Analysis on the minimal representation of $O(p,q)$ – III. ultrahyperbolic equations on $\mathbb{R}^{p-1,q-1}$

Toshiyuki KOBAYASHI

RIMS, Kyoto University, Sakyo-ku, Kyoto, 606-8502, Japan

and

Bent ØRSTED

*Department of Mathematics and Computer Science,
SDU-Odense University, Campusvej 55, DK-5230, Odense M, Denmark*

Abstract

For the group $O(p,q)$ we give a new construction of its minimal unitary representation via Euclidean Fourier analysis. This is an extension of the $q=2$ case, where the representation is the mass zero, spin zero representation realized in a Hilbert space of solutions to the wave equation. The group $O(p,q)$ acts as the Möbius group of conformal transformations on $\mathbb{R}^{p-1,q-1}$, and preserves a space of solutions of the ultrahyperbolic Laplace equation on $\mathbb{R}^{p-1,q-1}$. We construct in an intrinsic and natural way a Hilbert space of solutions so that $O(p,q)$ becomes a continuous irreducible unitary representation in this Hilbert space. We also prove that this representation is unitarily equivalent to the representation on $L^2(C)$, where C is the conical subvariety of the nilradical of a maximal parabolic subalgebra obtained by intersecting with the minimal nilpotent orbit in the Lie algebra of $O(p,q)$.

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Email addresses: `toshi@kurims.kyoto-u.ac.jp` (Toshiyuki KOBAYASHI), `orsted@imada.sdu.dk` (Bent ØRSTED).

1 Introduction

1.1 In this paper we study the symmetries of the ultrahyperbolic Laplace operator on a real finite-dimensional vector space equipped with a non-degenerate symmetric bilinear form. We shall work in coordinates so that the operator becomes

$$\square_{\mathbb{R}^{p-1,q-1}} \equiv \square_z := \frac{\partial^2}{\partial {z_1}^2} + \cdots + \frac{\partial^2}{\partial {z_{p-1}}^2} - \frac{\partial^2}{\partial {z_p}^2} - \cdots - \frac{\partial^2}{\partial {z_{p+q-2}}^2},$$

on $\mathbb{R}^n = \mathbb{R}^{p-1,q-1}$. In the case of Minkowski space ($q = 2$) we are studying the wave equation, which is well-known to have a conformally invariant space of solutions, see [14]. This corresponds to the fact that the equation $\square_{\mathbb{R}^{p-1,q-1}} f = 0$ in this case describes a particle of zero mass. Incidentally, it may also be interpreted as the bound states of the Hydrogen atom, namely each energy level corresponds to a K-type - for $(p, q) = (4, 2)$. This gives the angular momentum values by further restriction to $O(3)$. In general the indefinite orthogonal group $G = O(p, q)$ acts as the Möbius group of meromorphic conformal transformations on $\mathbb{R}^{p-1,q-1}$, leaving a space of solutions to $\square_{\mathbb{R}^{p-1,q-1}} f = 0$ invariant.

1.2 The main object of the present paper is to construct in an intrinsic and natural way a Hilbert space of solutions of $\square_{\mathbb{R}^{p-1,q-1}}$ so that the action of $O(p, q)$ becomes a continuous **unitary** irreducible representation in this Hilbert space for (p, q) such that $p, q \geq 2$ and $p + q > 4$ is even. From an algebraic view point of representation theory, our representations are:
i) minimal representations if $p+q \geq 8$ (i.e. the annihilator is the Joseph ideal).
ii) **not** spherical if $p \neq q$ (i.e. no non-zero K-fixed vector).
iii) **not** highest weight modules of $SO_0(p, q)$ if $p, q \geq 3$.

In a long history of representation theory of semisimple Lie groups, it is only quite recent that our representations for $p, q \geq 3$ have been paid attention, especially as minimal unitary representations; they were first discovered by Kostant [13] for $(p, q) = (4, 4)$ and generalized by Binegar-Zierau [3] as subrepresentations of degenerate principal series representations. There is also another algebraic approach to the same representations by using the theta correspondence for the trivial representation of $SL(2, \mathbb{R})$ by Huang-Zhu. Our previous papers [11] and [12] treated the same representation by geometric methods and with other points of view. We think that such various approaches reflect a rich structure of the minimal representations.

It is perhaps of independent interest that the (in some sense maximal group of) symmetries $O(p,q)$ of the space of solutions of $\Box_{\mathbb{R}^{p-1,q-1}} f = 0$ lead to such a natural Hilbert space. Our inner product $(\ , \ )_W$ defined by an integration over a non-characteristic hyperplane (see (1.5.1)) is a generalization of the one coming from energy considerations in the case of wave equations, and even the translation invariance of the inner product contains some new information about solutions.

It is also of independent interest from the representation theory of semisimple Lie groups that our representations are unitarily equivalent to the representations on $L^2(C)$, where C is the null cone of the quadratic form on $\mathbb{R}^{p-1,q-1}$. This result is proved via the Fourier transform in Theorem 4.9. Such L^2-realizations of "unipotent representations" is expected from the philosophy of the Kostant-Kirillov orbit method, but has not been proved except for some special cases of highest weight modules or spherical representations.

We have avoided most of the references to the theory of semisimple Lie groups and representation theory, and instead given direct constructions of the key objects, such as for example the minimal K-type; this is given as an explicit hypergeometric function, and we also calculate its Fourier transform in terms of a Bessel function. By application of explicit differential operators forming the Lie algebra of G we can generate the whole Hilbert space of solutions beginning from the minimal K-type.

1.3 For $q = 2$ (or $p = 2$) we are dealing with highest weight representations (when restricting to the identity component $SO_0(p, 2)$), and these have been studied by many authors, in particular in the physics literature. For a nice introduction to this representation and its construction via geometric quantization (and more), see [8]. In this case the K-types may be identified with energy levels of the bound states of the Hydrogen atom, and the smallest one with the bound state of lowest energy.

We can summarize the situation, covering both the classical Kepler problem and its quantization in case $q = 2$, as in the diagram below. Here the left-hand side represents the classical descriptions of respectively the Kepler problem and geodesic flow on the sphere; by "symplectic transform" we are alluding to the change of variables between these two Hamiltonian systems as presented in [8]. The right-hand side involves the quantizations of these two systems, where the wave-equation is considered as the quantization of geodesic flow, also to be thought of as geometric optics. The quantum analogue of the "symplectic transform" involves the Fourier transform. Finally we invoke conformal geometry and combine it with the Fourier transform, which in a different (and new as far as constructing Hilbert spaces and unitary actions) way appears in passing from the wave equation to the Fourier realization of solutions - this is the last arrow on the right-hand side.

$$
\begin{array}{rcl}
\text{Classical} & \Longrightarrow & \text{Quantum} \\
\boxed{\text{Kepler problem}} & \Longrightarrow & \boxed{\text{Hydrogen atom}} \\
\text{symplectic transform} \Downarrow & & \Downarrow \text{Fourier transform} \\
\boxed{\text{geometric optics}} & \Longrightarrow & \boxed{\text{(A) wave equation}} \\
 & & \text{conformal geometry} \\
 & & \Downarrow \text{Fourier transform} \\
 & & \boxed{\begin{array}{c}\text{(B) other realizations and} \\ \text{explicit inner products} \\ \text{of minimal representations}\end{array}}
\end{array}
$$

The main focus of this paper is on the boxes (A) and (B). In particular, we give an explicit inner product in the model (A) (Theorems 1.4 and 1.5) and construct via Fourier transform a new realization of the minimal representation (Theorem 1.6) for general p, q.

1.4 From now, suppose that $n := p + q - 2$ is an even integer greater than 2, and $p, q \geq 2$. Let us briefly state some of our main results in a more explicit way.

First, we find a formula of Green's function E_0 for the ultrahyperbolic Laplace operator $\square_{\mathbb{R}^{p-1,q-1}}$, in Proposition 4.2, namely, E_0 is given by a constant multiple of the imaginary part of the regularized Schwartz distribution:

$$e^{\frac{\sqrt{-1}\pi(q-1)}{2}}({x_1}^2 + \cdots + {x_{p-1}}^2 - {x_p}^2 - \cdots - {x_{p+q-1}}^2 + \sqrt{-1}0)^{1-\frac{n}{2}}.$$

See also the recent paper of Hörmander [9] for further details on distributions associated with this ultrahyperbolic equation. Then we construct solutions of $\square_{\mathbb{R}^{p-1,q-1}} f = 0$ by the integral transformation:

$$S : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \quad \varphi \mapsto E_0 * \varphi \qquad \text{(see (4.3.1))}.$$

The image $S(C_0^\infty(\mathbb{R}^n))$ turns out to be "large" in $\operatorname{Ker} \square_{\mathbb{R}^{p-1,q-1}}$ (see § 4.7, Remark (2)). On this image, we define a Hermitian form $(\ ,\ )_N$ by

$$(f_1, f_2)_N := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} E_0(y - x)\varphi_1(x)\overline{\varphi_2(y)}\, dxdy, \tag{1.4.1}$$

where $f_i = E_0 * \varphi_i$ $(i = 1, 2)$. Here is a part of Theorem 4.7, which is the first of our main results:

Theorem 1.4 $(\ ,\ )_N$ *is positive-definite on the image of* S*. Furthermore,* $O(p, q)$ *acts as an irreducible unitary representation on its Hilbert completion* $\mathcal{H}$*.*

We shall write $(\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}, \mathcal{H})$ for this unitary representation. We also prove that this representation is isomorphic to the minimal representation of $O(p, q)$, constructed previously by Kostant, Binegar-Zierau ([13], [3]) and also in our previous papers [11], [12] from different viewpoints. Thus, Theorem 1.4 may be regarded as a realization of the minimal representation (with an explicit inner product) in the solution space of the ultrahyperbolic equation.

1.5 The above definition of the inner product $(\ ,\ )_N$ (see (1.4.1)) uses the integral expression of solutions of $\Box_{\mathbb{R}^{p-1,q-1}} f = 0$. Can we write the inner product without knowing the preimage? Yes, the second of our main results is to give an intrinsic inner product on the same solution space by using the Cauchy data. For simplicity, we take $z_1 = 0$ as a non-characteristic hyperplane. Then, we decompose a solution

$$f = f_+ + f_-$$

such that $f_\pm(z_1, \ldots, z_n)$ is holomorphic with respect to the first variable z_1 in the complex domain of $\{z_1 \in \mathbb{C} : \pm \operatorname{Im} z_1 > 0\}$ of the z_1-variable. This is an expression of f as a hyperfunction, and such a pair (f_+, f_-) can be obtained by the convolution in the z_1-variable (see (6.2.3)):

$$f_\pm(z) = \frac{1}{2\pi\sqrt{-1}} \cdot \frac{\mp 1}{z_1 \pm \sqrt{-1}0} * f(z_1, \ldots, z_n),$$

where the integration makes sense for f with suitable decay at infinity. Then we define a Hermitian form

$$(f, f)_W := \frac{1}{\sqrt{-1}} \int_{\mathbb{R}^{n-1}} \left(f_+ \overline{\frac{\partial f_+}{\partial z_1}} - f_- \overline{\frac{\partial f_-}{\partial z_1}} \right) |_{z_1=0}\, dz_2 \cdots dz_n. \tag{1.5.1}$$

Then we shall prove that $(\ ,\ )_W$ is independent of the specific choice of a non-characteristic hyperplane, as follows from the (non-trivial) isometric invariance. Much more strongly, $(\ ,\ )_W$ is conformally invariant. A precise formulation for this is given in Theorem 6.2, which includes:

Theorem 1.5 $4\pi(\ ,\ )_W = (\ ,\ )_N$*. In particular,* $(\ ,\ )_W$ *is positive definite and* $O(p, q)$*-invariant.*

Hence, in place of Theorem 1.4, we can obtain the same irreducible unitary representation of $O(p, q)$ on the Hilbert completion of a space of solutions with respect to the inner product $(\ ,\ )_W$.

An interesting property of this inner product is its large invariance group. Even in the case of the usual wave equation ($q = 2$ case) our approach to the Hilbert space of solutions and the corresponding representation offers some new points of view. In this case, if we take the non-characteristic hyperplane as fixed time coordinate (namely, if z_1 stands for the time), then the translational invariance amounts to a remarkable "conservation law". Instead, we can take the non-characteristic hyperplane by fixing one of the space coordinates, and an analogous integration over the hypersurface (containing the time coordinate) still gives the same inner product ! As a final remark in § 6.7, we note the connection to the theory of conserved quantities for the wave equation ($q = 2$ case), such as the energy and others obtained by the action of the conformal group.

1.6 The Gelfand-Kirillov dimension of our representation $(\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}, \mathcal{H})$ is $p+q-3$. So, we may expect that the representation could be realized on a $(p+q-3)$-dimensional manifold. For this purpose, we define the null cone of the metric as

$$C := \{\zeta \in \mathbb{R}^n : {\zeta_1}^2 + \cdots + {\zeta_{p-1}}^2 - {\zeta_p}^2 - \cdots - {\zeta_n}^2 = 0\}.$$

The third of our main results is another realization of the unitary representation $(\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}, \mathcal{H})$ in a function space on a $(p+q-3)$-dimensional manifold C. The Fourier transform $\mathcal{F}$ maps solutions of $\Box_{\mathbb{R}^{p-1,q-1}} f = 0$ to distributions supported on the null cone C. Surprisingly, the inner product of our Hilbert space turns out to be simply the L^2-norm on C with respect to a canonical measure $d\mu$ (see (3.3.3)) ! Here is a part of Theorem 4.9: We regard $L^2(C)$ as a subspace of distributions by a natural injective map $T : L^2(C) \to \mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.6 $(2\pi)^{-\frac{n}{2}} T^{-1} \circ \mathcal{F}$ *is a surjective unitary operator from* $\mathcal{H}$ *to* $L^2(C)$.

Theorem 1.6 defines an irreducible unitary representation of $G = O(p, q)$ on $L^2(C)$, denoted by π, which is unitarily equivalent to $(\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}, \mathcal{H})$. Since the maximal parabolic subgroup $\overline{P^{\max}}$ of G (see § 2.7) acts on $\mathbb{R}^{p-1,q-1}$ as affine transformations, the restriction $\pi|_{\overline{P^{\max}}}$ has a very simple form, namely, the one obtained by the classical Mackey theory (see (3.3.5)). In this sense, Theorem 1.6 may be also regarded as an **extension theorem** of an irreducible unitary representation from the maximal parabolic subgroup $\overline{P^{\max}}$ to the whole group G.

1.7 The fourth of our main results is about the representation $(\pi, L^2(C))$ as a $(\mathfrak{g}, K)$-module on the Fourier transform side, especially to find an explicit vector in the minimal K-type.

In the realization on $L^2(C)$, the action $\pi(g)$ is not simple to describe except for $g \in \overline{P^{\max}}$. Instead, we consider the differential action $d\pi$ of the Lie algebra $\mathfrak{g}_0$ on smooth vectors of $L^2(C)$, which turns out to be given by differential operators at most of second order (see § 3.2). This makes the analogy with the metaplectic representation (where G is replaced by the symplectic group) a good one. Here we are recalling the fact, that the even part of the metaplectic representation may be realized as an L^2-space of functions on the cone generated by rank one projections in $\mathbb{R}^n$.

Moreover, by using a reduction formula of an Appell hypergeometric function, we find explicitly the Fourier transform of a Jacobi function multiplied by some conformal factor which equals to a scalar multiple of

$$\psi_{0,e}(\zeta) := |\zeta|^{\frac{3-q}{2}} K_{\frac{q-3}{2}}(2|\zeta|) d\mu \in \mathcal{S}'(\mathbb{R}^n).$$

Here $K_\nu(\zeta)$ is a modified Bessel function of the second kind. This vector $\psi_{0,e}(\zeta)$ corresponds to the bound state of lowest energy for $q = 2$ case . For general p, q, the K-span of $\psi_{0,e}(\zeta)$ generates the minimal K-type in the realization on $L^2(C)$.

We define a subspace U of $\mathcal{S}'(\mathbb{R}^n)$ to be the linear span of its iterative differentials

$$d\pi(X_1) \cdots d\pi(X_k)\psi_{0,e}(\zeta) \qquad (X_1, \ldots, X_k \in \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}).$$

What comes out of § 5 may be formulated in this way (combining with Theorem 4.9, see § 3.2 for notation): Suppose $p+q \in 2\mathbb{Z}$, $p+q > 4$ and $p \geq q \geq 2$.

Theorem 1.7 1) *$|\zeta|^{\frac{3-q}{2}} K_{\frac{q-3}{2}}(2|\zeta|)$ is a K-finite vector in $L^2(C)$.*
2) *U is an infinitesimally unitary $(\mathfrak{g}, K)$-module via $\widehat{\varpi}_{\frac{n-2}{2},\epsilon}$.*
3) *U is dense in the Hilbert space $T(L^2(C))$.*
4) *The completion of (2) defines an irreducible unitary representation of G on $T(L^2(C))$, and then also on $L^2(C)$.*

In the paper [4] one finds a similar construction of Hilbert spaces and unitary representations for Koecher-Tits groups associated with semisimple Jordan algebras under the assumption that the representations are spherical, and there also occur Bessel functions as spherical vectors. In our situation the representations are **not** spherical if $p \neq q$. Our approach is completely different from [4] that treats some spherical representations, and contrary to what is stated in [4] (p. 206) we show that for $G = O(p, q)$ ($p+q$ even) it is possible to extend the Mackey representation of the maximal parabolic subgroup to the whole group. Furthermore, even for $p = q$ case, our approach to Theorem 1.7 has an advantage that we give the exact constants normalizing the unitary correspondence between the minimal K-type in other realizations and the

Bessel function in our realization on $L^2(C)$ (see Theorem 5.5).

