Propagation of multiplicity-freeness property for holomorphic vector bundles

Dedicated to Joseph Wolf for his seventy-fifth birthday

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Abstract

We give a complete proof of a propagation theorem of multiplicity-free property from fibers to spaces of global sections for holomorphic vector bundles. The propagation theorem is formalised in three ways, aiming for producing various multiplicity-free theorems in representation theory for both finite- and infinite-dimensional cases in a systematic and synthetic manner.

The key geometric condition in our theorem is an orbit-preserving anti-holomorphic diffeomorphism on the base space, which brings us to the concept of visible actions on complex manifolds.

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1 Introduction

In representation theory, unitarity is an important concept, in particular, when we apply the classic philosophy — analysis and synthesis, namely, an attempt to understand things built up from the smallest ones. This is embodied by a theorem of Mautner and Teleman stating that any unitary representation π of a locally compact group G can be decomposed into the direct integral of irreducible unitary representations:

$$\pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi(\tau)\tau d\mu(\tau), \tag{1.1}$$

where $\widehat{G}$ denotes the set of equivalence classes of irreducible unitary representations ('*smallest objects*'), μ is a measure on $\widehat{G}$, and $m : \widehat{G} \to \mathbb{N} \cup \{\infty\}$ is a measurable function that stands for 'multiplicity'.

To find elements of $\widehat{G}$, basic results are unitarizability theorems established by Mackey [21] for L^2-induced representations in the 1950s and by Vogan [32] and Wallach [33] for cohomologically induced representations in the 1980s. These results may be thought of as a *propagation theory of unitarity* from fibers to spaces of sections (more generally, stalks to cohomologies).

Multiplicity-freeness is another important concept in representation theory that generalizes irreducibility. For a unitary representation π of G, we say that π is *multiplicity-free* if the ring of continuous G-intertwining endomorphisms is commutative. This condition implies that m is not greater than 1 almost everywhere with respect to the measure μ in the direct integral (1.1).

Multiplicity-free representations are a special class of representations, for which one could expect a simple and detailed study, and by which one could expect effective applications of representation theory. Multiplicity-free representations arise in broad range of mathematics in connection with expansions (Taylor series, Fourier expansion, spherical harmonics, the Gelfand–Tsetlin basis, ...) and the classical identities (the Capelli identity, various explicit formulae for special functions, ...), although we may not be aware of even the fact that the representation is there.

The aim of this paper is to prove a *propagation theorem of multiplicity-freeness* from fibers to spaces of sections for holomorphic vector bundles.

To state our main result, let H be a Lie group, and $\mathcal{V} \to D$ an H-equivariant holomorphic vector bundle. We naturally have a representation of H on the space $\mathcal{O}(D, \mathcal{V})$ of global holomorphic sections. Then the first form of our multiplicity-free theorem is stated briefly as follows (see Theorem 2.2 for details):

Theorem 1.1. *Any unitary representation of H which is realized in $\mathcal{O}(D, \mathcal{V})$ is multiplicity-free if the H-equivariant bundle $\mathcal{V} \to D$ satisfies the following* *three conditions:*

(1.2) (*Fiber*) *For every $x \in D$, the isotropy representation of H_x on the fiber $\mathcal{V}_x$ is multiplicity-free.*

(1.3) (*Base space*) *There exists an anti-holomorphic bundle endomorphism σ, which preserves every H-orbit on the base space D.*

(1.4) (*Compatibility*) *See* (2.2.3).

The compatibility condition (1.4) is less important because it is often automatically fulfilled by a natural choice of σ (see Remark 5.2.3 for an example of σ; see also [15, Appendix]). Thus the geometric condition (1.3) on the base space D is crucial for our propagation of the multiplicity-free property from fibers $\mathcal{V}_x$ to the space $\mathcal{O}(D, \mathcal{V})$ of sections.

The condition (1.3) with regard to holomorphic actions on complex manifolds has become the motivation that led us to the concept of *visible actions* in [13] . Recently, classification results on visible actions on complex manifolds have been obtained in various settings, see [16, 17, 26, 27, 28]. In this article, we use a variant of visible actions, namely, *S-visible actions*; see Definition 4.2.

The second form of our multiplicity-free theorem is formalized as Theorem 4.3 in terms of S-visible actions. Here, S is a slice of the H-action on the base space D. An old theorem of S. Kobayashi [11] (see also Wolf [34]) may be thought of as a *propagation theorem of irreducibility* from fibers to the space of sections when S is a singleton.

The third form of our multiplicity-free theorem is formalized in the setting where the bundle $\mathcal{V} \to D$ is associated to a principal bundle $K \to P \to D$ and to a representation (μ, V) of the structure group K. This is Theorem 5.3. This formulation is useful for actual applications, in particular, for branching problems (decompositions of irreducible representations when restricted to subgroups). In fact, this is the form that was used as a main machinery of [13, 14] in finding various multiplicity-free theorems in concrete settings, whereas the complete proof of Theorem 5.3 (stated as [13, Theorem 1.3] and [14, Theorem 2] loc.cit.) has been postponed until the present article.

Acknowledgement. A primitive case of Theorem 5.3 (the line bundle case) together with its application to branching problems for semisimple symmetric pairs (G, H) was announced in [12]. The heart of the proof of Theorem 2.2 is based on reproducing kernels, and was inspired by the original idea of Faraut–Thomas [6]. I thank J. Faraut for enlightening discussions, in particular, for explaining the idea of [6] in an early stage of this work.

Substantial part of its generalization in the present form was obtained during my visit to Harvard University in 2000–2001. I express my gratitude to W. Schmid who gave me a wonderful atmosphere for research. M. Duflo suggested me to use the terminology "propagation" for Theorem 1.1. Concrete applications of Theorem 1.1 and the theory of visible actions were presented in various occasions including the Oberwolfach workshops organized by A. Huckleberry, K.-H. Neeb and J. Wolf in 2000 and 2004 and at Winter School at Czech Republic organized by V. Souček in 2010.

A detailed account of the material of this article (the proof of the propagation theorem) was given in the graduate course lectures at Harvard University (2008, spring semester) and at the University of Tokyo (2008, fall semester), and also in a series of lectures at Functional Analysis X in Croatia (2008, summer). I express my deep gratitude to the organizers and to the participants for helpful and stimulating comments on various occasions. Special thanks are due to Ms. Suenaga for her help in preparing for the final manuscript.

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2 Complex geometry and multiplicity-free theorem

This section gives a first form of our multiplicity-free theorem. We may regard it as a propagation theorem of multiplicity-free property from fibers to spaces of sections in the setting where there may exist infinitely many orbits on base spaces. The main result of this section is Theorem 2.2. We shall reformulate it using visible actions in Section 4, and furthermore present its group-theoretic version in Section 5.

2.1 Equivariant holomorphic vector bundle

Let $\mathcal{V} = \amalg_{x \in D} \mathcal{V}_x \to D$ be a Hermitian holomorphic vector bundle over a connected complex manifold D. We denote by $\mathcal{O}(D, \mathcal{V})$ the space of holomorphic sections of $\mathcal{V} \to D$. It carries a Fréchet topology by the uniform convergence on compact sets.

Suppose a Lie group H acts on the bundle $\mathcal{V} \to D$ by automorphisms. This means that the action of H on the total space, denoted by $L_h : \mathcal{V} \to \mathcal{V}$, and the action on the base space, denoted simply by $h : D \to D$, $x \mapsto h{\cdot}x$, are both biholomorphic for $h \in H$, and that the induced linear map $L_h : \mathcal{V}_x \to \mathcal{V}_{h\cdot x}$ between the fibers is unitary for any $x \in D$. In particular, we have a unitary representation of the isotropy subgroup $H_x := \{h \in H : h \cdot x = x\}$ on the fiber $\mathcal{V}_x$.

The action of H on the bundle $\mathcal{V} \to D$ gives rise to a continuous representation on $\mathcal{O}(D, \mathcal{V})$ by the pull-back of sections, namely, $s \mapsto L_h s(h^{-1}\cdot\ )$ for $h \in H$ and $s \in \mathcal{O}(D, \mathcal{V})$.

Definition 2.1. Suppose π is a unitary representation of H defined on a Hilbert space $\mathcal{H}$. We will say π is *realized in* $\mathcal{O}(D, \mathcal{V})$ if there is an injective continuous H-intertwining map from $\mathcal{H}$ into $\mathcal{O}(D, \mathcal{V})$.

Let $\{U_\alpha\}$ be trivializing neighborhoods of D, and let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C})$ be the transition functions for the holomorphic vector bundle $\mathcal{V} \to D$. Then the anti-holomorphic vector bundle $\overline{\mathcal{V}} \to D$ is defined to be the complex vector bundle with the transition functions $\overline{g_{\alpha\beta}}$. We denote by $\overline{\mathcal{O}}(D, \overline{\mathcal{V}})$ the space of anti-holomorphic sections for $\overline{\mathcal{V}} \to D$.

