is the set of $\mathbb{C}$-algebra homomorphisms $\lambda : \mathbb{D}(X) \to \mathbb{C}$ such that the set $L^2(X_\Gamma, \mathcal{M}_\lambda)$ of weak solutions $f \in L^2(X_\Gamma)$ to the system

$$D_\Gamma f = \lambda(D) f \qquad \text{for all } D \in \mathbb{D}(X) \tag{$\mathcal{M}_\lambda$}$$

is nontrivial. In this note we wish to initiate the following general program.

A) Construct elements of $L^2(X_\Gamma, \mathcal{M}_\lambda)$, *i.e.* joint eigenfunctions on X_Γ corresponding to $\mathrm{Spec}_d(X_\Gamma)$.

B) Understand the behavior of $\mathrm{Spec}_d(X_\Gamma)$ under small deformations of Γ in G.

Problem B builds on the fact that for certain Clifford–Klein forms X_Γ, the proper discontinuity of the action of Γ on X is preserved under small deformations of Γ. The study of small deformations of Clifford–Klein forms in such a general setting was initiated in [10].

2. Main results

Let θ be a Cartan involution of G commuting with σ and let $K = G^\theta$ be the corresponding maximal compact subgroup of G, with Lie algebra $\mathfrak{k}$. In this note, we assume that

$$\mathrm{rank}\, G/H = \mathrm{rank}\, K/K\cap H, \tag{1}$$

where $\mathrm{rank}\, G/H$ (resp. $\mathrm{rank}\, K/K\cap H$) is the dimension of a maximal semisimple abelian subspace in the set of fixed points of $-d\sigma$ in $\mathfrak{g}$ (resp. $\mathfrak{k}$). We investigate Problems A and B for an important class of Clifford–Klein forms X_Γ, namely those that are *standard*, in the sense that Γ is contained in some closed reductive subgroup L of G acting properly on $X = G/H$. Note that if such an X_Γ is compact, then Γ must be a uniform lattice in L and the action of L on X must be cocompact. Here is our main result.

Theorem 2.1 *Under the rank assumption (1), the discrete spectrum* $\mathrm{Spec}_d(X_\Gamma)$ *is infinite for any standard compact Clifford–Klein form* X_Γ *of* X*; furthermore, there is an infinite subset of* $\mathrm{Spec}_d(X_\Gamma)$ *that is stable under small deformations of* Γ *in* G*. The same conclusion holds when* Γ *is convex cocompact in some reductive subgroup* L *of* G *with* $\mathrm{rank}_{\mathbb{R}} L = 1$.

Underlying Theorem 2.1 is the existence, due to [5] (properness) and [8] (compactness), of a neighborhood $\mathcal{U} \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion such that $X_{\varphi(\Gamma)} = \varphi(\Gamma)\backslash X$ is a Clifford–Klein form of X for all $\varphi \in \mathcal{U}$, with $X_{\varphi(\Gamma)}$ compact if X_Γ is. Theorem 2.1 states that, after possibly replacing $\mathcal{U}$ by some smaller neighborhood, there is an infinite set that is contained in $\mathrm{Spec}_d(X_{\varphi(\Gamma)})$ for all $\varphi \in \mathcal{U}$.

Recall that a discrete subgroup Γ of a reductive group L with $\mathrm{rank}_{\mathbb{R}} L = 1$ is said to be *convex cocompact* if it acts cocompactly on the convex hull of its limit set in the Riemannian symmetric space of L, this limit set being nonempty. Convex cocompact groups include uniform lattices, but also discrete groups of infinite covolume such as Schottky groups.