1.8 The paper is organized as follows: We begin by recalling some results from conformal geometry and facts about the conformal group, in particular in [11]. In section 3 we give the basic setup for a realization on the null cone via Fourier transform. Then we construct the intertwining operator from the minimal representation to the model treated here and calculate the new expression for the inner product (see Theorem 1.4). We show in Proposition 4.2 that the Green function of $\Box_{\mathbb{R}^{p-1,q-1}}$ has a Fourier transform equal to the invariant measure on the null-cone, allowing one more expression for the inner product (see Theorem 1.6); also we obtain from this an intertwining operator from test functions to solutions. Indeed, in section 4 Proposition 4.6 we prove that the Green function is up to a constant exactly the kernel in the Knapp-Stein intertwining integral operator between degenerate principal series representations at the parameters we study; this enables us to understand the unitarity of the minimal representation on the model $\mathbb{R}^{p-1,q-1}$ in an elementary and explicity way. Note that all normalizing constants are computed explicitly. Lemma 2.6 states the irreducibility and unitarizability, which we use; we give in [12], sections 7.6 and 8.3 independent proofs of these facts.

In section 5 we construct the lowest K-type as a modified Bessel function, whose concrete properties are important for K-type information about $L^2(C)$. The idea here is to use a classical formula on the Hankel transform due to Baily in 1930s, and then apply reduction formulae of an Appell hypergeometric function of two variables.

Section 6 contains formulae for the inner product $(\ ,\ )_W$ in terms of integration over a Cauchy hypersurface. Summarizing, we give five different realizations of the inner product together with the normalizations of these relative to each other. Namely, in addition to $(\ ,\ )_N$ and $(\ ,\ )_W$ we also define three more: $(\ ,\ )_M$ (coming from a pseudo-differential operator on $M = S^{p-1} \times S^{q-1}$), $(\ ,\ )_A$ (coming from a normalized Knapp-Stein intertwining operator), and finally $(\ ,\ )_C$, which is just $L^2(C)$. This is seen in the key diagram (see section 4.11)

$$\begin{array}{ccccccc} C_0^\infty(\mathbb{R}^n) & \xrightarrow{S} & \Psi^*_{\frac{n-2}{2}}(\widetilde{\Delta}_M) & \xrightarrow{\mathcal{F}} & \mathcal{S}'(\mathbb{R}^n) & \xhookleftarrow{T} & L^2(C) \\ & & \cap & & & & \\ & & \operatorname{Ker} \Box_{\mathbb{R}^{p-1,q-1}} & & & & \end{array}$$

where the spaces correspond to four different ways of generating solutions to our ultrahyperbolic equation. S will be an integral transform against the Green kernel (essentially, a Knapp-Stein intertwining operator with a specific parameter), and $\mathcal{F}$ the Fourier transform, mapping solutions to distributions supported on the null cone C. Correspondingly to the various ways of generating solutions, we write down explicitly the unitary inner product and its Hilbert space. We have tried to avoid the use of any semi-simple theory and stay within classical analysis on spheres and Euclidean spaces; still our treatment may also be of interest to people working with the classification of the unitary dual of semi-simple Lie groups, since we are providing new models of some unipotent representations. Tools like the standard Knapp-Stein intertwining operators become very natural to use here, also from the more elementary viewpoint, and the close connection between these and Green functions for ultrahyperbolic differential operators seems not to have been noticed before.

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2 Ultrahyperbolic equation on $\mathbb{R}^{p-1,q-1}$ and conformal group

2.1 As explained in the Introduction, we shall give a flat picture, the so-called N-picture, of the minimal representation, which is connected to classical facts about conformal geometry in $\mathbb{R}^n$. We shall give a unitary inner product in this realization (see Theorem 6.2) and also in its Fourier transform (Theorem 4.9), together with an explicit form of minimal K-type in this realization (see Theorem 5.5).

We shall assume $p+q \in 2\mathbb{N}$, $p \geq 2$, $q \geq 2$ and $(p,q) \neq (2,2)$. The parity condition $p+q \in 2\mathbb{N}$ is not necessary when we consider a representation of the parabolic subgroup $\overline{P^{\max}}$ or of the Lie algebra $\mathfrak{g}$. Indeed, it will be interesting to relax this parity condition in order to obtain an infinitesimally unitary representation, which does not integrate to a global unitary representation of G.

Throughout this paper, we let

$$n = p + q - 2.$$

This section is written in an elementary way, intended also for non-specialists of semisimple Lie groups. § 2.2 and § 2.6 review the needed results in [11].

2.2 We recall some basic fact of the distinguished representation of a conformal group (see [11], § 2). Let M be an n-dimensional manifold with pseudo-Riemannian metric g_M. We denote by Δ_M the Laplace operator on M, and by K_M the scalar curvature of M. The Yamabe operator is defined to be

$$\widetilde{\Delta}_M := \Delta_M - \frac{n-2}{4(n-1)} K_M.$$

Suppose (M, g_M) and (N, g_N) are pseudo-Riemannian manifolds. A local diffeomorphism $\Phi : M \to N$ is called a *conformal map* if there exists a positive-valued function Ω on M such that $\Phi^* g_N = \Omega^2 g_M$. For $\lambda \in \mathbb{C}$, we introduce a twisted pull-back

$$\Phi_\lambda^* : C^\infty(N) \to C^\infty(M), \ f \mapsto \Omega^\lambda \cdot f \circ \Phi. \tag{2.2.1}$$

Then the conformal quasi-invariance of the Yamabe operator is expressed by:

$$\Phi^*_{\frac{n+2}{2}} \widetilde{\Delta}_N = \widetilde{\Delta}_M \Phi^*_{\frac{n-2}{2}}. \tag{2.2.2}$$

Let G be a Lie group acting conformally on M. If we write the action as $x \mapsto L_h x$ $(h \in G, x \in M)$, we have a positive function $\Omega(h, x) \in C^\infty(G \times M)$ such that

$$L_h^* g_M = \Omega(h, \cdot)^2 \ g_M \qquad (h \in G).$$

We form a representation ϖ_λ of G, with parameter $\lambda \in \mathbb{C}$, on $C^\infty(M)$ as follows:

$$\varpi_\lambda(h^{-1}) f(x) = \Omega(h, x)^\lambda f(L_h x), \quad (h \in G, f \in C^\infty(M), x \in M). \tag{2.2.3}$$

Note that the right-hand side is given by the twisted pull-back $(L_h)^*_\lambda$ according to the notation (2.2.1). Then, Formula (2.2.2) implies that $\widetilde{\Delta}_M : C^\infty(M) \to C^\infty(M)$ is a G-intertwining operator from $\varpi_{\frac{n-2}{2}}$ to $\varpi_{\frac{n+2}{2}}$. Thus, we have constructed a distinguished representation of the conformal group:

Lemma 2.2 (see [11], Theorem 2.5) $\operatorname{Ker} \widetilde{\Delta}_M$ *is a representation space of the conformal group G of a pseudo-Riemannian manifold (M, g_M), through $\varpi_{\frac{n-2}{2}}$.*

If (N, g_N) is also a pseudo-Riemannian manifold on which the same group G acts conformally, then one can also define a representation $\varpi_{\lambda,N}$ on $C^\infty(N)$. Then the twisted pull-back Φ^*_λ is a G-intertwining operator.

2.3 Here is a setup on which we construct the minimal representation of $O(p, q)$ by applying Lemma 2.2. Let $p, q \geq 2$. We note $n = p + q - 2$. We write $\{e_0, \ldots, e_{n+1}\}$ for a standard basis of $\mathbb{R}^{p+q}$ and the corresponding coordinate as

$$(v_0, \ldots, v_{n+1}) = (x, y) = (v_0, z', z'', v_{n+1}),$$

where $x \in \mathbb{R}^p, y \in \mathbb{R}^q, z' \in \mathbb{R}^{p-1}, z'' \in \mathbb{R}^{q-1}$. The notation (x, y) will be used for $S^{p-1} \times S^{q-1}$, while (z', z'') for $\mathbb{R}^n = \mathbb{R}^{(p-1)+(q-1)}$. The standard norm on $\mathbb{R}^l$ will be written as $| \cdot |$ $(l = p - 1, p, q - 1, q)$.

We denote by $\mathbb{R}^{p,q}$ the pseudo-Riemannian manifold $\mathbb{R}^{p+q}$ equipped with the flat pseudo-Riemannian metric:

$$g_{\mathbb{R}^{p,q}} = {dv_0}^2 + \cdots + {dv_{p-1}}^2 - {dv_p}^2 - \cdots - {dv_{n+1}}^2. \tag{2.3.1}$$

We put two functions on $\mathbb{R}^{p+q}$ by

$$\nu : \mathbb{R}^{p+q} \to \mathbb{R}, \quad (x, y) \mapsto |x|, \tag{2.3.2}$$

$$\mu : \mathbb{R}^{p+q} \to \mathbb{R}, \quad (v_0, \ldots, v_{n+1}) \mapsto \frac{1}{2}(v_0 + v_{n+1}). \tag{2.3.3}$$

and define three submanifolds of $\mathbb{R}^{p,q}$ by

$$\begin{aligned}
\Xi &:= \{(x, y) \in \mathbb{R}^{p,q} : |x| = |y| \neq 0\}, \\
M &:= \{v \in \mathbb{R}^{p,q} : \nu(v) = 1\} \cap \Xi \qquad = S^{p-1} \times S^{q-1}, \\
N &:= \{v \in \mathbb{R}^{p,q} : \mu(v) = 1\} \cap \Xi \qquad \underset{\iota}{\xleftarrow{\sim}} \mathbb{R}^n.
\end{aligned}$$

where the bijection $\iota : \mathbb{R}^n \to N$ is given by

$$\iota : \mathbb{R}^n \to N, (z', z'') \mapsto (1 - \frac{|z'|^2 - |z''|^2}{4}, z', z'', 1 + \frac{|z'|^2 - |z''|^2}{4}). \tag{2.3.4}$$

We say a hypersurface L of Ξ is *transversal to rays* if the projection

$$\Phi : \Xi \to M, v \mapsto \frac{v}{\nu(v)} \tag{2.3.5}$$

induces a local diffeomorphism $\Phi|_L : L \to M$. Then, one can define a pseudo-Riemannian metric g_L of signature $(p-1, q-1)$ on L by the restriction of $g_{\mathbb{R}^{p,q}}$. In particular, M itself is transversal to rays, and the induced metric $g_{S^{p-1}\times S^{q-1}}$ equals $g_{S^{p-1}} \oplus (-g_{S^{q-1}})$, where $g_{S^{n-1}}$ denotes the standard Riemannian metric on the unit sphere S^{n-1}. Likewise, the induced pseudo-Riemannian metric on $\mathbb{R}^n$ through $\iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^{p,q}$ coincides with the standard flat pseudo-Riemannian metric $g_{\mathbb{R}^{p-1,q-1}}$ on $\mathbb{R}^n$.

2.4 Let $I_{p,q} := \operatorname{diag}(1, \ldots, 1, -1, \ldots, -1) \in GL(p + q, \mathbb{R})$. The indefinite orthogonal group

$$G = O(p, q) := \{g \in GL(p + q, \mathbb{R}) : {}^t g I_{p,q} g = I_{p,q}\},$$

acts isometrically on $\mathbb{R}^{p,q}$ by the natural representation, denoted by $v \mapsto g \cdot v$. This action stabilizes the light cone Ξ. We note that the multiplicative group $\mathbb{R}^\times_+ := \{r \in \mathbb{R} : r > 0\}$ also acts on Ξ as dilation, which commutes with the linear action of G. Then, using dilation, one can define an action of G on M, and also a meromorphic action on $\mathbb{R}^{p-1,q-1}$ as follows:

$$L_h : M \to M, \qquad v \mapsto \frac{h \cdot v}{\nu(h \cdot v)} \quad (h \in G), \tag{2.4.1}$$

$$L_h : \mathbb{R}^{p-1,q-1} \to \mathbb{R}^{p-1,q-1}, \quad z \mapsto \iota^{-1}\left(\frac{h \cdot \iota(z)}{\mu(h \cdot \iota(z))}\right) \quad (h \in G). \tag{2.4.2}$$

Then, both of these actions are conformal:

$$(L_h)^* g_M = \nu(h \cdot v)^{-2} g_M, \tag{2.4.3}$$

$$(L_h)^* g_{\mathbb{R}^{p-1,q-1}} = \mu(h \cdot \iota(z))^{-2} g_{\mathbb{R}^{p-1,q-1}}. \tag{2.4.4}$$

We note that (2.4.2) and (2.4.4) are well-defined if $\mu(h \cdot \iota(z)) \neq 0$. In fact, G acts only meromorphically on $\mathbb{R}^{p-1,q-1}$. An illustrative example for this feature is the linear fractional transformation of $SL(2,\mathbb{C})$ on $\mathbb{P}^1\mathbb{C} = \mathbb{C} \cup \{\infty\}$, which is a meromorphic action on $\mathbb{C}$. This example essentially coincides with (2.4.2) for $(p,q) = (3,1)$, since $SL(2,\mathbb{C})$ is locally isomorphic to $O(3,1)$ and $\mathbb{C} \simeq \mathbb{R}^2$.

2.5 The (meromorphic) conformal groups for the submanifolds M and N of Ξ are the same, namely, $G = O(p,q)$, while their isometry groups are different subgroups of G, as we shall see in Observation 2.5. In order to describe them, we define subgroups K, $M^{\max}$, $N^{\max}$, $A^{\max}$ and $\overline{N^{\max}}$ of G as follows:

First, we set

$$\begin{aligned}
m_0 &:= -I_{p+q}, \\
K &:= G \cap O(p+q) && = O(p) \times O(q), \\
M_+^{\max} &:= \{g \in G : g \cdot e_0 = e_0,\ g \cdot e_{n+1} = e_{n+1}\} && \simeq O(p-1,q-1), \\
M^{\max} &:= M_+^{\max} \cup m_0 M_+^{\max} && \simeq O(p-1,q-1) \times \mathbb{Z}_2.
\end{aligned}$$

The Lie algebra of G is denoted by $\mathfrak{g}_0 = \mathfrak{o}(p,q)$, which is given by matrices:

$$\mathfrak{g}_0 \simeq \{X \in M(p+q,\mathbb{R}) : X I_{p,q} + I_{p,q}{}^t X = O\}.$$

Next, we keep $n = p+q-2$ in mind and put

$$\varepsilon_j = \begin{cases} 1 & (1 \le j \le p-1), \\ -1 & (p \le j \le n), \end{cases} \tag{2.5.1}$$

and define elements of $\mathfrak{g}_0$ as follows:

$$\overline{N}_j := E_{j,0} + E_{j,n+1} - \varepsilon_j E_{0,j} + \varepsilon_j E_{n+1,j} \quad (1 \le j \le n), \tag{2.5.2)(a}$$

$$N_j := E_{j,0} - E_{j,n+1} - \varepsilon_j E_{0,j} - \varepsilon_j E_{n+1,j} \quad (1 \le j \le n), \tag{2.5.2)(b}$$

$$E := E_{0,n+1} + E_{n+1,0}, \tag{2.5.2)(c}$$

where E_{ij} denotes the matrix unit. Now, we define abelian subgroups of G by

$$\overline{N^{\max}} := \exp(\sum_{j=1}^n \mathbb{R}\overline{N}_j), \quad N^{\max} := \exp(\sum_{j=1}^n \mathbb{R}N_j), \quad A^{\max} := \exp(\mathbb{R}E).$$

For example, $M_+^{\max}$ is the Lorentz group and $M_+^{\max} N^{\max}$ is the Poincaré group if $(p,q) = (2,4)$. It is convenient to identify $\mathbb{R}^n$ with $\overline{N^{\max}}$ by putting

$$\overline{n}_a := \exp(\sum_{j=1}^{n} a_j \overline{N_j}) \in \overline{N^{\max}} \qquad \text{for } a = (a_1, \ldots, a_n) \in \mathbb{R}^n. \tag{2.5.3}$$

The geometric point here will be the following:

Observation 2.5 1) *On $S^{p-1} \times S^{q-1}$, G acts conformally, while K isometrically.*
2) *On $\mathbb{R}^{p-1,q-1}$, G acts meromorphically and conformally, while the motion group $M_+^{\max}\overline{N^{\max}}$ isometrically.*

2.6 Next, let us consider the pseudo-Riemannian manifold $M = S^{p-1} \times S^{q-1}$. It follows from (2.3.3) and (2.4.3) that we can define a representation $\varpi_{\lambda,M}$ of G on $C^\infty(M)$ by

$$(\varpi_{\lambda,M}(h^{-1})f)(v) := \nu(h \cdot v)^{-\lambda} f(L_h v).$$

The Yamabe operator on M is of the form:

$$\widetilde{\Delta}_M = \Delta_{S^{p-1}} - \Delta_{S^{q-1}} - (\frac{p-2}{2})^2 + (\frac{q-2}{2})^2 = \widetilde{\Delta}_{S^{p-1}} - \widetilde{\Delta}_{S^{q-1}}.$$

Applying Lemma 2.2, we obtain a representation of the conformal group $G = O(p,q)$, denoted by $(\varpi^{p,q}, V^{p,q})$, as a subrepresentation of $\varpi_{\frac{p+q-4}{2},M}$:

$$V^{p,q} := \operatorname{Ker} \widetilde{\Delta}_M = \{f \in C^\infty(M) : \widetilde{\Delta}_M f = 0\},$$
$$(\varpi^{p,q}(h^{-1})f)(v) := \nu(h \cdot v)^{-\frac{p+q-4}{2}} f(L_h v), \quad \text{for } h \in G, v \in M, f \in V^{p,q}.$$

The restriction of $\varpi^{p,q}$ from the conformal group to the isometry group gives useful knowledge on the representation $\varpi^{p,q}$. For this, we recall the classical theory of spherical harmonics, which is a generalization of Fourier series for S^1. For $p \geq 2$ and $k \in \mathbb{N}$, we define the space of spherical harmonics of degree k by

$$\begin{aligned} \mathcal{H}^k(\mathbb{R}^p) &= \{f \in C^\infty(S^{p-1}) : \Delta_{S^{p-1}} f = -k(k+p-2)f\}, \qquad (2.6.1)\\ &= \{f \in C^\infty(S^{p-1}) : \widetilde{\Delta}_{S^{p-1}} f = \left(\frac{1}{4} - (k + \frac{p-2}{2})^2\right) f\}. \end{aligned}$$

Then $O(p)$ acts irreducibly on $\mathcal{H}^k(\mathbb{R}^p)$ and the algebraic direct sum $\bigoplus_{k=0}^{\infty} \mathcal{H}^k(\mathbb{R}^p)$ is dense in $C^\infty(S^{p-1})$. We note that $\mathcal{H}^k(\mathbb{R}^2) \neq \{0\}$ only if $k = 0$ or 1.