Suppose σ is an anti-holomorphic diffeomorphism of D. Then the pull-back $\sigma^*\mathcal{V} = \amalg_{x\in D}\mathcal{V}_{\sigma(x)}$ is an anti-holomorphic vector bundle over D. In turn, $\overline{\sigma^*\mathcal{V}} \to D$ is a holomorphic vector bundle over D. The fiber at $x \in D$ is identified with $\overline{\mathcal{V}_{\sigma(x)}}$, the complex conjugate vector space of $\mathcal{V}_{\sigma(x)}$ (see Section 3.1).

The holomorphic vector bundle $\overline{\sigma^*\mathcal{V}}$ is isomorphic to $\mathcal{V}$ if and only if σ lifts to an anti-holomorphic endomorphism $\tilde{\sigma}$ of $\mathcal{V}$. In fact, such $\tilde{\sigma}$ induces a conjugate linear isomorphism $\tilde{\sigma}_x : \mathcal{V}_x \to \mathcal{V}_{\sigma(x)}$, which then defines a $\mathbb{C}$-linear isomorphism

$$\Psi_x : \mathcal{V}_x \to (\overline{\sigma^*\mathcal{V}})_x, \quad v \mapsto \overline{\tilde{\sigma}_x(v)} \tag{2.1.1}$$

via the identification $(\overline{\sigma^*\mathcal{V}})_x \simeq \overline{\mathcal{V}_{\sigma(x)}}$. Then $\Psi : \mathcal{V} \to \overline{\sigma^*\mathcal{V}}$ is an isomorphism of holomorphic vector bundles such that its restriction to the base space D is the identity. For simplicity, we shall use the letter σ in place of $\tilde{\sigma}$. For a Hermitian vector bundle $\mathcal{V}$, by a bundle endomorphism σ, we mean that σ_x is furthermore isometric (or equivalently, Ψ_x is unitary) for any $x \in D$.

2.2 Multiplicity-free theorem (first form)

The following is a first form of our multiplicity-free theorem:

Theorem 2.2. *Let $\mathcal{V} \to D$ be a Hermitian holomorphic vector bundle, on which a Lie group H acts by automorphisms. Assume*

(2.2.1) *the isotropy representation of H_x on the fiber $\mathcal{V}_x$ is multiplicity-free for any $x \in D$.*

We write its irreducible decomposition as $\mathcal{V}_x = \bigoplus_{i=1}^{n(x)} \mathcal{V}_x^{(i)}$. Assume furthermore that there exists an anti-holomorphic bundle endomorphism σ satisfying the following two conditions: for any $x \in D$,

(2.2.2) *there exists $h \in H$ such that $\sigma(x) = h \cdot x$, and*

(2.2.3) $\sigma_x(\mathcal{V}_x^{(i)}) = L_h(\mathcal{V}_x^{(i)})$ *for any i $(1 \le i \le n(x))$.*

Then any unitary representation that is realized in $\mathcal{O}(D, \mathcal{V})$ is multiplicity-free.

We shall give a proof of Theorem 2.2 in Section 3.

Remark 2.2.1. (1) The conditions (2.2.1)–(2.2.3) of Theorem 2.2 is local in the sense that the same conclusion holds if D' is an H-invariant open subset of D, and if the conditions (2.2.1)–(2.2.3) are satisfied for $x \in D'$. This is clear because the restriction map $\mathcal{O}(D, \mathcal{V}) \to \mathcal{O}(D', \mathcal{V}|_{D'})$ is injective and continuous.

(2) The proof in Section 3 shows that one can replace H_x with its arbitrary subgroup H'_x in (2.2.1). (Such a replacement makes (2.2.1) stronger, and (2.2.3) weaker.)

In the following two subsections we explain special cases of Theorem 2.2.

2.3 Line bundle case

We begin with the observation that the assumptions (2.2.1) and (2.2.3) are automatically fulfilled if $\mathcal{V}_x$ is irreducible, in particular, if $\mathcal{V} \to D$ is a line bundle. Hence we have the following.

Corollary 2.3. *In the setting of Theorem 2.2, assume $\mathcal{V} \to D$ is a line bundle. If there exists an anti-holomorphic bundle endomorphism satisfying (2.2.2), then any unitary representation that is realized in $\mathcal{O}(D, \mathcal{V})$ is multiplicity-free.*

This case was announced in [12] with a sketch of proof, and its applications are extensively discussed in [15] for the branching problems (i.e., the decomposition of the restriction of unitary representations) with respect to reductive symmetric pairs.

2.4 Trivial bundle case

If the vector bundle is the trivial line bundle $\mathcal{V} = D \times \mathbb{C}$, then any anti-holomorphic diffeomorphism on D lifts to an anti-holomorphic endomorphism of $\mathcal{V}$ by $(x, u) \mapsto (\sigma(x), \bar{u})$. Hence we have the following.

Corollary 2.4. *If there exists an anti-holomorphic diffeomorphism σ of D satisfying* (2.2.2)*, then any unitary representation which is realized in $\mathcal{O}(D)$ is multiplicity-free.*

This result was previously proved in Faraut and Thomas [6] under the assumption that $\sigma^2 = \mathrm{id}$.

2.5 Propagation of irreducibility

The strongest condition on group actions is transitivity. Transitivity on base spaces guarantees that even irreducibility propagates from fibers to spaces of sections. The following result is due to S. Kobayashi [11].

Proposition 2.5. *In the setting of Theorem 2.2, suppose that H acts transitively on D and that H_x acts irreducibly on $\mathcal{V}_x$ for some (equivalently, for any) $x \in D$. Then there exists at most one unitary representation π that can be realized in $\mathcal{O}(D, \mathcal{V})$. In particular, such π is irreducible if exists.*

Proof. This is an immediate consequence of Lemma 3.3 and Proposition 3.4 ($n(x) = 1$ case) below, which will be used in the proof of Theorem 2.2 in Section 3. □

We note that the condition (2.2.2) is much weaker than the transitivity of the action of the group H on D. Our geometric condition (2.2.2) brings us to the concept of visible actions, which we shall discuss in Section 4.

3 Proof of Theorem 2.2

This section is devoted entirely to the proof of Theorem 2.2.

3.1 Some linear algebra

We begin carefully with basic notations.

Given a complex Hermitian vector space V, we define a complex Hermitian vector space $\overline{V}$ as a collection of the symbol $\overline{v}$ ($v \in V$) equipped with a scalar multiplication $a\overline{v} := \overline{\overline{a}v}$ for $a \in \mathbb{C}$, and with an inner product $(\bar{u}, \bar{v}) := (v, u)$.

The complex dual space $V^\vee$ is identified with $\overline{V}$ by $\overline{V} \xrightarrow{\sim} V^\vee$, $\bar{v} \mapsto (\cdot, v)$. In particular, we have a natural isomorphism of complex vector spaces:

$$V \otimes \overline{V} \xrightarrow{\sim} \operatorname{End}(V). \tag{3.1.1}$$

Given a unitary map $A : V \to W$ between Hermitian vector spaces, we define a unitary map $\overline{A} : \overline{V} \to \overline{W}$ by $\overline{v} \mapsto \overline{Av}$. Then the induced map $A \otimes \overline{A} : V \otimes \overline{V} \to W \otimes \overline{W}$ gives rise to a complex linear isomorphism:

$$A_\sharp : \operatorname{End}(V) \to \operatorname{End}(W). \tag{3.1.2}$$

Then it is readily seen from the unitarity of A that

$$A_\sharp(\mathrm{id}_V) = \mathrm{id}_W\,. \tag{3.1.3}$$

In particular, if an endomorphism of V is diagonalized with respect to an orthogonal direct sum decomposition $V = \bigoplus_{i=1}^n V^{(i)}$, then we have the following formula of $A_\sharp$:

$$A_\sharp\left(\sum_{i=1}^n \lambda_i\, \mathrm{id}_{V^{(i)}}\right) = \sum_{i=1}^n \lambda_i\, \mathrm{id}_{A(V^{(i)})} \quad (\lambda_1, \dots, \lambda_n \in \mathbb{C}). \tag{3.1.4}$$

3.2 Reproducing kernel for vector bundles

This subsection summarizes some basic results on reproducing kernels for holomorphic vector bundles. The results here are standard for the trivial bundle case.