In order to describe the *stable discrete spectrum* of X_Γ in Theorem 2.1, let us briefly recall the structure of $\mathbb{D}(X)$ (see [4] for more details) and introduce some notation. Let $\mathfrak{g} = \mathfrak{h}+\mathfrak{q}$ be the decomposition of $\mathfrak{g}$ into eigenspaces of $d\sigma$, with respective eigenvalues $+1$ and -1, and let $\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C}+\mathfrak{q}_\mathbb{C}$ be its complexification. Fix a maximal semisimple abelian subspace $\mathfrak{j}_\mathbb{C}$ of $\mathfrak{q}_\mathbb{C}$ and let W be the Weyl group of the restricted root system $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ of $\mathfrak{j}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. The $\mathbb{C}$-algebra $\mathbb{D}(X)$ is naturally isomorphic to the subalgebra $S(\mathfrak{j}_\mathbb{C})^W$ of W-fixed points in the symmetric algebra $S(\mathfrak{j}_\mathbb{C})$; it is a polynomial ring in $r := \dim_\mathbb{C} \mathfrak{j}_\mathbb{C} = \operatorname{rank} G/H$ generators. In particular, the set of $\mathbb{C}$-algebra homomorphisms from $\mathbb{D}(X)$ to $\mathbb{C}$ naturally identifies with $\mathfrak{j}_\mathbb{C}^*/W$, where $\mathfrak{j}_\mathbb{C}^*$ is the dual vector space of $\mathfrak{j}_\mathbb{C}$. For any Clifford–Klein form X_Γ of X, we see $\operatorname{Spec}_d(X_\Gamma)$ as a subset of $\mathfrak{j}_\mathbb{C}^*/W$. Under the rank hypothesis (1), we may assume that $\mathfrak{j}_\mathbb{C}$ is the complexification of a maximal abelian subspace $\mathfrak{j}$ of $\sqrt{-1}\,\mathfrak{k}$, on which all restricted roots $\alpha \in \Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ take real values. We endow $\mathfrak{j}^*$ with a W-invariant inner product $\langle \cdot, \cdot \rangle$, fix a basis Ψ of $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$, defining a system $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ of positive roots, and let $\mathfrak{j}_+^*$ be the corresponding positive Weyl chamber in $\mathfrak{j}^*$, defined by $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Psi$. For $\lambda \in \mathfrak{j}_+^*$, we consider the natural "weighted distance" from λ to the walls of $\mathfrak{j}_+^*$ given by $d(\lambda) = \min_{\alpha\in\Psi} \langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle$. Let ρ (resp. ρ_c) be half the sum of roots in $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ (resp. in $\Sigma^+(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \cap \Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$), counted with multiplicities, and let Λ_+ be the intersection of $\mathfrak{j}_+^*$ with the lattice of $\mathfrak{j}^*$ generated by all highest weights of irreducible representations of K with nonzero $(K \cap H)$-fixed vectors. With this notation, here is a more precise statement of Theorem 2.1.

Theorem 2.2 *In the setting of Theorem 2.1, there are a constant $R > 0$ and a neighborhood $\mathcal{U} \subset \operatorname{Hom}(\Gamma, G)$ of the natural inclusion such that $\{\lambda \in 2\rho_c - \rho + \Lambda_+ :\ d(\lambda) \geq R\} \subset \operatorname{Spec}_d(X_{\varphi(\Gamma)})$ for all $\varphi \in \mathcal{U}$.*

We explicitly construct eigenfunctions $f \in L^2(X_{\varphi(\Gamma)}, \mathcal{M}_\lambda)$ for all $\lambda \in 2\rho_c-\rho+\Lambda_+$ with $d(\lambda)$ large enough and all homomorphisms φ that are close enough to the natural inclusion of Γ in G. These eigenfunctions actually satisfy $(\mathcal{M}_\lambda)$ as functions of class C^N where N is the maximal degree of r generators of $\mathbb{D}(X)$, and they belong to $L^p(X_{\varphi(\Gamma)})$ for all $1 \leq p \leq \infty$.

Using [12, Cor. 3.3.7], we see that Theorems 2.1 and 2.2 apply in particular to the following triples (G, H, L), where $n, p, q \geq 1$. In Example (vi), the group G_0 may be $\mathrm{SO}(p, 2q), \mathrm{SU}(p, q), \mathrm{Sp}(p, q), \mathrm{Sp}(n, \mathbb{R})$, $\mathrm{SO}^*(2n)$, or certain exceptional groups, and $\mathrm{Diag}(G_0)$ denotes the diagonal of $G_0 \times G_0$.