Now, we review a basic property of this representation $(\varpi^{p,q}, V^{p,q})$ on $M = S^{p-1} \times S^{q-1}$:

Lemma 2.6 *(see [3]; [11],§3) Assume* $p, q \geq 2$, $p+q \in 2\mathbb{N}$ *and* $(p,q) \neq (2,2)$.
1) $(\varpi^{p,q}, V^{p,q})$ *is an infinite dimensional irreducible representation of* G.
2) *(K-type formula)* $V^{p,q}$ *contains the algebraic direct sum*

$$\bigoplus_{\substack{a,b\in\mathbb{N} \\ a+\frac{p}{2}=b+\frac{q}{2}}} \mathcal{H}^a(\mathbb{R}^p) \otimes \mathcal{H}^b(\mathbb{R}^q) \tag{2.6.2}$$

as a dense subspace with respect to the Fréchet topology on $C^\infty(M)$.
3) G *preserves the norm on* $V^{p,q}$ *defined by*

$$\|F\|_M^2 := \|(\frac{1}{4} - \widetilde{\Delta}_{S^{p-1}})^{\frac{1}{4}} F\|_{L^2(M)}^2 = \sum_{a \geq \max(0, \frac{p-q}{2})} (a + \frac{q-2}{2}) \|F_{a,b}\|_{L^2(M)}^2,$$

if $F = \sum_a F_{a,b} \in V^{p,q}$ *with* $F_{a,b} \in \mathcal{H}^a(\mathbb{R}^p) \otimes \mathcal{H}^b(\mathbb{R}^q)$ *and* $b = a + \frac{p-q}{2}$. *Here,* $(\frac{1}{4} - \widetilde{\Delta}_{S^{p-1}})^{\frac{1}{4}}$ *is a pseudo-differential operator on* M, *which is equal to* $(\frac{1}{4} - \widetilde{\Delta}_{S^{q-1}})^{\frac{1}{4}}$ *on* $\operatorname{Ker} \widetilde{\Delta}_M$.

We write $(\ , \ )_M$ for the corresponding inner product. We denote by $\overline{V^{p,q}}$ the Hilbert completion of $V^{p,q}$, on which G acts as an irreducible unitary representation of G. We shall use the same notation $\varpi^{p,q}$ to denote this unitary representation.

If $p \geq q$ then $V^{p,q}$ contains the K-type of the form $\mathbf{1} \boxtimes \mathcal{H}^{\frac{p-q}{2}}(\mathbb{R}^q)$. This K-type is called a **minimal** K**-type** in the sense of Vogan, namely, its highest weight (with respect to a fixed positive root system of $\mathfrak{k}_0$) attains the minimum distance from the sum of negative roots of $\mathfrak{k}_0$ among all highest weights of K-types occurring in $\varpi^{p,q}$. Likewise for $p < q$.

Remark 1) If $p + q \geq 8$, $\varpi^{p,q}$ is called the **minimal representation** in the representation theory of semisimple Lie groups, in the sense that the annihilator is the Joseph ideal.
2) The formula (2.6.2) is regarded as a branching law from the conformal group G to the isometry subgroup K of the pseudo-Riemannian manifold $M = S^{p-1} \times S^{q-1}$ (see Observation 2.5). In [12], we generalized this branching law with respect to a non-compact reductive subgroup and proved the Parseval-Plancherel formula, in the framework of discretely decomposable restrictions [10].

2.7 Let us consider the flat pseudo-Riemannian manifold $\mathbb{R}^{p-1,q-1}$. The Yamabe operator on $\mathbb{R}^{p-1,q-1}$ is of the form:

$$\square_{\mathbb{R}^{p-1,q-1}} \equiv \square_z := \frac{\partial^2}{\partial {z_1}^2} + \cdots + \frac{\partial^2}{\partial {z_{p-1}}^2} - \frac{\partial^2}{\partial {z_p}^2} - \cdots - \frac{\partial^2}{\partial {z_{p+q-2}}^2},$$

because the scalar curvature on $\mathbb{R}^{p-1,q-1}$ vanishes. Since $G = O(p,q)$ acts on $\mathbb{R}^{p-1,q-1}$ as a (meromorphic) conformal transform by (2.4.4), we obtain a 'representation' with parameter $\lambda \in \mathbb{C}$ as in (2.2.3):

$$\varpi_{\lambda,\epsilon,\mathbb{R}^n}(g^{-1})f(z) = |\mu(g\iota(z))|^{-\lambda}\chi_\epsilon(\operatorname{sgn}(\mu(g\iota(z))))f(L_g z), \quad (g \in G). \quad (2.7.1)$$

Here, for $\epsilon = \pm 1$, we put

$$\chi_\epsilon : \mathbb{R}^\times \to \{\pm 1\}$$

by $\chi_1 \equiv 1$ and $\chi_{-1} = \operatorname{sgn}$. We may write $\varpi_{\lambda,\epsilon,\mathbb{R}^{p-1,q-1}}$ for $\varpi_{\lambda,\epsilon,\mathbb{R}^n}$ if we emphasize a view point of conformal geometry on the flat space $\mathbb{R}^{p-1,q-1}$.

We note that $C^\infty(M)$ is not stable by $\varpi_{\lambda,\epsilon,\mathbb{R}^{p-1,q-1}}(g^{-1})$ because L_g is meromorphic. To make (2.7.1) a representation, we need to consider suitable class of functions controlled at infinity. One method for this is to use a conformal compactification

$$\mathbb{R}^{p-1,q-1} \hookrightarrow (S^{p-1} \times S^{q-1})/\sim \mathbb{Z}_2,$$

and to take a twisted pull-back Ψ_λ^* from $C^\infty(M)$ by a conformal map Ψ. This method is easy, and we shall explain it soon in § 2.8 and § 2.9. The other is to find an inner product for specific parameter λ so that G acts as a continuous unitary representation on the Hilbert space. This is particularly non-trivial for a subrepresentation, and we shall consider it for $\operatorname{Ker} \square_{\mathbb{R}^{p-1,q-1}}$ in § 6.

Before taking a suitable class of functions, we first write a more explicit form of (2.7.1). First, we note that the maximal parabolic subgroup

$$\overline{P^{\max}} := A^{\max} M^{\max} \overline{N^{\max}} = (\mathbb{R}_+^\times \times O(p-1,q-1) \times \mathbb{Z}_2) \ltimes \mathbb{R}^n$$

acts transitively on the manifold $\iota(\mathbb{R}^n)$ as affine transformations. Furthermore, $M^{\max}\overline{N^{\max}}$ acts on $\iota(\mathbb{R}^n)$ as isometries (see Observation 2.5). Correspondingly, the representation $\varpi_{\lambda,\epsilon} \equiv \varpi_{\lambda,\epsilon,\mathbb{R}^n}$ given in (2.7.1) has a simple form when restricted to the subgroup $\overline{P^{\max}}$:

$$\begin{aligned}
(\varpi_{\lambda,\epsilon}(m)f)(z) &= f(m^{-1}z) && (m \in M_+^{\max}), && (2.7.2)(a)\\
(\varpi_{\lambda,\epsilon}(m_0)f)(z) &= \epsilon f(z), && && \\
(\varpi_{\lambda,\epsilon}(e^{tE})f)(z) &= e^{\lambda t} f(e^t z) && (t \in \mathbb{R}), && (2.7.2)(b)\\
(\varpi_{\lambda,\epsilon}(\overline{n}_a)f)(z) &= f(z - 2a) && (a \in \mathbb{R}^n). && (2.7.2)(c)
\end{aligned}$$

Second, we write an explicit formula of the differential action of (2.7.1). We define a linear map

$$\omega : \mathfrak{g}_0 \to C^\infty(\mathbb{R}^n)$$

by the Lie derivative of the conformal factor $\Omega(h,z) := \mu(h \cdot \iota(z))^{-1}$ (see (2.4.4)). For $Y = (Y_{i,j})_{0 \le i,j \le n+1} \in \mathfrak{g}_0$ and $z \in \mathbb{R}^n$, we have

$$\omega(Y)_z := \frac{d}{dt}\Big|_{t=0} \Omega(e^{tY}, z) = -Y_{0,n+1} - \frac{1}{2}\sum_{j=1}^{n}(Y_{0,j} + Y_{n+1,j})z_j. \quad (2.7.3)$$

We write the Euler vector field on $\mathbb{R}^n$ as

$$E_z = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}. \tag{2.7.4}$$

Then the differential $d\varpi_\lambda : \mathfrak{g}_0 \to \mathrm{End}(C^\infty(\mathbb{R}^n))$ is given by

$$\begin{aligned} d\varpi_\lambda(Y) &= -\lambda\omega(Y) - \omega(Y)E_z \\ &- \sum_{i=1}^{n} \left\{ (Y_{i,0} + Y_{i,n+1}) + \frac{|z'|^2 - |z''|^2}{4}(-Y_{i,0} + Y_{i,n+1}) + \sum_{j=1}^{n} Y_{i,j} z_j \right\} \frac{\partial}{\partial z_j} \end{aligned} \tag{2.7.5}$$

for $Y = (Y_{i,j})_{0 \le i,j \le n+1} \in \mathfrak{g}_0$ and $z \in \mathbb{R}^n$. In particular, we have

$$d\varpi_\lambda(N_j) = -\lambda\varepsilon_j z_j - \varepsilon_j z_j E_z + \frac{1}{2}(|z'|^2 - |z''|^2)\frac{\partial}{\partial z_j}, \qquad (1 \le j \le n). \tag{2.7.6}$$

2.8 We recall $M = S^{p-1} \times S^{q-1}$. This subsection relates the representation $\varpi_{\lambda,M}$ and $\varpi_{\lambda,\mathbb{R}^{p-1,q-1}}$ by the stereographic projection $\Psi^{-1} : M \to \mathbb{R}^{p-1,q-1}$ defined below.

We set a positive valued function $\tau : \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} \tau(z) \equiv \tau(z', z'') &:= \nu \circ \iota(z) \\ &= \left((1 - \frac{|z'|^2 - |z''|^2}{4})^2 + |z'|^2 \right)^{\frac{1}{2}} \\ &= \left((1 + \frac{|z'|^2 - |z''|^2}{4})^2 + |z''|^2 \right)^{\frac{1}{2}} \\ &= \left(1 + (\frac{|z'| + |z''|}{2})^2 \right)^{\frac{1}{2}} \left(1 + (\frac{|z'| - |z''|}{2})^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{2.8.1}$$

We define an injective diffeomorphism as a composition of $\iota : \mathbb{R}^{p-1,q-1} \hookrightarrow \Xi$ (see (2.3.4)) and $\Phi : \Xi \to M$ (see (2.3.5)):

$$\Psi : \mathbb{R}^{p-1,q-1} \to M, \quad z \mapsto \tau(z)^{-1}\iota(z). \tag{2.8.2}$$

The image of Ψ is

$$M_+ := \{u = (u_0, u', u'', u_{n+1}) \in M = S^{p-1} \times S^{q-1} : u_0 + u_{n+1} > 0\}. \tag{2.8.3}$$

Then, Ψ is a conformal map (see [11], Lemma 3.3, for example) such that

$$\Psi^* g_M = \tau(z)^{-2} g_{\mathbb{R}^{p-1,q-1}}. \tag{2.8.4}$$

The inverse of $\Psi : \mathbb{R}^{p-1,q-1} \to M_+$ is given by

$$\Psi^{-1}(u_0, u', u'', u_{n+1}) = (\frac{u_0 + u_{n+1}}{2})^{-1}(u', u'') = \mu(u)^{-1}(u', u''). \qquad (2.8.5)$$

Ψ^{-1} is nothing but a stereographic projection if $q = 1$. We note that Ψ induces a conformal compactification of the flat space $\mathbb{R}^{p-1,q-1}$:

$$\mathbb{R}^{p-1,q-1} \hookrightarrow (S^{p-1} \times S^{q-1})/\sim \mathbb{Z}_2.$$

Here $\sim \mathbb{Z}_2$ denotes the equivalence relation in the direct product space $S^{p-1} \times S^{q-1}$ defined by $u \sim -u$.

As in (2.2.1), we define the twisted pull-back by

$$\Psi_\lambda^* : C^\infty(M) \to C^\infty(\mathbb{R}^n), \quad F \mapsto \tau(z)^{-\lambda} F(\Psi(z)). \qquad (2.8.6)$$

Let

$$C^\infty(M)_\epsilon := \{f \in C^\infty(M) : f(-u) = \epsilon f(u), \ \text{ for any } u \in M\}.$$

Then $\Psi_\lambda^*|_{C^\infty(M)_\epsilon}$ is injective. The inverse map is given by

$$(\Psi_{\lambda,\epsilon}^*)^{-1} : \Psi_\lambda^*(C^\infty(M)_\epsilon) \to C^\infty(M)_\epsilon,$$
$$f \mapsto \begin{cases} |\frac{u_0+u_{n+1}}{2}|^{-\lambda} f(\Psi^{-1}(u)) & (u \in M_+) \\ \epsilon|\frac{u_0+u_{n+1}}{2}|^{-\lambda} f(\Psi^{-1}(-u)) & (u \in M_-). \end{cases} \qquad (2.8.7)$$

We note that $(\Psi_{\lambda,\epsilon}^*)^{-1} f$ makes sense for $f \in C_0^\infty(\mathbb{R}^n)$, since we have

$$C_0^\infty(\mathbb{R}^n) \subset \Psi_\lambda^*(C^\infty(M)_\epsilon).$$

Now, the representation $\varpi_{\lambda,\epsilon,\mathbb{R}^n}$ is well-defined on the following representation space: $\Psi_\lambda^*(C^\infty(M))$, a subspace of $C^\infty(\mathbb{R}^n)$ through $\varpi_{\lambda,M}$.

Then, by (2.2.2) (see [11], Proposition 2.6), we have:

Lemma 2.8 $\Psi_{\frac{n-2}{2}}^*(V^{p,q}) \subset \operatorname{Ker} \square_{\mathbb{R}^{p-1,q-1}}$, *where* $V^{p,q} = \operatorname{Ker} \widetilde{\Delta}_M$.

2.9 In the terminology of representation theory of semisimple Lie groups, Ψ_λ^* is a G-intertwining operator from the *K-picture* $(\varpi_{\lambda,M}, C^\infty(M)_\epsilon)$ to the *N-picture* $(\varpi_{\lambda,\epsilon,\mathbb{R}^n}, \Psi_\lambda^*(C^\infty(M)))$. To see this in an elementary way, we argue as follows: For $\nu \in \mathbb{C}$, we denote by the space

$$S^\nu(\Xi) := \{h \in C^\infty(\Xi) : h(t\xi) = t^\nu h(\xi), \ \text{ for any } \xi \in \Xi, t > 0\}, \qquad (2.9.1)$$

of smooth functions on Ξ of homogeneous degree ν. Then G acts on $S^\nu(\Xi)$ by left translations. Furthermore, for $\epsilon = \pm 1$, we put

$$S^{\nu,\epsilon}(\Xi) := \{h \in S^\nu(\Xi) : h(-\xi) = \epsilon h(\xi), \ \text{ for any } \xi \in \Xi\}. \qquad (2.9.2)$$

Then we have a direct sum decomposition

$$S^{\nu}(\Xi) = S^{\nu,1}(\Xi) + S^{\nu,-1}(\Xi),$$

on which G acts by left translations, respectively. Then $S^{\nu,\epsilon}(\Xi)$ corresponds to the degenerate principal series representation (see [11] for notation):

$$C^{\infty}\text{-}\operatorname{Ind}_{P^{\max}}^{G}(\epsilon \otimes \mathbb{C}_{\lambda}) \simeq S^{-\lambda-\frac{n}{2},\epsilon}(\Xi), \tag{2.9.3}$$

where $P^{\max} = M^{\max}A^{\max}N^{\max}$.