Suppose we are given a continuous embedding $\mathcal{H} \hookrightarrow \mathcal{O}(D, \mathcal{V})$ of a Hilbert space $\mathcal{H}$ into the Fréchet space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections of the holomorphic vector bundle $\mathcal{V} \to D$. The continuity implies in particular that for each $y \in D$ the point evaluation map:

$$\mathcal{H} \to \mathcal{V}_y, \quad f \mapsto f(y)$$

is continuous. Then by the Riesz representation theorem, there exists uniquely $K_{\mathcal{H}}(\cdot, y) \in \mathcal{H} \otimes \overline{\mathcal{V}_y}$ such that

$$(f, K_{\mathcal{H}}(\cdot, y))_{\mathcal{H}} = f(y) \quad \text{for any } f \in \mathcal{H}. \tag{3.2.1}$$

We take an orthonormal basis $\{\varphi_\nu\}$ of $\mathcal{H}$, and expand $K_{\mathcal{H}}$ as

$$K_{\mathcal{H}}(\cdot, y) = \sum_\nu a_\nu(y)\varphi_\nu(\cdot). \tag{3.2.2}$$

It follows from (3.2.1) that the coefficient $a_\nu(y)$ is given by

$$a_\nu(y) = (K_{\mathcal{H}}(\cdot, y), \varphi_\nu(\cdot))_{\mathcal{H}} = \overline{\varphi_\nu(y)},$$

and the expansion of $K_{\mathcal{H}}$ converges in $\mathcal{H}$. By the continuity $\mathcal{H} \hookrightarrow \mathcal{O}(D, \mathcal{V})$ again, (3.2.2) converges uniformly on each compact set for each fixed $y \in D$. Thus $K_{\mathcal{H}}(x, y)$ is given by the formula:

$$K_{\mathcal{H}}(x, y) \equiv K(x, y) = \sum_\nu \varphi_\nu(x)\overline{\varphi_\nu(y)} \in \mathcal{V}_x \otimes \overline{\mathcal{V}_y}\ , \tag{3.2.3}$$

and defines a smooth section of the exterior tensor product bundle $\mathcal{V} \boxtimes \overline{\mathcal{V}} \to D \times D$ which is holomorphic in the first argument and anti-holomorphic in the second. We will say $K_{\mathcal{H}}$ is the *reproducing kernel* of the Hilbert space $\mathcal{H} \subset \mathcal{O}(D, \mathcal{V})$.

For the convenience of the reader, we pin down basic properties of reproducing kernels for holomorphic vector bundles in a way that we use later.

Lemma 3.2. (1) *Let $K_i(x, y)$ be the reproducing kernels of Hilbert spaces $\mathcal{H}_i \subset \mathcal{O}(D, \mathcal{V})$ with inner products $(\ ,\ )_{\mathcal{H}_i}$, respectively, for $i = 1, 2$. If $K_1 \equiv K_2$, then the two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide and the inner products $(\ ,\ )_{\mathcal{H}_1}$ and $(\ ,\ )_{\mathcal{H}_2}$ are the same.*
(2) *If $K_1(x, x) = K_2(x, x)$ for all $x \in D$, then $K_1 \equiv K_2$.*

Proof. (1) Let us reconstruct the Hilbert space $\mathcal{H}$ from the reproducing kernel K. For each $y \in D$ and $v^* \in \mathcal{V}_y^* := \overline{\mathcal{V}_y}^\vee$, we define $\psi(y, v^*)$ by

$$\psi(y, v^*) := \langle K(\cdot, y), v^* \rangle \in \mathcal{H}.$$

Here, $\langle\ ,\ \rangle$ denotes the canonical pairing between $\overline{\mathcal{V}_y}$ and $\overline{\mathcal{V}_y}^\vee$. We claim that the $\mathbb{C}$-span of $\{\psi(y, v^*) : y \in D, v^* \in \mathcal{V}_y^*\}$ is dense in $\mathcal{H}$. This is because

$(f, \psi(y, v^*))_{\mathcal{H}} = \langle f(y), v^* \rangle$ for any $f \in \mathcal{H}$ by (3.2.1). Thus, the Hilbert space $\mathcal{H}$ is reconstructed from the pre-Hilbert structure

$$(\psi(y_1, v_1^*), \psi(y_2, v_2^*))_{\mathcal{H}} := \langle K(y_2, y_1), v_2^* \otimes \overline{v_1^*} \rangle. \tag{3.2.4}$$

(2) We denote by $\overline{D}$ the complex manifold endowed with the conjugate complex structure on the same real manifold D. Then $\overline{\mathcal{V}} \to \overline{D}$ is a holomorphic vector bundle, and we can form a holomorphic vector bundle $\mathcal{V} \boxtimes \overline{\mathcal{V}} \to D \times \overline{D}$. In turn, $K_i(\cdot, \cdot)$ are regarded as its holomorphic sections. As the diagonal embedding $\iota : D \to D \times \overline{D}, z \mapsto (z, z)$ is totally real, our assumption $(K_1 - K_2)|_{\iota(D)} \equiv 0$ implies $K_1 - K_2 \equiv 0$ by the unicity theorem of holomorphic functions. $\square$

3.3 Equivariance of the reproducing kernel

Next, suppose that the Hermitian holomorphic vector bundle $\mathcal{V} \to D$ is H-equivariant and that $(\pi, \mathcal{H})$ is a unitary representation of H realized in $\mathcal{O}(D, \mathcal{V})$. Let $K_{\mathcal{H}}$ be the reproducing kernel of the embedding $\mathcal{H} \hookrightarrow \mathcal{O}(D, \mathcal{V})$. We shall see how the unitarity of $(\pi, \mathcal{H})$ is reflected in the reproducing kernel $K_{\mathcal{H}}$.

We regard $K_{\mathcal{H}}(x, x) \in \mathcal{V}_x \otimes \overline{\mathcal{V}_x}$ as an element of $\mathrm{End}(\mathcal{V}_x)$ via the isomorphism (3.1.1). Then we have:

Lemma 3.3. *With the notation* (3.1.2) *applied to $L_h : \mathcal{V}_x \to \mathcal{V}_{h \cdot x}$, we have*

$$K_{\mathcal{H}}(h \cdot x, h \cdot x) = (L_h)_\sharp K_{\mathcal{H}}(x, x) \quad \textit{for any } h \in H.$$

In particular, $K_{\mathcal{H}}(x, x) \in \mathrm{End}_{H_x}(\mathcal{V}_x)$ for any $x \in D$.

Proof. Let $\{\varphi_\nu\}$ be an orthonormal basis of $\mathcal{H}$. Since $(\pi, \mathcal{H})$ is a unitary representation, $\{\pi(h)^{-1}\varphi_\nu\}$ is also an orthonormal basis of $\mathcal{H}$ for every fixed $h \in H$. Because the formula (3.2.3) of the reproducing kernel is valid for any orthonormal basis, we have

$$\begin{aligned} K_{\mathcal{H}}(x, y) &= \sum_\nu (\pi(h)^{-1}\varphi_\nu)(x)\overline{(\pi(h)^{-1}\varphi_\nu)(y)} \\ &= \sum_\nu L_{h^{-1}}\varphi_\nu(h \cdot x)\overline{L_{h^{-1}}\varphi_\nu(h \cdot y)} \\ &= (L_{h^{-1}} \otimes \overline{L_{h^{-1}}})K_{\mathcal{H}}(h \cdot x, h \cdot y) \end{aligned} \tag{3.3.1}$$

for any $x, y \in D$. Hence, $(L_h \otimes \overline{L_h})K_{\mathcal{H}}(x, y) = K_{\mathcal{H}}(h \cdot x, h \cdot y)$ and the lemma follows. $\square$

3.4 Diagonalization of the reproducing kernel

The reproducing kernel for a holomorphic vector bundle is a matrix valued section as we have defined in (3.2.3). The multiplicity-free property of the isotropy representation on the fiber diagonalizes the reproducing kernel:

Proposition 3.4. *Suppose* $(\pi, \mathcal{H})$ *is a unitary representation of* H *realized in* $\mathcal{O}(D, \mathcal{V})$*. Assume that the isotropy representation of* H_x *on the fiber* $\mathcal{V}_x$ *decomposes as a multiplicity-free sum of irreducible representations of* H_x *as* $\mathcal{V}_x = \bigoplus_{i=1}^{n} \mathcal{V}_x^{(i)}$*. (Here,* $n \equiv n(x)$ *may depend on* $x \in D$*.) Then the reproducing kernel is of the form*

$$K_{\mathcal{H}}(x, x) = \sum_{i=1}^{n} \lambda^{(i)}(x)\,\mathrm{id}_{\mathcal{V}_x^{(i)}}$$

for some complex numbers $\lambda^{(1)}(x), \dots, \lambda^{(n)}(x)$*.*

Proof. A direct consequence of Lemma 3.3 and Schur's lemma. □

3.5 Construction of an anti-linear isometry J

In the setting of Theorem 2.2, suppose that σ is an anti-holomorphic bundle endomorphism. We define a conjugate linear map

$$J : \mathcal{O}(D, \mathcal{V}) \to \mathcal{O}(D, \mathcal{V}), \quad f \mapsto \sigma^{-1} \circ f \circ \sigma, \tag{3.5.1}$$

namely, $Jf(x) := \sigma^{-1}(f(\sigma(x))$ for $x \in D$.

