COMPOSITION FORMULAS IN THE WEYL CALCULUS

TOSHIYUKI KOBAYASHI, BENT ØRSTED, MICHAEL PEVZNER, ANDRÉ UNTERBERGER

1. Introduction

No symbolic calculus of operators is more popular or better known than the Weyl calculus. It is the one that associates to a function $\mathfrak{S} = \mathfrak{S}(x, \xi)$ of $n+n$ variables, lying in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, the operator $\mathrm{Op}(\mathfrak{S})$, called the operator with symbol $\mathfrak{S}$, defined by the equation

$$(1.1) \qquad (\mathrm{Op}(\mathfrak{S})\, u)(x) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathfrak{S}(\frac{x+y}{2}, \eta)\, e^{2i\pi \langle x-y,\, \eta\rangle}\, u(y)\; dy\, d\eta \; :$$

such a linear operator extends as a continuous operator from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ while, in the case when $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, one can still define $\mathrm{Op}(\mathfrak{S})$ as a linear operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$; also, Op sets up an isometry from $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ onto the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. The sharp composition $\mathfrak{S}_1 \,\#\, \mathfrak{S}_2$ of two symbols, say lying in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, is that which makes the formula

$$(1.2) \qquad \mathrm{Op}(\mathfrak{S}_1)\, \mathrm{Op}(\mathfrak{S}_2) = \mathrm{Op}(\mathfrak{S}_1 \,\#\, \mathfrak{S}_2)\,,$$

in which the left-hand side denotes the usual composition of operators, valid.

The image of the Heisenberg representation is the group of unitary transformations $\exp\left(2i\pi\left(\langle \eta, Q\rangle - \langle y, P\rangle - t\right)\right)$ of $L^2(\mathbb{R}^n)$, as made meaningful by Stone's theorem, where the jth component of the vector $Q = (Q_1, \ldots, Q_n)$ is the multiplication by the jth coordinate x_j, $P = (P_1, \ldots, P_n)$ with $P_j = \frac{1}{2i\pi} \frac{\partial}{\partial x_j}$, and $y, \eta \in \mathbb{R}^n$, $t \in \mathbb{R}$. Introducing on $(\mathbb{R}^n \times \mathbb{R}^n)^2$ the symplectic form $[\, ,\,]$ such that

$$(1.3) \qquad [(x, \xi)\,,\, (y, \eta)] = -\langle x, \eta\rangle + \langle y, \xi\rangle\,,$$

let us use on $\mathbb{R}^n \times \mathbb{R}^n$ the *symplectic* Fourier transformation $\mathcal{F}$ defined by the equation

$$(1.4) \qquad (\mathcal{F}\, \mathfrak{S})(X) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathfrak{S}(Y)\, e^{-2i\pi\, [X,\, Y]}\, dY\,,$$

which commutes with all symplectic linear transformations of the variable in $\mathbb{R}^n \times \mathbb{R}^n$. Another, fully equivalent, way to define the Weyl calculus is by means of the equation

$$(1.5) \qquad \mathrm{Op}(\mathfrak{S}) = \int_{\mathbb{R}^n \times \mathbb{R}^n} (\mathcal{F}\, \mathfrak{S})(y, \eta)\, \exp\left(2i\pi\left(\langle \eta, Q\rangle - \langle y, P\rangle\right)\right) dy\, d\eta\,.$$

The first covariance rule of the Weyl calculus is the observation that

$$\begin{aligned}(1.6)\quad \exp\left(2i\pi\left(\langle \eta,\, Q\rangle - \langle y,\, P\rangle\right)\right)\, \mathrm{Op}(\mathfrak{S})\ \exp\left(-2i\pi\left(\langle \eta,\, Q\rangle - \langle y,\, P\rangle\right)\right)\\ = \mathrm{Op}((x,\,\xi)\mapsto \mathfrak{S}(x-y,\,\xi-\eta))\,.\end{aligned}$$

One way to emphasize this action on symbols of the group of translations of $\mathbb{R}^{2n}$ is to decompose in a systematic way the space of symbols $L^2(\mathbb{R}^{2n})$ with respect to this action. Now, the operators which commute with it are just the partial differential operators with constant coefficients: the generalized joint eigenfunctions of these are exactly the exponentials $X = (x,\,\xi) \mapsto e^{2i\pi\,[A,\,X]}$ with $A \in \mathbb{R}^{2n}$, and the sought-after decomposition of a symbol is provided by the symplectic Fourier transformation. On the other hand, if $A = (y,\,\eta)$, the operator with symbol $e^{2i\pi\,[A,\,X]}$ is none other than the operator $\exp\left(2i\pi\left(\langle \eta,\, Q\rangle - \langle y,\, P\rangle\right)\right)$, so that Heisenberg's commutation relation, expressed in Weyl's exponential version, takes the form

$$(1.7)\qquad e^{2i\pi\,[A^1,\,X]} \,\#\, e^{2i\pi\,[A^2,\,X]} = e^{i\pi\,[A^1,\,A^2]}\, e^{2i\pi\,[A^1+A^2,\,X]}\,.$$

Before coming to the point of the present work, let us briefly recall a few immediate consequences of this relation. First, one has (say, when $\mathfrak{S}_1$ and $\mathfrak{S}_2$ lie in $\mathcal{S}(\mathbb{R}^{2n})$), using (1.5), the integral composition formula

$$(1.8)\qquad (\mathfrak{S}_1 \,\#\, \mathfrak{S}_2)(X) = 2^{2n} \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} \mathfrak{S}_1(Y)\,\mathfrak{S}_2(Z)\, e^{-4i\pi\,[Y-X,\,Z-X]}\, dY\, dZ$$

or (a fully equivalent one)

$$(1.9)\qquad (\mathfrak{S}_1 \,\#\, \mathfrak{S}_2)(X) = \left[\,\exp(i\pi L)\,(\mathfrak{S}_1(Y)\,\mathfrak{S}_2(Z))\,\right]\,(Y = Z = X)$$

with (setting $Y = (y,\,\eta),\ Z = (z,\,\zeta)$)

$$(1.10)\qquad i\pi L = \frac{1}{4i\pi} \sum_{j=1}^{n} \left(-\frac{\partial^2}{\partial y_j\,\partial \zeta_j} + \frac{\partial^2}{\partial z_j\,\partial \eta_j} \right)\,.$$

Expanding the exponential into a series, one obtains the so-called Moyal formula

$$\begin{aligned}(1.11)\quad &(\mathfrak{S}_1 \,\#\, \mathfrak{S}_2)(x,\,\xi)\\ &= \sum \frac{(-1)^{|\alpha|}}{\alpha\,!\,\beta\,!} \left(\frac{1}{4i\pi}\right)^{|\alpha|+|\beta|} \left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial \xi}\right)^{\beta} \mathfrak{S}_1(x,\,\xi) \left(\frac{\partial}{\partial x}\right)^{\beta} \left(\frac{\partial}{\partial \xi}\right)^{\alpha} \mathfrak{S}_2(x,\,\xi)\,.\end{aligned}$$

This formula is an exact one in the case when the two operators under consideration are differential operators, which means exactly that their symbols (of course, not in $\mathcal{S}(\mathbb{R}^{2n})$) are polynomial with respect to the variables ξ, with coefficients depending on x in a smooth, but otherwise fairly arbitrary way; it is also exact when one of the two symbols is a polynomial in $(x,\, \xi)$.

As it turns out, this version of the composition formula is the only universally known one. Indeed, it has considerable importance in applications of pseudodifferential analysis to partial differential equations: classes of symbols for which the above formula, without being an exact one, still has some asymptotic value, provide a good proportion of the auxiliary operators needed for the solution of P.D.E. problems. In a conclusion, however, we shall illustrate on one example while this may sometimes fail and call instead for the composition formula which is the object of the present paper.

Our derivation of (1.8) was obtained as the result of pairing the concept of sharp composition of symbols with the decomposition of symbols according to the action by translations of the group $\mathbb{R}^{2n}$: the success of this point of view was essentially dependent on the fact that this action is an ingredient of the covariance formula (1.6). This takes us to the aim of the present paper: to take advantage of the other covariance property of the Weyl calculus – to be recalled now – and follow the same policy.

Recall that the metaplectic representation Met in $L^2(\mathbb{R}^n)$ is a certain unitary representation [15] of the twofold cover of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$, which consists of all linear transformations g of $\mathbb{R}^n \times \mathbb{R}^n$ such that $[gX,\, gY] = [X,\, Y]$ for every pair $(X,\, Y)$ of points of $\mathbb{R}^n \times \mathbb{R}^n$: it acts irreducibly on each of the two subspaces of $L^2(\mathbb{R}^n)$ consisting of functions with a given parity. Unitary transformations in the image of the metaplectic representation also act as automorphisms of the space $\mathcal{S}(\mathbb{R}^n)$ or of the space $\mathcal{S}'(\mathbb{R}^n)$: moreover, if such a unitary transformation U lies above $g \in \mathrm{Sp}(n,\mathbb{R})$, and if $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^{2n})$, one has the covariance formula

$$U\,\mathrm{Op}(\mathfrak{S})\,U^{-1} = \mathrm{Op}(\mathfrak{S} \circ g^{-1})\,. \tag{1.12}$$

In full analogy with the procedure adopted above in connection with the Heisenberg representation, we now start from a decomposition of the phase space representation $(g,\, \mathfrak{S}) \mapsto \mathfrak{S} \circ g^{-1}$ of $\mathrm{Sp}(n,\mathbb{R})$ in $L^2(\mathbb{R}^{2n})$ into irreducibles: this is just the same as decomposing functions in $L^2(\mathbb{R}^{2n})$ as integral superpositions of functions homogeneous of a given degree, and with a given parity.

Our main result is the formula which takes the place of (1.7): it decomposes the sharp product of two symbols h_1 and h_2, homogeneous of degrees $-n - i\lambda_1$ and $-n-i\lambda_2$ and with parities characterized by indices δ_1 and δ_2, as an integral superposition of functions homogeneous of degrees $-n - i\lambda$, with the parity $\delta \equiv \delta_1 + \delta_2$. It involves the integral kernel

$$|\,[Y,\, X]\,|_{\varepsilon_2}^{\frac{-n-i\lambda+i\lambda_1-i\lambda_2}{2}}\;|\,[X,\, Z]\,|_{\varepsilon_1}^{\frac{-n-i\lambda-i\lambda_1+i\lambda_2}{2}}\;|\,[Z,\, Y]\,|_{\varepsilon}^{\frac{-n+i\lambda+i\lambda_1+i\lambda_2}{2}}\,, \tag{1.13}$$

a product of three *signed* powers, obtained from the decomposition into homogeneous components with respect to the three variables of the integral kernel which occurs in the composition formula (1.8). Some preparation is needed in order to give this kernel a genuine meaning as a distribution, not only as a partially defined function. The principle of the proof of the new composition formula is simple, and relies on the decomposition of symbols into hyperplane waves, and the dual notion of rays. Its main difficulty lies in the singular nature of such distributions, which are nevertheless the only ones, sufficiently general, for which explicit computations are possible.