Lemma 2.9 1) *The restriction $S^{-\lambda,\epsilon}(\Xi) \to C^{\infty}(M)_{\epsilon}, h \mapsto h|_M$ induces the isomorphism of G-modules between $S^{-\lambda,\epsilon}(\Xi)$ and $(\varpi_{\lambda}, C^{\infty}(M)_{\epsilon})$ for any $\lambda \in \mathbb{C}$.*
2) *The restriction $S^{-\lambda,\epsilon}(\Xi) \to C^{\infty}(\mathbb{R}^n), h \mapsto h|_{\mathbb{R}^n}$ induces the isomorphism of G-modules between $S^{-\lambda,\epsilon}(\Xi)$ and $(\varpi_{\lambda,\epsilon,\mathbb{R}^n}, \Psi_{\lambda}^{*}(C^{\infty}(M)_{\epsilon}))$ for any $\lambda \in \mathbb{C}$.*

PROOF. See [11], Lemma 3.7.1 for (1). (2) follows from the commutative diagram:

$$\begin{array}{ccc} & S^{-\lambda,\epsilon}(\Xi) & \\ r_1 \swarrow & & \searrow r_2 \\ C^{\infty}(M)_{\epsilon} & \xrightarrow{\Psi_{\lambda}^{*}} & C^{\infty}(\mathbb{R}^n), \end{array} \tag{2.9.4}$$

because r_1 is bijective and r_2 is injective. $\square$

2.10 A natural bilinear form $\langle\ ,\ \rangle : S^{-\lambda-\frac{n}{2}}(\Xi) \times S^{\lambda-\frac{n}{2}}(\Xi) \to \mathbb{C}$ is defined by

$$\langle h_1, h_2 \rangle := \int_M h_1(b) h_2(b)\, db \tag{2.10.1}$$

$$= 2\int_{\mathbb{R}^n} h_1(\iota(z)) h_2(\iota(z))\, dz \qquad \text{(see (2.3.4))}. \tag{2.10.2}$$

Here, db is the Riemannian measure on $M = S^{p-1} \times S^{q-1}$. The second equation follows from $(h_1h_2)(\iota(z)) = \tau(z)^{-n}(h_1h_2)(\Psi(z))$ and the Jacobian for $\Psi : \mathbb{R}^n \to M$ is given by $\tau(z)^{-n}$ (see (2.8.4)). Then $\langle\ ,\ \rangle$ is K-invariant and $\overline{N^{\max}}$-invariant from (2.10.1) and (2.10.2), and thus G-invariant since G is generated by K and $\overline{N^{\max}}$.

3 Square integrable functions on the cone

3.1 In this section we shall study the irreducible unitary representation of the motion group $M_{+}^{\max}\overline{N^{\max}} \simeq O(p-1, q-1) \ltimes \mathbb{R}^{p+q-2}$ and the maximal parabolic subgroup $\overline{P^{\max}} = M^{\max}A^{\max}\overline{N^{\max}}$ on the space of solutions to our ultrahyperbolic equation $\square_{\mathbb{R}^{p-1,q-1}} f = 0$. This is a standard induced representation by the Mackey machine, and will be later extended to the minimal representation of $G = O(p,q)$ (see Theorem 4.9 (3)).

3.2 In the flat picture $\mathbb{R}^{p-1,q-1}$, our minimal representation $V^{p,q}$ of $O(p,q)$ can be realized in a subspace of $\operatorname{Ker} \square_{\mathbb{R}^{p-1,q-1}}$ (see Lemma 2.8). We shall study the representation space by means of the Fourier transform.

We normalize the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ by

$$(\mathcal{F}f)(\zeta) = \int_{\mathbb{R}^n} f(z) e^{\sqrt{-1}(z_1\zeta_1 + \cdots + z_n\zeta_n)} dz_1 \cdots dz_n,$$

and extends it to $\mathcal{S}'(\mathbb{R}^n)$, the space of the Schwartz distributions.

By composing the following two injective maps

$$C^\infty(M)_\epsilon \xrightarrow{\Psi^*_{\frac{n-2}{2}}} C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n) \xrightarrow{\mathcal{F}} \mathcal{S}'(\mathbb{R}^n),$$

we define a representation of G and $\mathfrak{g}$ on the image $\mathcal{F}\Psi^*_{\frac{n-2}{2}}(C^\infty(M))$, denoted by $\widehat{\varpi}_{\lambda,\epsilon} \equiv \widehat{\varpi}_{\lambda,\epsilon,\mathbb{R}^n}$, so that $\mathcal{F} \circ \Psi^*_{\frac{n-2}{2}}$ is a bijective G-intertwining operator from the representation space $(\varpi_{\lambda,M}, C^\infty(M)_\epsilon)$ to $(\widehat{\varpi}_{\lambda,\epsilon}, \mathcal{F}\Psi^*_{\frac{n-2}{2}}(C^\infty(M)_\epsilon))$. Then, it follows from (2.7.2) that the representation $\widehat{\varpi}_{\lambda,\epsilon}$ has a simple form when restricted to the subgroup $\overline{P^{\max}} = M^{\max}A^{\max}\overline{N^{\max}}$:

$$\begin{aligned}
(\widehat{\varpi}_{\lambda,\epsilon}(m)h)(\zeta) &= h({}^t m\zeta) && (m \in M^{\max}_+), && (3.2.1)(a)\\
(\widehat{\varpi}_{\lambda,\epsilon}(m_0)h)(\zeta) &= \epsilon h(\zeta), && && \\
(\widehat{\varpi}_{\lambda,\epsilon}(e^{tE})h)(\zeta) &= e^{(\lambda-n)t} h(e^{-t}\zeta) && (t \in \mathbb{R}), && (3.2.1)(b)\\
(\widehat{\varpi}_{\lambda,\epsilon}(\overline{n}_a)h)(\zeta) &= e^{2\sqrt{-1}(a_1\zeta_1 + \cdots + a_n\zeta_n)} h(\zeta) && (a \in \mathbb{R}^n). && (3.2.1)(c)
\end{aligned}$$

We remark that in the above formula, we regarded h as a function. The action of $A^{\max}$ on the space of distributions is slightly different by the contribution of the measure $d\zeta$:

$$(\widehat{\varpi}_{\lambda,\epsilon}(e^{tE})\phi)(\zeta) = e^{\lambda t}\phi(e^{-t}\zeta) \qquad (t \in \mathbb{R}), \qquad (3.2.1)(b')$$

if we write $\phi(\zeta) = h(\zeta)d\zeta \in \mathcal{S}'(\mathbb{R}^n)$.

The differential representation $d\widehat{\varpi}_{\lambda,\epsilon}$ of $\mathfrak{g}_0$ on $\mathcal{F}\Psi^*_{\frac{n-2}{2}}(C^\infty(M))$ is given by the following lemma:

Lemma 3.2 *We recall that E_ζ is the Euler operator (see (2.7.4)). With no-*

tation in (2.5.2), we have

$$
\begin{aligned}
d\widehat{\varpi}_{\lambda,\epsilon}(\overline{N}_j) &= 2\sqrt{-1}\zeta_j && (1 \le j \le n),\\
d\widehat{\varpi}_{\lambda,\epsilon}(N_j) &= \sqrt{-1}\left((\lambda-n)\varepsilon_j\frac{\partial}{\partial\zeta_j} - E_\zeta\varepsilon_j\frac{\partial}{\partial\zeta_j} + \frac{1}{2}\zeta_j\Box_\zeta\right) && (1 \le j \le n),\\
d\widehat{\varpi}_{\lambda,\epsilon}(E) &= \lambda - n - E_\zeta.
\end{aligned}
$$

PROOF. Lemma follows from the correspondence under the Fourier transform $\frac{\partial}{\partial z_j} \leftrightarrow -\sqrt{-1}\zeta_j$, $z_j \leftrightarrow -\sqrt{-1}\frac{\partial}{\partial\zeta_j}$, and therefore from $E_z \leftrightarrow -n - E_\zeta$, $P(z) \leftrightarrow -\Box_\zeta$, where $P(z) := {z_1}^2 + \cdots + {z_{p-1}}^2 - {z_p}^2 - \cdots - {z_{p+q-2}}^2$. $\Box$

Remark 1) *We note that $d\widehat{\varpi}_{\lambda,\epsilon}$ is independent of the signature $\epsilon = \pm 1$.*
2) *In Theorem 4.9, we shall find that $L^2(C)$, the Hilbert space of square integrable functions on the cone C in $\mathbb{R}^n$, is a G-invariant subspace of the Schwartz distributions $\mathcal{S}'(\mathbb{R}^n)$. Then, the action of the Lie algebra $\mathfrak{g}$ can be written in terms of differential operators along the cone C at most of second order.*

3.3 We define a quadratic form Q on $\mathbb{R}^n$ ($\simeq (\mathbb{R}^n)^*$) as the dual of $P(z)$ on $\mathbb{R}^n$ by

$$Q(\zeta) := {\zeta_1}^2 + \cdots + {\zeta_{p-1}}^2 - {\zeta_p}^2 - \cdots - {\zeta_{p+q-2}}^2 \qquad \text{for } \zeta \in \mathbb{R}^n = \mathbb{R}^{p+q-2} \quad (3.3.1)$$

and define a closed cone by

$$C := \{\zeta \in \mathbb{R}^n : Q(\zeta) = 0\}. \qquad (3.3.2)$$

It follows from Lemma 2.8 that the support of the distribution $\mathcal{F}\Psi^*_{\frac{n-2}{2}}F$ is contained in the cone C, for any $F \in V^{p,q}$. Surprisingly, $\mathcal{F}\Psi^*_{\frac{n-2}{2}}F$ becomes square integrable on C (see Theorem 4.9). As a preparation for the proof, we study a natural action on $L^2(C)$ of a parabolic subgroup $\overline{P^{\max}}$ in this subsection.

We take a differential $(n-1)$-form $d\mu$ on C such that

$$dQ \wedge d\mu = d\zeta_1 \wedge \cdots \wedge d\zeta_n.$$

Then the restriction $d\mu$ to the cone C defines a canonical measure (we use the same notation $d\mu$). Using polar coordinates on C: $\zeta = (s\omega, s\omega')$ with $s > 0$, $\omega \in S^{p-2}$, $\omega' \in S^{q-2}$, we write down the canonical measure $d\mu$ on C explicitly by

$$\int_C \phi d\mu = \frac{1}{2}\int_0^\infty \int_{S^{p-2}} \int_{S^{q-2}} \phi(s\omega, s\omega') s^{n-3}\, ds\, d\omega\, d\omega' \qquad (3.3.3)$$

for a test function ϕ. If $n > 2$, that is if $p+q > 4$, then the measure $d\mu$ defines a Schwartz distribution on $\mathbb{R}^n$, for which we shall also write $\delta(Q)$, the "delta function" supported on the cone C (see [7]).

For a measurable function ϕ on C, we define a norm of ϕ by

$$\|\phi\|^2 := \int_C |\phi|^2 d\mu \tag{3.3.4}$$

and denote by $L^2(C) \equiv L^2(C, d\mu)$ the Hilbert space of square integrable functions.

By the Mackey theory, we can define a natural representation π of the maximal parabolic subgroup $\overline{P^{\max}} = A^{\max} M^{\max} \overline{N^{\max}}$ on $L^2(C)$ if $p + q \in 2\mathbb{Z}$, by

$$\begin{aligned}
(\pi(e^{tE})\psi)(\zeta) &:= e^{-\frac{n-2}{2}t}\psi(e^{-t}\zeta), && (t \in \mathbb{R}) && (3.3.5)(\mathrm{a})\\
(\pi(m)\psi)(\zeta) &:= \psi({}^t m\zeta) && (m \in M_+^{\max}), && (3.3.5)(\mathrm{b})\\
(\pi(m_0)\psi)(\zeta) &:= (-1)^{\frac{p-q}{2}}\psi(\zeta) && && \\
(\pi(\overline{n}_a)\psi)(\zeta) &:= e^{2\sqrt{-1}(a_1\zeta_1 + \cdots + a_n\zeta_n)}\psi(\zeta) && (a \in \mathbb{R}^n). && (3.3.5)(\mathrm{c})
\end{aligned}$$

Proposition 3.3 1) *The representation $(\pi, L^2(C))$ of $\overline{P^{\max}}$ is unitary.*
2) *The representation π is still irreducible when restricted to the motion group $M_+^{\max}\overline{N^{\max}} \simeq O(p-1, q-1) \ltimes \mathbb{R}^{p+q-2}$. In particular, it is irreducible as a $\overline{P^{\max}}$-module.*

PROOF. (1) is straightforward from the definitions (3.3.4) and (3.3.5).
Let us prove (2). It follows from (3.3.5)(c) that any $\overline{N^{\max}}$-invariant closed subspace of $L^2(C)$ is of the form $L^2(C')$ where C' is a measurable subset of C. As $M_+^{\max}$ acts transitively on C, $L^2(C')$ is $M_+^{\max}$-invariant only if the measure of C' is either null or conull. Thus, $L^2(C')$ equals either $\{0\}$ or $L^2(C)$. Therefore, the unitary representation $L^2(C)$ is irreducible as an $M_+^{\max}\overline{N^{\max}}$-module. $\square$

3.4 It is not clear a priori if $(\pi, L^2(C))$ extends from $\overline{P^{\max}}$ to G. We shall prove in Theorem 4.9 that if $p, q \geq 2$ and $n(= p + q - 2) > 2$ then π extends to G as an irreducible unitary representation through an injective map $T : L^2(C) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, defined as follows:

By using the Cauchy-Schwarz inequality, we see the following map

$$T(\psi) : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}, \quad \varphi \mapsto \int_C \varphi\psi \, d\mu$$

is well-defined and continuous if $n > 2$, for each $\psi \in L^2(C)$. Thus we have a natural map

$$T : L^2(C) \longrightarrow \mathcal{S}'(\mathbb{R}^n), \quad \psi \mapsto \psi d\mu. \tag{3.4.1}$$

Clearly, T is injective. We shall regard $T(L^2(C))$ as a Hilbert space such that T is a unitary operator.

Lemma 3.4 *1) T is a $\overline{P^{\max}}$-intertwining operator from $(\pi, L^2(C))$ to $(\widehat{\varpi}_{\frac{n-2}{2},\epsilon}|_{\overline{P^{\max}}}, \mathcal{S}'(\mathbb{R}^n))$.*
2) $(\widehat{\varpi}_{\frac{n-2}{2},\epsilon}|_{\overline{P^{\max}}}, T(L^2(C)))$ is an irreducible unitary representation of $\overline{P^{\max}}$. It is still irreducible as an $M^{\max}\overline{N^{\max}}$-module.

PROOF. (1) follows directly from the definitions (3.2.1) and (3.3.5). (2) follows from (1) and Proposition 3.3 (2). $\square$

4 Green function and inner product

4.1 In this section, we shall construct solutions of the ultrahyperbolic equation $\square_{\mathbb{R}^{p-1,q-1}} f = 0$ by the integral transform given by convolution with the Green kernel. Then, we shall show that the Green kernel coincides with a special value of the Knapp-Stein intertwining operator for a degenerate principal series. This observations gives another expression of the inner product of the minimal representation of $O(p,q)$ by using the Green kernel (see Theorem 4.7), and also leads to a realization of the minimal representation on $L^2(C)$, the Hilbert space of square integrable functions on a cone C as will be discussed in § 6.

We put

$$P(x) = {x_1}^2 + \cdots + {x_{p-1}}^2 - {x_p}^2 - \cdots - {x_n}^2$$

for $x \in \mathbb{R}^n = \mathbb{R}^{p+q-2}$.

4.2 A distribution E satisfying $\square_{\mathbb{R}^{p-1,q-1}} E = \delta$ (Dirac's delta function) is called a *fundamental solution* of the ultra-hyperbolic Laplace operator $\square_{\mathbb{R}^{p-1,q-1}}$. Recall from [7], page 354, if n is even and $n > 2$ then

$$E = \frac{-\Gamma(\frac{n}{2}-1)e^{\frac{\sqrt{-1}\pi(q-1)}{2}}}{4\pi^{\frac{n}{2}}}(P(x)+\sqrt{-1}0)^{1-\frac{n}{2}}$$

is a fundamental solution of $\square_{\mathbb{R}^{p-1,q-1}}$, where $(P(x)+\sqrt{-1}0)^{\lambda}$ stands for the limit of the distribution $(P(x)+\sqrt{-1}R(x))^{\lambda}$ as a positive definite quadratic form $R(x)$ tends to 0. In view of the integral formula in [7], Chapter III, § 2.6

$$\mathcal{F}(P+\sqrt{-1}0)^{1-\frac{n}{2}} = \frac{4\pi^{\frac{n}{2}}e^{-\frac{\sqrt{-1}\pi(q-1)}{2}}}{\Gamma(\frac{n}{2}-1)}(Q-\sqrt{-1}0)^{-1},$$

we have readily the following formula for the Green function E_0 of $\square_{\mathbb{R}^{p-1,q-1}}$:

Proposition 4.2 *We define a distribution E_0 on $\mathbb{R}^n$ by*

$$E_0 = \frac{1}{2\pi\sqrt{-1}}(-E + \overline{E}). \tag{4.2.1}$$

Then $\Box_{\mathbb{R}^{p-1,q-1}} E_0 = 0$ and its Fourier transform is given by

$$\mathcal{F}E_0 = \frac{1}{2\pi\sqrt{-1}}((Q - \sqrt{-1}0)^{-1} - (Q + \sqrt{-1}0)^{-1}) = \delta(Q). \tag{4.2.2}$$

In the Minkowski case, i.e. $q = 2$, such a formula has been known ([14]), since the so-called two-point functions in the quantum field theory for a zero mass field exactly corresponds to $(Q + \sqrt{-1}0)^{-1}$ for negative frequency. In this case C naturally splits in two components, a forward and a backward light cone, and functions supported on the forward cone have Fourier transforms that extend to holomorphic functions on the corresponding tube domain, thus yielding a unitary highest weight representation of the connected group. The reproducing kernel of this representation is the Fourier transform of the measure on the forward cone, in analogy with what happens in Proposition 4.2.