	G	H	L
(i)	$\mathrm{SO}(2, 2n)$	$\mathrm{SO}(1, 2n)$	$\mathrm{U}(1, n)$
(ii)	$\mathrm{SO}(2, 4n)$	$\mathrm{U}(1, 2n)$	$\mathrm{SO}(1, 4n)$
(iii)	$\mathrm{SO}(4, 4n)$	$\mathrm{SO}(3, 4n)$	$\mathrm{Sp}(1, n)$
(iv)	$\mathrm{U}(2, 2n)$	$\mathrm{U}(1) \times \mathrm{U}(1, 2n)$	$\mathrm{Sp}(1, n)$
(v)	$\mathrm{SO}(8, 8)$	$\mathrm{SO}(7, 8)$	$\mathrm{Spin}(1, 8)$
(vi)	$G_0 \times G_0$	$\mathrm{Diag}(G_0)$	$G_0 \times \{1\}$
(vii)	$\mathrm{SO}(2, 2n) \times \mathrm{SO}(2, 2n)$	$\mathrm{Diag}(\mathrm{SO}(2, 2n))$	$\mathrm{SO}(1, 2n) \times \mathrm{U}(1, n)$
(viii)	$\mathrm{SO}(4, 4n) \times \mathrm{SO}(4, 4n)$	$\mathrm{Diag}(\mathrm{SO}(4, 4n))$	$\mathrm{SO}(3, 4n) \times \mathrm{Sp}(1, n)$
(ix)	$\mathrm{U}(2, 2n) \times \mathrm{U}(2, 2n)$	$\mathrm{Diag}(\mathrm{U}(2, 2n))$	$\mathrm{U}(1, 2n) \times \mathrm{Sp}(1, n)$
(x)	$\mathrm{SO}(8, 8) \times \mathrm{SO}(8, 8)$	$\mathrm{Diag}(\mathrm{SO}(8, 8))$	$\mathrm{SO}(7, 8) \times \mathrm{Spin}(1, 8)$
(xi)	$\mathrm{SO}(4, 4) \times \mathrm{SO}(4, 4)$	$\mathrm{Diag}(\mathrm{SO}(4, 4))$	$\mathrm{SO}(4, 1) \times \mathrm{Spin}(4, 3)$
(xii)	$\mathrm{SO}(4, 3) \times \mathrm{SO}(4, 3)$	$\mathrm{Diag}(\mathrm{SO}(4, 3))$	$\mathrm{SO}(4, 1) \times \mathrm{G}_{2(2)}$

(xiii)	$\mathrm{SO}^*(8) \times \mathrm{SO}^*(8)$	$\mathrm{Diag}(\mathrm{SO}^*(8))$	$\mathrm{U}(3,1) \times \mathrm{Spin}(1,6)$
(xiv)	$\mathrm{SO}^*(8) \times \mathrm{SO}^*(8)$	$\mathrm{Diag}(\mathrm{SO}^*(8))$	$(\mathrm{SO}^*(6) \times \mathrm{SO}^*(2)) \times \mathrm{Spin}(1,6)$

Note that in Examples (vii) to (xiv), which are of the form $(G, H, L) = (G_0 \times G_0, \mathrm{Diag}(G_0), H_0 \times L_0)$, if Γ_{H_0} (resp. Γ_{L_0}) is a uniform lattice of H_0 (resp. of L_0), then the compact Clifford–Klein form $(\Gamma_{H_0} \times \Gamma_{L_0})\backslash G/H$ identifies with $\Gamma_{L_0}\backslash G_0/\Gamma_{H_0}$ and is locally modelled on G_0. In Examples (ii), (vii), (xi) and (xii), small nonstandard deformations of standard compact Clifford–Klein forms of X can be obtained using a *bending construction* due to Johnson and Millson (see [5]). In Example (vi), small nonstandard deformations also exist for $G_0 = \mathrm{SO}(1, 2n)$ or $\mathrm{SU}(1, n)$ (see [10]).

An infinite subset of the stable discrete spectrum for standard compact Clifford–Klein forms of X may be found explicitly for $X = \mathrm{SO}(2, 4)/\mathrm{U}(1, 2)$ and for the 3-dimensional *anti-de Sitter space* $X = \mathrm{AdS}^3 = \mathrm{SO}(2, 2)/\mathrm{SO}(1, 2)$. By [7] and [13], the 3-dimensional compact anti-de Sitter manifolds (*i.e.* the 3-dimensional compact Lorentz manifolds with constant negative curvature) are the compact Clifford–Klein forms of AdS^3, up to finite covering, isometry, and renormalization of the metric. Using [6], we prove that Theorems 2.1 and 2.2 actually hold true for *all* these compact Clifford–Klein forms, not only standard ones.

Theorem 2.3 *The discrete spectrum of any 3-dimensional compact anti-de Sitter manifold M is infinite. Explicitly, there is an integer n_0 such that the discrete spectrum of the Laplacian Δ_M on M satisfies*

$$\mathrm{Spec}_d(\Delta_M) \supset \Big\{ \frac{1}{2} n(n+1) : \; n \in \mathbb{N}, \, n \geq n_0 \Big\}, \tag{2}$$

and (2) *still holds after a small deformation of the anti-de Sitter structure of M.*

Here we are using the normalization of the Lorentz metric by the Killing form. Theorem 2.3 holds more generally for any 3-dimensional anti-de Sitter manifold M satisfying some convex cocompactness property. We note that here $\mathbb{D}(X)$ is generated by the Laplacian, so that $\mathrm{Spec}_d(M)$ identifies with $\mathrm{Spec}_d(\Delta_M) \subset \mathbb{C}$. Since M is a Lorentz manifold, Δ_M is a hyperbolic operator. We may compare (2) with the following easy computation:

$$\mathrm{Spec}_d\big(\mathbb{P}^3(\mathbb{R}), \Delta_{\mathbb{P}^3(\mathbb{R})}\big) = \Big\{ -\frac{1}{2} n(n+1) : \; n \in \mathbb{N} \Big\}.$$

Theorems 2.1, 2.2, and 2.3 follow from a more general result that we prove for triples (G, H, Γ) that satisfy (1) and two conditions on the image of Γ by some Cartan projection of G (see Definition 3.3).

