PROPER ACTION ON A HOMOGENEOUS SPACE OF REDUCTIVE TYPE

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Abstract.

An action of L on a homogeneous space G/H is investigated where $L, H \subset G$ are reductive Lie groups.

A criterion of the properness of this action is obtained in terms of the little Weyl group of G. In particular, $\mathbb{R}$-rank$G$ = $\mathbb{R}$-rank$H$ iff Calabi-Markus phenomenon occurs, i.e. only finite subgroups of G can act properly discontinuously on G/H. Then by using cohomological dimension theory of a discrete group, $L\backslash G/H$ is proved compact iff $\mathrm{d}(G) = \mathrm{d}(L) + \mathrm{d}(H)$, where $\mathrm{d}(G)$ denotes the dimension of a Riemannian symmetric space associated with G, etc.

These results apply to the existence problem of lattice in G/H. Several series of classical pseudo-Riemannian homogeneous spaces are found to admit non-uniform lattice as well as uniform lattice, while some necessary condition for the existence of uniform lattice is obtained when rank$G$ = rank$H$.

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1. Introduction

Let H be a reductive subgroup in a real reductive linear group G. The purpose of this paper is to study a properly discontinuous action on a homogeneous space G/H of reductive type.

If H is compact, it is a famous result due to Borel and Harish-Chandra that G/H admits a uniform lattice (resp. nonuniform lattice), i.e. there is a discrete subgroup Γ in G acting properly discontinuously and freely on G/H so that $\Gamma\backslash G/H$ is compact (resp. noncompact but of finite volume) ([Bo], [B-H]). Now these theorems have been a foundation of abundant theory such as Eisenstein series in harmonic analysis on $L^2(\Gamma\backslash G/H)$.

However, unless H is compact, the action of a discrete group on G/H is *not* automatically properly discontinuous. In fact it sometimes occurs that only finite subgroup in G can act properly discontinuously on G/H. Calabi and Markus first found $SO(n+1,1)/SO(n,1)$ is such a case. Now some sufficient conditions on these *'Calabi-Markus phenomena'* have been obtained in a general case with a necessary condition in a very special case (see [C-M], [Wo], [Ku], [Wal]).

To study a properly discontinuous action on a reductive homogeneous space G/H, we will take the following approach (cf. Lemma(2.3)): *Find a reductive subgroup G' acting properly on G/H so that any discrete subgroup Γ of G' acts automatically properly discontinuously on G/H.* This approach was first partially carried out by R.S.Kulkarni ([Ku]), where he found that properties of the groups which can act properly on
$SO(p+1,q)/SO(p,q)$ dramatically depend on the conditions of p,q by making a detailed study of a quadratic form of type $(p+1,q)$.

The main results of this paper will be stated in §4. We introduce it briefly: Let G/H be a homogeneous space of reductive type (Definition(2.6)). First, we give a simple criterion to tell whether the action of a reductive subgroup G' on G/H is proper or not (Theorem(4.1)). As a corollary we show that Calabi-Markus phenomenon occurs in G/H if and only if $\mathbb{R}$-rank $G = \mathbb{R}$-rank H (Corollary(4.4)). Secondly, we obtain the necessary and sufficient condition that $G'\backslash G/H$ is compact under the assumption that the G'-action on G/H is proper (Theorem(4.7)). As a corollary of its proof, a certain necessary condition for the existence of uniform lattice is obtained when H has maximal rank in G (Proposition(4.10)). Finally, using these criteria in Theorem(4.1),(4.7), we find six series of non-Riemannian reductive homogeneous spaces which admit uniform lattices as well as non-uniform lattices (Proposition(4.9)).

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2. Notation and Preliminary Results

Let G be a topological group acting continuously on a topological space X; i.e. there is a continuous map $f : G \times X \to X$ which gives a homomorphisms of G into the group of homeomorphism of X. G is said to act *freely* iff $f(g, x) = x$ implies $g = e$ for every $x \in X$; *properly* iff $\{g \in G : f(g, S) \cap S \neq \emptyset\}$ is compact for every compact subset S in X; *properly discontinuously* iff G is discrete and acts properly on X. We shall often write $g \cdot x$ instead of $f(g, x)$.

When a discrete group Γ acts smoothly on a manifold X, $\Gamma\backslash X$ is a V-manifold in the sense of [Sa] if the action is properly discontinuously; $\Gamma\backslash X$ is a manifold if it is properly discontinuously and freely. The following lemma is well-known and elementary:

Lemma(2.1). *Let Γ be a group acting properly discontinuously on a locally compact Hausdorff space X. Then*
1) *$\Gamma\backslash X$ is also a locally compact Hausdorff space.*
2) *If $\Gamma\backslash X$ is compact, then Γ is finitely generated.*

In fact, $\{\gamma \in \Gamma : \gamma \cdot U \cap U \neq \emptyset\}$ (U is a relatively compact open set in X such that $X = \Gamma \cdot U$) gives a finite generator of Γ in (2).

When X is a homogeneous space G/H where H is a closed subgroup of G, our concern will be mainly restricted to the action of a subgroup of G on $X = G/H$ via the natural left action.

Definition(2.2). Let Γ be a discrete subgroup of G. Γ is called a *uniform lattice* in G/H iff Γ acts on G/H freely and properly discontinuously so that $\Gamma\backslash G/H$ is compact. When G/H carries a G-invariant measure, Γ is called a *lattice* in G/H iff Γ acts on G/H freely and properly discontinuously so that $\Gamma\backslash G/H$ is of finite volume; a *non-uniform lattice* in G/H iff Γ is a lattice but not a uniform lattice. Obviously these terminologies are consistent with the usual ones when $H = \{e\}$.