Lemma 3.5. *If the conditions* (2.2.1)–(2.2.3) *are satisfied, then* J *is an isometry from* $\mathcal{H}$ *onto* $\mathcal{H}$ *for any unitary representation* $(\pi, \mathcal{H})$ *realized in* $\mathcal{O}(D, \mathcal{V})$*.*

Proof. We define a Hilbert space $\widetilde{\mathcal{H}} := J(\mathcal{H})$, equipped with the inner product

$$(Jf_1, Jf_2)_{\widetilde{\mathcal{H}}} := (f_2, f_1)_{\mathcal{H}} \quad \text{for } f_1, f_2 \in \mathcal{H}.$$

Let us show that the reproducing kernel $K_{\widetilde{\mathcal{H}}}$ for $\widetilde{\mathcal{H}}$ coincides with $K_{\mathcal{H}}$. To see this, we take an orthonormal basis $\{\varphi_\nu\}$ of $\mathcal{H}$. Then $\{J\varphi_\nu\}$ is an orthonormal basis of $\widetilde{\mathcal{H}}$, and therefore

$$\begin{aligned}K_{\widetilde{\mathcal{H}}}(x,y) &= \sum_\nu J\varphi_\nu(x)\overline{J\varphi_\nu(y)} \\ &= \sum_\nu \sigma_x^{-1}\left(\varphi_\nu(\sigma(x))\right)\ \overline{\sigma_y^{-1}\left(\varphi_\nu(\sigma(y))\right)} \\ &= \left(\sigma_x^{-1}\otimes\overline{\sigma_y^{-1}}\right) K_{\mathcal{H}}(\sigma(x),\sigma(y)).\end{aligned}$$

For $x=y$, this formula can be restated as

$$K_{\widetilde{\mathcal{H}}}(x,x) = (\sigma_x^{-1})_\sharp K_{\mathcal{H}}(\sigma(x),\sigma(x)) \tag{3.5.2}$$

with the notation (3.1.2) applied to the unitary map $\sigma_x^{-1}: \mathcal{V}_{\sigma(x)} \to \mathcal{V}_x$. We fix $x \in D$, and take $h \in H$ such that $\sigma(x) = h\cdot x$ (see (2.2.2)). Then

$$K_{\widetilde{\mathcal{H}}}(x,x) = (\sigma_x^{-1})_\sharp K_{\mathcal{H}}(h\cdot x, h\cdot x) = (\sigma_x^{-1})_\sharp (L_h)_\sharp K_{\mathcal{H}}(x,x). \tag{3.5.3}$$

Here the last equality follows from Lemma 3.3.

Since the action of H_x on $\mathcal{V}_x$ is multiplicity-free, it follows from Proposition 3.4 that there exist complex numbers $\lambda^{(i)}(x)$ such that

$$K_{\mathcal{H}}(x,x) = \sum_i \lambda^{(i)}(x)\,\mathrm{id}_{\mathcal{V}_x^{(i)}}.$$

Then by (3.1.3) we have

$$(L_h)_\sharp K_{\mathcal{H}}(x,x) = \sum_i \lambda^{(i)}(x)\,\mathrm{id}_{L_h(\mathcal{V}_x^{(i)})}. \tag{3.5.4}$$

Furthermore, since $\sigma_x^{-1}(L_h(\mathcal{V}_x^{(i)})) = \mathcal{V}_x^{(i)}$ by the assumption (2.2.3), it follows from (3.1.4) that

$$(\sigma_x^{-1})_\sharp\left(\sum_i \lambda^{(i)}(x)\,\mathrm{id}_{L_h(\mathcal{V}_x^{(i)})}\right) = \sum_i \lambda^{(i)}(x)\,\mathrm{id}_{\mathcal{V}_x^{(i)}}. \tag{3.5.5}$$

Combining (3.5.3), (3.5.4) and (3.5.5), we get

$$K_{\widetilde{\mathcal{H}}}(x,x) = K_{\mathcal{H}}(x,x).$$

Then, by Lemma 3.2, the Hilbert space $\widetilde{\mathcal{H}}$ coincides with $\mathcal{H}$ and

$$(Jf_1, Jf_2)_{\mathcal{H}} = (Jf_1, Jf_2)_{\widetilde{\mathcal{H}}} = (f_2, f_1)_{\mathcal{H}} \quad \text{for } f_1, f_2 \in \mathcal{H}.$$

This is what we wanted to prove. $\square$

Remark 3.5.1. In terms of the bundle isomorphism $\Psi : \mathcal{V} \to \overline{\sigma^*\mathcal{V}}$ (see (2.1.1)), J is given by $(Jf)(x) = \Psi_x^{-1}(\overline{f(\sigma(x))})$. We note

$$J^2 = \mathrm{id} \quad \text{on } \mathcal{O}(D, \mathcal{V})$$

if $\sigma^2 = \mathrm{id}_{\mathcal{V}}$, or equivalently, if $\sigma^2 = \mathrm{id}_D$ and $\overline{\Psi_{\sigma(x)}} \circ \Psi_x = \mathrm{id}_{\mathcal{V}_x}$ for any $x \in D$. However, we do not use this condition to prove Theorem 2.2.

3.6 Proof of Theorem 2.2

As a final step, we need the following lemma which was proved in [6] under the assumption that $J^2 = \mathrm{id}$ and that $\mathcal{V} \to D$ is the trivial line bundle. For the sake of completeness, we give a proof here.

Lemma 3.6. *For $A \in \mathrm{End}_H(\mathcal{H})$, the adjoint operator A^* is given by*

$$A^* = JAJ^{-1}. \tag{3.6.1}$$

Proof. We divide the proof into two steps.
Step 1 (self-adjoint case). We may and do assume that $A - I$ is positive definite because neither the assumption nor the conclusion changes if we replace A by $A + cI$ ($c \in \mathbb{R}$). Here we note that $A + cI$ is positive definite if c is greater than the operator norm $\|A\|$.

From now, assume $A \in \mathrm{End}_H(\mathcal{H})$ is a self-adjoint operator such that $A - I$ is positive definite. We introduce a pre-Hilbert structure on $\mathcal{H}$ by

$$(f_1, f_2)_{\mathcal{H}_A} := (Af_1, f_2)_{\mathcal{H}} \quad \text{for } f_1, f_2 \in \mathcal{H}. \tag{3.6.2}$$

Since $A - I$ is positive definite, we have

$$(f, f)_{\mathcal{H}} \le (f, f)_{\mathcal{H}_A} \le \|A\|(f, f)_{\mathcal{H}} \quad \text{for } f \in \mathcal{H}.$$

Therefore $\mathcal{H}$ is still complete with respect to the new inner product $(\ ,\ )_{\mathcal{H}_A}$. The resulting Hilbert space will be denoted by $\mathcal{H}_A$.

If $f_1, f_2 \in \mathcal{H}$ and $g \in H$, then

$$\begin{aligned}(\pi(g)f_1, \pi(g)f_2)_{\mathcal{H}_A} &= (A\pi(g)f_1, \pi(g)f_2)_{\mathcal{H}} \\ &= (\pi(g)Af_1, \pi(g)f_2)_{\mathcal{H}} = (Af_1, f_2)_{\mathcal{H}} = (f_1, f_2)_{\mathcal{H}_A}.\end{aligned}$$

Therefore π also defines a unitary representation on $\mathcal{H}_A$. Applying Lemma 3.5 to both $\mathcal{H}_A$ and $\mathcal{H}$, we have

$$\begin{aligned}(Af_1, f_2)_{\mathcal{H}} &= (f_1, f_2)_{\mathcal{H}_A} = (Jf_2, Jf_1)_{\mathcal{H}_A} = (AJf_2, Jf_1)_{\mathcal{H}} \\ &= (Jf_2, A^*Jf_1)_{\mathcal{H}} = (Jf_2, JJ^{-1}A^*Jf_1)_{\mathcal{H}} = (J^{-1}A^*Jf_1, f_2)_{\mathcal{H}}.\end{aligned}$$

Hence, $A = J^{-1}A^*J$.

Step 2 (general case). Suppose $A \in \mathrm{End}_H(\mathcal{H})$. Then A^* also commutes with $\pi(g)$ $(g \in H)$ because π is unitary. We put $B := \frac{1}{2}(A + A^*)$ and $C := \frac{\sqrt{-1}}{2}(A^* - A)$. Then both B and C are self-adjoint operators commuting with $\pi(g)$ $(g \in H)$. It follows from Step 1 that $B^* = JBJ^{-1}$ and $C^* = JCJ^{-1}$. Since J is conjugate-linear, we have $(\sqrt{-1}C)^* = J(\sqrt{-1}C)J^{-1}$. Hence, $A = B + \sqrt{-1}C$ also satisfies $A^* = JAJ^{-1}$. □

Proof of Theorem 2.2. Let A, $B \in \mathrm{End}_H(\mathcal{H})$. By Lemma 3.6, we have

$$AB = J^{-1}(AB)^*J = J^{-1}B^*JJ^{-1}A^*J = BA.$$

Therefore the ring $\mathrm{End}_H(\mathcal{H})$ is commutative. □

4 Visible actions on complex manifolds

This section analyzes the geometric condition (2.2.2) on the complex manifold D. We shall introduce the concept of S-visible actions, with which Theorem 2.2 is reformulated in a simpler manner (see Theorem 4.3).