In the one-dimensional case, the integral kernel above reduces to a function

$$(1.14)\quad J(x,y,z) = |x-y|_{\varepsilon_2}^{\frac{-1-i\lambda+i\lambda_1-i\lambda_2}{2}}\, |z-x]\, |_{\varepsilon_1}^{\frac{-1-i\lambda-i\lambda_1+i\lambda_2}{2}}\, |y-z|_{\varepsilon}^{\frac{-1+i\lambda+i\lambda_1+i\lambda_2}{2}}$$

of three real variables, and the composition formula was treated along these lines in [12, section 17]. It is true that the proof, in the higher-dimensional case, is actually, for the main part, a reduction to the one-dimensional case: but signed powers of linear forms with exponents lying on the line $-n+i\mathbb{R}$, the consideration of which is necessary for spectral-theoretic reasons, are more singular distributions when $n \geq 2$, which has made some technical improvements necessary. It may be interesting to recall briefly what can be done in the one-dimensional case in relation to automorphic distribution theory.

In the automorphic situation, the integral kernel (1.14) enables one to build new non-holomorphic modular forms from given pairs of such. In [11], one of this paper's authors introduced the notion of automorphic distribution: this is a distribution in $\mathbb{R}^2$ invariant under *linear* changes of coordinates associated to elements of some arithmetic subgroup of $SL(2,\mathbb{R})$, for instance $SL(2,\mathbb{Z})$. This concept is equivalent — in a non-trivial way — to the Lax-Phillips notion of *pairs* of non-holomorphic modular forms, as introduced in their scattering theory [7] for the automorphic wave equation. Automorphic distributions can be taken as symbols in the Weyl calculus and, at the price of important difficulties, the one-dimensional case of the analysis of sharp-products in the present paper can be developed in the automorphic environment. Things are more interesting, in some sense, since besides a continuous part, in which Eisenstein distributions serve as generalized eigenfunctions, the automorphic Euler operator has a discrete spectrum, and the corresponding eigendistributions are cusp-distributions. Finding the appropriate composition formulas calls for the explicit computation of integrals of $J(x,y,z)$ against three non-holomorphic modular forms, in the realization of these as distributions on the line invariant under representations taken from the principal series of the arithmetic subgroup of $SL(2,\mathbb{R})$ under consideration: this has been completed up to some large extent, for the case of the full modular group, in [12] (*cf.* in particular section 16), and it provides a pseudodifferential-theoretic approach to such notions as L-functions, convolution L-functions, etc... As a preparation for automorphic pseudodifferential analysis, and in view of other applications as well, either to arithmetic or to quantization theory, a study of the integral kernel (1.14) had been made in [11]. It has also been considered recently in [8], in the automorphic case (for its own sake, not in connection with pseudodifferential analysis), and we take it from the references there that, outside the automorphic environment, it had already appeared in [9]: note that the objects called automorphic distributions in [8] are not the same as those in [11, 12] (they are close to what was called modular distributions in [11]).

Obviously, it would be of great interest to push the present composition formula for n-dimensional pseudodifferential analysis up to an automorphic environment, despite the great difficulties experienced with automorphic pseudodifferential analysis in the one-dimensional case. In any case, linking pseudodifferential analysis to harmonic analysis, then to modular form theory (also the subject of [13], though the connection between these domains is different there) is certain to bring rewards in the future. In a non-automorphic environment, the basic idea put forward in the present paper, namely that of building composition formulas from the pairing of covariance with the decomposition of representations into irreducibles, may also [12, section 19] be of use whenever some symbolic calculus of operators is examined, thus finding its place within quantization theory in general.

Contents

2. Decomposing the action of the symplectic group on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$

Consider the linear space $\mathbb{R}^n \times \mathbb{R}^n$ with its canonical symplectic form (1.3) and measure $dx\, d\xi$: we also set, when convenient, $X = (x,\, \xi)$. The symplectic group $G = \mathrm{Sp}(n, \mathbb{R})$ is the group of linear transformations g of $\mathbb{R}^n \times \mathbb{R}^n$ which preserve the symplectic form, *i.e.*, satisfy the identity $[gX,\, gY] = [X,\, Y]$ for any pair X, Y of points of $\mathbb{R}^{2n}$. The *phase space representation* of G in $L^2(\mathbb{R}^{2n})$ is defined by the action $(g,\, h) \mapsto g\,.\,h$ such that $(g\,.\,h)(X) = h(g^{-1}X)$. It is unitary, and since all linear transformations on $\mathbb{R}^n \times \mathbb{R}^n$ preserve the parity of functions and commute with the Euler operator

$$2i\pi\mathcal{E} = \sum \left[x_j \frac{\partial}{\partial x_j} + \xi_j \frac{\partial}{\partial \xi_j} \right] + n \tag{2.1}$$

(the additional constant turns $\mathcal{E}$ into a formally self-adjoint operator on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$), the (extension of the) phase space representation under study preserves the linear space of functions on $\mathbb{R}^{2n}\backslash\{0\}$ homogeneous of a given degree, and with a given parity.

Given $h \in L^2(\mathbb{R}^{2n})$, we first decompose it into its even and odd parts. Then, setting for every real number $s \neq 0$ and $\alpha \in \mathbb{C}$

$$|s|_0^\alpha = |s|^\alpha \,, \qquad |s|_1^\alpha = \langle s \rangle^\alpha = |s|^\alpha \operatorname{sign} s \,, \tag{2.2}$$

we may write

$$h = \sum_{\delta=0,1} \int_{-\infty}^{\infty} h_{i\lambda,\delta}\, d\lambda \,, \tag{2.3}$$

provided we set

$$h_{i\lambda,\delta}(X) = \frac{1}{4\pi} \int_{-\infty}^{\infty} |t|_\delta^{n-1+i\lambda}\, h(tX)\, dt \,. \tag{2.4}$$

Then, $h_{i\lambda,\delta}$ is homogeneous of degree $-n - i\,\lambda$ and has the parity associated to δ: we shall refer to the pair $(-n - i\lambda,\, \delta)$ as the *type* of $h_{i\lambda,\delta}$. More generally, we may consider on $\mathbb{R}^{2n}\backslash\{0\}$ functions of type $(-n - \nu,\, \delta)$ for an arbitrary complex parameter ν.

So as to cut down, as is needed, the dimension by 1, one may realize functions of a given type as sections of some appropriate line bundle over the projective space $P_{2n-1}(\mathbb{R})$. We first need to introduce the so-called tautological bundle $E_{\mathbb{C}}$ over $P_{2n-1}(\mathbb{R})$, the fibre of which above a point $p(\theta)$ (p being the canonical map: $\mathbb{R}^{2n}\backslash\{0\} \to P_{2n-1}(\mathbb{R})$) is the complex line $\mathbb{C}\,\theta$ in $\mathbb{C}^{2n}$. Incidentally, note that the total space of the real line analogue $E_{\mathbb{R}}$ of this bundle is just the blown-up space $\widehat{\mathbb{R}^{2n}}$ which is used consistently for desingularization purposes, as will be the case in next section.

A canonical set of charts of $P_{2n-1}(\mathbb{R})$ is obtained in the following way: given a vector $S \in \mathbb{R}^{2n}\backslash\{0\}$, set $\Omega_S = \{\theta \in \mathbb{R}^{2n} \colon [\theta,\, S] \neq 0\}$ and, in $\omega_S = p(\Omega_S)$, take the chart $p(\theta) \mapsto \frac{\theta}{[\theta,\, S]}$, which identifies ω_S with the affine hyperplane $\mathcal{M}_S = \{X \in \mathbb{R}^{2n} \colon [X,\, S] = 1\}$. Above $\mathcal{M}_S$, a section of $E_{\mathbb{C}}$ can be identified with a complex-valued function f_S, associating to such a function the section $X \mapsto f_S(X)\, X$. Note that, if $X \in \mathcal{M}_S$ satisfies $[X,\, T] \neq 0$ for some new vector $T \in \mathbb{R}^{2n}\backslash\{0\}$, the points $X \in \mathcal{M}_S$ and $\frac{X}{[X,\, T]} \in \mathcal{M}_T$ are truly the images, under the charts associated with S and T, of the same point in $P_{2n-1}(\mathbb{R})$. Identifying

$f_S(X)\,X$ with $f_T(Y)\,Y$, where we have set $Y = \frac{X}{[X,\,T]}$, leads to the compatibility condition

$$f_T\left(\frac{X}{[X,\,T]}\right) = [X,\,T]\;f_S(X)\,, \tag{2.5}$$

which defines the transition functions of the line bundle $E_{\mathbb{C}}$.

More generally, given $(\mu,\,\delta)$ with $\mu \in \mathbb{C}$ and $\delta = 0$ or 1, define the signed power $|(E_{\mathbb{C}})|_\delta^\mu$ of $E_{\mathbb{C}}$ by taking the corresponding signed powers of the transition functions: then, a section of the line bundle $|(E_{\mathbb{C}})|_\delta^\mu$ is associated to a set (f_S) of functions, f_S defined in $\mathcal{M}_S$, satisfying the requirement that

$$f_T\left(\frac{X}{[X,\,T]}\right) = |\,[X,\,T]\,|_\delta^\mu\;f_S(X) \tag{2.6}$$

whenever $X \in \mathcal{M}_0$ and $[X,\,T] \neq 0$. Then, a function h of type $(-n-\nu,\,\delta)$ can be identified with the section of $|(E_{\mathbb{C}})|_\delta^{n+\nu}$ characterized by the fact that, for every $S \in \mathbb{R}^{2n}\backslash\{0\}$, f_S is the restriction of h to $\mathcal{M}_S$. Conversely, any function f in $\mathcal{M}_S$ uniquely lifts as a function $f^\sharp$ in the part of $\mathbb{R}^{2n}\backslash\{0\}$ consisting of vectors θ such that $[\theta,\,S] \neq 0$, to wit the one defined by the equation

$$f^\sharp(\theta) = |\,[\theta, S]\,|_\delta^{-n-\nu}\;f\left(\frac{\theta}{[\theta, S]}\right). \tag{2.7}$$

The representation $\pi_{\nu,\delta}$ from the full, non-unitary principal series of $\mathrm{Sp}(n,\mathbb{R})$ is by definition the restriction of the phase space representation of $\mathrm{Sp}(n,\mathbb{R})$ (again, this is defined by the assignment $(g,\,h) \mapsto h \circ g^{-1}$) to the space of functions in $\mathbb{R}^{2n}\backslash\{0\}$ of type $(-n-\nu,\delta)$. It will be convenient — but there is a price to pay — not to have to change the hyperplane $\mathcal{M}_S$ consistently, and we denote as $\mathcal{M}_0$ the one which should really be denoted as $\mathcal{M}_{e_1}$ (where e_1 is the first vector from the canonical basis of $\mathbb{R}^n \times \mathbb{R}^n$), *i.e.*, the one consisting of vectors $X = (x\,;\,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $\xi_1 = 1$. Starting from (2.7) and using the fact that $f^\sharp$ is of type $(-n-\nu,\,\delta)$, together with the relation $[g^{-1}X,\,e_1] = [X,\,ge_1]$, one obtains the relation

$$(\pi_{\nu,\delta}(g)\,f)(X) = |\,[X,\,ge_1]\,|_\delta^{-n-\nu}\;f\left(\frac{g^{-1}X}{[X,\,ge_1]}\right). \tag{2.8}$$