Our approach of a properly discontinuous action is based on the following simple observation:

Lemma(2.3). *Let a real Lie group G act on a locally compact space X and Γ be a uniform lattice in G. Then*
1) *The G-action on X is proper iff the Γ-action is properly discontinuous.*
2) *$G\backslash X$ is compact iff $\Gamma\backslash X$ is compact.*

Proof. 1) Suppose Γ acts properly discontinuously. Take a compact subset $C = C^{-1}$ in G so that $G = C \cdot \Gamma$. Then for any compact subset S in X, $\{g \in G : g \cdot S \cap S \neq \emptyset\} \subset C \cdot \Gamma_{CS}$, where $\Gamma_{CS} := \{\gamma \in \Gamma : \gamma(C \cdot S) \cap (C \cdot S) \neq \emptyset\}$ is a finite set. Thus

the G-action is proper. The 'only if' part is nothing but the definition.
2) Suppose $G\backslash X$ is compact. We can choose finite relatively open sets U_j in X so that $X = \bigcup_j G \cdot U_j = \bigcup_j \Gamma \cdot (C \cdot U_j)$, showing $\Gamma\backslash X$ is compact. The converse statement is clear. □

Suppose G is a real Lie group. We denote by G_o the identity component of G, by $\mathfrak{g}$ the Lie algebra of G, by $\mathrm{Ad} : G \to GL(\mathfrak{g})$ the adjoint representation of G, by $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ the adjoint representation of $\mathfrak{g}$. This notation will be applied to groups denoted by other Roman letters in the same way without comment. For $S \subset G$ and $\mathfrak{t} \subset \mathfrak{g}$, $N_S(\mathfrak{t})$, $Z_S(\mathfrak{t})$ denote the normalizer, centralizer of $\mathfrak{t}$ in S respectively. Similar notations will be used when S is a subset in $\mathfrak{g}$.

Definition(2.4). By a *real reductive linear group*, we will mean a real Lie group G (not necessarily connected), contained in a connected complex reductive Lie group $G_{\mathbb{C}}$ whose Lie algebra $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{g} \underset{\mathbb{R}}{\otimes} \mathbb{C}$.

If G is a real reductive linear group, there is a Cartan involution θ of G such that the mapping
$$K \times \mathfrak{p} \to G,\ (k, X) \mapsto k \cdot \exp(X)$$
gives a surjective diffeomorphism. Here $K \overset{\text{def}}{=} \{g \in G : \theta g = g\}$ is a maximal compact subgroup of G and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the $+1$ and -1 eigenspace decomposition for θ (we denote by the same letter the differential of θ as usual). The ambiguity of the choice of a Cartan involution $\theta_{|[G,G]}$ is just by inner automorphisms of G (cf. [He] Ch.VI Theorem 2.1). We set

$$\mathrm{d}(G) := \dim(G/K) = \dim_{\mathbb{R}} \mathfrak{p} \tag{2.5}$$

Clearly this definition is independent of the choice of a Cartan involution.

We will use standard results concerning real reductive linear groups as needed when these are trivial consequences of the corresponding ones for connected cases, although the references may treat only connected semisimple Lie groups.

Definition-Lemma(2.6). Let H be a closed subgroup in a real reductive linear group G (Definition(2.4)). We call H *reductive in* G and G/H *a homogeneous space of reductive type* when the following equivalent conditions are satisfied:
(2.6.1) There is a Cartan involution θ of G such that H has a polar decomposition $H = (H \cap K) \cdot \exp(\mathfrak{h} \cap \mathfrak{p})$.
(2.6.2) H is of finitely many connected components and the adjoint representation $\mathrm{ad}|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ is completely reducible.

It is easy to see that H is a real reductive linear group in the sense of Definition(2.4) if H is reductive in G. Our definition of reductive subgroups are slightly stronger than the usual one by excluding the case H has infinitely many connected components.

Let H be reductive in G. Retain notations as in (2.6.1). Fix a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, which we denote by $\langle\ ,\ \rangle$. We can and do choose this form so that the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is orthogonal for $\langle\ ,\ \rangle$, and $\langle\ ,\ \rangle$ is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. This form will be restricted to $\mathfrak{p}$ without change of notation. Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Then we have

Lemma(2.7). *Suppose G/H is a homogeneous space of reductive type. Then we have*

$$G = K \exp(\mathfrak{q} \cap \mathfrak{p}) H.$$

Furthermore, G/H is diffeomorphic to the fiber bundle

$$(K/H \cap K) \underset{\mathrm{Ad}_{|H\cap K}}{\times} (\mathfrak{q} \cap \mathfrak{p}),$$

with the base $K/H \cap K$ and fibers $\mathfrak{q} \cap \mathfrak{p}$, by the map

$$f : K \times (\mathfrak{q} \cap \mathfrak{p}) \ni (k, Z) \mapsto k \exp(Z) H \in G/H.$$

The above lemma is well-known at least when G/H is a semisimple symmetric space. But we shall give a proof in Appendix for the sake of completeness.

Let $\mathfrak{a}$ be any maximal abelian subspace in $\mathfrak{p}$. All such subspaces are mutually conjugate by an element of K_o. The dimension $\dim \mathfrak{a}$ is called the *real rank* of G, denoted by $\mathbb{R}$-rank G. A subspace in $\mathfrak{g}$ conjugate to $\mathfrak{a}$ in G is called a *maximally split abelian subspace*. Let A be the connected group with Lie algebra $\mathfrak{a}$, M' (resp. M) the normalizer (resp. centralizer) of $\mathfrak{a}$ in K. Let

$$\mathfrak{g}(\mathfrak{a}; \alpha) \overset{\text{def}}{=} \{X \in \mathfrak{g}; [H, X] = \alpha(H)X \ \text{ for any } \ H \in \mathfrak{a}\}$$

for $\alpha \in \mathfrak{a}^*$, and

$$\Sigma \equiv \Sigma(\mathfrak{g}, \mathfrak{a}) \overset{\text{def}}{=} \{\alpha \in \mathfrak{a}^*; \mathfrak{g}(\mathfrak{a}; \alpha) \neq 0\} \setminus \{0\}$$

be the restricted root system for $(\mathfrak{g}, \mathfrak{a})$. Let $W \equiv W(\mathfrak{g}; \mathfrak{a}) \simeq M'/M$ be the corresponding Weyl group of Σ (often called the little Weyl group). The simultaneous diagonalization of $\mathrm{ad}(\mathfrak{a})|_{\mathfrak{g}}$ gives the decomposition

$$\mathfrak{g} = \mathfrak{g}(\mathfrak{a}; 0) + \sum_{\alpha \in \Sigma} \mathfrak{g}(\mathfrak{a}; \alpha),$$
$$\mathfrak{g}(\mathfrak{a}; 0) = \mathfrak{m} + \mathfrak{a}.$$

Let $p_\alpha : \mathfrak{g} \to \mathfrak{g}(\mathfrak{a}; \alpha)$ be the corresponding projection for each element $\alpha \in \Sigma \cup \{0\}$. The subset $\mathfrak{a}' = \{H \in \mathfrak{a} : \alpha(H) \neq 0 \text{ for any } \alpha \in \Sigma\}$ of regular elements consists of the complement of finitely many hyperplanes, and (the closures of) its components are called (closed) Weyl chambers.