4.1 Visible actions on complex manifolds

Suppose a Lie group H acts holomorphically on a connected complex manifold D.

Definition 4.1. We say the action is *S-visible* if there exists a subset S of D such that

(4.1.1) $D' := H \cdot S$ is open in D,

and also exists an anti-holomorphic diffeomorphism σ of D' satisfying the following two conditions:

(4.1.2) $\sigma|_S = \mathrm{id}$,

(4.1.3) σ preserves every H-orbit in D'.

Remark 4.1.1. The above condition is *local* in the sense that we may replace S by its subset S' in Definition 4.1 as far as $H \cdot S'$ is open in D.

Remark 4.1.2. By the definition of D', it is obvious that

(4.1.4) S meets every H-orbit in D'.

Thus Definition 4.1 is essentially the same with *strong visibility* in the sense of [14, Definition 3.3.1]. In fact, the difference is only an additional requirement that S be a smooth submanifold in [14]. We note that if S is a smooth submanifold in Definition 4.1, then S is totally real by the condition (4.1.2), and consequently, the H-action becomes *visible* in the sense of [13] (see [14, Theorem 4.3]).

4.2 Compatible automorphism

Retain the setting of Definition 4.1. Suppose σ is the anti-holomorphic diffeomorphism of D'. Twisting the original H-action by σ, we can define another holomorphic action of H on D' by

$$D' \to D',\ x \mapsto \sigma(h \cdot \sigma^{-1}(x)).$$

If this action can be realized by H, namely, if there exists a group automorphism $\tilde{\sigma}$ of H such that

$$\tilde{\sigma}(h) \cdot x = \sigma(h \cdot \sigma^{-1}(x)) \quad \text{for any } x \in D',$$

we say $\tilde{\sigma}$ is *compatible* with σ. This condition is restated simply as

$$\tilde{\sigma}(h) \cdot \sigma(y) = \sigma(h \cdot y) \quad \text{for any } y \in D'. \tag{4.2.1}$$

Definition 4.2. We say an S-visible action has a *compatible automorphism* of the transformation group H if there exists an automorphism $\tilde{\sigma}$ of the group H satisfying the condition (4.2.1).

We remark that the condition (4.1.3) follows from (4.1.1) and (4.1.2) if there exists $\tilde{\sigma}$ satisfying (4.2.1). In fact, any H-orbit in D' is of the form $H \cdot x$ for some $x \in S$, and then

$$\sigma(H \cdot x) = \tilde{\sigma}(H) \cdot \sigma(x) = H \cdot x$$

by (4.1.2) and (4.2.1).

Suppose $\mathcal{V} \to D$ is an H-equivariant holomorphic vector bundle. If there is a compatible automorphism $\tilde{\sigma}$ of H with an anti-holomorphic diffeomorphism σ on D, then we have the following isomorphism:

$$(\overline{\sigma^*\mathcal{V}})_{h\cdot y} \simeq \overline{\mathcal{V}}_{\sigma(h\cdot y)} = \overline{\mathcal{V}}_{\tilde{\sigma}(h)\cdot\sigma(y)} \quad \text{for } h \in H \text{ and } y \in D.$$

Therefore we can let H act equivariantly on the holomorphic vector bundle $\overline{\sigma^*\mathcal{V}} \to D$ by defining the left translation on $\overline{\sigma^*\mathcal{V}}$ as

$$L_h^\sigma : (\overline{\sigma^*\mathcal{V}})_y \to (\overline{\sigma^*\mathcal{V}})_{h\cdot y}$$

via the identification with the left translation $\overline{L_{\tilde{\sigma}(h)}} : \overline{\mathcal{V}}_{\sigma(y)} \to \overline{\mathcal{V}}_{\tilde{\sigma}(h)\cdot\sigma(y)}$. Then the two H-equivariant holomorphic vector bundles $\mathcal{V}$ and $\overline{\sigma^*\mathcal{V}}$ are isomorphic if and only if σ lifts to an anti-holomorphic bundle endomorphism σ (we use the same letter) which respects the H-action in the sense that

$$L_{\tilde{\sigma}(h)} \circ \sigma = \sigma \circ L_h \quad \text{on } \mathcal{V} \qquad \text{for any } h \in H. \tag{4.2.2}$$

4.3 Propagation of multiplicity-free property

By using the concept of S-visible actions, we give a second form of our main theorem as follows:

Theorem 4.3. *Let $\mathcal{V} \to D$ be an H-equivariant Hermitian holomorphic vector bundle. Assume the following three conditions are satisfied:*

(4.3.1) (*Base space*) *The action on the base space D is S-visible with a compatible automorphism of the group H (Definition 4.2).*

(4.3.2) (*Fiber*) *The isotropy representation of H_x on $\mathcal{V}_x$ is multiplicity-free for any $x \in S$.*

We write its irreducible decomposition as

$$\mathcal{V}_x = \bigoplus_{i=1}^{n(x)} \mathcal{V}_x^{(i)}.$$

(4.3.3) (*Compatibility*) *σ lifts to an anti-holomorphic endomorphism (we use the same letter σ) of the H-equivariant Hermitian holomorphic vector bundle $\mathcal{V}$ such that* (4.2.2) *holds and*

(4.3.3)(a) $$\sigma_x(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)} \quad \textit{for } 1 \le i \le n(x),\ x \in S.$$

Then any unitary representation which is realized in $\mathcal{O}(D, \mathcal{V})$ is multiplicity-free.

The difference between the conditions of Theorem 4.3 with the previous conditions (2.2.1) and (2.2.3) in Theorem 2.2 is the following: The conditions (4.3.2) and (4.3.3)(a) are imposed only on the slice S, while the conditions in Theorem 2.2 were imposed on the whole base space D (or at least its open subset).

Remark 4.3.1. We can sometimes find a slice S such that the isotropy subgroup H_x is independent of *generic* $x \in S$. Bearing this in mind, we set

$$\begin{aligned} H_S &:= \bigcap_{x \in S} H_x \\ &= \{g \in H : gx = x \quad \text{for any } x \in S\}. \end{aligned}$$

Theorem 4.3 still holds if we replace H_x with H_S (see also Remark 2.2.1(2)).

Proof. We shall reduce Theorem 4.3 to Theorem 2.2 by using the H-equivariance of the bundle endomorphism σ. Let us show that the conditions (2.2.1), (2.2.2) and (2.2.3) are satisfied for the H-invariant open subset $D' := H \cdot S$ of D.

First we observe that the condition (4.1.3) implies (2.2.2) because $\sigma(x) \in \sigma(H \cdot x) = H \cdot x$ for any $x \in D'$.

Next, take any element $x \in D'$ and write $x = h \cdot x_0$ ($h \in H$, $x_0 \in S$). We set

$$\mathcal{V}_x^{(i)} := L_h(\mathcal{V}_{x_0}^{(i)}) \quad (1 \le i \le n(x_0)).$$

Through the group isomorphism $H_{x_0} \xrightarrow{\sim} H_x, l \mapsto hlh^{-1}$ and the left translation $L_h : \mathcal{V}_{x_0} \to \mathcal{V}_x$, we get the isomorphism between the two isotropy representations, $H_{x_0} \to GL(\mathcal{V}_{x_0})$ and $H_x \to GL(\mathcal{V}_x)$, because $L_{hlh^{-1}} = L_h \circ L_l \circ L_h^{-1}$ ($l \in H_{x_0}$). In particular, the direct sum

$$\mathcal{V}_x = \bigoplus_{i=1}^{n(x_0)} \mathcal{V}_x^{(i)}$$

gives a multiplicity-free decomposition of irreducible representations of H_x. Hence the condition (2.2.1) is satisfied for all $x \in D'$.

Finally, we set $g := \tilde{\sigma}(h)h^{-1} \in H$. As $\sigma(x_0) = x_0$, we have

$$\sigma(x) = \sigma(h \cdot x_0) = \tilde{\sigma}(h) \cdot \sigma(x_0) = \tilde{\sigma}(h) \cdot x_0 = g \cdot x.$$

Besides, we have for any i $(1 \le i \le n(x) = n(x_0))$,

$$\begin{aligned}
\sigma_x(\mathcal{V}_x^{(i)}) &= \sigma_x(L_h(\mathcal{V}_{x_0}^{(i)})) \\
&= L_{\tilde{\sigma}(h)}(\sigma_{x_0}(\mathcal{V}_{x_0}^{(i)})) && \text{by (4.2.2)} \\
&= L_{\tilde{\sigma}(h)}\big(\mathcal{V}_{x_0}^{(i)}\big) && \text{by (4.3.3)(a)} \\
&= L_{\tilde{\sigma}(h)h^{-1}} L_h(\mathcal{V}_{x_0}^{(i)}) \\
&= L_g(\mathcal{V}_x^{(i)}).
\end{aligned}$$

Hence the condition (2.2.3) holds for any $x \in D'$. Therefore all the assumptions of Theorem 2.2 are satisfied for the open subset D'. Now, Theorem 4.3 follows from Theorem 2.2 and Remark 2.2.1 (1). $\square$

5 Multiplicity-free theorem for associated bundles

This section provides a third form of our multiplicity-free theorem (see Theorem 5.3). It is intended for actual applications to group representation theory, especially to branching problems. The idea here is to reformalize the geometric condition of Theorem 4.3 (second form) in terms of the representation of the structure group of an equivariant principal bundle.