As an example, when $n = 1$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, starting from $X = \left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)$, so that $g^{-1}X = \left(\begin{smallmatrix} dx-b \\ -cx+a \end{smallmatrix}\right)$, one obtains, after one has abbreviated $f\left(\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)\right)$ as $f^\flat(x)$, the relation

$$(\pi_{\nu,\delta}(g)\,f)^\flat(x) = |-cx+a|_\delta^{-1-\nu}\;f^\flat\Big(\frac{dx-b}{-cx+a}\Big)\,. \tag{2.9}$$

Still specializing, for the time being, in the hyperplane $\mathcal{M}_0$, we set

$$X = (x\,;\,\xi) = (x_1, x_*\,;\,\xi_1, \xi_*)\,, \tag{2.10}$$

and denote as $h^\flat_{i\lambda,\delta}$ the restriction of $h_{i\lambda,\delta}$ to $\mathcal{M}_0$ (it is the same as the function which would have been denoted as $(h_{i\lambda,\delta})_{e_1}$ in the less specialized setting above). One has the reciprocal equations

$$h^\flat_{i\lambda,\delta}(x\,;\,\xi_*) = h_{i\lambda,\delta}(x\,;\,1,\xi_*)\,,$$

$$h_{i\lambda,\delta}(x\,;\,\xi) = |\xi_1|_\delta^{-n-i\lambda}\,h^\flat_{i\lambda,\delta}\left(\frac{x}{\xi_1}\,;\,\frac{\xi_*}{\xi_1}\right)\,. \tag{2.11}$$

Remark 2.1. Under the preceding pair of equations, the functions $h_{i\lambda,\delta}$ and $h^\flat_{i\lambda,\delta}$ are virtually indistinguishable, *once the type* $(-n-i\lambda,\,\delta)$ *has been fixed.* Using the second notion will be useful in connection with all concepts using integrals, such as integral operators, norms,... However, the first point of view is more intrinsic, and is especially useful (since some singularities could lie "at infinity" relative to the chosen hyperplane $\mathcal{M}_0$) when, as will be the case in Section 4, we need to extend the representation $\pi_{\nu,\delta}$ or the intertwining operator to be introduced below to a distribution setting.

Proposition 2.1. *The space* $L^2(\mathbb{R}^{2n})$ *can be decomposed as the Hilbert direct integral*

$$L^2(\mathbb{R}^{2n}) \sim \bigoplus_{\delta=0,1}\int^{\oplus} \mathcal{H}_{i\lambda,\delta}\,d\lambda\,, \tag{2.12}$$

if one denotes as $\mathcal{H}_{i\lambda,\delta}$ *the inverse image under the map* $h_{i\lambda,\delta}\mapsto h^\flat_{i\lambda,\delta}$ *of the space* $L^2(\mathcal{M}_0;\,dx\,d\xi_*)$ *: the decomposition is provided by (2.3), and it commutes with the phase space representation of* G *in* $L^2(\mathbb{R}^{2n})$ *.*

Proof. What remains to be done is proving the equation

$$\|\,h\,\|^2_{L^2(\mathbb{R}^{2n})} = 4\pi\sum_{\delta=0,1}\int_{-\infty}^{\infty}\|\,h^\flat_{i\lambda,\delta}\,\|^2_{L^2(\mathcal{M}_0)}\,d\lambda\,, \tag{2.13}$$

using on $\mathcal{M}_0$ the measure $dx\,d\xi_*$. Indeed, with $h_{(\delta)} = h_{\rm even}$ or $h_{\rm odd}$ according to the parity of δ , set

$$\phi_X(s) = e^{2\pi ns}\,h_{(\delta)}(e^{2\pi s}X)\,, \qquad s\in\mathbb{R}\,,\ X\in\mathbb{R}^{2n}\backslash\{0\}\,, \tag{2.14}$$

so that

$$\hat{\phi}_X(\lambda) = h_{i\lambda,\delta}(X)\,. \tag{2.15}$$

The one-dimensional Fourier inversion formula then yields (2.3) (of course, using the Mellin transform rather than coupling a Fourier transform with the change of variable $t = e^{2\pi s}$ would be more natural: the choice really depends on your familiarity with the inversion formula in both cases). Next, using (2.11) and the Plancherel formula for the Fourier transformation,

$$\begin{aligned}
\| h_{(\delta)} \|^2_{L^2(\mathbb{R}^{2n})} &= 4\pi \int_{-\infty}^{\infty} e^{2\pi s}\, ds \int_{\mathbb{R}^{2n-1}} |h_{(\delta)}(x\,;\, e^{2\pi s}, \xi_*)|^2\, dx\, d\xi_* \\
&= 4\pi \int_{-\infty}^{\infty} ds \int_{\mathbb{R}^{2n-1}} |\phi_{(x\,;\,1,\xi_*)}(s)|^2\, dx\, d\xi_* \\
&= 4\pi \int_{\mathbb{R}^{2n-1}} dx\, d\xi_* \int_{-\infty}^{\infty} |\hat{\phi}_{(x\,,1,\xi_*)}(s)|^2\, ds \\
&= 4\pi \int_{\mathbb{R}^{2n-1}} dx\, d\xi_* \int_{-\infty}^{\infty} |h_{i\lambda,\delta}(x\,;\,\xi_*)|^2\, ds\,, \qquad (2.16)
\end{aligned}$$

which proves (2.13).

□

The decomposition above gives right to the series $(\pi_{i\lambda,\delta})_{\lambda\in\mathbb{R},\delta=0,1}$ of representations of G in $L^2(\mathcal{M}_0)$, a special case of the representations $\pi_{\nu,\delta}$ already considered; it suffices to set

$$\pi_{i\lambda,\delta}(g)\, h^\flat_{i\lambda,\delta} = f^\flat_{i\lambda,\delta} \tag{2.17}$$

if $h \in L^2(\mathbb{R}^{2n})$, $g \in G$, $f = h \circ g^{-1}$. Each representation $\pi_{i\lambda,\delta}(g)$ is unitary as a consequence of Proposition 2.1: to show that $\| \pi_{i\lambda,\delta}(g)\, h^\flat_{i\lambda,\delta} \| = \| h^\flat_{i\lambda,\delta} \|$ for every λ such that $h^\flat_{i\lambda,\delta} \in L^2(\mathcal{M}_0)$, not only almost every λ, it suffices to start from a dense space of functions h such that $h^\flat_{i\lambda,\delta}$ depends in a continuous way on λ, which is ensured for instance when h lies in $\mathcal{S}(\mathbb{R}^{2n})$. Recall (*cf.* Remark 2.1) that we also set $\pi_{i\lambda,\delta}(g)\, h_{i\lambda,\delta} = f_{i\lambda,\delta}$.

In Section 7, it will be proved that most representations $\pi_{i\lambda,\delta}$ are irreducible.

Remark 2.2. When integrating on $\mathcal{M}_S$, we shall have to worry a lot about singularities: but we shall never have to worry about the contribution to integrals of the part of this hyperplane away from some compact subset because, in reality, we shall be dealing with integrals on the compact space $P_{2n-1}(\mathbb{R})$ and (say, with the help of partitions of unity), we could always, replacing the integral under consideration by a finite sum of integrals taken on *distinct* hyperplanes, replace for each term the integral by the integral taken on some compact subset of the corresponding hyperplane.

The (symplectic) Fourier transform of a function homogeneous of degree $-n-i\lambda$ with a given parity is homogeneous of degree $-n+i\lambda$, and has the same parity, so that, given $h \in L^2(\mathbb{R}^{2n})$, one has

$$\mathcal{F}\, h_{i\lambda,\delta} = (\mathcal{F}h)_{-i\lambda,\delta}\ : \tag{2.18}$$

consequently, the representations $\pi_{i\lambda,\delta}$ and $\pi_{-i\lambda,\delta}$ are unitarily equivalent.

Definition 2.2. The (unitary) intertwining operator $\theta_{i\lambda,\delta}$ is the one characterized by the validity of the equation

$$\theta_{i\lambda,\delta}\, h_{i\lambda,\delta} = (\mathcal{F}h)_{-i\lambda,\delta} \tag{2.19}$$

for every $h \in L^2(\mathbb{R}^{2n})$. We also set (*cf.* Remark 2.1)

$$\theta_{i\lambda,\delta}\, h^\flat_{i\lambda,\delta} = (\mathcal{F}h)^\flat_{-i\lambda,\delta}\,. \tag{2.20}$$

The proof that $\theta_{i\lambda,\delta}$ preserves the L^2-norm for every λ, not only almost every λ, is the same as the one which, in connection with the definition of $\pi_{i\lambda,\delta}$, followed (2.17). It is easy to make the unitary intertwining operator $\theta_{i\lambda,\delta}$ associated to (2.18) explicit in terms of the coordinates on $\mathcal{M}_0$. Indeed, starting from (2.11), one can write

$$\begin{aligned} (2.21)\quad & (\mathcal{F}\,h)^\flat_{-i\lambda,\delta}(x\,;\,\xi_*) = (\mathcal{F}\,h_{i\lambda,\delta})(x\,;1,\xi_*) \\ &= \int |\eta_1|_\delta^{-n-i\lambda}\, h^\flat_{i\lambda,\delta}\Big(\frac{y}{\eta_1};\frac{\eta_*}{\eta_1}\Big)\, \exp\left(2i\pi\,[x_1\eta_1 + \langle x_*,\eta_*\rangle - y_1 - \langle y_*,\xi_*\rangle]\right) dy d\eta_1 d\eta_* \\ &= \int |\eta_1|_\delta^{n-1-i\lambda}\, h^\flat_{i\lambda,\delta}(y;\eta_*) \exp\left(2i\pi\eta_1[x_1 + \langle x_*,\eta_*\rangle - y_1 - \langle y_*,\xi_*\rangle]\right)\, dy d\eta_1 d\eta_*. \end{aligned}$$

Making a one-dimensional Fourier transformation explicit, this gives another approach to the intertwining operator $\theta_{i\lambda,\delta}$ from $\pi_{i\lambda,\delta}$ to $\pi_{-i\lambda,\delta}$: the operator $\theta_{i\lambda,\delta}$ is defined *formally* as the operator with integral kernel

(2.22)

$$k_{i\lambda,\delta}(x,\xi_*\,;\,y,\eta_*) = i^\delta\,\pi^{\frac12-n+i\lambda}\,\frac{\Gamma(\frac{n-i\lambda+\delta}{2})}{\Gamma(\frac{1-n+i\lambda+\delta}{2})}\;|x_1 - y_1 + \langle x_*,\eta_*\rangle - \langle y_*,\xi_*\rangle|_\delta^{-n+i\lambda}\,.$$

Note that, while Definition 2.2 is a rigorous definition of the intertwining operator, (2.22) can only be used after some preparation, which will be done in Section 3.