For $Y \in \mathfrak{a}$, $t \in \mathbb{R}$, put

$$\Xi(Y; t) \overset{\text{def}}{=} \{\alpha \in \Sigma \cup \{0\} : \alpha(Y) \geq t\}.$$

It easy to see that $\Xi(Y;t) = \Xi(Z;t)$ for all $t \in \mathbb{R}$ if and only if $Y = Z\,(\in \mathfrak{a})$ when the center of G is compact. Set

$$\begin{aligned}
\mathfrak{p}(Y) &\overset{\text{def}}{=} \sum_{\alpha\in\Xi(Y;0)} \mathfrak{g}(\mathfrak{a};\alpha),\\
\mathfrak{l}(Y) &\overset{\text{def}}{=} Z_{\mathfrak{g}}(Y) = \sum_{\alpha(Y)=0} \mathfrak{g}(\mathfrak{a};\alpha),\\
\mathfrak{n}(Y) &\overset{\text{def}}{=} \sum_{\alpha(Y)>0} \mathfrak{g}(\mathfrak{a};\alpha),\\
P(Y) &\overset{\text{def}}{=} N_G(\mathfrak{p}(Y)),\\
L(Y) &\overset{\text{def}}{=} N_{P(Y)}(\mathfrak{l}(Y)) = P(Y)\cap\theta P(Y),\\
N(Y) &\overset{\text{def}}{=} \exp(\mathfrak{n}(Y)).
\end{aligned}$$

Then $\mathfrak{p}(Y)$ (resp. $P(Y)$) is a parabolic subalgebra (resp. parabolic subgroup) with Levi decomposition

$$\mathfrak{p}(Y) = \mathfrak{n}(Y) + \mathfrak{l}(Y) \quad (\text{resp. } P(Y) = N(Y)\cdot L(Y)\,). \tag{2.8}$$

Since G is contained in a connected complexification $G_{\mathbb{C}}$,

$$L(Y) \simeq Z_G(Y). \tag{2.9}$$

Then the following lemma is well-known (cf. [War] Ch.1):

Lemma(2.10). *Let $Y, Z \in \mathfrak{a}$ be contained in the same closed Weyl chamber for $\Sigma(\mathfrak{g},\mathfrak{a})$. If $a \in G$ satisfies $\mathrm{Ad}(a)\,\mathfrak{p}(Y) = \mathfrak{p}(Z)$, then*

$$\mathfrak{p}(Y) = \mathfrak{p}(Z) \quad \text{and} \quad a \in P(Y) = P(Z).$$

Lemma(2.11). *Let $Y \in \mathfrak{a}$ and $a \in P(Y)$. Then*

$$\left(\bigoplus_{\beta(Y)=t} p_\beta\right)\left(\mathrm{Ad}(a)\sum_{\alpha(Y)=t}\mathfrak{g}(\mathfrak{a};\alpha)\right) = \sum_{\alpha(Y)=t}\mathfrak{g}(\mathfrak{a};\alpha),$$

$$\left(\bigoplus_{\beta(Y)=t} p_\beta\right)\left(\mathrm{Ad}(a)\sum_{\alpha(Y)>t}\mathfrak{g}(\mathfrak{a};\alpha)\right) = 0.$$

Proof of Lemma(2.11). Let $a = n\cdot l$ $(n \in N(Y), l \in L(Y))$ be a Levi decomposition corresponding to (2.8). Since l centralizes Y by (2.9), we have

$$\mathrm{Ad}(l)\sum_{\alpha(Y)=t}\mathfrak{g}(\mathfrak{a};\alpha) = \sum_{\alpha(Y)=t}\mathfrak{g}(\mathfrak{a};\alpha)$$

for any $t \in \mathbb{R}$. So we may and do assume $a = n \in N(Y)$ to prove the lemma. If $X_\alpha \in \mathfrak{g}(\mathfrak{a};\alpha)$ $(\alpha \in \Sigma \cup \{0\})$ and $n \in N(Y)$, we have

$$\mathrm{Ad}(n)\,X_\alpha - X_\alpha \in \sum_\beta \mathfrak{g}(\mathfrak{a};\alpha+\beta),$$

where the sum is taken over the roots of $\mathfrak{n}(Y)$ for $\mathfrak{a}$, namely $\{\beta \in \Sigma : \beta(Y) > 0\}$. Now the lemma follows. $\square$

3. A Lemma (Abelian Case)

Throughout this section G is a real reductive linear group and we retain notations as in §2. We shall show a key lemma(3.1) to the criterion of the properness of the action of a reductive subgroup on a homogeneous space of reductive type (see Theorem(4.1)).

Lemma(3.1). *Let $\mathfrak{a}_1$, $\mathfrak{a}_2$ be two subspaces in $\mathfrak{a}$, and denote by A_1, A_2 the analytic subgroups corresponding to $\mathfrak{a}_1$, $\mathfrak{a}_2$ respectively. Then the following two conditions are equivalent:*
(3.1.1) *For any compact subset S in G, $SA_1S^{-1} \cap A_2$ is compact.*
(3.1.2) *For any $w \in W(\mathfrak{g}, \mathfrak{a})$, $w \cdot \mathfrak{a}_1 \cap \mathfrak{a}_2 = \{0\}$.*

Proof. We may and do assume that the center of G is compact. Since the Weyl group $W(\mathfrak{g}, \mathfrak{a}) \simeq N_K(\mathfrak{a})/Z_K(\mathfrak{a}) = M'/M$, it is easy to see that (3.1.2) follows from (3.1.1).

Conversely, assume $SA_1S^{-1} \cap A_2$ is not compact with some compact subset S in G. Then the following claim holds:

Claim(3.2). *With the notation in Lemma(3.1), if there is a compact subset S in G such that $SA_1S^{-1} \cap A_2$ is not compact, then there are sequences*

$$a_n, b_n \in G,\ Y_n \in \mathfrak{a}_1,\ Z_n \in \mathfrak{a}_2,\ t_n \in \mathbb{R}_+, \quad (n \in \mathbb{N})$$

and there are

$$a, b \in G,\ Y \in \mathfrak{a}_1,\ Z \in \mathfrak{a}_2,\ w \in W(\mathfrak{g}.\mathfrak{a})$$

such that

$$\lim_{n\to\infty} t_n = \infty$$
$$\lim_{n\to\infty} a_n = a,\ \lim_{n\to\infty} b_n = b,$$
$$\lim_{n\to\infty} Y_n = Y,\ \lim_{n\to\infty} Z_n = Z,$$
$$a_n = \exp(t_n Z_n)\, b_n \exp(-t_n w \cdot Y_n),$$

where $w \cdot Y$ and Z are contained in the same closed Weyl chamber for $\Sigma(\mathfrak{g}, \mathfrak{a})$.