3. Ideas of proofs

For simplicity we assume that G is a real form of a simply connected complex Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let (H^d, G^d, K^d) be the dual triple of (K, G, H), *i.e.* the triple of connected reductive Lie groups with the same complexified Lie algebras and such that G^d/K^d is a Riemannian symmetric space. There is an injective homomorphism [2] from the set $\mathcal{A}_K(X)$ of K-finite analytic functions on $X = G/H$ into the set $\mathcal{A}_{H^d}(G^d/K^d)$ of H^d-finite analytic functions on G^d/K^d. For $\lambda \in 2\rho_c - \rho + \Lambda_+$, Flensted-Jensen [2] introduced the function $\psi_\lambda \in \mathcal{A}_K(X)$ whose image in $\mathcal{A}_{H^d}(G^d/K^d)$ is given by

$$g^d K^d \longmapsto \int_{K^d \cap H^d} e^{-\langle \lambda + \rho, \zeta((g^d)^{-1} \ell) \rangle} \, d\ell,$$

where $G^d = K^d A^d N^d$ is an Iwasawa decomposition of G^d with $A^d = \exp \mathfrak{j}$, and $\zeta : G^d \to \mathfrak{j}$ is given by $g^d \in K^d e^{\zeta(g^d)} N^d$ for all $g^d \in G^d$. Assuming (1), he proved that $\psi_\lambda \in L^2(X, \mathcal{M}_\lambda)$ for $d(\lambda)$ large enough. To establish Theorems 2.1, 2.2, and 2.3 we prove the following, where $\overline{x}$ is the image of $x \in X$ in $X_{\varphi(\Gamma)}$.

Proposition 3.1 *In the setting of Theorems 2.1 or 2.3, there is a constant $R > 0$ and a neighborhood $\mathcal{U} \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion such that for all $\lambda \in 2\rho_c - \rho + \Lambda_+$ with $d(\lambda) \geq R$ and all $\varphi \in \mathcal{U}$, the eigenfunction*

$$\psi_\lambda^{\varphi(\Gamma)} \ : \ \overline{x} \longmapsto \sum_{\gamma\in\Gamma} \psi_\lambda(\varphi(\gamma)\cdot x)$$

on $X_{\varphi(\Gamma)}$ is well-defined, nonzero, L^p for all $1 \leq p \leq \infty$, and of class C^m whenever $d(\lambda) \geq (m+1)R$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition associated with θ and let $\mathfrak{b}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. Fix a system $\Sigma^+(\mathfrak{g}^{\sigma\theta}, \mathfrak{b})$ of positive restricted roots and let $\overline{\mathfrak{b}_+} \subset \mathfrak{b}$ be the corresponding closed positive Weyl chamber, so that the decomposition $G = K \exp(\overline{\mathfrak{b}_+}) H$ holds. We define a map $\nu : G \to \overline{\mathfrak{b}_+}$ by $g \in K e^{\nu(g)} H$ for all $g \in G$. Proposition 3.1 relies on the following uniform asymptotic estimates for ψ_λ, which we establish by building on the work of Flensted-Jensen [2] and Matsuki–Oshima [14]. Here $\|\cdot\|$ denotes any fixed norm on $\mathfrak{b}$.

Lemma 3.2 *Under the rank assumption (1), there is a constant $\varepsilon > 0$ such that for any $\lambda \in 2\rho_c - \rho + \Lambda_+$,*

(i) $\psi_\lambda(eH) = 1$ and $|\psi_\lambda(gH)| \leq \cosh(\varepsilon\|\nu(g)\|)^{-d(\lambda+\rho)}$ for all $g \in G$,

(ii) for any $D \in \mathbb{D}(X)$, the function $g \mapsto D\psi_\lambda(gH)\, e^{\varepsilon d(\lambda+\rho)\|\nu(g)\|}$ is bounded on G.