While $X = (x\,;\,\xi)$ (or $Y = (y\,;\,\eta)$, ...) will always denote a generic point in $\mathbb{R}^{2n}$, we shall draw attention to points $(x\,;\,1,\xi_*) = (x_1,x_*\,;\,1,\xi_*)$ of $\mathcal{M}_0$ by denoting them as X_*: similarly, $Y_* = (y\,;\,1,\eta_*)$. Given $X_* \in \mathcal{M}_0$, we set $X_{**} = (x_*\,;\,\xi_*)$, so that one can also identify X_* with $(x_1,\,X_{**})$. We abbreviate the measure $dx\,d\xi_*$ on $\mathcal{M}_0$ as $dm(X_*)$. On $\mathbb{R}^{2n-2}$, one can also consider the symplectic form obtained from an appropriate restriction of the one available on $\mathbb{R}^{2n}$, *i.e.*, set

$$[X_{**},\,Y_{**}] = -\langle x_*,\,\eta_*\rangle + \langle y_*,\,\xi_*\rangle\,, \tag{2.23}$$

while, on $\mathcal{M}_0$, one must define

$$\begin{aligned}[X_*,\, Y_*] &= [\,((x_1, x_*)\,;\,(1, \xi_*))\,,\,((y_1, y_*)\,;\,(1, \eta_*))\,]\\ &= -x_1 + y_1 - \langle x_*, \eta_*\rangle + \langle y_*, \xi_*\rangle\,. \end{aligned} \tag{2.24}$$

One may then rewrite (2.22) as

$$(\theta_{i\lambda,\delta}\, f)(X_*) = i^{\delta}\, \pi^{\frac{1}{2}-n+i\lambda}\, \frac{\Gamma(\frac{n-i\lambda+\delta}{2})}{\Gamma(\frac{1-n+i\lambda+\delta}{2})} \int_{\mathcal{M}_0} |\,[Y_*,\, X_*]\,|_{\delta}^{-n+i\lambda}\, f(Y_*)\, dm(Y_*)\,. \tag{2.25}$$

The intertwining operator may be better understood after some transformation. Denote as $\mathcal{F}_1$ the usual Fourier transformation as applied when emphasis is set on the first variable only of a function of several variables. Given a function f on $\mathcal{M}_0$, write it as $h^{\flat}_{i\lambda,\delta}$, which, according to (2.11), is possible in a unique way for a given pair $(i\lambda, \delta)$, so that the left-hand side of (2.21) is just $(\theta_{i\lambda,\delta}\, f)(x;\, \xi_*)$ according to (2.18). Starting from (2.21), one can then write, if $n \geq 2$,

$$\begin{aligned} &(\mathcal{F}_1\, \theta_{i\lambda,\delta}\, f)(t, x_*\,;\, \xi_*) = (\mathcal{F}_1\, \theta_{i\lambda,\delta}\, f)(t,\, X_{**})\\ =\ & |t|_{\delta}^{n-1-i\lambda} \int_{\mathcal{M}_0} f(y_1,\, Y_{**})\, \exp\left(-2i\pi\, t\, [y_1 + [X_{**},\, Y_{**}]\right) dy_1\, dY_{**}\\ =\ & |t|_{\delta}^{n-1-i\lambda} \int_{\mathbb{R}^{2n-2}} (\mathcal{F}_1 f)(t,\, Y_{**})\, \exp\left(-2i\pi\, t\, [X_{**},\, Y_{**}]\right) dY_{**}\,. \end{aligned} \tag{2.26}$$

In this definition of the intertwining operator, $\theta_{i\lambda,\delta}$ appears as the "product" of a one-dimensional intertwining operator with respect to the first variable and of a Fourier transformation in $\mathbb{R}^{2n-2}$: only, some rescaling, by the variable dual to the first one, is performed with respect to the last $2n-2$ variables. As a straightforward application of this equation, note the formula, in which $\delta_2 :\,= \delta_1 + \delta$,

$$(\mathcal{F}_1\, \theta_{i\lambda_1,\delta_1}\, \theta_{i\lambda,\delta}\, f)(t,\, X_{**}) = |t|_{\delta_2}^{-i(\lambda_1+\lambda)}\, (\mathcal{F}_1\, f)(t,\, X_{**}) \;: \tag{2.27}$$

hence, the composition of the two intertwining operators under consideration reduces to an intertwining operator with respect to the first variable, with integral kernel

$$\begin{aligned} &((x_1,\, X_{**}),\, (y_1,\, X_{**})) \mapsto\\ &\qquad i^{\delta_2}\, \pi^{-\frac{1}{2}+i(\lambda_1+\lambda)}\, \frac{\Gamma(\frac{1-i(\lambda_1+\lambda)+\delta_2}{2})}{\Gamma(\frac{i(\lambda_1+\lambda)+\delta_2}{2})}\, |x_1 - y_1|_{\delta_2}^{-1+i(\lambda_1+\lambda)}\; \delta(X_{**} - Y_{**})\,. \end{aligned} \tag{2.28}$$

At this point, it may be useful to clarify the respective roles of the coordinates ξ_1 and x_1, as they occur in what precedes. Isolating the coordinate ξ_1 is tantamount to singling out the affine hyperplane $\mathcal{M}_0$, the equation of which is

$[X,\, e_1] = 1$, while $[X,\, e_1] = \xi_1$ generally. The expression $\frac{\partial f}{\partial x_1}$, for $f \in C^\infty(\mathcal{M}_0)$, is then the image of f under a canonical operator on $\mathcal{M}_0$, since it may be thought of as the Poisson bracket of the function $X \mapsto \xi_1$ with an arbitrary smooth extension of f to the whole of $\mathbb{R}^{2n}$. One may interpret the convolution operator the integral kernel of which is given in (2.28) as a function (a signed power, of course), in the sense of functional calculus, of the operator $\frac{1}{2i\pi} \frac{\partial}{\partial x_1}$. On the other hand, the coordinate x_1 is not intrinsically attached to $\mathcal{M}_0$: with the help of a well-chosen symplectic transformation preserving the coordinate ξ_1 , it can be transformed to the sum of x_1 and of an arbitrary linear combination of $x_2,\, \dots,\, x_n,\, \xi_1,\, \dots,\, \xi_n$.

Note if $f \in L^2(\mathcal{M}_0)$ the relation

$$\overline{\pi_{i\lambda,\delta}(g)\, f} = \pi_{-i\lambda,\delta}(g)\, \bar{f} \tag{2.29}$$

from which, polarizing the identity which expresses that $\pi_{i\lambda,\delta}$ is unitary, we obtain the identity

$$\int_{\mathcal{M}_0} f_2(X)\, f_1(X_*)\, dm(X_*) = \int_{\mathcal{M}_0} (\pi_{-i\lambda,\delta}(g)\, f_2)(X_*)\, (\pi_{i\lambda,\delta}(g)\, f_1)(X_*)\, dm(X_*) \tag{2.30}$$

involving a pair $(f_1,\, f_2)$ of functions in $L^2(\mathcal{M}_0)$: this can also be regarded as a particular case of (2.27), to the effect that the inverse of the isometry $\theta_{i\lambda,\delta}$ is $\theta_{-i\lambda,\delta}$. Assuming convergence, one can extend (2.30) as

$$\int_{\mathcal{M}_0} f_2(X_*)\, f_1(X_*)\, dm(X_*) = \int_{\mathcal{M}_0} (\pi_{-\nu,\delta}(g)\, f_2)(X_*)\, (\pi_{\nu,\delta}(g)\, f_1)(X_*)\, dm(X_*)\, . \tag{2.31}$$

We now introduce the integral kernel obtained from the decomposition into homogeneous components of the integral kernel $e^{4i\pi[Y,\, X]}\, e^{4i\pi[X,\, Z]}\, e^{4i\pi[Z,\, Y]}$ which occurs in the composition formula (1.8). Consider on $(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$ the (almost everywhere defined only) function

$$(Y,\, Z;\, X) \mapsto |\, [Y,\, X]\, |_{\varepsilon_2}^{\alpha_1}\ |\, [X,\, Z]\, |_{\varepsilon_1}^{\alpha_2}\ |\, [Z,\, Y]\, |_{\varepsilon}^{\alpha_3}\, , \tag{2.32}$$

where the exponents and indices of parity are given. It is of type $(\alpha_1 + \alpha_3,\, \varepsilon + \varepsilon_2 \bmod 2)$, *resp.* $(\alpha_2 + \alpha_3,\, \varepsilon + \varepsilon_1 \bmod 2)$, *resp.* $(\alpha_1 + \alpha_2,\, \varepsilon_1 + \varepsilon_2 \bmod 2)$ with respect to Y, *resp.* Z, *resp.* X .

Given a triple (ν_1, ν_2, ν) of complex numbers, and a triple $(\delta_1, \delta_2, \delta)$ of numbers equal to 0 or 1 , satisfying the relation $\delta \equiv \delta_1 + \delta_2 \bmod 2$, the system of equations

$$\varepsilon_2 + \varepsilon \equiv \delta_1, \quad \varepsilon_1 + \varepsilon \equiv \delta_2, \quad \varepsilon_1 + \varepsilon_2 \equiv \delta \tag{2.33}$$

for $\varepsilon,\, \varepsilon_1,\, \varepsilon_2$ mod 2 has two solutions, obtained as

$$\varepsilon \equiv j + \delta \quad \varepsilon_1 \equiv j + \delta_1 \quad \varepsilon_2 \equiv j + \delta_2 \tag{2.34}$$

with $j = 0$ or 1. Then, the types of the function above with respect to Y, Z, X will be $(-n+\nu_1,\delta_1)$, $(-n+\nu_2,\delta_2)$ and $(-n-\nu,\delta)$ if and only if

(2.35)
$$\alpha_1 = \frac{-n-\nu+\nu_1-\nu_2}{2}\,, \quad \alpha_2 = \frac{-n-\nu-\nu_1+\nu_2}{2}\,, \quad \alpha_3 = \frac{-n+\nu+\nu_1+\nu_2}{2}\,.$$

Hence, provided that (2.33) is satisfied, the integral kernel

(2.36)
$$J^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(Y,\,Z;\,X) = |\,[Y,\,X]\,|_{\varepsilon_2}^{\frac{-n-\nu+\nu_1-\nu_2}{2}}\,|\,[X,\,Z]\,|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}}\,|\,[Z,\,Y]\,|_{\varepsilon}^{\frac{-n+\nu+\nu_1+\nu_2}{2}}$$

in $(\mathbb{R}^{2n}\backslash\{0\}) \times (\mathbb{R}^{2n}\backslash\{0\}) \times (\mathbb{R}^{2n}\backslash\{0\})$ satisfies the covariance relation

$$\begin{aligned}(2.37)\quad \pi_{\nu,\delta}(g)\, &\left(X \mapsto J^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(Y,Z;\,X)\right)\\ &= \left[\,\pi_{-\nu_1,\delta_1}(g^{-1}) \otimes \pi_{-\nu_2,\delta_2}(g^{-1})\right]\,((Y,\,Z) \mapsto J^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(Y,Z;\,X))\,.\end{aligned}$$

We may also restrict this integral kernel to $\mathcal{M}_0 \times \mathcal{M}_0 \times \mathcal{M}_0$: the relation of covariance is preserved, though with a slightly different understanding (*cf.* (2.17)). In next section, we shall see, after we have given the integral kernel so obtained a meaning in an appropriate distribution sense, not only as a partially defined function, that if one denotes as $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}$ the associated operator, thought of as being defined by the equation

(2.38)
$$(\mathbf{J}^{\varepsilon_1,\varepsilon_2;\varepsilon}_{\nu_1,\nu_2;\nu}(f_1,\,f_2))(X_*) = \int_{\mathcal{M}_0\times\mathcal{M}_0} J^{\varepsilon_1,\varepsilon_2;\varepsilon}_{\nu_1,\nu_2;\nu}(Y_*,Z_*;X_*)\,f_1(Y_*)f_2(Z_*)\,dm(Y_*)\,dm(Z_*),$$

one has the covariance identity

$$(2.39)\qquad \pi_{\nu,\delta}(g)\,(\mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(f_1,\,f_2)) = \mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}\,(\pi_{\nu_1,\delta_1}(g)\,f_1,\,\pi_{\nu_2,\delta_2}(g)\,f_2)\,,$$

formally immediate from (2.37) and (2.31). In the case when $f_1 = (h_1)^{\flat}_{\nu_1,\delta_1}$ and $f_2 = (h_2)^{\flat}_{\nu_2,\delta_2}$, we can, and shall sometimes, write $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}((h_1)_{\nu_1,\delta_1},\,(h_2)_{\nu_2,\delta_2})$ for $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(f_1,\,f_2)$. Also, as explained in Remark 2.1, the result can be regarded as a function in $\mathbb{R}^{2n}\backslash\{0\}$ of type $(-n-\nu,\,\delta)$ rather than, again, as being defined only on $\mathcal{M}_0$.