4.3 In order to give the integral expression of solutions $\Box_{\mathbb{R}^{p-1,q-1}} f = 0$, we define a convolution map by the Green kernel:

$$S : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n), \quad \varphi \mapsto E_0 * \varphi. \tag{4.3.1}$$

Fix $\nu \in \mathbb{C}$. We consider the representation $\varpi_{\nu,\epsilon} \equiv \varpi_{\nu,\epsilon,\mathbb{R}^n}$ of G on a subspace of $C^\infty(\mathbb{R}^n)$ (see § 2). The restriction to $\overline{P^{\max}}$ stabilizes $C_0^\infty(\mathbb{R}^n)$, as follows from (2.7.2).

Lemma 4.3 1) $\mathrm{Image}\, S \subset \mathrm{Ker}\, \Box_{\mathbb{R}^{p-1,q-1}}$.
2) *S is an intertwining operator of $\overline{P^{\max}}$-modules between on one side $\varpi_{\nu+2,\epsilon}|_{\overline{P^{\max}}}$ and on the other side $\varpi_{\nu,\epsilon}|_{\overline{P^{\max}}}$ for any $\nu \in \mathbb{C}$ and $\epsilon = \pm 1$.*

PROOF. (1) As $\Box_{\mathbb{R}^{p-1,q-1}} E_0 = 0$, we have $\mathrm{Image}\, S \subset \mathrm{Ker}\, \Box_{\mathbb{R}^{p-1,q-1}}$.
The proof of (2) is direct from (2.7.2). We illustrate it by the action of $e^{tE} \in A^{\max}$:

$$\begin{aligned} S(\varpi_{\nu+2,\epsilon}(e^{tE})\varphi)(y) &= \int_{\mathbb{R}^n} E_0(y - z) e^{(\nu+2)t} \varphi(e^t z)\, dz \\ &= \int_{\mathbb{R}^n} E_0(e^t y - e^t z) e^{(\nu+2)t + (n-2)t} \varphi(e^t z)\, dz \\ &= e^{\nu t}(S\varphi)(e^t y) \\ &= \varpi_{\nu,\epsilon}(e^{tE})(S\varphi)(y). \end{aligned}$$

This shows that S intertwines the action of $A^{\max}$. The case for the action of $M^{\max}\overline{N^{\max}}$ is similar and easier. $\Box$

We shall see that S extends to a G-intertwining operator for $\nu = \frac{n}{2} - 1$ in Proposition 4.6.

4.4 Recall the notation in § 2.9. Let

$$[u, v] := u_0 v_0 + \cdots + u_{p-1} v_{p-1} - u_p v_p - \cdots - u_{n+1} v_{n+1}$$

for $u, v \in \mathbb{R}^{n+2} = \mathbb{R}^{p+q}$. The Knapp-Stein intertwining operator

$$A_{\lambda,\epsilon} : S^{-\lambda-\frac{n}{2},\epsilon}(\Xi) \to S^{\lambda-\frac{n}{2},\epsilon}(\Xi)$$

is given by the integral operator with kernel function

$$\psi_{\lambda-\frac{n}{2},\epsilon}(u_0 - u_{n+1}) = \psi_{\lambda-\frac{n}{2},\epsilon}([u, \xi_0]) \tag{4.4.1}$$

on Ξ. Here, we put $\xi_0 := {}^t(1, 0, \ldots, 0, 1) \in \Xi$ and $\psi_{\nu,\epsilon}$ is a distribution (or a hyperfunction) of one variable is defined by

$$\psi_{\nu,\epsilon}(y) := \frac{1}{\Gamma(\frac{2\nu+3-\epsilon}{4})} |y|^\nu \chi_\epsilon(\operatorname{sgn} y).$$

Via the bijection r_1 in the commutative diagram (2.9.4),

$$A_{\lambda,\epsilon} : C^\infty(M)_\epsilon \to C^\infty(M)_\epsilon$$

is written as

$$(A^M_{\lambda,\epsilon} f)(u) \equiv (A_{\lambda,\epsilon} f)(u) := \int_M \psi_{\lambda-\frac{n}{2},\epsilon}([u, v]) \ f(v) \ dv \quad (u \in M)$$

in the compact picture $M \simeq S^{p-1} \times S^{q-1}$.

The Gamma factor in the definition of $\psi_{\nu,\epsilon}(y)$ exactly cancels the poles of the distribution $|y|^\nu \chi_\epsilon(\operatorname{sgn} y)$ of one variable y with meromorphic parameter ν. This means that the distribution $\psi_{\nu,\epsilon}([u, v])$ of multi-variables makes sense for any $\nu \in \mathbb{C}$ when restricted to the open set $\{(u, v) \in M \times M : u \neq \pm v\}$, where $[u, dv] + [v, du] \neq 0$. Then $\psi_{\nu,\epsilon}([u, v])$ continues meromorphically as a distribution on $M \times M$ with possible poles only at $\lambda = 0, -1, \ldots,$ whose residues are distributions supported on $\{(u, v) \in M \times M : u = \pm v\}$.

In view of the normalization of our parameter (see Lemma 2.9), $A_{\lambda,\epsilon}$ is a G-intertwining operator from $\varpi_{\lambda+\frac{n}{2},\varepsilon}$ to $\varpi_{-\lambda+\frac{n}{2},\varepsilon}$ for $\lambda \neq 0, -1, -2, \ldots$. What we need is the case $\lambda = 1$ and we recall from [11], § 3.9 (basically since we know the composition series of the induced representations and the eigenvalue of $A_{\lambda,\epsilon}$ on each K-type):

Lemma 4.4 *Let* $p \equiv q \mod 2$, $p, q \geq 2$ *and* $(p,q) \neq (2,2)$. *We put*

$$\epsilon := (-1)^{\frac{p-q}{2}} = \begin{cases} 1 & p-q \equiv 0,4 \mod 8, \\ -1 & p-q \equiv 2,6 \mod 8. \end{cases} \tag{4.4.2)(a}$$

$$\delta := (-1)^{[\frac{q-p}{4}]} = \begin{cases} 1 & p-q \equiv 0,2 \mod 8, \\ -1 & p-q \equiv 4,6 \mod 8. \end{cases} \tag{4.4.2)(b}$$

1) *The image* $(\varpi_{\frac{n}{2}-1,\varepsilon}, A_{1,\epsilon}(C^\infty(M)_\epsilon))$ *is a dense subrepresentation of* $(\varpi^{p,q}, V^{p,q})$, *where we recall* $V^{p,q} = \operatorname{Ker} \widetilde{\Delta}_M$.

Let us define a Hermitian form $(\ ,\ )_A$ *on the same image* $A_{1,\epsilon}(C^\infty(M)_\epsilon)$ *by*

$$(F_1, F_2)_A := \delta\langle A_{1,\epsilon}\varphi_1, \overline{\varphi_2}\rangle = \delta\langle \varphi_1, \overline{A_{1,\epsilon}\varphi_2}\rangle, \tag{4.4.3}$$

for $F_i = A_{1,\epsilon}\varphi_i$, $\varphi_i \in C^\infty(M)_\epsilon$ $(i = 1,2)$, *where* $\langle\ ,\ \rangle$ *is the bilinear form given as the integral over* M *(see (2.10.1)).*
2) *The Hermitian form* $(\ ,\ )_A$ *is well-defined (namely, independent of the choice of* φ_i*) and* G*-invariant under the action of* $\varpi_{\frac{n}{2}-1}$.
3) *In comparison with the inner product* $(\ ,\ )_M$ *given in Lemma 2.6, we have*

$$(F_1, F_2)_A = c_1(F_1, f_2)_M \qquad \text{for any } F_1, F_2 \in A_{1,\epsilon}(C^\infty(M)_\epsilon), \tag{4.4.4}$$

where we put (see [11], (3.9.6))

$$c_1 := \frac{\Gamma(\frac{n-1-\epsilon}{4})}{2\pi^{\frac{n+1}{2}}}.$$

In particular, the Hermitian form $(\ ,\ )_A$ *is positive definite and the completion of a pre-Hilbert space* $(A_{1,\epsilon}(C^\infty(M)_\epsilon), (\ ,\ )_A)$ *coincides with the Hilbert space* $\overline{V^{p,q}}$ *given in § 2.6.*

4.5 In the flat picture $\mathbb{R}^{p-1,q-1}$, we have

$$-\xi_0 + \xi_{n+1} = |z'|^2 - |z''|^2 = P(z)$$

for $\xi = \iota(z)$ by (2.3.4). Then, via the injection r_2 in the commutative diagram (2.9.4), the Knapp-Stein intertwining operator

$$A^{\mathbb{R}^n}_{\lambda,\varepsilon} : \Psi^*_{\lambda+\frac{n}{2}}(C^\infty(M)_\varepsilon) \to \Phi^*_{-\lambda+\frac{n}{2}}(C^\infty(M)_\varepsilon)$$

is given by the convolution:

$$A^{\mathbb{R}^n}_{\lambda,\epsilon}\varphi \equiv A_{\lambda,\epsilon}\varphi := 2\psi_{\lambda-\frac{n}{2},\epsilon}(P(z)) * \varphi \tag{4.5.1}$$

when restricted to $C_0^\infty(\mathbb{R}^n) \subset \Psi^*_{\lambda+\frac{n}{2}}(C^\infty(M)_\epsilon)$. We are interested in the case $\lambda = 1$. Then we have

$$A^{\mathbb{R}^n}_{1,\epsilon} \circ \Psi^*_{\frac{n+2}{2}} = \Psi^*_{\frac{n-2}{2}} \circ A^M_{1,\epsilon}. \tag{4.5.2}$$

Lemma 4.5 *Retain the setting of Lemma 4.4. We put*

$$h(y) := e^{\frac{\sqrt{-1}\pi(q-1)}{2}}(y+\sqrt{-1}0)^{1-\frac{n}{2}}. \tag{4.5.3}$$

Then we have

$$\psi_{1-\frac{n}{2},\epsilon}(y) = \delta \frac{\Gamma(\frac{-1+n+\epsilon}{4})}{2\pi\sqrt{-1}}(h(y) - \overline{h(y)}). \tag{4.5.4}$$

PROOF. Solving $(y \pm \sqrt{-1}0)^\nu = y_+^\nu + e^{\pm\sqrt{-1}\pi\nu} y_-^\nu$, we have

$$\psi_{\nu,\epsilon}(y) = \frac{(e^{-\sqrt{-1}\nu\pi} - \epsilon)(y+\sqrt{-1}0)^\nu - (e^{\sqrt{-1}\nu\pi} - \epsilon)(y-\sqrt{-1}0)^\nu}{-2\sqrt{-1}\sin\pi\nu\Gamma(\frac{2\nu+3-\epsilon}{4})}.$$

Then, Lemma 4.5 follows from a residue calculation:

$$\lim_{\nu\to 1-\frac{n}{2}} \frac{e^{-\sqrt{-1}\nu\pi} - \epsilon}{\sin\pi\nu\ \Gamma(\frac{2\nu+3-\epsilon}{4})} = \lim_{a\to 0} \frac{e^{-\sqrt{-1}a\pi} - (-1)^q}{\sin\pi a\ \Gamma(\frac{5-n-\epsilon}{4} + \frac{a}{2})} = \frac{-\delta e^{\frac{\sqrt{-1}\pi(q-1)}{2}}\Gamma(\frac{-1+n+\epsilon}{4})}{\pi}.$$

4.6 Our key observation is that the special value of the Knapp-Stein operator is given by the Green function up to a scalar constant. In particular, the image of $C_0^\infty(\mathbb{R}^n)$ under the integral transform S satisfies not only the ultra-hyperbolic equation $\square_{\mathbb{R}^{p-1,q-1}} f = 0$ but also a certain decay condition at infinity that matches a conformal compactification of $\mathbb{R}^{p-1,q-1}$ (see § 2.8).

Proposition 4.6 *Retain the notation as in Proposition 4.2 and Lemma 4.4. We recall that* $\delta, \epsilon = \pm 1$ *are determined by* $p - q \mod 8$ *as in (4.4.2) and define the constant* c_2 *by*

$$c_2 := \frac{4\delta\pi^{\frac{n}{2}}\Gamma(\frac{-1+n+\epsilon}{4})}{\Gamma(\frac{n}{2}-1)}.$$

Then we have:

$$\psi_{1-\frac{n}{2},\epsilon}(P(z)) = c_2 E_0, \tag{4.6.1}$$
$$A_{1,\epsilon}^{\mathbb{R}^n}|_{C_0^\infty(\mathbb{R}^n)} = 2c_2 S. \tag{4.6.2}$$

In particular, $S(C_0^\infty(\mathbb{R}^n)) \subset \Psi^*_{\frac{n-2}{2}}(V^{p,q})$.

PROOF. The first formula follows from Lemma 4.5 and from the definition of E_0 (see (4.2.1)). The second formula then follows from the definition (4.5.1). Then, in view of (4.5.2) and (4.6.2), S extends to a G-intertwining operator between $(\varpi_{\frac{n+2}{2}}, \Psi^*_{\frac{n+2}{2}}(C^\infty(M)_\epsilon))$ and $(\varpi_{\frac{n-2}{2}}, \Psi^*_{\frac{n-2}{2}}(C^\infty(M)_\epsilon))$. Then we have

$$S(C_0^\infty(\mathbb{R}^n)) \subset A_{1,\epsilon}^{\mathbb{R}^n}(\Psi^*_{\frac{n+2}{2}}(C^\infty(M)_\epsilon)) = \Psi^*_{\frac{n-2}{2}}(A_{1,\epsilon}^M(C^\infty(M)_\epsilon)) \subset \Psi^*_{\frac{n-2}{2}}(V^{p,q}).$$

because $C_0^\infty(\mathbb{R}^n) \subset \Psi^*_{\frac{n+2}{2}}(C^\infty(M)_\epsilon)$. $\square$

4.7 We are ready to introduce an inner product on $\operatorname{Ker} \square_{\mathbb{R}^{p-1,q-1}}$ by using the Green function E_0 (see (4.2.1)), so that our minimal representation is realized here as a unitary representation.

We define a Hermitian form on $S(C_0^\infty(\mathbb{R}^n)) \subset \operatorname{Ker} \square_{\mathbb{R}^{p-1,q-1}}$ by

$$(f_1, f_2)_N := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} E_0(y-x)\varphi_1(x)\overline{\varphi_2(y)}\, dx\, dy \tag{4.7.1}$$

for $f_i = S\varphi_i = E_0 * \varphi_i$ $(1 \le i \le 2)$. The right-hand side of (4.7.1) does not depend on the choice of φ_i because of the formula

$$(f_1, f_2)_N = (f_1, \varphi_2)_{L^2(\mathbb{R}^n)} = (\varphi_1, f_2)_{L^2(\mathbb{R}^n)}.$$

We recall from Proposition 4.6 and Lemma 2.8 the inclusive relations:

$$S(C_0^\infty(\mathbb{R}^n)) \subset A^{\mathbb{R}^n}_{1,\epsilon}(\Psi^*_{\frac{n+2}{2}}(C^\infty(M)_\epsilon)) \subset \Psi^*_{\frac{n-2}{2}}(V^{p,q}) \subset \operatorname{Ker} \square_{\mathbb{R}^{p-1,q-1}} \subset C^\infty(\mathbb{R}^n). \tag{4.7.2}$$

Theorem 4.7 *Let $p \equiv q \mod 2$, $p, q \ge 2$ and $(p,q) \ne (2,2)$. Recall $n = p+q-2$ and $\epsilon = (-1)^{\frac{p-q}{2}}$.*
1) The Hermitian form $(\ ,\ )_N$ is positive definite on $S(C_0^\infty(\mathbb{R}^n))$.
2) The Hermitian form $(\ ,\ )_N$ is invariant under $\omega_{\frac{n-2}{2},\epsilon,\mathbb{R}^n}(G)$.

Let $\mathcal{H}$ be the completion of the pre-Hilbert space $(S(C_0^\infty(\mathbb{R}^n)), (\ ,\ )_N)$.
3) The Fréchet representation $\omega_{\frac{n-2}{2},\epsilon,\mathbb{R}^n}$ of G on $\Psi^*_{\frac{n-2}{2}}(C^\infty(M)_\epsilon)$ induces a representation of G on the Hilbert space $\mathcal{H}$, which we shall denote by $(\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}, \mathcal{H})$.
4) The unitary representation $(\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}, \mathcal{H})$ is irreducible.
5) The map (see (2.8.7) for the definition)

$$\begin{array}{ccc} (\Psi^*_{\frac{n-2}{2},\epsilon})^{-1} : S(C_0^\infty(\mathbb{R}^n)) & \to & V^{p,q} \\ \cap & & \cap \\ C^\infty(\mathbb{R}^n) & & C^\infty(M) \end{array}$$

extends uniquely to a unitary isomorphism between $(\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}, \mathcal{H})$ and the minimal representation $(\varpi^{p,q}, \overline{V^{p,q}})$ up to a scalar constant. More precisely, for any $f_1, f_2 \in \mathcal{H}$, we have

$$(f_1, f_2)_N = 2(F_1, F_2)_M, \tag{4.7.3}$$

where we put

$$F_i := (\Psi^*_{\frac{n-2}{2},\epsilon})^{-1} f_i, \quad (i = 1, 2).$$

Remark 1) We shall give a different proof of the first statement of Theorem 4.7 in Theorem 4.9 by using the Fourier transform of the Green kernel.

