The Restriction of $A_{\mathfrak{q}}(\lambda)$ to Reductive Subgroups II

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§ 1. **Introduction.** In this paper we continue the investigation of the restriction of irreducible unitary representations of real reductive groups, with emphasis on the discrete decomposability. We recall that a representation π of a reductive Lie group G on a Hilbert space V is G-*admissible* if (π, V) is decomposed into a discrete Hilbert direct sum with finite multiplicities of irreducible representations of G. The same terminology is used for a $(\mathfrak{g}, K)$-module on a pre-Hilbert space, if its completion is G-admissible.

Let H be a reductive subgroup of a real reductive Lie group G, and (π, V) an irreducible unitary representation of G. The restriction $(\pi_{|H}, V)$ is decomposed uniquely into irreducible unitary representations of H, which may involve a continuous spectrum if H is noncompact. In [5],[6], we have posed a problem to single out the triplet (G, H, π) such that the restriction of $(\pi_{|H}, V)$ is H-admissible, together with some application to harmonic analysis on homogeneous spaces. The purpose of this paper is to give a new insight of such a triplet (G, H, π) from view points of algebraic analysis. In particular, we will give a sufficient condition on the triplet (G, H, π) for the H-admissible restriction as a generalization of [5],[6] to arbitrary H, and also present an obstruction for the H-admissible restriction.

§ 2. **A sufficient condition for discrete decomposability.** Let K be a compact Lie group. We write $\mathfrak{k}_0$ for the Lie algebra of K, and $\mathfrak{k}$ for its complexification. Analogous notation is used for other groups. Take a Cartan subalgebra $\mathfrak{t}_0^c$ of $\mathfrak{k}_0$. The weight lattice L in $\sqrt{-1}(\mathfrak{t}_0^c)^*$ is the additive subgroup of $\sqrt{-1}(\mathfrak{t}_0^c)^*$ consisting of differentials of the weights of finite dimensional representations of K. Let $\bar{C} \subset \sqrt{-1}(\mathfrak{t}_0^c)^*$ be a dominant Weyl chamber. We write K_0 for the identity component of K, and $\widehat{K_0}$ for the unitary dual of K_0. The Cartan-Weyl theory of finite dimensional representations establishes a bijection:

$$L \cap \bar{C} \xrightarrow{\sim} \widehat{K_0},\ \lambda \mapsto F(K_0, \lambda).$$

Suppose X is a K-module (possibly, of infinite dimension) which carries an algebraic action of K. The K_0-multiplicity function of X is given by

$$m \equiv m_X : L \cap \bar{C} \to \boldsymbol{N} \cup \infty,$$
$$m(\lambda) := \dim \operatorname{Hom}_{K_0}(F(K_0, \lambda), X).$$

The asymptotic K-support $T(X) \subset \bar{C}$ was introduced in [3] as follows:

$S(X) := \{\lambda \in L \cap \bar{C} : m_X(\lambda) \neq 0\}$,

$T(X) := \{\lambda \in \bar{C} : V \cap S(X)$ is not relatively compact for any open cone V containing $\lambda\}$.

Hereafter we assume a growth condition on m_X: there are constants $A, R > 0$ such that

(2.1) $\quad m_X(\lambda) \leq A \exp(R|\lambda|)$ for any $\lambda \in L \cap \bar{C}$.

This condition assures that the character of the representation X is a hyperfunction on K, whose singularity spectrum we can estimate in terms of $T(X)$.

Suppose H is a closed subgroup of K. Let $\mathrm{pr}_{K\to H} : \mathfrak{k}^* \to \mathfrak{h}^*$ be the projection dual to the inclusion of Lie algebras $\mathfrak{h} \hookrightarrow \mathfrak{k}$. Put $\mathfrak{h}^{\perp} := \operatorname{Ker}(\mathrm{pr}_{K\to H} : \mathfrak{k}^* \to \mathfrak{h}^*)$. We set

(2.2) $\quad \bar{C}(\mathfrak{h}) := \bar{C} \cap \mathrm{Ad}^*(K)\mathfrak{h}^{\perp} \subset \sqrt{-1}(\mathfrak{t}_0^c)^*$.

Note that $\bar{C}(\mathfrak{k}) = \{0\}$ and $\bar{C}(0) = \bar{C}$.

Theorem 2.3. *Let X be a K-module satisfying* (2.1). *If a closed subgroup H of K satisfies*

$$T(X) \cap \bar{C}(\mathfrak{h}) = \{0\},$$

then the restriction $X_{|H}$ is H-admissible.