3. The integral kernel $J^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(Y,\,Z\,;\,X)$

In all this section, we deal with functions of a given type in their realization as functions on $\mathcal{M}_0$. Rather than trying to define $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(f_1,\,f_2)$, as in (2.38), as a function of X_*, we lower our requirements, only trying to define the expression

$$\begin{aligned}(3.1)\quad & \langle \mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(f_1,\, f_2)\,,\, f\,\rangle \\ &= \int_{\mathcal{M}_0\times\mathcal{M}_0\times\mathcal{M}_0} J^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(Y_*,\, Z_*;\, X_*)\, f_1(Y_*)\, f_2(Z_*)\, f(X_*)\, dm(Y_*)\, dm(Z_*)\, dm(Z_*)\end{aligned}$$

for appropriate triples $(f_1,\, f_2,\, f)$. This is of course tantamount to a reinterpretation of $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}$ as a distribution of some kind, a notion dependent on that of C^∞- vectors of the representations π_{ν_1,δ_1}, π_{ν_2,δ_2}, $\pi_{-\nu,\delta}$ involved (the sign change in the last subscript is an effect of duality: *cf.* (2.30)).

First, we observe that, though the representation $\pi_{\nu,\delta}$ is not unitary unless ν is pure imaginary, it is still useful to regard it as a representation in some Hilbert space, to wit the one defined by the equation

$$(3.2)\qquad \|\, f\,\|_\nu^2 = \int_{\mathcal{M}_0} |f(X_*)|^2\; |X_*|^{2\,\mathrm{Re}\,\nu}\, dm(X_*)\ :$$

here, $|X_*|^2 = |x|^2 + 1 + |\xi_*|^2$ when $X_* = (x\,;\,1,\xi_*)$. We now show that, for any given $g \in \mathrm{Sp}(n,\mathbb{R})$, the transformation $\pi_{\nu,\delta}(g)$ is a bounded endomorphism of the Hilbert space $\mathcal{H}_\nu$ thus defined. First,

$$(3.3)\qquad Y\colon = \frac{g^{-1}X}{[X,\, ge_1]}\quad \text{lies in } \mathcal{M}_0 \text{ if } X \in \mathbb{R}^{2n} \text{ and } [X,\, ge_1] \neq 0\ :$$

indeed, recall that $\xi_1 = [X,\, e_1]$ if $X = (x\,;\,\xi)$ and that $[X,\, ge_1] = [g^{-1}X,\, e_1]$. Recalling the recipe, just before (2.30), which served as a definition of $\pi_{\nu,\delta}(g)$, we first extend f, initially defined on $\mathcal{M}_0$, as a function $f^\sharp$ in $\mathbb{R}^{2n}\backslash\{0\}$, setting

$$(3.4)\qquad f^\sharp(x\,;\,\xi_1,\xi_*) = |\xi_1|_\delta^{-n-\nu}\, f(\frac{x}{\xi_1}\,;\,1,\frac{\xi_*}{\xi_1})\,,$$

so that

$$(3.5)\qquad f^\sharp(g^{-1}\,.\,(x\,;\,\xi_1,\xi_*)) = |\,[X,\, ge_1]\,|_\delta^{-n-\nu}\, f\left(\frac{g^{-1}X}{[X,\, ge_1]}\right)\,,$$

and

$$(3.6)\qquad (\pi_{\nu,\delta}(g)\, f)(X_*) = |\,[X_*,\, ge_1]\,|_\delta^{-n-\nu}\, f(Y_*)$$

with $Y_* = \frac{g^{-1}X_*}{[X_*,\, ge_1]}$. The next thing to do is to compute the Jacobian $\frac{dm(Y_*)}{dm(X_*)}$ when X_* lies in $\mathcal{M}_0$: to this effect, the simplest way is to use the unitarity of $\pi_{0,\delta}$, to wit the relation

$$(3.7)\qquad \int_{\mathcal{M}_0} |\,[X_*,\, ge_1]\,|^{-2n}\, |f(Y_*)|^2\, dm(X_*) = \int_{\mathcal{M}_0} |f(X_*)|^2\, dm(X_*)\,,$$

finding

$$(3.8)\qquad dm(Y_*) = |\,[X_*,\, ge_1]\,|^{-2n}\, dm(X_*)\,.$$

Then, with the help of the same change of variables, one has more generally

$$\begin{aligned}\|\,\pi_{\nu,\delta}(g)\,f\,\|_\nu^2 &= \int_{\mathcal{M}_0} |\,[X_*,\,ge_1]\,|^{-2n-2\,\mathrm{Re}\,\nu}\,|f(Y_*)|^2\ |X_*|^{2\,\mathrm{Re}\,\nu}\,dm(X_*)\\ &= \int_{\mathcal{M}_0} |\,[X_*,\,ge_1]\,|^{-2\,\mathrm{Re}\,\nu}\,|f(Y_*)|^2\ |X_*|^{2\,\mathrm{Re}\,\nu}\,dm(Y_*)\\ &= \int_{\mathcal{M}_0}\left(\frac{|X_*|}{|g^{-1}X_*|}\right)^{2\,\mathrm{Re}\,\nu}|f(Y_*)|^2\ |Y_*|^{2\,\mathrm{Re}\,\nu}\,dm(Y_*)\,,\end{aligned}\tag{3.9}$$

an expression which we want to bound in terms of $\|\,f\,\|_\nu^2$. It suffices to observe that the ratio $\left(\frac{|X_*|}{|g^{-1}X_*|}\right)^{2\,\mathrm{Re}\,\nu}$ is bounded for $X_*\in\mathcal{M}_0$, the bound depending of course on g . Hence, $\pi_{\nu,\delta}$ is a representation by means of bounded operators in $\mathcal{H}_\nu$.

This makes it possible, in the usual way, to define the space of C^∞ vectors of the given representation. Recalling that the Lie algebra of the symplectic group consists of block-matrices $\left(\begin{smallmatrix}A & B\\ C & -A'\end{smallmatrix}\right)$ with B and C symmetric, one sees that the space of infinitesimal operators of the phase space representation of $\mathrm{Sp}(n,\mathbb{R})$ in $L^2(\mathbb{R}^{2n})$ is generated by the vector fields $\xi_j\,\frac{\partial}{\partial x_k}+\xi_k\,\frac{\partial}{\partial x_j}$, $x_j\,\frac{\partial}{\partial x_k}-\xi_k\,\frac{\partial}{\partial \xi_j}$, $x_j\,\frac{\partial}{\partial \xi_k}+x_k\,\frac{\partial}{\partial \xi_j}$, the values of which at each point $(x\,;\,\xi)$ with $\xi_1=1$ generate the linear subspace of $\mathbb{R}^{2n}$ tangent to $\mathcal{M}_0$. It follows that the space of C^∞- vectors of the representation $\pi_{\nu,\delta}$ consists of C^∞ functions in the usual sense. This condition is of course not sufficient: there are conditions “at infinity” best rephrased by simply changing the hyperplane $\mathcal{M}_0$ to an appropriate finite collection of hyperplanes $\mathcal{M}_S$, as will be seen for instance in the proof of Lemma 4.1.

Proposition 3.1. *When* $\mathrm{Re}\,\nu_1=\mathrm{Re}\,\nu_2=n$ *and* $\mathrm{Re}\,\nu=-n$ *, the function* $J^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(Y_*,\,Z_*;\,X_*)$ *as defined in (2.36) is a bounded function. One can extend its meaning as a distribution in* $\mathcal{M}_0\times\mathcal{M}_0\times\mathcal{M}_0$ *, holomorphic with respect to* ν_1,ν_2,ν *in the open subset of* $\mathbb{C}^3$ *defined, recalling (2.33) and (2.34), by the conditions*

$$\frac{n+\nu-\nu_1+\nu_2}{2}\neq\varepsilon_2+1,\,\varepsilon_2+3,\,\ldots;\quad \frac{n+\nu+\nu_1-\nu_2}{2}\neq\varepsilon_1+1,\,\varepsilon_1+3,\,\ldots\,;\quad \frac{n-\nu-\nu_1-\nu_2}{2}\neq\varepsilon+1,\,\varepsilon+3,\,\ldots\,,\tag{3.10}$$

together with the fact that at least one of three following conditions should hold:

$$3n+\nu-\nu_1-\nu_2\neq\begin{cases}1,\,3,\,\ldots\\ 2j+2,\,2j+6\,\ldots\end{cases}\qquad \text{and } n+\nu\neq\delta+1,\,\delta+3,\,\ldots\tag{3.11}$$

or any of the conditions obtained from (3.11) by changing $(\nu,\nu_1,\nu_2;\,\delta,\delta_1,\delta_2)$ *to* $(-\nu_1,-\nu,\nu_2;\,\delta_1,\delta,\delta_2)$ *or to* $(-\nu_2,\nu_1,-\nu;\,\delta_2,\delta_1,\delta)$ *. When* $n=1$ *, one can delete the condition* $3+\nu-\nu_1-\nu_2\neq 1,\,3,\,\ldots$ *from (3.11).*

Something entirely similar holds after one has replaced $\mathcal{M}_0$ *by* $\mathcal{M}_S$ *for an arbitrary* $S \in \mathbb{R}^{2n} \backslash \{0\}$ *. In view of the inclusion* $C^\infty(\pi_{\nu,\delta}) \subset C^\infty(\mathcal{M}_0)$ *and of Remark 2.2, this will automatically make it a continuous trilinear form on the space of* $(f_1,\, f_2,\, f) \in C^\infty(\pi_{\nu_1,\delta_1}) \times C^\infty(\pi_{\nu_2,\delta_2}) \times C^\infty(\pi_{-\nu,\delta})$ *. Setting, when* ν_1, ν_2, ν *satisfy (3.10) and (3.11), and* f_1, f_2, f *are* C^∞ *functions with compact support in* $\mathcal{M}_0$ *,*

$$\text{(3.12)} \quad \mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(f_1,\, f_2;\, f)$$
$$= \int_{\mathcal{M}_0 \times \mathcal{M}_0 \times \mathcal{M}_0} J^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(Y_*,\, Z_*;\, X_*)\, f_1(Y_*)\, f_2(Z_*)\, f(X_*)\, dm(Y_*)\, dm(Z_*)\, dm(Z_*)\,,$$

one has the covariance relation

$$\text{(3.13)} \qquad \mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}\, (\pi_{\nu_1,\delta_1}(g)\, f_1,\, \pi_{\nu_2,\delta_2}(g)\, f_2;\, \pi_{-\nu,\delta}(g)\, f) = \mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(f_1,\, f_2;\, f)$$

for every symplectic transformation g *such that the transformed versions of* f_1, f_2, f *also have compact support in* $\mathcal{M}_0$ *.*

Proof. The "integral" on the right-hand side of (3.12) is of course a usual notation for what is in effect the result of testing a certain distribution on the function $f_1 \otimes f_2 \otimes f$. Before coming to the proof, let us indicate that one should not worry about the condition of compact support: in the way explained in Remark 2.2, one can dispense with it, only replacing the domain of integration $\mathcal{M}_0 \times \mathcal{M}_0 \times \mathcal{M}_0$ by a finite collection of domains $\mathcal{M}_S \times \mathcal{M}_S \times \mathcal{M}_S$.