2) As we shall see in the proof of Theorem 4.7, $S(C_0^\infty(\mathbb{R}^n))$ is dense in $\Psi^*_{\frac{n-2}{2}}(V^{p,q})$ with respect to the above inner product.
3) We can realize naturally the Hilbert space $\mathcal{H}$ as a subspace of the Schwartz distributions $\mathcal{S}'(\mathbb{R}^n)$, namely, $\mathcal{F}^{-1}T(L^2(C))$, as we shall prove in Theorem 4.9.

4.8 In order to prove Theorem 4.7, we need:

Lemma 4.8 *Retain the notation of Theorem 4.7.*
1) *For any $f_1, f_2 \in S(C_0^\infty(\mathbb{R}^n))$, we have*

$$(f_1, f_2)_N = c_3(F_1, F_2)_A, \tag{4.8.1}$$

where

$$c_3 := \frac{\pi^{\frac{n}{2}}\ \Gamma(\frac{-1+n+\epsilon}{4})}{\Gamma(\frac{n}{2}-1)}.$$

We extend the Hermitian form $(\ ,\ )_N$ from $S(C_0^\infty(\mathbb{R}^n))$ to $A_{1,\epsilon}^{\mathbb{R}^n}(\Psi^*_{\frac{n+2}{2}}(C^\infty(M)_\epsilon))$ by using the right side of (4.8.1) (recall the inclusive relation (4.7.2)).
2) *The Hermitian form $(\ ,\ )_N$ is positive definite on $A_{1,\epsilon}^{\mathbb{R}^n}(\Psi^*_{\frac{n+2}{2}}(C^\infty(M)_\epsilon))$, in which $S(C_0^\infty(\mathbb{R}^n))$ is dense. In particular, the Hilbert space $\mathcal{H}$ (see Theorem 4.7) coincides with the completion of the pre-Hilbert space $A_{1,\epsilon}(\Psi^*_{\frac{n+2}{2}}(C^\infty(M)_\epsilon))$.*

We first finish the proof of Theorem 4.7, and then give a proof of Lemma 4.8.

Proof of Theorem 4.7 (1) is clear from Lemma 4.8 (1) and Lemma 4.4. Next, let us prove (4.7.3). We consider the G-intertwining operator

$$(\Psi^*_{\frac{n-2}{2},\epsilon})^{-1} : \Psi^*_{\frac{n-2}{2}}\left(A_{1,\epsilon}^{M}(C^\infty(M)_\epsilon)\right) \to A_{1,\epsilon}^{M}(C^\infty(M)_\epsilon) \subset V^{p,q},$$

or equivalently by (4.5.2),

$$(\Psi^*_{\frac{n-2}{2},\epsilon})^{-1} : A_{1,\epsilon}^{\mathbb{R}^n}\left(\Psi^*_{\frac{n+2}{2}}(C^\infty(M)_\epsilon)\right) \to V^{p,q}.$$

Combining (4.4.4) and (4.8.1), we have

$$(f_1, f_2)_N = c_3(F_1, F_2)_A = c_1 c_3 (F_1, F_2)_M = 2(F_1, F_2)_M,$$

where the second equality follows from a classical formula of the Gamma function:

$$2^{2z-1}\Gamma(z)\ \Gamma(z+\frac{1}{2}) = \sqrt{\pi}\ \Gamma(2z).$$

Thus, we have proved (4.7.3), especially, $(\Psi^*_{\frac{n-2}{2},\epsilon})^{-1}$ is an isometry up to scalar. Then $(\Psi^*_{\frac{n-2}{2},\epsilon})^{-1}$ extends to an isometric (up to scalar) G-intertwining operator

$\mathcal{H} \to \overline{V^{p,q}}$. This map is surjective because $(\varpi^{p,q}, \overline{V^{p,q}})$ is an irreducible unitary representation of G (see Lemma 2.6). All other statements are now clear. $\square$

Proof of Lemma 4.8 1) Suppose $f_i = S\varphi_i$, $\varphi_i \in C_0^\infty(\mathbb{R}^n)$ $(i = 1, 2)$. Recall the constant c_2 in Proposition 4.6, we have

$$\begin{aligned}
(f_1, f_2)_N &= \int_{\mathbb{R}^n} (S\varphi_1)(x)\overline{\varphi_2(x)}dx \\
&= \frac{1}{2c_2}\int_{\mathbb{R}^n} (A_{1,\epsilon}^{\mathbb{R}^n}\varphi_1)(x)\ \overline{\varphi_2(x)}\ dx \\
&= \frac{1}{4c_2}\int_M A_{1,\epsilon}^M((\Psi^*_{\frac{n+2}{2},\epsilon})^{-1}\varphi_1)(b)\ \overline{(\Psi^*_{\frac{n+2}{2},\epsilon})^{-1}\varphi_2(b)}\ db. \qquad (4.8.2)
\end{aligned}$$

We put

$$\psi_i := \frac{1}{c_2}(\Psi^*_{\frac{n+2}{2},\epsilon})^{-1}\varphi_i \ \in C^\infty(M)_\epsilon, \quad (i = 1, 2).$$

Then $F_i = A_{1,\epsilon}^M\psi_i$ by (4.5.2). Therefore,

$$(4.8.2) = \frac{1}{4c_2}c_2^2\int_M (A_{1,\epsilon}^M\psi_1)(b)\ \overline{\psi_2(b)}\ db = \frac{\delta c_2}{4}(F_1, F_2)_A$$

which equals the right side of (4.8.1).
2) It is enough to show that $S(C_0^\infty(\mathbb{R}^n))$ is dense in a pre-Hilbert space $A_{1,\epsilon}(\Psi^*_{\frac{n+2}{2}}(C^\infty(M)_\epsilon))$ because the inner product $(\ ,\ )_A$ is positive definite from Lemma 4.4.

Suppose $f = A_{1,\epsilon}\varphi$ $(\varphi \in \Psi^*_{\frac{n+2}{2}}(C^\infty(M)_\epsilon))$ is orthogonal to $S(C_0^\infty(\mathbb{R}^n))$ with respect to the inner product $(\ ,\ )_A$. This means that $(f, A_{1,\epsilon}\phi)_A = 0$ for any $\phi \in C_0^\infty(\mathbb{R}^n)$. Then we have

$$\int_M (\Psi^*_{\frac{n-2}{2},\epsilon})^{-1}f(b)\ \overline{(\Psi^*_{\frac{n-2}{2},\epsilon})^{-1}\phi(b)}\ db = 0 \qquad \text{for any } \phi \in C_0^\infty(\mathbb{R}^n).$$

Since $(\Psi^*_{\frac{n-2}{2},\epsilon})^{-1}(C_0^\infty(\mathbb{R}^n))$ is dense in

$$L^2(M)_\epsilon := \{f \in L^2(M) : f(-u) = \epsilon f(u)\},$$

we have $(\Psi^*_{\frac{n-2}{2},\epsilon})^{-1}f = 0$, and thus $f = 0$. $\square$

4.9 We recall $\Psi^*_{\frac{n-2}{2}}(V^{p,q}) \subset \Psi^*_{\frac{n-2}{2}}(C^\infty(M)_\epsilon) \subset \mathcal{S}'(\mathbb{R}^n)$, on which we can define the Fourier transform $\mathcal{F}$. We consider the following maps (see § 4.3, § 3.2, § 3.4):

$$C_0^\infty(\mathbb{R}^n) \xrightarrow{S} \Psi^*_{\frac{n-2}{2}}(V^{p,q}) \xrightarrow{\mathcal{F}} \mathcal{S}'(\mathbb{R}^n) \xhookleftarrow{T} L^2(C)$$

Here is a description of the Fourier transform of the minimal unitary representation $(\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}, \mathcal{H})$ of $G = O(p,q)$ which is obtained as the completion of a pre-Hilbert space $(S(C_0^\infty(\mathbb{R}^n)), (\ ,\ )_N)$ (see Theorem 4.7).

Theorem 4.9 1) *$\mathcal{F}S(C_0^\infty(\mathbb{R}^n))$ is contained in $T(L^2(C))$. Furthermore, it is dense in the Hilbert space $T(L^2(C))$.*
2) *$T^{-1} \circ \mathcal{F} : S(C_0^\infty(\mathbb{R}^n)) \to L^2(C)$ extends uniquely to a unitary map $\mathcal{H} \to L^2(C)$ up to a scalar constant. This constant is given explicitly by*

$$(2\pi)^n (f,f)_N = \|T^{-1} \circ \mathcal{F} f\|^2_{L^2(C)}. \tag{4.9.1}$$

3) *The $\overline{P^{\max}}$-module $(\pi, L^2(C))$ extends to an irreducible unitary representation of G, denoted by the same letter π, so that $(2\pi)^{-\frac{n}{2}} T^{-1} \circ \mathcal{F}$ gives a unitary equivalence between $(\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}, \mathcal{H})$ and $(\pi, L^2(C))$.*
4) *$T^{-1} \circ \mathcal{F} \circ \Psi^*_{\frac{n-2}{2}} : V^{p,q} \to L^2(C)$ induces a unitary equivalence between irreducible unitary representations $(\varpi^{p,q}, \overline{V^{p,q}})$ and $(\pi, L^2(C))$, up to a scalar constant given by:*

$$(\phi,\phi)_M = 2(2\pi)^n \|T^{-1} \circ \mathcal{F} \circ \Psi^*_{\frac{n-2}{2}} \phi\|^2_{L^2(C)}. \tag{4.9.2}$$

PROOF. If $f = E_0 * \varphi$ $(\varphi \in C_0^\infty(\mathbb{R}^n))$, then it follows from the integration formula of the Green function (see Proposition 4.2) that its Fourier transform is given by

$$\mathcal{F}f = \mathcal{F}(E_0 * \varphi) = (\mathcal{F}E_0)(\mathcal{F}\varphi) = (\mathcal{F}\varphi)\delta(Q) = T((\mathcal{F}\varphi)|_C).$$

Since $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have $(\mathcal{F}\varphi)|_C \in L^2(C)$. Hence, $\mathcal{F}S(C_0^\infty(\mathbb{R}^n))$ is contained in $T(L^2(C))$. Then we have (4.9.1), as follows from

$$\|(\mathcal{F}\varphi)|_C\|^2_{L^2(C)} = ((\mathcal{F}E_0)(\mathcal{F}\varphi), \mathcal{F}\varphi) = (\mathcal{F}(E_0 * \varphi), \mathcal{F}\varphi) = (2\pi)^n (f,f)_N. \tag{4.9.3}$$

We note that (4.9.3) gives a different proof that $(\ ,\ )_N$ is a positive definite Hermitian form on $S(C_0^\infty(\mathbb{R}^n))$ (see Remark after Theorem 4.7).

It follows from Lemma 3.4 that $T^{-1} \circ \mathcal{F}$ is an $\overline{P^{\max}}$-intertwining operator from $(\varpi_{\frac{n-2}{2},\epsilon,\mathbb{R}^n}|_{\overline{P^{\max}}}, S(C_0^\infty(\mathbb{R}^n)))$ to $(\pi, L^2(C))$. This map is isometric up to a scalar by (4.9.1). Then, it extends naturally to an $\overline{P^{\max}}$-intertwining operator from $(\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}|_{\overline{P^{\max}}}, \mathcal{H})$ to $(\pi, L^2(C))$, which is surjective because $(\pi, L^2(C))$ is irreducible (see Proposition 3.3). Hence we have proved (1), (2) and (3). The statement (4) follows from (2) and Theorem 4.7. Thus, we have proved Theorem 4.9. $\square$

4.10 By semisimple theory, it is known that a minimal representation is still irreducible when restricted to any maximal parabolic subgroup. In particular, $\varpi^{p,q}$ is irreducible as an $\overline{P^{\max}}$-module. In our case, this fact can be strengthened as follows:

Corollary 4.10 *The restriction of the minimal representation $\varpi^{p,q}$ (equivalently, $\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}$) to $M_+^{\max}\overline{N^{\max}} \simeq O(p-1,q-1) \ltimes \mathbb{R}^{p+q-2}$ is still irreducible.*

PROOF. Theorem 4.7 and Theorem 4.9 show that $\varpi^{p,q}$, $\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}$ and π are unitary equivalent to one another. Now, Corollary follows from Proposition 3.3. $\square$

4.11 For the convenience of the reader, we summarize the maps used in the proofs.

$$\begin{array}{ccccccc}
C^\infty(M) & \xrightarrow{\Psi^*_{\frac{n-2}{2}}} & C^\infty(\mathbb{R}^n)\cap \mathcal{S}'(\mathbb{R}^n) & \xrightarrow{\mathcal{F}} & \mathcal{S}'(\mathbb{R}^n) & & \\
\cup & & \cup & & \cup & & \\
V^{p,q} & \xrightarrow{\sim} & \Psi^*_{\frac{n-2}{2}}(V^{p,q}) & \xrightarrow{\sim} & \mathcal{F}\Psi^*_{\frac{n-2}{2}}(V^{p,q}) & & \\
\cap\,\text{dense} & & \cap\,\text{dense} & & \cap\,\text{dense} & & \\
(\varpi^{p,q}, \overline{V^{p,q}}) & \xrightarrow{\sim} & (\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}, \mathcal{H}) & \xrightarrow{\sim} & T(L^2(C)) & \underset{T}{\xleftarrow{\sim}} & (\pi, L^2(C)).
\end{array} \tag{4.11.1}$$

In the last line, we have written also the notation for unitary representations.

5 Bessel function and an integral formula of spherical functions

5.1 In this section, we shall compute explicitly the lowest K-type of our minimal representation in the N-picture, i.e. find it as a solution to $\square_{\mathbb{R}^{p-1,q-1}} f = 0$, and also its Fourier transform as a function on C; this turns out to be written in terms of a Bessel function (see Theorem 5.5). Note that except when $p = q$ we are not dealing with a spherical representation of G (namely, there is no non-zero K-fixed vector in our representation). At the end of this section, we reformulate the equivalent realizations of the minimal representation found in the previous section, using now the minimal K-type to understand the different pictures.

5.2 Without loss of generality, we may and do assume $p \geq q$ in this section. Instead of K-fixed vectors, our idea here is to focus on an $O(p)\times O(q-1)$-fixed vector. Then, it follows from Lemma 2.6 that such a vector, which we shall denote by F_0, is unique in our minimal representation $(\varpi^{p,q}, V^{p,q})$ up to a scalar multiple, and is contained in the minimal K-type of the form $\mathbf{1} \boxtimes \mathcal{H}^{\frac{p-q}{2}}(\mathbb{R}^q)$. (We note that this K-type is not one dimensional if $p \neq q$).

We shall find an explicit formula of the Fourier transform of this vector F_0 after a conformal change of coordinates. We start with the following classical lemma, for which we give a proof for the sake of the completeness. We take a coordinate $(u_{p+1}, \ldots, u_{p+q})$ in $\mathbb{R}^q$ and realize $O(q-1)$ in $O(q)$ such that it stabilizes the last coordinate u_{p+q}.

Lemma 5.2 *For any $l \in \mathbb{N}$, $O(q-1)$-invariant spherical harmonics of degree l form a one dimensional vector space. More precisely, we have*

$$\mathcal{H}^l(\mathbb{R}^q)^{O(q-1)} \simeq \mathbb{C}\ {}_2F_1(\frac{-l}{2}, \frac{q-2+l}{2}; \frac{q-1}{2}; {u_{p+1}}^2 + \cdots + {u_{p+q-1}}^2).$$

PROOF. In terms of the polar coordinate of S^{q-1}:

$$\Phi_{++} : S^{q-2} \times (0, \frac{\pi}{2}) \to S^{q-1},\ (y, \theta) \mapsto ((\sin\theta)y, \cos\theta), \tag{5.2.1}$$

the Laplace-Beltrami operator on S^{q-1} takes the form:

$$\Delta_{S^{q-1}} = \frac{\partial^2}{\partial\theta^2} + (q-2)\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\Delta_{S^{q-2}}. \tag{5.2.2}$$

If $F \in \mathcal{H}^l(\mathbb{R}^q)$ is $O(q-1)$-invariant, then $F \circ \Phi_{++}(\theta, y)$ depends only on θ, for which we write $h(\theta)$. Then $h(\theta)$ is an even function satisfying:

$$(\frac{d^2}{d\theta^2} + (q-2)\cot\theta\frac{d}{d\theta} + l(l+q-2))h(\theta) = 0.$$

Since $h(\theta)$ is regular at $\theta = 0$, it is a scalar multiple of the Jacobi function:

$$\begin{aligned}\varphi^{\frac{q-3}{2}, -\frac{1}{2}}_{\sqrt{-1}(l+\frac{q-2}{2})}(\sqrt{-1}\theta) &= {}_2F_1(\frac{-l}{2}, \frac{q-2+l}{2}; \frac{q-1}{2}; \sin^2\theta) \\ &= {}_2F_1(-l, q-2+l; \frac{q-1}{2}; \frac{1-\cos\theta}{2}).\end{aligned}$$

Thus, we have proved the lemma. □

5.3 In view of Lemma 2.6, the special case of Lemma 5.2 with $l = \frac{p-q}{2}$ yields:

Proposition 5.3 *Suppose $p \geq q \geq 2$, $p+q \in 2\mathbb{N}$ and $(p,q) \neq (2,2)$. Let $(u_1, \ldots, u_{p+q})$ be the coordinate of $M = S^{p-1} \times S^{q-1}$ in $\mathbb{R}^{p+q}$. We define a function $F_0 : M \to \mathbb{C}$ by*

$$F_0(u_1, \ldots, u_{p+q}) := {}_2F_1(\frac{q-p}{4}, \frac{p+q-4}{4}; \frac{q-1}{2}; {u_{p+1}}^2 + \cdots + {u_{p+q-1}}^2). \tag{5.3.1}$$

Then F_0 is an $O(p) \times O(q-1)$-invariant analytic function on M satisfying the Yamabe equation $\widetilde{\Delta}_M F_0 = 0$. Conversely, any such function is a scalar multiple of F_0.