Theorem 5.3 is used as a main machinery in [13, 14] (referred to as [13, Theorem 1.3] and [14, Theorem 2], of which we have postponed the proof to this article) for various multiplicity-free theorems including the following cases:

- tensor product representations of $GL(n)$ [13, Theorem 3.6],
- branching problems for $GL(n) \downarrow GL(n_1) \times GL(n_2) \times GL(n_3)$ ([13, Theorem 3.4]),
- Plancherel formulae for vector bundles over Riemannian symmetric spaces ([14, Theorems 21 and 30]).

5.1 Automorphisms on equivariant principal bundles

We begin with the setting where a Hermitian holomorphic vector bundle $\mathcal{V}$ over a connected complex manifold D is given as the associated bundle

$\mathcal{V} \simeq P \times_K V$ to the following data (P, K, μ, V):

K is a Lie group,
$\varpi : P \to D$ is a principal K-bundle,
V is a finite dimensional Hermitian vector space,
$\mu : K \to GL_{\mathbb{C}}(V)$ is a unitary representation.

Suppose that a Lie group H acts on P from the left, commuting with the right action of K. Then H acts also on the Hermitian vector bundle $\mathcal{V} \to D$ by automorphisms.

We take $p \in P$, and set $x := \varpi(p) \in D$. If $h \in H_x$, then $\varpi(hp) = h \cdot x = x = \varpi(p)$. Therefore there is a unique element of K, denoted by $i_p(h)$, such that

$$hp = p\, i_p(h). \tag{5.1.1}$$

The correspondence $h \mapsto i_p(h)$ gives rise to a Lie group homomorphism $i_p : H_x \to K$. We set

$$H_{(p)} := i_p(H_x). \tag{5.1.2}$$

Then $H_{(p)}$ is a subgroup of K.

Definition 5.1. By an automorphism of the H-equivariant principal K-bundle $\varpi : P \to D$, we mean that there exist a diffeomorphism $\sigma : P \to P$ and Lie group automorphisms $\sigma : K \to K$ and $\sigma : H \to H$ (by a little abuse of notation, we use the same letter σ) such that

(5.1.3) $\sigma(hpk) = \sigma(h)\sigma(p)\sigma(k) \quad (h \in H, k \in K, p \in P)$.

The condition (5.1.3) immediately implies that

(5.1.4) σ induces an action (denoted again by σ) on $P/K \simeq D$,

(5.1.5) the induced action σ on D is compatible with $\sigma \in \mathrm{Aut}(H)$ (see (4.2.1) for the definition).

We write P^σ for the set of fixed points by σ, that is,

$$P^\sigma := \{p \in P : \sigma(p) = p\}.$$

Then we have:

Lemma 5.1. *$\sigma(H_{(p)}) = H_{(p)}$ if $p \in P^\sigma$.*

Proof. Take $h \in H_x$. Applying σ to the equations $h \cdot x = x$ ($\in D$) and $hp = pi_p(h)$ ($\in P$), we have $\sigma(h) \cdot x = x$ and $\sigma(h)p = p\sigma(i_p(h))$ from (5.1.3). Hence $\sigma(h) \in H_x$ and $i_p(\sigma(h)) = \sigma(i_p(h))$. Therefore $\sigma(H_x) \subset H_x$ and $\sigma(H_{(p)}) \subset H_{(p)}$. Likewise, $\sigma^{-1}(H_x) \subset H_x$ and $\sigma^{-1}(H_{(p)}) \subset H_{(p)}$. Hence we have proved $\sigma(H_x) = H_x$ and $\sigma(H_{(p)}) = H_{(p)}$. $\square$

5.2 Multiplicity-free theorem

For a representation μ of K, we denote by $\mu^\vee$ the contragredient representation of μ. It is isomorphic to the conjugate representation $\overline{\mu}$ if μ is unitary.

Proposition 5.2. *Retain the setting of Subsection 5.1. Assume that there exist an automorphism σ of the H-equivariant principal K-bundle $\varpi : P \to D$ such that*

(5.2.1) *the induced action of σ on D is anti-holomorphic,*

and a subset B of P^σ satisfying the following two conditions:

(5.2.2) *HBK contains a non-empty open subset of P.*

(5.2.3) *The restriction $\mu|_{H_{(b)}}$ is multiplicity-free as an $H_{(b)}$-module for any $b \in B$.*

We write its irreducible decomposition as $\mu|_{H_{(b)}} \simeq \bigoplus_{i=1}^{n} \nu_b^{(i)}$. Further, we assume:

(5.2.4) (a) *$\mu \circ \sigma \simeq \mu^\vee$ as K-modules.*

(5.2.4) (b) *For any $b \in B$ and i, $\nu^{(i)} \circ \sigma \simeq \nu^{(i)\vee}$ as $H_{(b)}$-modules.*

Then any unitary representation of H that is realized in $\mathcal{O}(D, \mathcal{V})$ is multiplicity-free.

The proof of Proposition 5.2 is given in Section 6.

Remark 5.2.1. Loosely, the conditions (5.2.2) and (5.2.3) mean that the holomorphic bundle $\mathcal{V} \to D$ cannot be 'too large', with respect to the transformation group H. The remaining condition (5.2.4) is often automatically fulfilled (e.g., Corollary 5.4).

Remark 5.2.2. As in Remark 2.2.1, Proposition 5.2 still holds if $H_{(b)}$ is replaced by its arbitrary subgroup $H'_{(b)}$ for each $b \in B$ in (5.2.3) and (5.2.4) (b).

Remark 5.2.3. For a connected compact Lie group K, the condition (5.2.4) (a) is satisfied for any finite-dimensional representation μ of K if we take $\sigma \in \mathrm{Aut}(K)$ to be a *Weyl involution*. We recall that σ is a Weyl involution if there exists a Cartan subalgebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{k}$ of K such that $d\sigma = -\mathrm{id}$ on $\mathfrak{t}$. It is noteworthy that any simply-connected compact Lie group admits a Weyl involution.

5.3 Multiplicity-free theorem (third form)

In the assumption of Proposition 5.2, the subgroups $H_{(b)}$ may depend on b (see (5.2.3) and (5.2.4) (b)). For actual applications, we give a weaker but simpler form by taking just one subgroup M instead of a family of subgroups $H_{(b)}$.

For a subset B of P, we define the following subgroup $M_H(B)$ of K:

(5.3.1)
$$\begin{aligned} M_H(B) &:= \{k \in K : \text{ for each } b \in B, \text{ there is } h \in H \text{ such that } hb = bk\} \\ &= \bigcap_{b \in B} K_{Hb}\ , \end{aligned}$$

where K_{Hb} denotes the isotropy subgroup at Kb in the left coset space $H\backslash P$, which is acted on by K from the right. Then $M_H(B)$ is σ-stable if $B \subset P^\sigma$, as is readily seen from (5.1.3).

Theorem 5.3. *Assume that there exist an automorphism σ of the H-equivariant principal K-bundle $\varpi : P \to D$ satisfying* (5.2.1) *and a subset B of P^σ with the following three conditions* (5.3.2) – (5.3.4)*: Let $M := M_H(B)$.*

(5.3.2) *HBK contains a non-empty open subset of P.*

(5.3.3) *The restriction $\mu|_M$ is multiplicity-free.*

We shall write its irreducible decomposition as $\mu|_M \simeq \bigoplus_{i=1}^{n} \nu^{(i)}$.

(5.3.4) (a) *$\mu \circ \sigma \simeq \mu^\vee$ as representations of K.*

(5.3.4) (b) *$\nu^{(i)} \circ \sigma \simeq {\nu^{(i)}}^\vee$ as representations of M for any i $(1 \le i \le n)$.*

Then any unitary representation of H which is realized in $\mathcal{O}(D, \mathcal{V})$ is multiplicity-free.

Remark 5.3.1. Theorem 5.3 still holds if we replace M with an arbitrary σ-stable subgroup of $M_H(B)$ to verify the conditions (5.3.3) and (5.3.4) (b).

Assuming Proposition 5.2, we first complete the proof of Theorem 5.3.

Proof of Theorem 5.3. In view of Proposition 5.2 and Remark 5.2.2, it is sufficient to show that $M_H(B) \subset H_{(b)}$ for all $b \in B$.