When $\mathrm{Re}\,\nu_1 = \mathrm{Re}\,\nu_2 = n$ and $\mathrm{Re}\,\nu = -n$, all exponents in definition (2.36) of $J^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(Y_*,\, Z_*;\, X_*)$ have real part zero, so that the first point is obvious. To define when possible, in the distribution sense, complex powers of possibly vanishing functions can often be done by using Hironaka's desingularisation theorem, in particular, when necessary (this will be the case here because we wish to find the poles as they appear in conditions (3.10) and (3.11)) explicit blow-up transformations: the idea was used in general, and applied toward a shorter proof of a classical theorem in partial differential equations, in [1, 3]. We shall use it here, following its use in the one-dimensional case in [8]. Recall that one can define the direct image of a distribution under any C^∞ proper map. Our point is to give products of signed powers of the three functions

$$\begin{aligned} \ell_1 &:= [Y_*,\, X_*] = x_1 - y_1 + \langle x_*,\, \eta_* \rangle - \langle y_*,\, \xi_* \rangle \\ \ell_2 &:= [X_*,\, Z_*] = z_1 - x_1 + \langle z_*,\, \xi_* \rangle - \langle x_*,\, \zeta_* \rangle\,, \\ \ell_3 &:= [Z_*,\, Y_*] = y_1 - z_1 + \langle y_*,\, \zeta_* \rangle - \langle z_*,\, \eta_* \rangle \end{aligned} \tag{3.14}$$

a meaning for generic values of the parameters. Note that it is not necessary to desingularize fully the variety of zeros of the product $\ell_1 \ell_2 \ell_3$, only to reach a situation in which we are dealing locally with products of signed powers of functions with linearly independent differentials at common zeros.

Considering only the partial derivatives with respect to x_1, y_1, z_1, one observes that a linear relation between the differentials of these three functions cannot hold unless it consists in the fact that the sum of the three differentials is zero: computing then the partial derivatives with respect to ξ_*, η_*, ζ_*, finally with respect to x_*, y_*, z_*, one sees that the three differentials are linearly dependent if and only if $X_{**} = Y_{**} = Z_{**}$ with the notation of Section 2.

In the open set where this condition is not satisfied, one can complete the set of three functions under consideration into a local coordinate system in $\mathbb{R}^{2n}$, and the proposition follows in this case from the following well-known fact from the theory of distributions in one variable [10]: the function $\nu \mapsto |x|_\delta^{-1-\nu}$, a locally summable function if $\operatorname{Re}\nu < 0$, extends as a distribution-valued holomorphic function of ν for $\nu \neq \delta, \delta+2, \dots$. This gives the distribution $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}$ a (local) meaning provided that $\frac{n+\nu+\nu_1-\nu_2}{2} \neq \varepsilon_1+1, \varepsilon_1+3, \dots$, $\frac{n+\nu-\nu_1+\nu_2}{2} \neq \varepsilon_2+1, \varepsilon_2+3, \dots$ and $\frac{n-\nu-\nu_1-\nu_2}{2} \neq \varepsilon+1, \varepsilon+3, \dots$.

When the condition $X_{**} = Y_{**} = Z_{**}$ is satisfied, saying that $[Z_*, Y_*]$ is zero is the same as saying that $y_1 = z_1$, and there are two analogous statements related to the last two equations. At points where none of the three functions under consideration vanishes, there is of course no problem. Near points where only, say, the first function $[Z_*, Y_*]$ vanishes, it can be taken as one of a set of local coordinates, and the distribution under examination makes sense whenever $\frac{n-\nu-\nu_1-\nu_2}{2} \neq \varepsilon+1, \varepsilon+3, \dots$. The only problem remains near points at which $X_{**} = Y_{**} = Z_{**}$ and $x_1 = y_1 = z_1$ *i.e.*, $X_* = Y_* = Z_*$. We thus need to tame the three functions under consideration near a point such as (X_*^0, X_*^0, X_*^0), and there is no loss of generality in assuming that $X_*^0 = e_{n+1}$, the $(n+1)$th vector from the canonical basis of $\mathbb{R}^n \times \mathbb{R}^n$, since a symplectic transformation preserving the linear form $X \mapsto \xi_1$ can take us to this case.

We first replace the triple $(Y_*, Z_*, X_*) \in \mathcal{M}_0 \times \mathcal{M}_0 \times \mathcal{M}_0$ by the set of points $(T_1, T_2\,; x_1\,; Y_{**}, Z_{**}, X_{**})$ in $\mathbb{R}^2 \times \mathbb{R} \times (\mathbb{R}^{2n-2})^3$, with

$$T_1 = \ell_1(Y_*, Z_*, X_*)\,, \quad T_2 = \ell_2(Y_*, Z_*, X_*)\,. \tag{3.15}$$

That these equations define, near (X_*^0, X_*^0, X_*^0), an admissible new set of coordinates, follows the fact that ℓ_1 and ℓ_2 have linearly independent partial differentials with respect to the pair (y_1, z_1). Next, we blow up the (T_1, T_2)-plane around 0, replacing it by the subspace $\widehat{\mathbb{R}^2}$ of $P_1(\mathbb{R}) \times \mathbb{R}^2$ consisting of pairs (τ, T) such that, in the case when $T \neq 0$, τ is the image of T under the canonical projection map $p\colon \mathbb{R}^2\backslash\{0\} \to P_1(\mathbb{R})$. Generally setting $\tau = p(\theta)$, the domain ω_j of $P_1(\mathbb{R})$ characterized by the condition $\theta_j \neq 0$ gives rise to the domain Ω_j of $\widehat{\mathbb{R}^2}$ consisting of pairs (τ, T) such that either $T_j \neq 0$ and $p(T) = \tau$ or $T = 0$ and $\tau \in \omega_j$.

The domains Ω_1 and Ω_2 cover $\widehat{\mathbb{R}^2}$ and taking in Ω_1 the set of coordinates

$$(\tau_2,\, T_1) = \left(\frac{\theta_2}{\theta_1},\, T_1\right)\,, \tag{3.16}$$

and in Ω_2 the set of coordinates

$$(\tau_1,\, T_2) = \left(\frac{\theta_1}{\theta_2},\, T_2\right)\,, \tag{3.17}$$

one turns $\widehat{\mathbb{R}^2}$ into a smooth manifold. The projection map $\phi\colon (\tau,\, T) \mapsto T$ is proper since the inverse image of a point $T \neq 0$ reduces to the point $(p(T),\, T)$, while that of 0 is $\Sigma = P_1(\mathbb{R}) \times \{0\}$.

In Ω_1, one has $\ell_1 = T_1,\, \ell_2 = \tau_2 T_1$, so that the pullbacks in $\widehat{\mathbb{R}^2} \times \mathbb{R} \times (\mathbb{R}^{2n-2})^3$ of the three functions under consideration express themselves as

$$\begin{aligned} \ell_1^\natural &= T_1 \\ \ell_2^\natural &= \tau_2\, T_1\,, \\ \ell_3^\natural &= -(1+\tau_2)\, T_1 + [X_{**},\, Y_{**} - Z_{**}] - [Y_{**},\, Z_{**}]\,. \end{aligned} \tag{3.18}$$

The differentials of $\ell_1^\natural$ and $\ell_2^\natural$ are not linearly independent when $T_1 = 0$, but the differentials of T_1 and τ_2 are, which is sufficient as a start. We must now insert a lemma, in order to take care of the extra terms in $\ell_3^\natural$.

Lemma 3.2. *Consider on $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ the function*

$$F(Y,\, Z,\, X) = [X,\, Y - Z] - [Y,\, Z]\,, \tag{3.19}$$

which is critical exactly at points $(-X^0,\, -X^0,\, X^0)$, where it vanishes. Consider the blow-up $\widehat{\mathbb{R}^{6n}}$ of $\mathbb{R}^{6n}$ at such a point, and the pullback $\widetilde{F}$ in $\widehat{\mathbb{R}^{6n}}$ of the function F. Locally around any point lying in the inverse image of $(-X^0,\, -X^0,\, X^0)$, one can find two smooth real-valued functions R and S such that $\widetilde{F}$ expresses itself as $R\, S^2$.

Proof. First, observe the identity

$$F(-X^0 + Y,\, -X^0 + Z,\, X^0 + X) = F(Y,\, Z,\, X)\,, \tag{3.20}$$

so that there is no loss of generality in assuming that $X^0 = 0$. The space $\widehat{\mathbb{R}^{6n}}$ obtained as the result of blowing up $\mathbb{R}^{6n}$ around 0 is covered by a family $(\Omega_j)_{1 \leq j \leq 6n}$ of open sets with the following properties: for each j, there is a function S_j taken from the set of canonical coordinates of one of the three vectors $Y,\, Z,\, X$ such that, within Ω_j, the equation $S_j = 0$ defines the inverse image $P_{6n-1}(\mathbb{R}) \times \{0\}$ of $0 \in \mathbb{R}^{6n}$; next, there is a set of smooth vector-valued functions $\dot{Y},\, \dot{Z},\, \dot{X}$, each of which has $2n$ components, such that the identities $Y = S_j\, \dot{Y},\, Z = S_j\, \dot{Z},\, X = S_j\, \dot{X}$ hold, and such that, deleting from the set of components of the vectors $\dot{Y},\, \dot{Z},\, \dot{X}$ the coordinate which, of necessity, is the constant 1 , one obtains a family of functions which, when completed by the function S_j , constitutes an admissible set of coordinates in Ω_j . Then, one may write

$$\widetilde{F}(S_j,\, \dot{Y},\, \dot{Z},\, \dot{X}) = S_j^2\, (\, [\dot{X},\, \dot{Y} - \dot{Z}] - [\dot{Y},\, \dot{Z}]\,)\,, \tag{3.21}$$

and it suffices to observe that the second factor is a function without critical point. Indeed, assuming for instance that the coordinate S_j has been taken from the components of Y (it would be fully similar if it had been taken from any of the other two remaining vectors), the equation $(\dot{Y})_j = 1$ shows that the partial derivatives of $\tilde{\phi}$ with respect to the coordinates in $\dot{X}$ or $\dot{Z}$ "conjugate with respect to the symplectic form" to $(\dot{Y})_j$ are not zero.