Remark If $p+q$ is odd, then F_0 in the right side of (5.3.1) still gives a solution $\widetilde{\Delta}_M F = 0$ on an open dense set of M such that $u_{p+q} \neq 0$. Furthermore, F_0 is a continuous function on M. However, it does not solve the Yamabe equation as a distribution on M.

5.4 We recall $\tau(z', z'')$ is a conformal factor defined in (2.8.1). Let us define $f_0 := \Psi^*_{\frac{n-2}{2}} F_0$, namely,

$$f_0(z', z'') := \tau(z', z'')^{-\frac{p+q-4}{2}} {}_2F_1\Big(\frac{q-p}{4}, \frac{p+q-4}{4}; \frac{q-1}{2}; \frac{|z''|^2}{\tau(z', z'')^2}\Big). \qquad (5.4.1)$$

We note that $|\tau(z', z'')| \geq |z''|$ for any $(z', z'') \in \mathbb{R}^{p-1,q-1}$. The equality holds if and only if $|z'|^2 - |z''|^2 = -4$.

The following Proposition is immediate from Lemma 2.8 and Proposition 5.3:

Proposition 5.4 *With the same assumption on p, q in Proposition 5.3, we have:*
1) f_0 is a real analytic function on $\mathbb{R}^n$ that solves $\square_{\mathbb{R}^{p-1,q-1}} f_0 = 0$.
2) f_0 is $O(p) \times O(q-1)$-invariant.

We say F_0 is the *generating function* of $V^{p,q} = \operatorname{Ker} \widetilde{\Delta}_M$, and f_0 is that of $(\varpi^{\min}_{\mathbb{R}^{p-1,q-1}}, \mathcal{H})$.

Remark More strongly than Proposition 5.4, one can prove that f_0 is a real analytic solution of $\square_{\mathbb{R}^{p-1,q-1}} f_0 = 0$ if $p+q > 4$ by using Proposition 5.6, where we do not assume that $p+q$ is even. The real analyticity is not obvious from the expression (5.4.1) in the neighborhood of the hypersurface of $|z'|^2 - |z''|^2 = -4$.

5.5 We recall the definitions of Bessel functions:

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}z)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)} \qquad \text{(Bessel function)},$$

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)} \qquad \text{(modified Bessel function of the first kind)},$$

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} \qquad \text{(modified Bessel function of the second kind)}.$$

Then K_ν satisfies

$$\left(z^2 \frac{d^2}{dz^2} + z\frac{d}{dz} - (z^2 + \nu^2) \right) K_\nu(z) = 0.$$

The asymptotic of the functions K_ν is well-known, for example $z^{-\nu}K_\nu(z)$ decays exponentially as $z \to +\infty$.

Theorem 5.5 *We put* $|\zeta| := ({\zeta_1}^2 + \cdots + {\zeta_n}^2)^{\frac{1}{2}}$ *for* $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$. *Let* $F_0 \in C^\infty(M)$ *be the generating function of* $V^{p,q} = \operatorname{Ker} \widetilde{\Delta}_M$ *(see Proposition 5.3). Then*

$$(\mathcal{F}\Psi^*_{\frac{n-2}{2}} F_0)(\zeta) = (2\pi)^{\frac{p+q-2}{2}} 2^{-\frac{p-5}{2}} \frac{\Gamma(\frac{q-1}{2})}{\Gamma(\frac{p+q-4}{2})} |\zeta|^{\frac{3-q}{2}} K_{\frac{q-3}{2}}(2|\zeta|)\delta(Q).$$

Note that the $|\zeta|^{\frac{3-q}{2}} K_{\frac{q-3}{2}}(2|\zeta|)$ belongs to the Hilbert space $L^2(C)$ if $p+q > 4$ by the asymptotic behaviour of the Bessel function K_ν and by the explicit form of $\delta(Q)$ in (3.3.3).

5.6 Theorem 5.5 follows from the following Proposition:

Proposition 5.6 *We write* $\mathcal{F}^{-1}$ *for the inverse Fourier transform. With notation in (5.4.1), we have*

$$\mathcal{F}^{-1}\left(|\zeta|^{\frac{3-q}{2}} K_{\frac{q-3}{2}}(2|\zeta|)\delta(Q)\right)(z) = \frac{\Gamma(\frac{p+q-4}{2})}{2^{\frac{q+3}{2}} \pi^{\frac{p+q-2}{2}} \Gamma(\frac{q-1}{2})} f_0(z). \tag{5.6.1}$$

PROOF. Let $\phi(r)$ be a function of one variable, which will be taken later to be $r^{\frac{3-q}{2}} K_{\frac{q-3}{2}}(2r)$. Then, it follows from (3.3.3) that for $z = (z', z'') \in \mathbb{R}^{p-1,q-1}$,

$$\begin{aligned} &2\mathcal{F}^{-1}\left(\phi(|\zeta|)\delta(Q)\right)(z) \\ &= (2\pi)^{-(p+q-2)} \int_0^\infty \int_{S^{p-2}} \int_{S^{q-2}} \phi(r) e^{-\sqrt{-1}((z', r\omega)+(z'', r\eta))}\, r^{p+q-5} dr\, d\omega\, d\eta. \end{aligned}$$

Using the formula $\int_{S^{m-1}} e^{\sqrt{-1}t(\eta,\omega)} d\omega = (2\pi)^{\frac{m}{2}} t^{1-\frac{m}{2}} J_{\frac{m}{2}-1}(t)$, we have

$$\begin{aligned} =&(2\pi)^{-\frac{p+q-2}{2}} \int_0^\infty \phi(r)\ (r|z'|)^{\frac{3-p}{2}} J_{\frac{p-3}{2}}(r|z'|)(r|z''|)^{\frac{3-q}{2}} J_{\frac{q-3}{2}}(r|z''|) r^{p+q-5}\ dr \\ =&(2\pi)^{-\frac{p+q-2}{2}} |z'|^{\frac{3-p}{2}} |z''|^{\frac{3-q}{2}} \int_0^\infty \phi(r) J_{\frac{p-3}{2}}(r|z'|) J_{\frac{q-3}{2}}(r|z''|) r^{\frac{p+q-4}{2}}\ dr. \end{aligned}$$

Now, put $\phi(r) := r^{\frac{3-q}{2}} K_{\frac{q-3}{2}}(2r)$. We use the following formula of the Hankel transform due to Bailey [2] (see also [6], § 19.6 (8))

$$
\begin{aligned}
&\int_0^\infty t^{\lambda-1} J_\mu(at) J_\nu(bt) K_\rho(ct) dt \\
=&\frac{2^{\lambda-2} a^\mu b^\nu \Gamma(\frac{1}{2}(\lambda+\mu+\nu-\rho))\Gamma(\frac{1}{2}(\lambda+\mu+\nu+\rho))}{c^{\lambda+\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)} \\
&\times F_4(\frac{1}{2}(\lambda+\mu+\nu-\rho), \frac{1}{2}(\lambda+\mu+\nu+\rho); \mu+1, \nu+1; -\frac{a^2}{c^2}, -\frac{b^2}{c^2}).
\end{aligned}
$$

Here, F_4 is the Appell hypergeometric function of two variables, defined by

$$
F_4(a, b; c, d; x, y) = \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(a)_{i+j}(b)_{i+j}}{i!j!(c)_i(d)_j} x^i y^j.
$$

Then we have:

$$
\begin{aligned}
&\mathcal{F}^{-1}\left(\phi(|\zeta|)\delta(Q)\right)(z) \\
=&(2\pi)^{-\frac{p+q-2}{2}} |z'|^{\frac{3-p}{2}} |z''|^{\frac{3-q}{2}} \int_0^\infty K_{\frac{q-3}{2}}(2r) J_{\frac{p-3}{2}}(r|z'|) J_{\frac{q-3}{2}}(r|z''|) r^{\frac{p-1}{2}}\ dr \\
=&(2\pi)^{-\frac{p+q-2}{2}} 2^{\frac{p-3}{2}} \frac{\Gamma(\frac{p+q-4}{2})}{\Gamma(\frac{q-1}{2})} F_4(\frac{p-1}{2}, \frac{p+q-4}{2}; \frac{p-1}{2}, \frac{q-1}{2}; \frac{-|z'|^2}{4}, \frac{-|z''|^2}{4}).
\end{aligned}
$$

5.7 Then, the proof of Proposition 5.6 will be finished by showing the following reduction formula:

Lemma 5.7 *Let $\tau(z', z'')$ be the conformal factor defined in (2.8.1). We have*

$$
\begin{aligned}
&F_4(\frac{p-1}{2}, \frac{p+q-4}{2}; \frac{p-1}{2}, \frac{q-1}{2}; \frac{-|z'|^2}{4}, \frac{-|z''|^2}{4}) \\
=&\tau(z', z'')^{-\frac{p+q-4}{2}} {}_2F_1(\frac{q-p}{4}, \frac{p+q-4}{4}; \frac{q-1}{2}; \frac{|z''|^2}{\tau(z', z'')^2}).
\end{aligned}
\tag{5.7.1}
$$

Proof of Lemma 5.7 We recall a reduction formula of Appell's hypergeometric functions (see [5], § 5.10, (8)):

$$
\begin{aligned}
&F_4(\alpha, \beta; 1+\alpha-\beta, \beta; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}) \\
=&(1-y)^\alpha {}_2F_1(\alpha, \beta; 1+\alpha-\beta; \frac{-x(1-y)}{1-x})
\end{aligned}
\tag{5.7.2}
$$

and a quadratic transformation for hypergeometric functions (see [5], § 2.11 (32)):

$$ {}_2F_1(\alpha,\beta;1+\alpha-\beta;z) = (1-z)^{-\alpha}{}_2F_1(\frac{\alpha}{2},\frac{\alpha+1-2\beta}{2};1+\alpha-\beta;\frac{-4z}{(1-z)^2}). \tag{5.7.3} $$

Combining (5.7.2) with (5.7.3) for $\alpha = \frac{p+q-4}{2}$ and $\beta = \frac{p-1}{2}$, and using the symmetry of a and b; (c,x) and (d,y) in $F_4(a,b;c,d;x,y)$, we have

$$ \begin{aligned} &F_4(\frac{p-1}{2},\frac{p+q-4}{2};\frac{p-1}{2},\frac{q-1}{2};\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}) \\ &= \left(\frac{(1-x)(1-y)}{1-xy}\right)^{\frac{p+q-4}{2}} {}_2F_1(\frac{p+q-4}{4},\frac{q-p}{4};\frac{q-1}{2};\frac{4y(1-x)(1-y)}{(1-xy)^2}). \end{aligned} $$

If we put

$$ \frac{|z'|^2}{4} = \frac{x}{(1-x)(1-y)}, \quad \frac{|z''|^2}{4} = \frac{y}{(1-x)(1-y)} $$

then a simple computation shows

$$ \tau(z',z'')^2 = \left(\frac{1-xy}{(1-x)(1-y)}\right)^2, \quad \frac{|z''|^2}{\tau(z',z'')^2} = \frac{4y(1-x)(1-y)}{(1-xy)^2}. $$

Thus, Lemma 5.7 is proved. $\square$

5.8 Using the generating function F_0 we may recover the whole representation by letting the Lie algebra of G act. Let us see how our previous results may be reformulated: It follows from the definition of $\widehat{\varpi}_{\frac{n-2}{2},\epsilon}$ (see § 3.2) that the linear map

$$ \mathcal{F}\circ\Psi^*_{\frac{n-2}{2}} : C^\infty(M) \to \mathcal{S}'(\mathbb{R}^n) $$

induces a natural intertwining map from $(\varpi^{p,q}, V^{p,q})$ to $(\widehat{\varpi}_{\frac{n-2}{2},\epsilon}, \mathcal{F}\Psi^*_{\frac{n-2}{2}}(V^{p,q}))$ as G-modules and also as $\mathfrak{g}$-modules. Here again $\epsilon = (-1)^{\frac{p-q}{2}}$, $V^{p,q} = \operatorname{Ker}\widetilde{\Delta}_M$ $(\subset C^\infty(M))$ and $\Psi^*_{\frac{n-2}{2}}(V^{p,q}) \subset \operatorname{Ker}\square_{\mathbb{R}^{p-1,q-1}}$.

For $b\in\mathbb{R}^n, m\in O(p-1,q-1)$, we define a function on the cone C by

$$ \psi_{b,m}(\zeta) := e^{\sqrt{-1}\langle b,\zeta\rangle}|m\zeta|^{\frac{3-q}{2}}K_{\frac{q-3}{2}}(2|m\zeta|). \tag{5.8.1} $$

In particular, we have

$$ \psi_{0,e}(\zeta) = |\zeta|^{\frac{3-q}{2}}K_{\frac{q-3}{2}}(2|\zeta|). $$

Here we give explicit functions which are dense in the minimal representations.

Theorem 5.8 *Suppose $p \geq q \geq 2$, $p+q \in 2\mathbb{N}$, and $(p,q) \neq (2,2)$.*
1) *$\psi_{0,e}(\zeta)$ is a K-finite vector of $(\pi, L^2(C))$. It belongs to the minimal K-type of π.*
2) *$\psi_{b,m}$ is a K-finite vector and $\mathbb{C}-\mathrm{span}\{\psi_{b,m} : b \in \mathbb{R}^n, m \in O(p-1,q-1)\}$ is a dense subspace of the minimal representation $(\pi, L^2(C))$.*
2′) *$\mathbb{C}-\mathrm{span}\{\psi_{b,m}\delta(Q) : b \in \mathbb{R}^n, m \in O(p-1,q-1)\}$ is a dense subspace of the minimal representation $(\widehat{\varpi}_{\frac{n-2}{2},\epsilon}, \mathcal{F}\Psi^*_{\frac{n-2}{2}}(\overline{V^{p,q}}))$.*

PROOF. This follows by combining Theorem 4.9 and Lemma 5.5, and by using the Mackey theory we have an irreducible representation of the parabolic group. $\square$

5.9 The advantage in the realization on $L^2(C)$ is that the action for $\overline{P^{\max}}$ and the inner product are easily described. On the other hand, the action of K is not easy to describe, and especially, the K-finiteness in the statement (2) is non-trivial.

Let $U(\mathfrak{g})$ be the enveloping algebra of the complexified Lie algebra $\mathfrak{g}$. We define a subspace of $\mathcal{S}'(\mathbb{R}^n)$ by

$$U := d\widehat{\varpi}_{\frac{n-2}{2},\epsilon}(U(\mathfrak{g}))(\psi_{0,e}\delta(Q)). \tag{5.9.1}$$

That is, U is the linear span of a Bessel function $|\zeta|^{\frac{3-q}{2}} K_{\frac{q-3}{2}}(2|\zeta|)d\mu$ on the cone C and its iterative differentials corresponding to the action of the Lie algebra $\mathfrak{g}$.

We have seen that $\mathcal{F}\Psi^*_{\frac{n-2}{2}}(V^{p,q}) \subset T(L^2(C))$ in Theorem 4.9. We may restate in this way:

Theorem 5.9 1) *U is an infinitesimally unitary $(\mathfrak{g}, K)$-module via $\widehat{\varpi}_{\frac{n-2}{2},\epsilon}$.*
2) *U is dense in the Hilbert space $T(L^2(C))$.*
3) *The completion of (1) defines an irreducible unitary representation of G on $T(L^2(C))$, and then also on $L^2(C)$. This gives an extension of π from $\overline{P^{\max}}$ to G.*

This has already been done by Theorem 4.9 and the irreducibility of the minimal representation.

One of non-trivial parts of the above assertion is to show

$$U \cap T(L^2(C)) \neq \{0\}$$

which was proved in Theorem 5.5.

6 Explicit inner product on solutions $\square_{\mathbb{R}^{p-1,q-1}} f = 0$

6.1 The aim of this section is to provide an explicit inner product on a certain subspace (see (6.2.1)) of solutions of the ultrahyperbolic equation $\square_{\mathbb{R}^{p-1,q-1}} f = 0$, such that its Hilbert completion gives the unitarization of the minimal representation of $O(p,q)$.

Roughly speaking, the inner product will be given in terms of the integration over a hyperplane after convoluting a distribution along the normal direction.

We assume $n > 2$. We fix $i \in \{1, 2, \dots, n\}$ once and for all. The hyperplane on which we integrate will be $\{z = (z_1, \dots, z_n) \in \mathbb{R}^n : z_i = 0\}$, for which we simply write $\mathbb{R}^{n-1}$.

6.2 Let $C^{(i)} := \{\zeta \in C : \zeta_i \neq 0\}$, an open dense subset of the null cone C (see (3.3.2)). We note that $C_0^\infty(C \setminus \{0\})$ is dense in $L^2(C)$. We define a subspace of solutions of $\operatorname{Ker} \square_{\mathbb{R}^{p-1,q-1}}$ by

$$W := \mathcal{F}^{-1} \circ T(C_0^\infty(C^{(i)})). \tag{6.2.1}$$

Here, we recall $T : L^2(C) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is the embedding via the measure $d\mu$ on the cone C. By the Paley-Wiener theorem for compactly supported distributions, W consists of real analytic solutions of $\square_{\mathbb{R}^{p-1,q-1}} f = 0$.