To deduce Proposition 3.1 from Lemma 3.2, we consider a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ containing $\mathfrak{b}$. We fix a system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ of positive restricted roots and let $\overline{\mathfrak{a}_+} \subset \mathfrak{a}$ be the corresponding closed positive Weyl chamber, so that the Cartan decomposition $G = K \exp(\overline{\mathfrak{a}_+}) K$ holds. By the *properness criterion* of Benoist [1] and Kobayashi [9], the Cartan projection $\mu : G \to \overline{\mathfrak{a}_+}$, defined by $g \in K e^{\mu(g)} K$ for all $g \in G$, controls the properness of the action of any closed subgroup of G on G/H. We introduce the following two conditions, where $\|\cdot\|$ denotes any norm on $\mathfrak{a}$ extending that of $\mathfrak{b}$, inducing a distance $\mathrm{dist}_\mathfrak{a}$ on $\mathfrak{a}$, and $\ell_F : \Gamma \to \mathbb{N}$ is the word length with respect to F.

Definition 3.3 *Let $c, C > 0$. A subgroup Γ of G with finite generating subset F is said to satisfy*

- *the* angle condition *with constants (c, C) if $\mathrm{dist}_\mathfrak{a}(\mu(\gamma), \mu(H)) \geq c\,\|\mu(\gamma)\| - C$ for all $\gamma \in \Gamma$,*
- *the* QI condition *with constants (c, C) if $\|\mu(\gamma)\| \geq c\,\ell_F(\gamma) - C$ for all $\gamma \in \Gamma$.*

In the setting of Theorems 2.1 or 2.3, there are constants $c, C > 0$ and a neighborhood $\mathcal{U} \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion such that for all $\varphi \in \mathcal{U}$, the group $\varphi(\Gamma)$ satisfies both the angle condition with constants (c, C) (by [5] and [6]) and the QI condition with constants (c, C) (as was first proved by Guichard [3]). Proposition 3.1 follows from this, together with Lemma 3.2 and the following inequality.

Lemma 3.4 *There is a constant $C_0 > 0$ such that $\|\nu(g)\| \geq C_0\, \mathrm{dist}_\mathfrak{a}(\mu(g), \mu(H))$ for all $g \in G$.*

Detailed proofs will appear elsewhere.

References

[1] Y. Benoist, *Actions propres sur les espaces homogènes réductifs*, Ann. Math. 144 (1996), p. 315–347.

[2] M. Flensted-Jensen, *Discrete series for semisimple symmetric spaces*, Ann. Math. 111 (1980), p. 253–311.

[3] O. Guichard, *Groupes plongés quasi-isométriquement dans un groupe de Lie*, Math. Ann. 330 (2004), p. 331–351.

[4] S. Helgason, *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions*, Mathematical Surveys and Monographs 83, American Mathematical Society, Providence, RI, 2000.

[5] F. Kassel, *Deformation of proper actions on reductive homogeneous spaces*, arXiv:0911.4247.

[6] F. Kassel, *Quotients compacts d'espaces homogènes réels ou p-adiques*, PhD thesis, Université Paris-Sud 11, November 2009, see `http://www.math.u-psud.fr/~kassel/`.

[7] B. Klingler, *Complétude des variétés lorentziennes à courbure constante*, Math. Ann. 306 (1996), p. 353–370.

[8] T. Kobayashi, *Proper action on a homogeneous space of reductive type*, Math. Ann. 285 (1989), p. 249–263.

[9] T. Kobayashi, *Criterion for proper actions on homogeneous spaces of reductive groups*, J. Lie Theory 6 (1996), p. 147–163.

[10] T. Kobayashi, *Deformation of compact Clifford–Klein forms of indefinite-Riemannian homogeneous manifolds*, Math. Ann. 310 (1998), p. 394–408.

[11] T. Kobayashi, *Hidden symmetries and spectrum of the Laplacian on an indefinite Riemannian manifold*, in *Spectral analysis in geometry and number theory*, p. 73–87, Contemp. Math. 484, Amer. Math. Soc., Providence, RI, 2009.

[12] T. Kobayashi, T. Yoshino, *Compact Clifford–Klein forms of symmetric spaces — revisited*, Pure and Applied Mathematics Quaterly 1 (2005), p. 591–653.

[13] R. S. Kulkarni, F. Raymond, *3-dimensional Lorentz space-forms and Seifert fiber spaces*, J. Differential Geom. 21 (1985), p. 231–268.

[14] T. Matsuki, T. Oshima, *A description of discrete series for semisimple symmetric spaces*, in *Group representations and systems of differential equations*, p. 331–390, Adv. Stud. Pure Math. 4, North-Holland, Amsterdam, 1984.