$\square$

End of proof of Proposition 3.1. Applying Lemma 3.2 with $n-1$ substituted for n , we may rewrite (3.18), more precisely the pullbacks of the three functions there to a new blown-up space, as

$$\begin{aligned} \ell_1^{\natural\natural} &= T_1 \\ \ell_2^{\natural\natural} &= \tau_2\, T_1\,, \\ \ell_3^{\natural\natural} &= -(1+\tau_2)\, T_1 + R\, S^2\,, \end{aligned} \tag{3.22}$$

where the four functions $T_1,\, \tau_2,\, R,\, S$ have linearly independent differentials.

The differential $d\ell_3^{\natural\natural}$ is a linear combination of $d\ell_1^{\natural\natural}$ and $d\ell_3^{\natural\natural}$ exactly at points where $S = 0$, but let us not forget the origin (3.16) of the coordinate T_1 , which implies that there is no loss of generality in assuming that we are near a point where $T_1 = 0$ as well.

In the open set where $1 + \tau_2$ does not vanish, we may take $\ell_3^{\natural\natural}$ to the form $-T_1 + R\, S^2$, and we blow up the plane of the variables $T_1,\, S$ around 0 : this amounts, with new variables, to setting in appropriate domains either $S = T_1\, S'$ or $T_1 = S\, T_1'$, finding either $-T_1 + R\, S^2 = T_1\, (-1 + R\, T_1\, S'^2)$ or $-T_1 + R\, S^2 = S(-T_1' + R\, S)$. In the first case we are dealing with a pair of functions, the first of which is T_1 and the second is the product of T_1 by a function which, at points where it vanishes, has a differential linearly independent from dT_1 . In the second case, we still have to desingularize the pair of functions $(S\, T_1',\; S(-T_1' + R\, S)$ or, setting aside the factors S in the product of signed powers to be analyzed, the triple of functions $(S,\, T_1',\, -T_1' + R\, S)$. Again, we blow up the $(T_1',\, S)$ - space, which amounts to setting either $S = T_1'\, S''$, in which case the triple becomes $(T_1'\, S''\, ,\; T_1'\, ,\; T_1'(-1 + R\, S''))$, or $T_1' = S\, T_1''$, in which case the triple becomes $(S\, ,\; S\, T_1'',\; S\, (-T_1'' + R))$, a satisfactory situation.

Finally, we must place ourselves near a point where T_1 and $1+\tau_2$ vanish. We may then forget about $\ell_2^{\natural\natural}$ entirely, and we blow up the variables $T_1,\ 1+\tau_2,\ S$ near 0. In local charts, this makes up one of the three following possibilities:

$$\begin{aligned} 1+\tau_2 &= T_1\,\sigma_2\,,\ \ S = T_1\,S'\,, & \ell_3^{\natural\natural\natural} &= T_1^2\,(-\sigma_2 + R\,S'^2)\,,\\ T_1 &= (1+\tau_2)\,T_1'\,,\ \ S = (1+\tau_2)\,S'\,, & \ell_3^{\natural\natural\natural} &= (1+\tau_2)^2\,(-T_1' + R\,S'^2)\,,\\ T_1 &= S\,T_1'\,,\ \ 1+\tau_2 = S\,\sigma_2\,, & \ell_3^{\natural\natural\natural} &= S^2\,(-\sigma_2\,T_1' + R)\,. \end{aligned} \tag{3.23}$$

In the first (*resp.* third) case, a product of signed powers of T_1 and $\ell_3^{\natural\natural\natural}$ becomes a product of signed powers of T_1 and $-\sigma_2 + R\,S'^2$ (*resp.* a product of signed powers of S, of T_1' and $-\sigma_2\,T_1' + R$), a satisfactory situation since we are dealing in each case with two functions with linearly independent differentials. This is not the case on the second line, in which, after leaving the factors $1+\tau_2$ aside, we have to consider the pair of functions T_1' and $-T_1' + R\,S'^2$: these do not have linearly independent differentials; however, this pair can be desingularized since we are back to the situation examined above, relative to the pair $(T_1,\ -T_1 + R\,S^2)$.

We are now in a position to define locally the distribution $\mathbf{J}_{\nu_1,\nu_2;\,\nu}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}$ as the direct image, under a proper map, of a distribution of the kind

$$|\ell_1^{\natural}|_{\varepsilon_2}^{\frac{-n-\nu+\nu_1-\nu_2}{2}}\ |\ell_2^{\natural}|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}}\ |\ell_3^{\natural}|_{\varepsilon}^{\frac{-n+\nu+\nu_1+\nu_2}{2}}\,, \tag{3.24}$$

where the factors $\ell_1^{\natural},\ \ell_2^{\natural},\ \ell_3^{\natural}$ really denote the initial functions ℓ_1, ℓ_2, ℓ_3 after they have been pulled back in one of the appropriate ways just described: only, we here dispense with the collection of $\natural$ supercripts which has been used before in order to keep track of the number of blow-ups needed. In case the reader should worry about it, the fact that the subscript ε_2 should be associated to ℓ_1, not ℓ_2, is not a blunder: the index δ_1 is actually that which must be associated to ℓ_1, and we recall (2.33). The important fact is that, in local charts, the functions $\ell_1^{\natural},\ \ell_2^{\natural},\ \ell_3^{\natural}$ are all built as powers of the same set of functions with linearly independent differentials. Recall from (2.35) that

$$\alpha_1 = \frac{-n-\nu+\nu_1-\nu_2}{2}\,,\quad \alpha_2 = \frac{-n-\nu-\nu_1+\nu_2}{2}\,,\quad \alpha_3 = \frac{-n+\nu+\nu_1+\nu_2}{2}\,. \tag{3.25}$$

To find the poles, as a distribution-valued function of ν_1, ν_2, ν, of the distribution (3.24), we must go back to the desingularizing operations and keep track of the signed powers involved in each case, starting from the fact that $|f|_\delta^{-1-\mu}$ makes sense as a distribution, assuming that f has no critical zero, when $\mu \neq \delta,\ \delta+2,\ \dots$. As already said, when none of the three functions ℓ_1, ℓ_2, ℓ_3 vanishes, there is of course no condition on the exponents involved, and when just one

of them vanishes (the case discussed between (3.14) and (3.15)), we must assume
(3.26)
$$-\alpha_1 \neq \varepsilon_2+1,\, \varepsilon_2+3,\, \dots; \quad -\alpha_2 \neq \varepsilon_1+1,\, \varepsilon_1+3,\, \dots; \quad -\alpha_3 \neq \varepsilon+1,\, \varepsilon+3,\, \dots\,.$$

Next, we go to our discussion following (3.22). Forgetting the factors without zeros, the product of signed powers we are led to is of one of the following species, in which we introduce the new letter $V,\, S'',\, T_1'',\, \dots$ for each of the functions, with differentials independent from the other ones at points where they vanish, such as $-1+R\,T_1\,S'^2$, which have appeared in the discussion:

$$|T_1|_{\varepsilon_2}^{\alpha_1}\ \ |\tau_2\,T_1|_{\varepsilon_1}^{\alpha_2}\ \ |T_1\,V|_{\varepsilon}^{\alpha_3} \qquad \text{or}$$
$$|T_1'\,S''\,T_1'|_{\varepsilon_2}^{\alpha_1}\ \ |\tau_2\,T_1'\,S''\,T_1'|_{\varepsilon_1}^{\alpha_2}\ \ |T_1'\,S''\,T_1'\,V|_{\varepsilon}^{\alpha_3} \qquad \text{or}$$
$$|S^2\,T_1''|_{\varepsilon_2}^{\alpha_1}\ \ |\tau_2\,S^2\,T_1''|_{\varepsilon_1}^{\alpha_2}\ \ |S^2\,V|_{\varepsilon}^{\alpha_3} \qquad \text{or}$$
$$|T_1|_{\varepsilon_2}^{\alpha_1}\ \ |\tau_2\,T_1|_{\varepsilon_1}^{\alpha_2}\ \ |T_1^2\,V|_{\varepsilon}^{\alpha_3} \qquad \text{or}$$
$$|S\,T_1'|_{\varepsilon_2}^{\alpha_1}\ \ |S\,T_1'|_{\varepsilon_1}^{\alpha_2}\ \ |S^2\,V|_{\varepsilon}^{\alpha_3} \qquad \text{or}$$
$$|1+\tau_2|_{\varepsilon_2}^{\alpha_1}\ \ |1+\tau_2|_{\varepsilon_1}^{\alpha_2}\ \ (1+\tau_2)^2|_{\varepsilon}^{\alpha_3}\ \ |T_1''\,S''\,T_1''|_{\varepsilon_2}^{\alpha_1}\ \ |T_1''\,S''\,T_1''|_{\varepsilon_1}^{\alpha_2}\ \ |T_1''\,S''\,T_1''\,V|_{\varepsilon}^{\alpha_3} \qquad \text{or}$$
$$|1+\tau_2|_{\varepsilon_2}^{\alpha_1}\ \ |1+\tau_2|_{\varepsilon_1}^{\alpha_2}\ \ (1+\tau_2)^2|_{\varepsilon}^{\alpha_3}\ \ |S'^2\,T_1''|_{\varepsilon_2}^{\alpha_1}\ \ |S'^2\,T_1''|_{\varepsilon_1}^{\alpha_2}\ \ |S'^2\,V|_{\varepsilon}^{\alpha_3}\,. \tag{3.27}$$

Besides, we must not forget that all these local forms are only available in some domains above parts of Ω_1, not Ω_2 (*cf.* (3.16)), so we must complete the preceding list with the one obtained from it by exchanging the two pairs $(\varepsilon_2,\, \nu_1)$ and $(\varepsilon_1,\, \nu_2)$. All lines are treated in the same way: let us consider the last one, which happens to make all possible demands on the exponents, and let us rewrite it as

$$|1+\tau_2|_{\varepsilon_1+\varepsilon_2 \bmod 2}^{\alpha_1+\alpha_2}\ \ |1+\tau_2|^{\alpha_3}\ \ |S'|^{2\,(\alpha_1+\alpha_2+\alpha_3)}\ \ |T_1''|_{\varepsilon_1+\varepsilon_2 \bmod 2}^{\alpha_1+\alpha_2}\ \ |V|_{\varepsilon}^{\alpha_3}\,. \tag{3.28}$$

Since $\varepsilon_1+\varepsilon_2+\varepsilon \equiv j \mod 2$, this can be written as

$$|1+\tau_2|_{j}^{\alpha_1+\alpha_2+\alpha_3}\ \ |1+\tau_2|_{\varepsilon}^{\alpha_3}\ \ |S'|^{2\,(\alpha_1+\alpha_2+\alpha_3)}\ \ |T_1''|_{\varepsilon_1+\varepsilon_2 \bmod 2}^{\alpha_1+\alpha_2}\ \ |V|_{\varepsilon}^{\alpha_3}\,. \tag{3.29}$$

Now, one has
(3.30)
$$\alpha_1+\alpha_2+\alpha_3 = \frac{-3n-\nu+\nu_1+\nu_2}{2}\,, \quad \alpha_1+\alpha_2 = -n-\nu\,, \quad \varepsilon_1+\varepsilon_2 \equiv j+\varepsilon \equiv \delta \mod 2\,,$$

so that, besides the conditions (3.26), it suffices to assume moreover that

$$\frac{3n+\nu-\nu_1-\nu_2}{2} \neq j+1,\, j+3,\, \dots\,, \quad 3n+\nu-\nu_1-\nu_2 \neq 1,\, 3,\, \dots\,, \tag{3.31}$$

and that $n+\nu \neq \delta+1,\, \delta+3,\, \dots\,$.