Now, let us apply Theorem (2.3) to some standard $(\mathfrak{g}, K)$-modules. Suppose that G is a real reductive linear Lie group and that K is a maximal compact subgroup of G. A dominant element $a \in \sqrt{-1}\,\mathfrak{t}_0^c$ defines a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, where $\mathfrak{l}$, $\mathfrak{u}$ are the sum of eigenspaces of $\mathrm{ad}(a)$ with 0, positive eigenvalues, respectively. Let L be the centralizer of a in G. Zuckerman introduced the cohomological parabolic induction $\mathscr{R}_{\mathfrak{q}}^j \equiv (\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^j$ $(j \in \boldsymbol{N})$, which is a covariant functor from the category of metaplectic $(\mathfrak{l}, (L \cap K)^{\sim})$-modules to that of $(\mathfrak{g}, K)$-modules, as a generalization of the Borel-Weil-Bott construction of finite dimensional representations of compact groups. In particular, we write $A_{\mathfrak{q}}(\lambda) := (\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^S(C_\lambda)$ for a metaplectic unitary character C_λ in the good range of parameter (see [8] Definition 2.5), where $S := \dim_{\boldsymbol{C}}(\mathfrak{u} \cap \mathfrak{k})$. Then $A_{\mathfrak{q}}(\lambda)$ is an irreducible unitarizable $(\mathfrak{g}, K)$-module (see [7] Theorem 6.8), and we write $\overline{A_{\mathfrak{q}}(\lambda)} \in \hat{G}$ for its completion.

The K-module structure of the alternating sum $\sum_j (-1)^j (\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^j (W)_{|K}$ is known as a generalized Blattner formula (see [7] Theorem 6.34). Its proof also gives an upper estimate of each term $(\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^j (W)_{|K}$, which leads us to:

Theorem 2.4. *If W is a finite dimensional metaplectic $(\mathfrak{l}, (L \cap K)^\sim)$-module, then the restriction $(\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^j (W)_{|K}$ satisfies* (2.1) and

$$T((\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^j (W)_{|K}) \subset \boldsymbol{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle \cap \bar{C} \ (j \in \boldsymbol{N}).$$

Here, we recall $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ (Levi decomposition) and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (Cartan decomposition), and we define a closed cone by

$$\boldsymbol{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle := \{ \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p}, \mathfrak{t}^c)} n_\beta \beta : n_\beta \geq 0 \} \subset \sqrt{-1}(\mathfrak{t}_0^c)^*.$$

Corollary 2.5. *In the setting of Theorem* (2.3), *if*

$$\bar{C}(\mathfrak{h}) \cap \boldsymbol{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle = \{0\},$$

then the restriction $(\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^j (W)_{|H}$ is H-admissible for any finite dimensional metaplectic $(\mathfrak{l}, (L \cap K)^\sim)$-module W and for any $j \in \boldsymbol{N}$. In particular, $\overline{A_{\mathfrak{q}}(\lambda)}_{|H}$ is decomposed discretely into irreducible unitary representations of H.

As a special case of Corollary (2.5), we obtain a new and unified proof of some of the main results in [5],[6], where we imposed some assumptions on a subgroup H.

First, suppose that $H \subset K$ is a symmetric pair defined by an involution $\sigma \in \mathrm{Aut}(K)$. Take a maximal abelian subspace $\mathfrak{a}_0$ in $\{Y \in \mathfrak{k}_0 : \sigma(Y) = -Y\}$ and extend it to a Cartan subalgebra $\mathfrak{t}_0^c$ of $\mathfrak{k}_0$. We take a dominant Weyl chamber $\bar{C}$ so that $\sqrt{-1}\,\mathfrak{a}_0^* \cap \bar{C}$ is also a dominant Weyl chamber for the restricted root system $\Sigma(\mathfrak{k}, \mathfrak{a})$.

Corollary 2.6 (cf. [5] Theorem 1.2; [6] Theorem 3.2). *Retain the above setting. If*

$$\sqrt{-1}\mathfrak{a}_0^* \cap \bar{C} \cap \boldsymbol{R}_+ \langle \mathfrak{u} \cap \mathfrak{p} \rangle = \{0\},$$

then the restriction $(\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^j (W)_{|H}$ $(j \in \boldsymbol{N})$ is H-admissible for any finite dimensional metaplectic $(\mathfrak{l}, (L \cap K)^\sim)$-module W.

Next, suppose K is (locally) isomorphic to a direct product $K_1 \times K_2$. We note that the Cartan subalgebra $\mathfrak{t}_0^c$ is also decomposed into a direct sum $\mathfrak{t}_0^c = (\mathfrak{t}_1^c)_0 + (\mathfrak{t}_2^c)_0$.