The proof of this claim will be given soon. Let us continue the proof of the lemma. Using the above claim and its notations, we have

$$\begin{aligned} \mathrm{Ad}(a_n)X_\alpha &= \mathrm{Ad}(\exp(t_n Z))\mathrm{Ad}(b_n)\mathrm{Ad}(\exp(-t_n\, w \cdot Y_n))X_\alpha \\ &= \sum_\beta \exp t_n(\beta(Z) - \alpha(w \cdot Y_n))\ p_\beta(\mathrm{Ad}(b_n)X_\alpha). \end{aligned}$$

for any root vector $X_\alpha \in \mathfrak{g}(\mathfrak{a}; \alpha)$ with $\alpha \in \Sigma \cup \{0\}$. Since the set $\{\mathrm{Ad}(b_n)X_\alpha; n \in \mathbb{N}\}$ is bounded, we have

$$\mathrm{Ad}(a)X_\alpha \in \sum_{\beta \in \Xi(Z;\alpha(w\cdot Y))} \mathfrak{g}(\mathfrak{a}; \beta).$$

Therefore for any $t \in \mathbb{R}$,

$$\mathrm{Ad}(a) \sum_{\alpha \in \Xi(w \cdot Y; t)} \mathfrak{g}(\mathfrak{a}; \alpha) \subset \sum_{\beta \in \Xi(Z; t)} \mathfrak{g}(\mathfrak{a}; \beta).$$

Similarly the equation

$$a_n^{-1} = \exp(t_n w \cdot Y_n) b_n^{-1} \exp(-t_n Z_n)$$

leads to

$$\mathrm{Ad}(a^{-1}) \sum_{\beta \in \Xi(Z; t)} \mathfrak{g}(\mathfrak{a}; \beta) \subset \sum_{\alpha \in \Xi(w \cdot Y; t)} \mathfrak{g}(\mathfrak{a}; \alpha).$$

Thus for any $t \in \mathbb{R}$,

$$\mathrm{Ad}(a) \sum_{\alpha \in \Xi(w \cdot Y; t)} \mathfrak{g}(\mathfrak{a}; \alpha) = \sum_{\beta \in \Xi(Z; t)} \mathfrak{g}(\mathfrak{a}; \beta). \tag{3.3}$$

In particular putting $t = 0$ in (3.3), we have

$$\mathrm{Ad}(a)\, \mathfrak{p}(w \cdot Y) = \mathfrak{p}(Z).$$

As $w \cdot Y$ and Z are contained in the same closed Weyl chamber, we have

$$a \in P(w \cdot Y) = P(Z),$$

from Lemma(2.10). Now operating the projection

$$\bigoplus_{\beta(w \cdot Y) = t} p_\beta : \mathfrak{g} \longrightarrow \sum_{\beta(w \cdot Y) = t} \mathfrak{g}(\mathfrak{a}; \beta)$$

to the both sides of (3.3) for fixed $t \in \mathbb{R}$, we get

$$\sum_{\alpha(w \cdot Y) = t} \mathfrak{g}(\mathfrak{a}; \alpha) = \sum_{\beta(Z) \geq t = \beta(w \cdot Y)} \mathfrak{g}(\mathfrak{a}; \beta).$$

from Lemma(2.11). This equation implies

$$\{\alpha \in \Sigma \cup \{0\} : \alpha(w \cdot Y) = t\} \subset \Xi(Z; t).$$

In particular we have $\Xi(w \cdot Y; t) \subset \Xi(Z; t) \quad$ for all $t \in \mathbb{R}$. The converse inclusion is obtained in the same way by operating $\bigoplus_{\beta(Z)=t} p_\beta$ to the equation $\sum_{\alpha \in \Xi(w \cdot Y; t)} \mathfrak{g}(\mathfrak{a}; \alpha) = \mathrm{Ad}(a)^{-1} \sum_{\beta \in \Xi(Z; t)} \mathfrak{g}(\mathfrak{a}; \beta)$. Hence we have proved

$$\Xi(w \cdot Y; t) = \Xi(Z; t) \quad \text{for all } t \in \mathbb{R}.$$

This relation implies $Z = w \cdot Y$ and so the condition (3.1.2) holds. The proof of Lemma(3.1) is now completed except for showing Claim(3.2). $\square$

Proof of Claim(3.2). Replacing S by KSK if necessary, we may assume S is bi-K-invariant. From the assumption and the compactness of S, we can choose sequences

$$a_n, b_n \in S,\ Y_n \in \mathfrak{a}_1,\ Z_n \in \mathfrak{a}_2,\ t_n \in \mathbb{R}_+,\ u_n \in \mathbb{R}_+, \quad (n \in \mathbb{N})$$

such that

$$\lim_{n\to\infty} u_n = \infty$$

$$\lim_{n\to\infty} a_n = a,\ \lim_{n\to\infty} b_n = b,$$

$$\lim_{n\to\infty} Y_n = Y,\ \lim_{n\to\infty} Z_n = Z,$$

$$a_n = \exp(t_n Z_n)\, b_n \exp(-u_n Y_n),$$

with some $a, b \in G$, $Y \in \mathfrak{a}_1 \setminus \{0\}$, $Z \in \mathfrak{a}_2 \setminus \{0\}$. Choose a positive system Σ^+ of Σ making Z a dominant element. Take $m_w \in N_K(\mathfrak{a})$ representing an element $w \in W(\mathfrak{g}; \mathfrak{a})$ such that $w{\cdot}Y$ is dominant for Σ^+. Replacing a_n, b_n by $a_n m_w^{-1}$, $b_n m_w^{-1}$ (with the same notations), we have

$$a_n = \exp(t_n Z_n) b_n \exp(-u_n\, w \cdot Y_n).$$

Therefore for each root vector $X_\alpha \in \mathfrak{g}(\mathfrak{a}; \alpha)$,

$$\operatorname{Ad}(a_n) X_\alpha = \sum_{\beta \in \Sigma \cup \{0\}} \exp(t_n \beta(Z_n) - u_n \alpha(Y_n))\ p_\beta(\operatorname{Ad}(b_n) X_\alpha). \tag{3.4}$$