To see this, take any $k \in M_H(B)$. By the definition (5.3.1), there exists $h \in H$ such that $hb = bk$. Then $h \in H_{\varpi(b)}$. Since $i_b(h) \in K$ is characterized by the property $hb = b\, i_b(h)$ (see (5.1.1)), k coincides with $i_b(h)$. Hence $k = i_b(h) \in i_b(H_{\varpi(b)}) = H_{(b)}$ (see (5.1.2)). Thus we have proved $M_H(B) \subset H_{(b)}$ for all $b \in B$. $\square$

5.4 Line bundle case

In general, the condition (5.3.2) tends to be fulfilled if B is large, while the condition (5.3.3) tends to be fulfilled if B is small (namely, if M is large). However, we do not have to consider the condition (5.3.3) if $\mathcal{V} \to D$ is a line bundle. Hence, by taking B to be maximal, that is, by setting $B := P^\sigma$, we get:

Corollary 5.4. *Suppose we are in the setting of Subsection 5.1. Suppose furthermore that K is connected and* $\dim \mu = 1$. *Assume that there exists an automorphism σ of the H-equivariant principal K-bundle $\varpi : P \to D$ satisfying* (5.2.1) *and the following two conditions:*

(5.4.1) $\quad d\sigma = -\mathrm{id}$ *on the center $\mathfrak{c}(\mathfrak{k})$ of the Lie algebra $\mathfrak{k}$ of K.*

(5.4.2) $\quad HP^\sigma K$ *contains a non-empty open subset of P.*

Then any unitary representation which can be realized in $\mathcal{O}(D, \mathcal{V})$ is multiplicity-free.

Proof of Corollary. As we mentioned, we apply Theorem 5.3 with $B := P^\sigma$. The condition (5.3.3) is trivially satisfied because $\dim \mu = 1$.

Let us show $\mu \circ \sigma = \mu^\vee$. We write $K = [K, K] \cdot C$, where $[K, K]$ is the commutator subgroup and $C = \exp(\mathfrak{c}(\mathfrak{k}))$. Since $[K, K]$ is semisimple, it acts trivially on the one-dimensional representations $\mu \circ \sigma$ and $\mu^\vee$. By (5.4.1),

$\mu \circ \sigma(e^X) = \mu(e^{-X}) = \mu^\vee(e^X)$ for any $X \in \mathfrak{c}(\mathfrak{k})$. Hence $\mu \circ \sigma = \mu^\vee$ both on $[K, K]$ and C. Therefore the condition (5.3.4) (a) holds. Then (5.3.4) (b) also holds. Therefore Corollary follows from Theorem 5.3. $\square$

5.5 Multiplicity-free branching laws

So far, we have not assumed that P has a group structure. Now, we consider the case that P is a Lie group which we denote by G, and that H and K are closed subgroups of G. This framework enables us to apply Theorem 5.3 to the *restriction* of representations of G (constructed on G/K) to its subgroup H. Applications of Corollary 5.5 include multiplicity-free branching theorems of highest weight representations for both finite- and infinite-dimensional cases (see [13, 14, 15]).

We denote the centralizer of B in $H \cap K$ by

$$Z_{H\cap K}(B) := \{l \in H \cap K : lbl^{-1} = b \text{ for any } b \in B\}.$$

Corollary 5.5. *Suppose $D = G/K$ carries a G-invariant complex structure, and $\mathcal{V} = G \times_K V$ is a G-equivariant holomorphic vector bundle over D associated to a unitary representation $\mu : K \to GL(V)$. We assume there exist an automorphism σ of the Lie group G stabilizing H and K such that the induced action on $D = G/K$ is anti-holomorphic, and a subset B of G^σ satisfying the conditions* (5.3.2), (5.3.3), *and* (5.3.4) (a) *and* (b) *for $P := G$ and $M := Z_{H\cap K}(B)$. Then any unitary representation of H which can be realized in the G-module $\mathcal{O}(D, \mathcal{V})$ is multiplicity-free.*

Proof. Since $Z_{H\cap K}(B)$ is contained in $M_H(B)$ by the definition (5.3.1), Corollary 5.5 is a direct consequence of Theorem 5.3 and Remark 5.3.1. $\square$

6 Proof of Proposition 5.2

This section gives a proof of Proposition 5.2 by showing that all the conditions of Theorem 4.3 are fulfilled. Then the proof of our third form (Theorem 5.3) will be completed.

6.1 Verification of the condition (4.3.1)

Suppose we are in the setting of Proposition 5.2. Then HBK contains a non-empty open subset of P, and consequently $\varpi(HBK)$ contains a non-empty open subset, say W, of D. By taking the union of H-translates of W, we get an H-invariant open subset $D' := H \cdot W$ of D. We set

$$S := D' \cap \varpi(B).$$

Then $D' = H \cdot S$. Besides, $\sigma|_S = \mathrm{id}$ because $B \subset P^\sigma$. Thus the H-action on D is S-visible with a compatible automorphism σ of H by (5.1.3) in the sense of Definition 4.2. Thus the condition (4.3.1) holds for D'.

6.2 Verification of the condition (4.3.2)

Next, let us prove that $\mathcal{V}_x$ is multiplicity-free as an H_x-module for all x in S.

Let $\mathcal{V} \simeq P \times_K V$ be the associated bundle, and $P \times V \to \mathcal{V}$, $(p, v) \mapsto [p, v]$ by the natural quotient map. For $p \in P$ we set $x := \varpi(p) \in D$. Then we can identify the fiber $\mathcal{V}_x$ with V by the bijection

$$\iota_p : V \xrightarrow{\sim} \mathcal{V}_x, \quad v \mapsto [p, v]. \tag{6.2.1}$$

Via the bijection (6.2.1) and the group homomorphism $i_p : H_x \to H_{(p)}$, the isotropy representation of H_x on $\mathcal{V}_x$ factors through the representation $\mu : H_{(p)} \to GL(V)$, namely, the following diagram commutes for any $l \in H_x$:

$$\begin{array}{ccc} V & \xrightarrow[\iota_p]{\sim} & \mathcal{V}_x \\ {\scriptstyle \mu(i_p(l))}\big\downarrow & & \big\downarrow{\scriptstyle L_l} \\ V & \xrightarrow[\iota_p]{\sim} & \mathcal{V}_x \end{array} \tag{6.2.2}$$

Now, suppose $x \in S$. We take $b \in B$ such that $x = \varpi(b)$.

According to (5.2.3), we decompose V as a multiplicity-free sum of irreducible representations of $H_{(b)}$, for which we write

$$\mu = \bigoplus_{i=1}^{n} \nu_b^{(i)}, \quad V = \bigoplus_{i=1}^{n} V_b^{(i)}. \tag{6.2.3}$$

Then it follows from (6.2.2) that if we set $\mathcal{V}_x^{(i)} := \iota_b(V_b^{(i)})$, then

$$\mathcal{V}_x = \bigoplus_{i=1}^{n} \mathcal{V}_x^{(i)} \tag{6.2.4}$$

is an irreducible decomposition as an H_x-module. Hence (4.3.2) is verified.

6.3 Verification of the condition (4.3.3)

Third, let us construct an isomorphism $\Psi : \mathcal{V} \to \overline{\sigma^* \mathcal{V}}$. According to the assumption (5.2.4) (a), there exists a K-intertwining isomorphism, denoted by $\psi : V \to \overline{V}$, between the two representations μ and $\overline{\mu \circ \sigma}$. As the vector bundle $\mathcal{V} \to D$ is associated to the data (P, K, μ, V), so is the vector bundle $\overline{\sigma^* \mathcal{V}} \to D$ to the data $(P, K, \overline{\mu \circ \sigma}, \overline{V})$. Hence the map

$$P \times V \to P \times \overline{V}, \quad (p, v) \mapsto (p, \psi(v))$$

induces the bundle isomorphism

$$\Psi : \mathcal{V} \xrightarrow{\sim} \overline{\sigma^* \mathcal{V}}. \tag{6.3.1}$$

In other words, the conjugate linear map defined by

$$\varphi : V \to V, \quad v \mapsto \overline{\psi(v)} \tag{6.3.2}$$

satisfies

$$\mu(\sigma(k)) \circ \varphi = \varphi \circ \mu(k) \quad \text{for } k \in K.$$

Hence we can define an anti-holomorphic endomorphism of $\mathcal{V}$ by

$$\mathcal{V} \to \mathcal{V}, \quad [p, v] \mapsto [\sigma(p), \varphi(v)].$$

This endomorphism, denoted by the same letter σ, is a lift of the anti-holomorphic map $\sigma : D \to D$, and satisfies (4.2.2) because of (5.1.3).

Besides, for $x = \varpi(p)$, we have

$$\iota_{\sigma(p)} \circ \varphi = \sigma_x \circ \iota_p \, . \tag{6.3.3}$$

Finally, let us verify the condition (4.3.3)(a).