Using an interpretation of the Dirac delta function in terms of hyperfunctions:

$$\delta(z_i) = \frac{1}{2\pi\sqrt{-1}} \left(\frac{1}{z_i - \sqrt{-1}0} - \frac{1}{z_i + \sqrt{-1}0} \right),$$

we decompose $f \in W$ as

$$f(z) = f_+^{(i)}(z) + f_-^{(i)}(z), \tag{6.2.2}$$

where $f_\pm^{(i)}(z)$ is defined by the convolution in the z_i-variable:

$$f_\pm^{(i)}(z) := \frac{1}{2\pi\sqrt{-1}} \cdot \frac{\mp 1}{z_i \pm \sqrt{-1}0} * f(z). \tag{6.2.3}$$

We shall see later that the decomposition (6.2.2) makes sense not only for $f \in W$ but also for any $f \in \mathcal{F}^{-1} \circ T(L^2(C))$ (see Lemma 6.5). We set

$$(f, f)_W := \frac{1}{\sqrt{-1}} \int_{\mathbb{R}^{n-1}} \left(f_+^{(i)} \overline{\frac{\partial f_+^{(i)}}{\partial z_i}} - f_-^{(i)} \overline{\frac{\partial f_-^{(i)}}{\partial z_i}} \right) |_{z_i=0} \, dz_1 \cdots \widehat{dz_i} \cdots dz_n. \tag{6.2.4}$$

Theorem 6.2 *Fix any $i \in \{1, 2, \dots, n\}$.*
1) *The formula (6.2.4) defines a positive definite Hermitian form on W, a*

subspace of solutions of the ultrahyperbolic operator $\Box_{\mathbb{R}^{p-1,q-1}} = \frac{\partial^2}{\partial {z_1}^2} + \cdots + \frac{\partial^2}{\partial {z_{p-1}}^2} - \frac{\partial^2}{\partial {z_p}^2} - \cdots - \frac{\partial^2}{\partial {z_{p+q-2}}^2}$ *on* $\mathbb{R}^{p-1,q-1}$.
2) *The inner product (6.2.4) is independent of the choice of* $i \in \{1, 2, \ldots, n\}$.
3) *The action of* $G = O(p,q)$ *preserves the inner product (6.2.4), so that the Hilbert completion* $\overline{W}$ *of* W *defines a unitary representation of* $G = O(p,q)$.
4) *The resulting unitary representation is unitarily equivalent to the minimal representation* $(\varpi^{p,q}, \overline{V^{p,q}})$. *The* G*-intertwining operator* $\Psi^*_{\frac{n-2}{2}} : \overline{V^{p,q}} \to \overline{W}$ *gives a unitary equivalence up to a scalar constant.*
5) $c\mathcal{F}^{-1} \circ T : L^2(C) \to \overline{W}$ *is a unitary* G*-intertwining operator, if we put* $c = 2^{\frac{n+2}{2}} \pi^{\frac{n+1}{2}}$.

This Theorem gives a new formulation of the Hilbert space of the minimal representation purely in terms of intrinsic objects in the flat space $\mathbb{R}^{p-1,q-1}$ where the differential equation is the classical ultrahyperbolic one. It generalizes the $q = 2$ case where an inner product was known in terms of integration of Cauchy data - here one could interpret the inner product in terms of the energy generator. The interesting property about the inner product is its large invariance group; even translational invariance amounts to a remarkable "conservation law", and we may also note that the integration over a coordinate hyperplane can be replaced by integration over any non-characteristic hyperplane (since such a hyperplane is conjugate to either $z_1 = 0$ or $z_n = 0$ by $O(p-1, q-1) \ltimes \mathbb{R}^{p-1,q-1}$), or even the image of such a hyperplane under conformal inversion.

The strategy of the proof of Theorem 6.2 is as follows: We recall from Theorem 4.7 that the Hilbert space $\mathcal{H}$ is the completion of the space $S(C_0^\infty(\mathbb{R}^n))$ with respect to another inner product $(\ ,\ )_N$ (S is an integral transform by the Green kernel). Since $C_0^\infty(C \setminus \{0\})$ is dense in $L^2(C)$ and since $T^{-1} \circ \mathcal{F} : \mathcal{H} \to L^2(C)$ is an isomorphism of Hilbert spaces (up to a scalar) by Theorem 4.9 (1), W is a dense subspace of the Hilbert space $(\mathcal{H}, (\ ,\ )_N)$. In light of this, the key ingredient of the proof of Theorem 6.2 is to give a formula of $(\ ,\ )_W$ by means of $(\ ,\ )_{L^2(C)}$. We shall prove:

$$2(2\pi)^{n+1}(f, f)_W = \|T^{-1} \circ \mathcal{F} f\|^2_{L^2(C)} \quad \text{for any } f \in W. \tag{6.2.5}$$

Once we prove (6.2.5), it follows from (4.9.1) that

$$4\pi(\ ,\ )_W = (\ ,\ )_N$$

on the subspace W. In particular, we have

$$\mathcal{H} = \overline{W},$$

and all other statements of Theorem 6.2 on our inner product $(\ ,\ )_W$ are clear from the corresponding results on the inner product $(\ ,\ )_N$ proved in

Theorem 4.7 and Theorem 4.9 (e.g. Theorem 6.2 (4) corresponds to Theorem 4.7 (5); Theorem 6.2 (5) to Theorem 4.9 (3)).

The rest of this section is devoted to the proof of the formula (6.2.5).

6.3 We define an open subset of the cone C by

$$C_{\pm}^{(i)} := \{\zeta \in C : \pm\zeta_i > 0\}.$$

Then

$$C^{(i)} := C_{+}^{(i)} \cup C_{-}^{(i)}$$

is an open dense subset of the cone C, and we have a direct sum decomposition of the Hilbert space:

$$L^2(C) = L^2(C_{+}^{(i)}) \oplus L^2(C_{-}^{(i)}).$$

We define the Heaviside function $Y_{\pm}^{(i)}(\zeta)$ of the variable ζ_i by

$$Y_{\pm}^{(i)}(\zeta_1, \dots, \zeta_n) = \begin{cases} 1 & \text{if } \pm\zeta_i > 0, \\ 0 & \text{if } \pm\zeta_i \le 0, \end{cases} \qquad \text{for } \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n.$$

For $\phi \in C_0^{\infty}(C^{(i)})$, we put

$$\phi_{\pm}^{(i)} := Y_{\pm}^{(i)} \phi.$$

Then $\operatorname{Supp} \phi_{\pm}^{(i)} \subset C_{\pm}^{(i)}$, and we have

$$\phi = \phi_{+}^{(i)} + \phi_{-}^{(i)}, \tag{6.3.1}$$

$$\|\phi\|_{L^2(C)}^2 = \|\phi_{+}^{(i)}\|_{L^2(C)}^2 + \|\phi_{-}^{(i)}\|_{L^2(C)}^2. \tag{6.3.2}$$

6.4 Let us take the $n-1$ variables $\zeta_1, \dots, \widehat{\zeta_i}, \dots, \zeta_n$ as a coordinate on $C_{\pm}^{(i)}$. Then we have

$$\zeta_i = \pm\sqrt{Q^{(i)}(\zeta_1, \dots, \widehat{\zeta_i}, \dots, \zeta_n)} \tag{6.4.1}$$

on $C_{\pm}^{(i)}$, respectively, if we put

$$Q^{(i)}(\zeta_1, \dots, \widehat{\zeta_i}, \dots, \zeta_n) := -\varepsilon_i \left({\zeta_1}^2 + {\zeta_2}^2 + \cdots \widehat{\pm{\zeta_i}^2} \pm \cdots - {\zeta_{n-1}}^2 - {\zeta_n}^2 \right), \tag{6.4.2}$$

where $\varepsilon_i = \pm 1$ is the signature of ${\zeta_i}^2$ in the quadratic form $Q(\zeta)$ as in (2.5.1). We note that $Q^{(i)}(\zeta_1, \dots, \widehat{\zeta_i}, \dots, \zeta_n) \ge 0$ on the cone C, and the map

$$(\zeta_1, \dots, \widehat{\zeta_i}, \dots, \zeta_n) \mapsto (\zeta_1, \dots, \pm\sqrt{Q^{(i)}}, \dots, \zeta_n)$$

gives a bijection from $\{(\zeta_1, \ldots, \widehat{\zeta_i}, \ldots, \zeta_n) \in \mathbb{R}^n : Q^{(i)}(\zeta_1, \ldots, \widehat{\zeta_i}, \ldots, \zeta_n) \neq 0\}$ onto $C^{(i)}$. By substituting (6.4.1) into $\phi_\pm^{(i)}$, we put

$$\varphi_\pm^{(i)}(\zeta_1, \ldots, \widehat{\zeta_i}, \ldots, \zeta_n) := \phi_\pm^{(i)}(\zeta_1, \ldots, \pm\sqrt{Q^{(i)}}, \ldots, \zeta_n).$$

Since the measure $d\mu$ on the cone C is of the form $\frac{1}{2\sqrt{Q^{(i)}}} d\zeta_1 \cdots \widehat{d\zeta_i} \cdots d\zeta_n$, we have

$$\|\phi_\pm^{(i)}\|_{L^2(C)}^2 = \int_{\mathbb{R}^{n-1}} \frac{|\varphi_\pm^{(i)}(\zeta_1, \ldots, \widehat{\zeta_i}, \ldots, \zeta_n)|^2}{2\sqrt{Q^{(i)}}} d\zeta_1 \cdots \widehat{d\zeta_i} \cdots d\zeta_n \tag{6.4.3}$$

$$\begin{aligned} \mathcal{F}^{-1} T \phi_\pm^{(i)}(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} & e^{-\sqrt{-1}(z_1\zeta_1 + \cdots + \widehat{z_i\zeta_i} + \cdots + z_n\zeta_n)} \qquad (6.4.4) \\ & \times e^{\mp\sqrt{-1} z_i \sqrt{Q^{(i)}}} \frac{\varphi_\pm^{(i)}(\zeta_1, \ldots, \widehat{\zeta_i}, \ldots, \zeta_n)}{2\sqrt{Q^{(i)}}} d\zeta_1 \cdots \widehat{d\zeta_i} \cdots d\zeta_n. \end{aligned}$$

From (6.4.4) we have the following:

Lemma 6.4 *We write $\mathcal{F}_{\mathbb{R}^k}$ for the Fourier transform in $\mathbb{R}^k$ ($k = n-1, n$). Then,*

$$\mathcal{F}_{\mathbb{R}^n}^{-1} T \phi_\pm^{(i)}|_{z_i=0} = \frac{1}{4\pi} \mathcal{F}_{\mathbb{R}^{n-1}}^{-1} \left(\frac{\varphi_\pm^{(i)}}{\sqrt{Q^{(i)}}} \right) (z_1, \ldots, \widehat{z_i}, \ldots, z_n),$$

$$\frac{\partial}{\partial z_i}|_{z_i=0} \mathcal{F}_{\mathbb{R}^n}^{-1} T \phi_\pm^{(i)} = \frac{\mp\sqrt{-1}}{4\pi} \mathcal{F}_{\mathbb{R}^{n-1}}^{-1} \left(\varphi_\pm^{(i)} \right) (z_1, \ldots, \widehat{z_i}, \ldots, z_n).$$

6.5 We recall the Fourier transform of the Riesz potential

$$\int_{-\infty}^{\infty} e^{\sqrt{-1}x\xi} \xi_+^\lambda \, dx = \sqrt{-1} e^{\frac{\sqrt{-1}\lambda\pi}{2}} \Gamma(\lambda+1)(\xi + \sqrt{-1}0)^{-\lambda-1}$$

for a meromorphic parameter λ. Letting $\lambda = 0$, we have

$$\mathcal{F}^{-1}(Y_\pm^{(i)}) = \frac{\mp\sqrt{-1}}{2\pi}(z_i \mp \sqrt{-1}0)^{-1} \delta(z_1, \ldots, \widehat{z_i}, \ldots, z_n),$$

where $\delta(z_1, \ldots, \widehat{z_i}, \ldots, z_n)$ is the Dirac delta function of $n-1$ variables. Then we have

$$\begin{aligned} \mathcal{F}^{-1}(Y_\pm^{(i)} \cdot T\phi) &= (\mathcal{F}^{-1} Y_\pm^{(i)}) * (\mathcal{F}^{-1} T\phi) \\ &= \frac{\mp\sqrt{-1}}{2\pi}(z_i \mp \sqrt{-1}0)^{-1} \delta(z_1, \ldots, \widehat{z_i}, \ldots, z_n) * f(z) \\ &= \frac{\pm 1}{2\pi\sqrt{-1}}(z_i \mp \sqrt{-1}0)^{-1} * f(z). \end{aligned}$$

Here the first two convolutions are for n variables $z_1, \ldots, z_n$, while the last one for only $z_i \in \mathbb{R}$. In view of the definition of $\phi_\pm^{(i)}$ and $f_\pm^{(i)}$, we have proved the following

Lemma 6.5 *If $f = \mathcal{F}^{-1}T\phi$ then $f_\pm^{(i)} = \mathcal{F}^{-1}T\phi_\pm^{(i)}$.*

6.6 We are now ready to complete the proof of Theorem 6.2. By using Lemma 6.4 and Lemma 6.5, the Plancherel formula for $\mathbb{R}^{n-1}$, and the integration formula (6.4.3), respectively, we have

$$\begin{aligned}
(f_\pm^{(i)}|_{z_i=0}, \frac{\partial f_\pm^{(i)}}{\partial z_i}|_{z_i=0})_{L^2(\mathbb{R}^{n-1})} &= \frac{\pm\sqrt{-1}}{16\pi^2}(\mathcal{F}_{\mathbb{R}^{n-1}}^{-1}(\frac{\varphi_\pm^{(i)}}{\sqrt{Q^{(i)}}})), \mathcal{F}_{\mathbb{R}^{n-1}}^{-1}(\varphi_\pm^{(i)}))_{L^2(\mathbb{R}^{n-1})} \\
&= \frac{\pm\sqrt{-1}}{16\pi^2(2\pi)^{n-1}}(\frac{\varphi_\pm^{(i)}}{\sqrt{Q^{(i)}}}, \varphi_\pm^{(i)})_{L^2(\mathbb{R}^{n-1})} \\
&= \frac{\pm\sqrt{-1}}{2(2\pi)^{n+1}}\|\phi_\pm^{(i)}\|^2_{L^2(C)}.
\end{aligned}$$

Hence,

$$(f, f)_W = \frac{1}{2(2\pi)^{n+1}}\left(\|\phi_+^{(i)}\|^2_{L^2(C)} + \|\phi_-^{(i)}\|^2_{L^2(C)}\right) = \frac{1}{2(2\pi)^{n+1}}\|\phi\|^2_{L^2(C)}.$$

This finishes the proof of the formula (6.2.5) and hence Theorem 6.2. □

6.7 The main content of Theorem 6.2 is to give yet another realization of the inner product and of the Hilbert space. This is very close to the form most known in the case of the wave equation, where one integrates Cauchy data on the zero time hypersurface to get the inner product. Note the connection to the theory of conserved quantities for the wave equation - we end the paper by making more explicit this final remark:

When $p = 2$ let us introduce time and space coordinates (t, x) by

$$(t, x) = (t, x_1, \ldots, x_{n-1}) = (z_1, \ldots, z_n)$$

and the dual variable k to x so that positive-energy solutions to the wave equation are given by the Fourier transform ($i = \sqrt{-1}$)

$$u^+(t, x) = \int_{\mathbb{R}^{n-1}} e^{i(kx - t|k|)}\frac{\varphi^+(k)}{|k|}dk$$

and similarly for negative-energy solutions

$$u^-(t, x) = \int_{\mathbb{R}^{n-1}} e^{i(kx + t|k|)}\frac{\varphi^-(k)}{|k|}dk$$

where kx denotes the usual scalar product, $|k|$ the Euclidean length, and $dk = dk_1 \cdots dk_{n-1}$. Any solution is the sum of two such: $u = u^+ + u^-$. The energy of the wave u is given by

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^{n-1}} (|u_t|^2 + |\nabla u|^2) dx$$

and is a conserved quantity, i.e. independent of which constant-time hyperplane we integrate over. It is easy to see that cross-terms drop out, so that on the Fourier transform side we obtain

$$\mathcal{E}(u) = (2\pi)^{n-1} \int_{\mathbb{R}^{n-1}} (|\varphi^+(k)|^2 + |\varphi^-(k)|^2) dk$$

for the energy. Note that this energy only differs from our inner product by a density factor of $|k|$, and that it may be thought of as an integral over C. In terms of our conformally invariant inner product (6.2.4) this is up to a constant just

$$(u, |H|u) = (u^+, Hu^+) - (u^-, Hu^-),$$

where $H = i\partial_t$ is the energy generator (infinitesimal time translations). In the same way, we have the analogous "conserved quantities" for the ultrahyperbolic equation and the inner product (6.2.4), namely: Let $H_j = i\partial_{z_j}$ be the generator of translations in the coordinate z_j, then for a solution f in the Hilbert space

$$\mathcal{E}_j(f) = (f, |H_j|f)$$

is invariant under translations in the coordinate z_j. Furthermore, the quantity $\mathcal{E}_j(f)$ can be expressed in terms of an integral of local quantities. In particular we may use this to study uniqueness and decay properties of solutions to $\Box_{\mathbb{R}^{p-1,q-1}} f = 0$. Since this is outside the scope of the present paper, we shall not do so here; but note that one easy consequence is the fact, that if a solution and its normal derivative vanish on a coordinate hyperplane, then it is identically zero - a classical fact about the energy (time zero hyperplane).

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