Let us show that $\{\frac{t_{n(k)}}{u_{n(k)}} : n \in \mathbb{N}\}$ is bounded from 0 and ∞. In fact, suppose there were subsequences $n(k)$ $(k \in \mathbb{N})$ such that $\lim\limits_{k\to\infty} \frac{t_{n(k)}}{u_{n(k)}} = \infty$. Choose $\beta \in \Sigma$ such that $\beta(Z) > 0$ which exists because $Z \neq 0$. Then the existence of the limit of (3.4) as $n(k) \to \infty$ requires that $p_\beta(\operatorname{Ad}(b) X_\alpha) = 0$ for any $\alpha \in \Sigma \cup \{0\}$, which leads to a contradiction because $\operatorname{Ad}(b)$ is invertible. Similarly, suppose there were subsequences $n(k)$ $(k \in \mathbb{N})$ such that $\lim\limits_{k\to\infty} \frac{t_{n(k)}}{u_{n(k)}} = 0$. Choose $\alpha \in \Sigma$ such that $\alpha(Y) > 0$ which exists because $Y \neq 0$. Then the existence of the limit of (3.4) as $n(k) \to \infty$ requires that $\operatorname{Ad}(a) X_\alpha = \lim\limits_{k\to\infty} \operatorname{Ad}(a_{n(k)}) X_\alpha = 0$, which leads to a contradiction because $\operatorname{Ad}(a)$ is invertible. Thus we have shown that $\{\frac{t_n}{u_n} : n \in \mathbb{N}\}$ is bounded from 0 and ∞. Therefore by taking subsequences we may assume $\lim\limits_{n\to\infty} \frac{t_n}{u_n} = C$ with some positive constant C. Replacing u_n, Y_n and Y by t_n, $\frac{u_n}{t_n} Y_n$ and $\frac{1}{C} Y$ respectively, we get the claim. $\square$

4. Main Results

We consider the situation that a reductive subgroup H_1 in G acts on a homogeneous space G/H_2 of reductive type. First we give a criterion of the properness of this action.

Theorem(4.1). *Let H_1, H_2 be reductive subgroups in a real reductive linear group G (Definition (2.6)). Let $\mathfrak{a}(H_1)$, $\mathfrak{a}(H_2)$ and $\mathfrak{a}$ be maximally split abelian subspaces in $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{g}$ respectively. Fix $g_i \in G$ such that $\mathfrak{a}_i \overset{\mathrm{def}}{=} \mathrm{Ad}(g_i)\mathfrak{a}(H_i) \subset \mathfrak{a} \quad (i = 1, 2)$. Then the following three conditions on $\{H_1, H_2\}$ are equivalent:*
(4.1.1) *H_1 acts on G/H_2 properly.*
(4.1.2) *H_2 acts on G/H_1 properly.*
(4.1.3) *For any $w \in W(\mathfrak{g}; \mathfrak{a})$, $w \cdot \mathfrak{a}_1 \cap \mathfrak{a}_2 = \{0\}$.*

Proof. First note that any of the condition (4.1.1)-(4.1.3) is independent of the choice of g_i and does not change by replacing H_i by $g_i H_i g_i^{-1}$. Therefore we can and do assume that $g_i = e$ $(i = 1, 2)$ and that there is a Cartan involution θ such that $\theta H_i = H_i$ $(i = 1, 2)$.

Since the role of H_i $(i = 1, 2)$ are the same in (4.1.3), we only have to prove that (4.1.2) is equivalent to (4.1.3). From the definition of a proper action, (4.1.2) is equivalent to the compactness of the set

$$\{h \in H_2 : hSH_1 \cap SH_1 \neq \emptyset\} = SH_1S^{-1} \cap H_2$$

for every compact set S in G. In this condition we only have to treat the case S is bi-K-invariant. Put $A_i := \exp(\mathfrak{a}_i)$. If S is bi-K-invariant, the Cartan decomposition of H_i ([He] Ch.IX Thm.1.1) gives

$$\begin{aligned} SH_1S^{-1} \cap H_2 &= S(K \cap H_1)A_1(K \cap H_1)S^{-1} \cap (K \cap H_2)A_2(K \cap H_2) \\ &= (K \cap H_2)(SA_1S^{-1} \cap A_2)(K \cap H_2). \end{aligned}$$

Using the Cartan decomposition again, $SH_1S^{-1} \cap H_2$ is compact iff $SA_1S^{-1} \cap A_2$ is compact. Hence the proof of Theorem reduces to the preceding Lemma(3.2) in an abelian case. □

As an immediate corollary to Theorem(4.1), we have

Corollary(4.2). *Let G' be a reductive subgroup in G acting properly on a homogeneous space G/H of reductive type. Then*

$$\mathbb{R}\text{-rank} G' + \mathbb{R}\text{-rank} H \le \mathbb{R}\text{-rank} G.$$

This estimate is best possible. That is

Corollary(4.3). *Let G/H be a homogeneous space of reductive type. Then there exists a subgroup G' reductive in G such that G' acts properly on G/H and that $\mathbb{R}\text{-rank} G' + \mathbb{R}\text{-rank} H = \mathbb{R}\text{-rank} G$.*

In fact Theorem(4.1) guarantees a trivial choice of G': we can take $G' := \exp \mathfrak{b}$, where $\mathfrak{b}$ is any ($\mathbb{R}$-rank $G - \mathbb{R}$-rank H)-dimensional subspace complementary to $W(\mathfrak{g};\mathfrak{a}) \cdot \mathfrak{a}(H)$ in $\mathfrak{a}$.

It does depend on H and G whether a larger subgroup (with the same real rank) than the above G' can act properly on G/H or not . Although it is interesting to classify the maximal ones among such subgroups based on Theorem(4.1), we shall not go into here. Anyway, now we can tell explicitly when Calabi-Markus phenomenon occurs in a homogeneous space of reductive type:

Corollary(4.4). *Let G/H be a homogeneous space of reductive type. Then the following three conditions are equivalent:*
(4.4.1) *A subgroup which can act properly discontinuously and freely on G/H must be finite.*
(4.4.1)$'$ *Only finite subgroup can act properly discontinuously on G/H.*
(4.4.2) $\mathbb{R}$-rank $G = \mathbb{R}$-rank H.