Corollary 2.7 (cf. [6] Corollary 4.4; [4] Proposition 4.1.3). *In the setting as above, if a θ-stable parabolic subalgebra $\mathfrak{q}$ is given by $a \in \sqrt{-1}(\mathfrak{t}_1^c)_0$, then the restriction $(\mathscr{R}_{\mathfrak{q}}^{\mathfrak{g}})^j (W)_{|K_1} (j \in \boldsymbol{N})$ is K_1-admissible for any finite dimensional metaplectic $(\mathfrak{l}, (L \cap K)^\sim)$-module W.*

We note that Corollaries (2.6),(2.7) are deduced from Corollary (2.5) by using $\bar{C}(\mathfrak{h}) = \sqrt{-1}\,\mathfrak{a}_0^* \cap \bar{C}$, $\bar{C}(\mathfrak{k}_1) = \sqrt{-1}(\mathfrak{t}_2^c)_0^* \cap \bar{C}$, respectively.

Remark 2.8. The above corollaries (2.4),(2.5),(2.6) are valid if we replace H by any reductive subgroup H' containing H, because of Corollary (1.3) in [6].

§ 3. A necessary condition for discrete decomposability. In § 2, we have given a sufficient condition that the restriction of a $(\mathfrak{g}, K)$-module X has an H-admissible restriction with respect to a subgroup H. Conversely, we will find a necessary condition in terms of associated varieties of $\mathfrak{g}$-modules in this section.

We recall that the associated variety of a $(\mathfrak{g}, K)$-module X of finite length is defined by

$$\mathscr{V}(X) \equiv \mathscr{V}_G(X) = \mathrm{Supp}_{S(\mathfrak{g})}(\mathrm{gr}(X)) \subset \mathfrak{g}^*,$$

as the support in $\mathfrak{g}^*$ of the associated graded module $\mathrm{gr}(X)$ over the symmetric algebra $S(\mathfrak{g})$, with respect to a good filtration (see [1]). It is known that $\mathscr{V}(X)$ is a subset of the nilpotent cone $\mathscr{N}^* \equiv \mathscr{N}^*(\mathfrak{g}) \subset \mathfrak{g}^*$.

Let H be a closed subgroup that is reductive in G. We fix a Cartan involution θ of G which makes H stable so that $H \cap K$ is a maximal compact subgroup of H. Write the projection $\mathrm{pr}_{G \to H} : \mathfrak{g}^* \to \mathfrak{h}^*$ as before.

Theorem 3.1. *Suppose X is a $(\mathfrak{g}, K)$-module of finite length. Assume that the restriction $X_{|H}$ is H-admissible. Let Y be any $(\mathfrak{h}, H \cap K)$-module occurring as a direct summand of X. Then we have*

$$\mathrm{pr}_{G \to H}(\mathscr{V}_G(X)) \subset \mathscr{V}_H(Y).$$

This theorem gives rise to an obstruction for the admissibility of the restriction of a unitary representation.

Corollary 3.2. *Suppose X is a $(\mathfrak{g}, K)$-module of finite length. Assume that the restriction $X_{|H}$ is H-admissible. Then*

$$\mathrm{pr}_{G \to H}(\mathscr{V}_G(X)) \subset \mathscr{N}^*(\mathfrak{h}).$$

Applying Corollary (3.2) to $X = A_{\mathfrak{q}}(\lambda)$, we have:

Corollary 3.3. *Let us identify* $\mathfrak{g}^*$ *with* $\mathfrak{g}$ *via the Killing form. Assume a* θ*-stable parabolic subalgebra* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ *of* $\mathfrak{g}$ *satisfies*

$$\mathrm{pr}_{G \to H}(\mathrm{Ad}(K_C)(\mathfrak{u} \cap \mathfrak{p})) \not\subset \mathcal{N}^*(\mathfrak{h}).$$

Then the restriction of $A_{\mathfrak{q}}(\lambda) \in \hat{G}$ *to* H *is not* H*-admissible.*

Remark 3.4. If $H = K$, then the assumption of Theorem (3.1) is always satisfied. In this special case, Theorem (3.1) implies a well-known result $\mathrm{pr}_{G \to K}(\mathcal{V}_G(X)) = \{0\}$ (see [9] Corollary 5.13) because the associated variety of a finite dimensional representation is zero. In a general case where H is non-compact, $\mathrm{pr}_{G \to H}(\mathcal{V}_G(X))$ is not necessarily $\{0\}$.

Finally, we mention a useful information about $\hat{H}$ occurring as direct summands of the restriction $X_{|H}$, as an elementary application of associated varieties. This helps us to understand a strange phenomenon about the direct summands occurring in the restriction of $A_{\mathfrak{q}}(\lambda)_{|H}$, which was pointed out in [6] Introduction.

Theorem 3.5. *Suppose* X *is an irreducible* $(\mathfrak{g}, K)$*-module. Assume that* X *is* H*-admissible. Let* Y_1, Y_2 *be any* irreducible $(\mathfrak{h}, H \cap K)$*-module occurring as a direct summand of the restriction* X *to* $(\mathfrak{h}, H \cap K)$*. Then we have*

$$\mathcal{V}_H(Y_1) = \mathcal{V}_H(Y_2).$$

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