Step 1. First, let us show

$$\varphi(V_b^{(i)}) = V_b^{(i)} \quad \text{for } 1 \le i \le n. \tag{6.3.4}$$

Bearing the inclusion $H_{(b)} \subset K$ in mind, we consider the representation $\overline{\mu \circ \sigma} : K \to GL(\overline{V})$ and its subrepresentation realized on $\psi(V_b^{(i)})$ $(\subset \overline{V})$ as an $H_{(b)}$-module. Then this is isomorphic to $(\nu_b^{(i)}, V_b^{(i)})$ as $H_{(b)}$-modules because $\psi : V \to \overline{V}$ intertwines the two representations μ and $\overline{\mu \circ \sigma}$ of K. On the other hand, it follows from the irreducible decomposition (6.2.3) that the representation $\overline{\mu \circ \sigma}$ when restricted to the subspace $\overline{V_b^{(i)}}$ is isomorphic to $\overline{\nu_b^{(i)} \circ \sigma}$ as $H_{(b)}$-modules. By our assumption (5.2.4) (b), $\nu_b^{(i)}$ is isomorphic to $\overline{\nu_b^{(i)} \circ \sigma}$, which occurs in $\overline{V}$ exactly once. Therefore the two subspaces $\psi(V_b^{(i)})$ and $\overline{V_b^{(i)}}$ must coincide. Hence we have (6.3.4) by (6.3.2).

Step 2. Next we show that (4.3.3)(a) holds for $x = \varpi(b)$ if $b \in B$. We note that $\sigma(b) = b$ and $\sigma(x) = x$. Then it follows from (6.3.3) and (6.3.4) that

$$\sigma_x \circ \iota_b(V_b^{(i)}) = \iota_{\sigma(b)} \circ \varphi(V_b^{(i)}) = \iota_{\sigma(b)}(V_b^{(i)}) = \iota_b(V_b^{(i)}).$$

Since $\mathcal{V}_x^{(i)} = \iota_b(V_b^{(i)})$, we have proved $\sigma_x(\mathcal{V}_b^{(i)}) = \mathcal{V}_b^{(i)}$. Hence (4.3.3)(a) holds.

Thus all the conditions of Theorem 4.3 hold for D'. Therefore Proposition 5.2 follows from Theorem 4.3 and Remark 2.2.1 (2). Hence the proof of Theorem 5.3 is completed. □

7 Concluding remarks

7.1 Applications in concrete settings

The application of our multiplicity-free theorem ranges from finite-dimensional representations to infinite-dimensional ones, from the discrete spectrum to the continuous spectrum, and from classical groups to exceptional groups. Although concrete applications are not the main issue of this article, let us mention some of them (see [14, 15] for details on this topic).

The first paper [12] in this direction (i.e., multiplicity-free theorem for the line bundle case) already demonstrated that there are fairly many *new* multiplicity-free representations for which explicit decomposition formulae were not known at that time. (See [1, 20, 24] and references therein in the finite-dimensional cases and to [15, 30] in the infinite-dimensional cases for some of new explicit branching laws.)

More generally, Theorem 5.3 gives a systematic and synthetic proof of the multiplicity-free property including the Plancherel formula for Riemannian symmetric spaces due to É. Cartan and I. M. Gelfand, its extension to line bundles and certain vector bundles (A. Deitmar [4], see also [14, Theorem 30]), and even its *deformation* which traces back to the *canonical representation* of Vershik–Gelfand–Graev in the $SL(2, \mathbb{R})$ case ([5], [14, Example 8.3.3] for more general groups); the Hua–Kostant–Schmid K-type formula

[14, 29], and its generalization to semisimple symmetric pairs due to the author ([12], see also [15]). These are examples in the *infinite-dimensional* case, to which our propagation theorem of the multiplicity-free property applies. On the other hand, there are also various examples of multiplicity-free representations in the *finite-dimensional* case, where combinatorial argument is often involved. It is noteworthy that some of (apparently, quite complicated) multiplicity-free representations in the finite-dimensional case can be constructed geometrically as a special case of our propagation theorem. For example, we see in [13] that Theorem 5.3 gives us a new and simple geometric construction of all pairs (π_1, π_2) of irreducible finite-dimensional representations of $GL(n)$ for which the tensor product of two representations $\pi_1 \otimes \pi_2$ is multiplicity-free (they exhaust all such cases in view of the classification due to Stembridge [31] by a combinatorial method in the spirit of case-by-case).

Least but not last, Theorem 5.3 has raised also a set of new problems concerning *analysis on multiplicity-free representations* beyond (algebraic) branching laws, see [15, Section 1.8] for a short summary of developments made by Ben Saïd, van Dijk, Hille, Ørsted, Neretin, Zhang, and by the author among others in the last decade.

7.2 Visible actions and coisotropic actions

There are the following three concepts on group actions in different geometric settings:

- (Complex geometry) (S-)visible actions (Definition 4.1).
- (Symplectic geometry) coisotropic actions.
- (Riemannian geometry) polar actions.

See [13, 14] for more details about visible actions on complex manifolds; Guillemin and Sternberg [7] or Huckleberry and Wurzbacher [10] for coisotropic actions on symplectic manifolds; and Heintze et al [9] or Podestà–Thorbergsson [25] for polar actions on Riemannian manifolds. It should be noted that Lie groups G are usually assumed to be compact for coisotropic actions and also for polar actions in the literature, whereas we allow G to be non-compact for visible actions in [13, 14] so that we can apply this concept to the study of infinite dimensional representations of G.

We may compare the above three concepts assuming that the manifold is Kähler so that it is endowed with complex, symplectic, and Riemannian structures simultaneously. It should be noted that, according to [10], J. Wolf first suggested the terminology "multiplicity-free actions" for coisotropic actions in the symplectic setting. Further study on coisotropic actions and multiplicity-free representations of compact Lie groups may be found in [10]. The relation of visible actions with coisotropic actions and polar actions is discussed in [14, Section 4].

A special case is given by linear actions on Hermitian vector spaces.

Proposition 7.2. *Suppose $\tau : G \to GL(V)$ is a unitary representation of a compact Lie group G on a finite dimensional Hermitian vector space V. Then the following two conditions* (i) *and* (ii) *are equivalent. Further, the condition* (iii) *implies* (i) *and* (ii).

(i) *G acts strongly visibly on V as a complex manifold.*

(ii) *G acts coisotropically on V as a symplectic manifold.*

(iii) *G acts polarly on V as a Riemannian manifold.*

The equivalence (i) $\Longleftrightarrow$ (ii) follows from the classification of multiplicity-free linear actions by V. Kac (irreducible case), C. Benson–G. Ratcliff and A. Leahy (reducible cases; see [2] and references therein), Huckleberry and Wurzbacher [10], and the classification of multiplicity-free linear visible actions by A. Sasaki [26, 28]. The last statement follows from Dadok [3]. The converse of the last statement does not hold. A counterexample is the natural action of $U(3) \times Sp(n)$ on $\mathbb{C}^3 \otimes \mathbb{C}^{2n}$, which is not polar but is strongly visible and coisotropic.

7.3 Generalization of the main theorem

So far we have assumed that the base space D and the fiber V are finite dimensional, and discussed representations realized in the space of holomorphic sections for an equivariant holomorphic bundle $\mathcal{V} \to D$. It may be interesting to consider an analog of the propagation theorem of multiplicity-free property (Theorem 2.2) in a more general setting. Among others, we raise the following two cases for generalization.

1. Visible actions on infinite dimensional complex manifolds.

It is plausible that our framework and its idea would work in the infinite dimensional settings by careful analysis (see [23], for example). A generalization to infinite dimensional setting applies to the following objects:

the fiber V,

the complex manifold D (base space),

the group G.

Here, we have in mind also an application to branching problems of representations of infinite dimensional Lie groups, e.g., an infinite dimensional analogue of [15, Theorem A] for those are constructed by a generalized Borel–Weil theorem.

2. Dolbeault cohomologies for equivariant holomorphic vector bundles.

The point here is to replace the space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections by the Dolbeault cohomology group $H^j(D, \mathcal{V})$. In this setting, we highlight irreducible unitary representations corresponding to a "geometric quantization" of elliptic orbits $\mathcal{O}_\lambda$. For real reductive groups, these representations are realized in the Dolbeault cohomology groups for equivariant holomorphic line bundles $\mathcal{L}_\lambda$ over $\mathcal{O}_\lambda$, giving the maximal globalization of Zuckerman derived functor modules $A_{\mathfrak{q}}(\lambda)$. (The localization of their contragredient representations are $K_{\mathbb{C}}$-equivariant sheaves of $\mathcal{D}$-modules supported on closed $K_{\mathbb{C}}$-orbits on the generalized flag variety $G_{\mathbb{C}}/Q$ by the Hecht–Miličić–Schmid–Wolf duality theorem [8].) A generalization of our propagation theorem would yield an interesting family of multiplicity-free branching problems, see [19, Conjecture 4.2].

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Note added in proof: Regarding to a potential generalization of our propagation theorem raised in Subsection 7.3, one may find recent progress in: K.-H. Neeb, Holomorphic realization of unitary representations of Banach-Lie groups, in the same volume.