Proof. If $k := \mathbb{R}$-rank $G - \mathbb{R}$-rank $H > 0$, then there is a subgroup Γ isomorphic to $\mathbb{Z}^k$ consisting of semisimple elements in G such that Γ acts properly discontinuously and freely on G/H by Corollary(4.3). Thus (4.4.1) $\to$ (4.4.2) is proved. (4.4.1)$'$ $\to$ (4.4.1) is trivial. Although (4.4.2) $\to$ (4.4.1)$'$ is immediately deduced from a known sufficient condition for Calabi-Markus phenomena ([Ku] Theorem A.1.2, see also [Wo]), we review it because it is elementary but instructive: Suppose $\mathbb{R}$-rank$G$ = $\mathbb{R}$-rank$H$. Then a maximally split abelian subspace $\mathfrak{a}(H)$ in $\mathfrak{h}$ is also a maximal one in $\mathfrak{g}$, and thus $G = K \exp \mathfrak{a}(H) K = KHK$, which implies that $\{g \in G; g(K/H\cap K) \cap (K/H\cap K) \neq \phi \text{ in } G/H\} = G$. Since $K/H \cap K$ is compact, this implies (4.4.1)$'$. $\square$

Example(4.5). Only finite subgroup in $GL(n,\mathbb{C})$ can act properly discontinuously on $GL(n,\mathbb{C})/GL(n,\mathbb{R})$.

Remark(4.6).

1) Kulkarni proved the necessity of (4.4.2) in a special case: that is, when $G/H = SO(p+1,q)/SO(p,q)$, (4.4.1) holds if $p < q$. One should note that $p < q$ is equivalent to $\mathbb{R}$-rank$G > \mathbb{R}$-rank$H$ in this case. But our proof is different from his method ([Ku]).
2) The above corollary does not exclude the existence of an infinite *abstract* group acting properly discontinuously on G/H. For instance $G/H = SL(2,\mathbb{R})/A$, (A is the diagonal matrix group) is diffeomorphic to $S^1 \times \mathbb{R}$, and so admitting a properly discontinuous and free action of $\mathbb{Z}$.
3) When G/H is a semisimple symmetric space, the real rank condition in Corollary(4.4) is equivalent to the condition that the 'dual symmetric space' G^d/H^d has a nonempty discrete series (ref. [FJ]).

Following the approach stated in Introduction, we want a reductive subgroup G' as large as possible acting on G/H. The extreme case $-$ where $G'\backslash G/H$ is compact $-$ is characterized by a simple condition:

Theorem(4.7). *Let H_i $(i = 1, 2)$ be reductive subgroups in a real reductive linear group G* (Definition(2.6))*. Under the equivalent conditions* (4.1.1)-(4.1.3)*, the following four conditions are equivalent:*
(4.7.1) *$H_1\backslash G/H_2$ is compact.*
(4.7.2) *There exists a discrete group Γ_1 in H_1 so that $\Gamma_1\backslash G/H_2$ is a compact smooth manifold.*
(4.7.2)$'$ *There exists a discrete group Γ_2 in H_2 so that $H_1\backslash G/\Gamma_2$ is a compact smooth manifold.*
(4.7.3) $\mathrm{d}(G) = \mathrm{d}(H_1) + \mathrm{d}(H_2)$.

The proof of Theorem(4.7) together with Proposition(4.10) below will be given at the end of §5. Theorem(4.1) and Theorem(4.7) give a method to get examples of a (not necessarily Riemannian) homogeneous space admitting a uniform lattice: namely, *find the triplet $\{G, H_1, H_2\}$ which satisfies the criteria (4.1.3) and (4.7.3), and then there exists a uniform lattice in G/H_i for $i =$ 1, 2.* Notice that if H_2 is compact, we can always choose $H_1 = G$. The next one is also rather stupid.

Example(4.8). Let G' be a real reductive linear group. Then $G/H_2 := \underbrace{G' \times G' \times \cdots \times G'}_{n-times} /\Delta G'$, where $\Delta G' := \{(g, \ldots, g) \in G : g \in G'\}$, admits a uniform lattice because we can choose $H_1 := G' \times G' \times \cdots \times G' \times \{e\}$.

The following examples are remarkable:

Proposition(4.9). *Either of the following triplets $\{G, H_1, H_2\}$ satisfies both* (4.1.1)-(4.1.3) *and* (4.7.1)-(4.7.3)*. Therefore G/H_i $(i = 1, 2)$ admits a uniform lattice as well as a non-uniform lattice.*

1) $G = U(2, 2n),\ H_1 = Sp(1, n),\ H_2 = U(1) \times U(1, n)$,
2) $G = SO(2, 2n),\ H_1 = U(1, n),\ H_2 = SO(1, 2n)$,
3) $G = SO(4, 4n),\ H_1 = Sp(1, n),\ H_2 = SO(3, 4n)$.

This Proposition can be easily checked by the criteria (4.1.3) and (4.7.3). Among these examples, G/H_2 in 2) and 3) were previously obtained in Theorem 6.1 in [Ku].

Finally we give another evidence (cf. Corollary(4.4)) that not so many homogeneous spaces of reductive type have uniform lattices:

Proposition(4.10). *Let H be a maximal rank reductive subgroup in a real reductive linear group G. Then G/H admits a uniform lattice only if*

$$\mathrm{rank} K = \mathrm{rank} H \cap K.$$

Recall that the rank of a reductive group G, denoted by rank G, is the dimension of a Cartan subalgebra of $\mathfrak{g} \otimes \mathbb{C}$ over $\mathbb{C}$. The above result is somewhat stronger than the one which K.Ono and the author have recently obtained (see Corollary 5 in [K-O]).

Example(4.11). *$Sp(2n,\mathbb{R})/Sp(n,\mathbb{C})$ has no uniform lattice but admits a properly discontinuous and free action of a subgroup isomorphic to $\mathbb{Z}^n$.*

5. Application of Cohomological Dimension Theory

First we review some notations concerning the cohomology of an abstract group. General references are [Bi], [C-E] and [Ser].

Let Γ be an abstract group, R a commutative ring with $1 \neq 0$, $R[\Gamma]$ the corresponding group ring. For each left $R[\Gamma]$-module A, the cohomology groups of Γ with coefficients in A are defined by

$$H^q(\Gamma, A) \stackrel{\text{def}}{=} Ext^q_{\mathbb{Z}[\Gamma]}(\mathbb{Z}, A) \simeq Ext^q_{R[\Gamma]}(R, A), \tag{5.1}$$

where R (resp. $\mathbb{Z}$) is regarded as a left $R[\Gamma]$ (resp. $\mathbb{Z}[\Gamma]$) module with trivial action of Γ. For the second isomorphism in (5.1), see [C-E] Ch.X §3.4. The cohomological dimension of Γ over R denoted by $cd_R(\Gamma)$ is the projective dimension of R as a left $R[\Gamma]$-module. Equivalently,

$$cd_R(\Gamma) = \sup\{n \in \mathbb{N} : H^n(\Gamma; A) \neq 0 \text{ for some left } R[\Gamma]-\text{module } A\}.$$

Following Serre (see [Ser]), we call Γ *virtually torsionless* iff Γ has a torsionless subgroup of finite index. Then the following result is due to Selberg ([Sel] Lemma 8):

Lemma(5.2). *A finitely generated matrix group is virtually torsionless.*

From now on, we shall restrict ourselves to the case when $R = \mathbb{R}$, the field of real numbers. This suffices for our application in this paper.

In this case, $cd_{\mathbb{R}}\Gamma' = cd_{\mathbb{R}}\Gamma$ for any subgroup Γ' of finite index in Γ (see [Ser] Thm.1, [Bi] §5.4).

Lemma(5.3). *Let G/H be a homogeneous space of reductive type* (Definition(2.6)). *Let Γ be a virtually torsionless discrete subgroup of G. Set*

$$S := \dim(K/H \cap K),\ N := cd_{\mathbb{R}}\Gamma.$$

1) (Serre) $N < \infty$
2) *If Γ acts properly discontinuously on G/H, then there is a subgroup Γ' of finite index in Γ such that Γ' acts properly discontinuously and freely on G/H and that there is a natural isomorphism*

   $$H^N(\Gamma'; A) \simeq H^{N+S}(\Gamma' \backslash G_o/H_o; A) \tag{5.3.1}$$

   for any left $\mathbb{R}[\Gamma']$-module A. Here A is regarded as a local coefficient system on $\Gamma' \backslash G_o/H_o$ in the right hand.

Proof. 1) is proved in [Ser]. In fact the spectral sequence (5.4) below collapses when $H = K$.

Suppose Γ act properly discontinuously on G/H. Take any torsion free subgroup Γ' of finite index in $\Gamma \cap G_o$. Then Γ' also acts freely on G/H (and G_o/H_o) because the action is properly discontinuous. There is a well-known first quadrant spectral sequence corresponding to the covering $G_o/H_o \to \Gamma' \backslash G_o/H_o$ ([C-E] Ch.XVI §9)

$$E_n^{p,q} \Longrightarrow H^{p+q}(\Gamma' \backslash G_o/H_o; A), \tag{5.4}$$

with E_2 term

$$E_2^{p,q} \simeq H^p(\Gamma'; H^q(G_o/H_o; A))$$

The differential d_n has bidegree $(n, 1-n)$. Here the action of Γ' on $H^q(G_o/H_o; A)$ is the diagonal one induced from the action of $\Gamma \times \Gamma$ on $H^q(G_o/H_o; A)$. Since Γ' is contained in G_o, this action agrees the action on the second factor A alone. As Lemma(2.7) assures that G_o/H_o has the same homotopy type with $K_o/H_o \cap K_o$, we have the following Γ'-module isomorphism:

$$H^q(G_o/H_o; A) \simeq H^q(K_o/H_o \cap K_o; A) \simeq \begin{cases} 0, & \text{if } q > S \\ A, & \text{if } q = S. \end{cases}$$

Then the spectral sequence (5.4) and the definition of $cd_{\mathbb{R}}\Gamma' (= cd_{\mathbb{R}}\Gamma = N)$ yield (5.3.1). $\square$

Now the same argument in [Ser] Prop.18 (cf. [Ku] Thm.2.1) leads to a cohomological restriction on Γ from the topology of a smooth manifold $\Gamma' \backslash G_o/H_o$ by (5.3.1). Recall that we have defined $\mathrm{d}(G)$ by the dimension of the Riemannian symmetric space associated to G ((2.5)).

Corollary(5.5). *Let Γ be a discrete subgroup in G acting properly discontinuously on a homogeneous space G/H of reductive type.*

1) *If $\Gamma \backslash G/H$ is compact, then*
   a) *Γ is virtually torsionless,*
   b) $cd_{\mathbb{R}}\Gamma = \mathrm{d}(G) - \mathrm{d}(H)$.
      Fix a torsion free subgroup Γ' of finite index in $\Gamma \cap G_o$.
   c) $\dim_{\mathbb{R}} H^j(\Gamma'; \mathbb{R}) < \infty$ *for all $j \in \mathbb{Z}$,*
   d) $\chi(\Gamma')\, \chi(K_o/H_o \cap K_o) = \chi(\Gamma' \backslash G/H)$.
      Here $\chi(\Gamma') \overset{\text{def}}{=} \sum_{j=0}^{N} (-1)^j \dim_{\mathbb{R}} H^j(\Gamma'; \mathbb{R})$, $\chi(M)$ denotes the Euler number of a compact orientable manifold M.
2) *If $\Gamma \backslash G/H$ is noncompact and if Γ is virtually torsionless, then* $cd_{\mathbb{R}}\Gamma \leq \mathrm{d}(G) - \mathrm{d}(H) - 1$.

Proof. a) is a direct consequence of Lemma(2.1) and Lemma(5.2). b) and 2) is deduced from (5.3.1) and from the following well-known Lemma(5.6) (This is proved by the Poincaré duality). Notice that $S = \dim(G/H) - (\mathrm{d}(G) - \mathrm{d}(H))$ in the notation of Lemma(5.3). c) is shown by induction from N to 0 by using the spectral sequence (5.4), and d) is the Euler-Poincaré principle. $\square$

Lemma(5.6). *For any local system $\mathcal{S}$ on a noncompact manifold M, $H^j(M, \mathcal{S}) = 0$ if $j \geq \dim M$.*

Now we are ready to prove Proposition(4.10).

Proof of Proposition(4.10). The proof is essentially the same as in Corollary 5 in [K-O]. What we must show is only the fact that G/H is a θ-stable homogeneous space in the sense of [K-O], Definition(3.3). This follows from the fact that an analytic subgroup $H_{\mathbb{C}}$ in $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{h} \otimes \mathbb{C}$ is closed because of the rank condition rank$H =$ rank$G$. □

Finally let us prove Theorem(4.7) as an application of Corollary(5.5).

Proof of Theorem(4.7). Since a real reductive linear group has a uniform lattice ([Bo]), (4.7.1), (4.7.2) and (4.7.2)$'$ are equivalent from Lemma(2.3). Fix any torsion free discrete subgroup Γ_1 in H_1. Apply Corollary(5.5) with $G = H_1$ and $H = \{e\}$, we have $cd_{\mathbb{R}}\Gamma_1 \leq \mathrm{d}(H_1)$ and the equality holds iff $\Gamma_1 \backslash H_1$ is compact. Apply Corollary(5.5) again with $H = H_2$, we have $cd_{\mathbb{R}}\Gamma_1 \leq \mathrm{d}(G) - \mathrm{d}(H_2)$ and the equality holds iff $\Gamma_1 \backslash G/H_2$ is compact. If (4.7.2) holds, then $\mathrm{d}(G) = cd_{\mathbb{R}}\Gamma_1 + \mathrm{d}(H_2) \leq \mathrm{d}(H_1) + \mathrm{d}(H_2)$. On the other hand, the assumption (4.1.3) implies that $\mathrm{d}(G) \geq \mathrm{d}(H_1) + \mathrm{d}(H_2)$, showing (4.7.3). Conversely if (4.7.3) holds, any uniform lattice Γ_1 in H_1 is cocompact in G/H_2 because $cd_{\mathbb{R}}\Gamma_1 = \mathrm{d}(H_1) = \mathrm{d}(G) - \mathrm{d}(H_2)$. Now the Theorem follows. □

6. Appendix

Let H be a closed subgroup reductive in a real reductive linear group G (Definition (2.6)). Fix a Cartan involution θ of G such that $H = (H \cap K)\exp(\mathfrak{h} \cap \mathfrak{p})$.

Lemma(6.1). *Retain notations as above. Then the mapping*

$$\pi : (\mathfrak{h} \cap \mathfrak{p}) + (\mathfrak{q} \cap \mathfrak{p}) \ni (X, Y) \mapsto \exp X \exp Y \cdot o \in G/K$$

gives a surjective diffeomorphism.

Proof. The identity mapping of H into G induces an inclusion $H/H \cap K \hookrightarrow G/K$. Since H is a closed subgroup in G, $H/H \cap K$ is also a closed submanifold in G/K. Identifying as usual the tangent space $T(G/K)_o$ with $\mathfrak{p}$, we have $T(H/H \cap K)_o \simeq \mathfrak{h} \cap \mathfrak{p}$. As the geodesics through o have the form $\exp(tX) \cdot o$ ($t \in \mathbb{R}$) where X is a general vector in $\mathfrak{p}$, this is tangent to $H/H \cap K$ at o if and only if $X \in \mathfrak{h} \cap \mathfrak{p}$. Since H acts on G/K isometrically, it follows that $H/H \cap K = \exp(\mathfrak{h} \cap \mathfrak{p}) \cdot o$ is totally geodesic in G/K. Fix an element X of $\mathfrak{h} \cap \mathfrak{p}$, and put $p = \exp(X) \cdot o \in G/K$ and $S^{\perp}(p) \overset{\mathrm{def}}{=} \exp(X)\exp(\mathfrak{q} \cap \mathfrak{p}) \cdot o \subset G/K$. Pull back the tangent space at p by $L^{-1}_{\exp(X)*} : T(G/K)_p \to \mathfrak{p}$, and we have

$$L^{-1}_{\exp(X)*}(T(H/H \cap K)_p) = A_X(\mathfrak{h} \cap \mathfrak{p}),$$

$$L^{-1}_{\exp(X)*}(T(S^{\perp}(p))_p) = \mathfrak{q} \cap \mathfrak{p},$$

where $A_X \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(\mathrm{ad}X)^{2n}}{(2n+1)!}{}_{|\mathfrak{p}} \in GL(\mathfrak{p})$ ([He] Ch.1 Theorem 13.3 and Ch.4 Theorem 4.1). Since $X \in \mathfrak{h} \cap \mathfrak{p}$, $\mathrm{ad}(X)^2$ preserves $\mathfrak{h} \cap \mathfrak{p}$ and therefore $A_X(\mathfrak{h} \cap \mathfrak{p}) = \mathfrak{h} \cap \mathfrak{p}$. As $\mathfrak{h} \cap \mathfrak{p}$ and $\mathfrak{q} \cap \mathfrak{p}$ are orthogonal to each other with respect to $\langle\ ,\ \rangle$, $S^{\perp}(p)$ is a submanifold made of the geodesics in G/K which are perpendicular to $H/H \cap K$ at p. Thus we have (see [He] Ch.1 Theorem 14.6)

$$\begin{aligned} G/K &= \coprod_{X \in \mathfrak{h} \cap \mathfrak{p}} S^{\perp}(p) \\ &= \coprod_{X \in \mathfrak{h} \cap \mathfrak{p}} \exp(X) \exp(\mathfrak{q} \cap \mathfrak{p}) \cdot o. \end{aligned}$$

Hence π is bijective. □

Now let us prove Lemma(2.7).

Proof of Lemma(2.7). From Lemma(6.1), we have

$$\begin{aligned} G &= K \exp(\mathfrak{q} \cap \mathfrak{p}) \exp(\mathfrak{h} \cap \mathfrak{p}) \\ &= K \exp(\mathfrak{q} \cap \mathfrak{p}) \exp(\mathfrak{h} \cap \mathfrak{p}) \,(H \cap K) \\ &= K \exp(\mathfrak{q} \cap \mathfrak{p})\, H. \end{aligned}$$

Let $f(k_1, Z_1) = f(k_2, Z_2)$ for $k_1, k_2 \in K$ and $Z_1, Z_2 \in \mathfrak{q} \cap \mathfrak{p}$. Then there are $X \in \mathfrak{h} \cap \mathfrak{p}$ and $h \in H \cap K$ such that $k_1 \exp(Z_1) \exp(X) h = k_2 \exp(Z_2)$. Therefore $k_1 h \exp(\mathrm{Ad}(h)^{-1}(Z_1)) \exp(\mathrm{Ad}(h)^{-1}(X)) = k_2 \exp(Z_2)$. Since $\mathrm{Ad}(h)^{-1}(Z_1) \in \mathfrak{q} \cap \mathfrak{p}$ and $\mathrm{Ad}(h)^{-1}(X) \in \mathfrak{h} \cap \mathfrak{p}$, we have $k_1 h = k_2$, $\mathrm{Ad}(h)^{-1}(Z_1) = Z_2$ and $X = 0$, from the uniqueness of the decomposition $G = K \exp(\mathfrak{q} \cap \mathfrak{p}) \exp(\mathfrak{h} \cap \mathfrak{p})$. □

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