These conditions are clearly invariant under the exchange of pairs $(\varepsilon_2,\, \nu_1)$ and $(\varepsilon_1,\, \nu_2)$. They are not fully necessary: the reason for this is that, in our desingularisation procedure, we have started with giving the pair $(\ell_1,\, \ell_2)$ special consideration, while we might just as well started from giving the pair $(\ell_2,\, \ell_3)$ or $(\ell_3,\, \ell_1)$ special consideration. This takes us to the assumptions in Proposition

3.1, not forgetting that in the one-dimensional case, the desingularization process stops at (3.18).

The rest of the proof is trivial.

□

We shall also need the following result, in the same spirit as Proposition 3.1, though of course its proof presents no difficulty.

Proposition 3.3. *Set, assuming* $-\rho \neq \delta+1,\, \delta+3,\, \dots$ *and* $\rho \neq \delta,\, \delta+2,\, \dots$,

$$c(\rho,\delta) = (-i)^{\delta}\, \pi^{-\frac{1}{2}-\rho}\, \frac{\Gamma(\frac{\rho+1+\delta}{2})}{\Gamma(\frac{-\rho+\delta}{2})}\,, \tag{3.32}$$

so that one should have, in one dimension,

$$(\mathcal{F}\,(|s|_{\delta}^{\rho}))(\sigma) = c(\rho,\delta)\, |\sigma|_{\delta}^{-\rho-1} \tag{3.33}$$

(of course, we are using here the usual Fourier transformation, with integral kernel $e^{-2i\pi s\sigma}$ *: there is no symplectic Fourier transformation on an odd-dimensional space). Recalling (2.22), consider the integral kernel*

$$k_{\nu,\delta}(x,\xi_*\,;\,y,\eta_*) = (-1)^{\delta}\; c(n-1-\nu,\delta)\; |x_1 - y_1 + \langle x_*,\eta_*\rangle - \langle y_*,\xi_*\rangle|_{\delta}^{-n+\nu}\,. \tag{3.34}$$

When $-n < \mathrm{Re}\,\nu < 1-n$ *, this is the integral kernel of an operator* $\theta_{\nu,\delta}$ *well-defined, in the weak sense, from the space of* C^∞ *vectors of the representation* $\pi_{\nu,\delta}$ *to the dual of that space (which contains the space of* C^∞ *vectors of the representation* $\pi_{-\nu,\delta}$*). As an operator-valued function of* ν *,* $\theta_{\nu,\delta}$ *extends as a holomorphic function in* $\mathbb{C}\backslash P$ *, where the set* P *consists of the values* ν *such that* $-n+\nu = \delta,\, \delta+2,\, \dots$ *or* $n-\nu = \delta+1,\, \delta+3,\, \dots$ *. The operator* $\theta_{\nu,\delta}$ *is an intertwiner from the representation* $\pi_{\nu,\delta}$ *to the representation* $\pi_{-\nu,\delta}$ *. When* $\nu \in i\mathbb{R}$ *, it coincides with the one introduced in another way in Definition 2.2.*

The latter way to define the operator $\theta_{i\lambda,\delta}$ has the advantages, especially in the version (2.19), that on one hand it continues to be meaningful after $\nu \in \mathbb{C}$ has been substituted for $i\lambda$, on the other hand that it extends to a (tempered) distribution setting: but this requires that the homogeneous functions, or distributions, under consideration, should have a well-defined meaning as distributions in $\mathbb{R}^{2n}$, not only as functions, or distributions, in $\mathbb{R}^{2n}\backslash\{0\}$.

4. Hyperplane waves and rays

We decompose here symbols as integral superpositions of homogeneous hyperplane waves, also of homogeneous rays, by which we mean homogeneous measures carried by straight lines through the origin of $\mathbb{R}^{2n}$. With the help of such decompositions, we shall transform, in this section, the triple product studied in Section 3 in a way crucial towards the proof of the main theorem.

Consider the transformation $\mathcal{G}$, a rescaled version of the symplectic Fourier transformation (also a unitary involution of $L^2(\mathbb{R}^{2n})$) defined as

$$(\mathcal{G}h)(X) = 2^n \int_{\mathbb{R}^{2n}} h(Y)\, e^{-4i\pi[X,\,Y]}\, dy \ : \tag{4.1}$$

part of our interest in this transformation [11, p. 120] is that, for every $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^{2n})$, the distribution $\mathcal{G}\,\mathfrak{S}$ is the Weyl symbol of the operator $u \mapsto \mathrm{Op}(\mathfrak{S})\,\check{u}$, where $\check{u}(x) = u(-x)$. If a symbol $h = h(x\,;\,\xi)$ depends only on ξ_1, say $h(x\,;\,\xi) = \phi(\xi_1)$, it is immediate that $(\mathcal{G}h)(x\,;\,\xi) = 2\,\hat{\phi}(-2x_1)\,\delta(x_*)\,\delta(\xi)$: in other words, $\mathcal{G}h$ is the measure carried by the line $\{te_1 \colon t \in \mathbb{R}\}$, with density $2\,\hat{\phi}(-2t)\,dt$. More generally, if $S \in \mathbb{R}^{2n}\backslash\{0\}$, setting $S = ge_1$ with $g \in \mathrm{Sp}(n,\mathbb{R})$, the $\mathcal{G}$- transform of the hyperplane wave $X \mapsto \phi([X,\,S])$ is the measure carried by the line $\{tS \colon t \in \mathbb{R}\}$, with density $2\,\hat{\phi}(-2t)\,dt$.

In particular, for any $\rho \in \mathbb{C},\ -\rho \neq \delta+1,\ \delta+3,\ \dots$, we shall denote as $\mu_S(\rho,\,\delta)$ the measure carried by the line $\{tS \colon t \in \mathbb{R}\}$, with density $|t|_\delta^\rho\,dt$. Recalling the definition (3.32) of $c(\rho,\delta)$, we have, provided that $n+\nu \neq \delta+1,\ \delta+3,\ \dots$ and $-n-\nu \neq \delta,\ \delta+2,\ \dots$,

$$\mathcal{G}(X \mapsto |\,[X,\,S]\,|_\delta^{-n-\nu}) = (-1)^\delta\, 2^\nu\, c(-n-\nu,\delta)\ \mu_S(n-1+\nu,\,\delta)\,. \tag{4.2}$$

Note that the measure $\mu_S(\rho,\,\delta)$ is a homogeneous distribution of type $(\rho+1-2n,\,\delta)$ (do not forget that, in $\mathbb{R}^{2n-1}$, the Dirac mass at the origin is homogeneous of degree $1-2n$).

Let us first decompose functions in $\mathcal{S}(\mathbb{R}^{2n})$ into homogeneous hyperplane waves. Start from the continuation of (2.4), to wit

$$h_{\nu,\delta}(X) = \frac{1}{4\pi} \int_{-\infty}^{\infty} |t|_\delta^{n-1+\nu}\, h(tX)\, dt\,, \tag{4.3}$$

where the integral converges for every $X \neq 0$ provided that $\mathrm{Re}\,\nu > -n$. In this case, the function $h_{\nu,\delta}$ is, as we now show, a C^∞ vector of the representation $\pi_{\nu,\delta}$. With $X_* = (x\,;\,1,\xi_*)$, one has for every N the inequality $|h(tX_*)| \leq C\,(1+|t|)^{-N}\,(1+|x|+|\xi_*|)^{-N}$ for some constant C : then, with the norm defined in (3.2), one has $\|\,X_* \mapsto h(tX_*)\,\|_\nu \leq C\,(1+|t|)^{-N}$, from which one obtains, since $\mathrm{Re}\,(n-1+\nu) > -1$, that the function $h_{\nu,\delta}$ lies in the Hilbert space $\mathcal{H}_\nu$ defined in association with this norm. That it is a C^∞ vector of the representation $\pi_{\nu,\delta}$ follows from the fact that this representation corresponds, under the transformation (4.3) from h to $h_{\nu,\delta}$, to the phase space representation of $\mathrm{Sp}(n,\mathbb{R})$ in $\mathcal{S}(\mathbb{R}^{2n})$.

In the case when, moreover, $\operatorname{Re}\nu < 1-n$, one may write

$$\begin{aligned} h_{\nu,\delta}(X) &= 2^n \int_{-\infty}^{\infty} |t|_\delta^{n-1+\nu}\,dt \int_{\mathbb{R}^{2n}} e^{-4i\pi\, t\,[X,\,S]}\,(\mathcal{G}h)(S)\,dS \\ &= \frac{2^{-\nu}}{4\pi}\,c(n-1+\nu,\delta) \int_{\mathbb{R}^{2n}} |\,[X,\,S]\,|_\delta^{-n-\nu}\,(\mathcal{G}h)(S)\,dS\,, \end{aligned} \tag{4.4}$$

which leads to the decomposition of h into homogeneous hyperplane waves if coupled with the equation

$$h = \sum_{\delta=0,1} \int_{\operatorname{Re}\nu=a} h_{\nu,\delta}\,\frac{d\nu}{i}\,, \tag{4.5}$$

in which $-n < a < 1-n$. From (2.3), however, the line of integration we are particularly interested in is the pure imaginary line, for which this decomposition is just the spectral decomposition of h relative to the (self-adjoint) operator $\mathcal{E}$ in $L^2(\mathbb{R}^{2n})$. Starting from (4.4) and moving the set of values of ν, we certainly reach, for fixed S, poles of the distribution-valued function $\nu \mapsto |\,[X,\,S]\,|_\delta^{-n-\nu}$, at points $\nu = -n+\delta+1,\ \nu = -n+\delta+3,\ \dots$, but these poles are simple, and disappear after multiplication by the factor $c(n-1+\nu,\delta)$, as seen from (3.32). This makes it possible to continue the decomposition of h into homogeneous hyperplane waves up to the spectral line.

Starting from $\mathcal{G}h$ in place of h and noting that $(\mathcal{G}h)_{-\nu,\delta} = \mathcal{G}\,h_{\nu,\delta}$, one obtains also, if $\operatorname{Re}\nu < n$,

$$\begin{aligned} h_{\nu,\delta} &= \frac{2^{\nu}}{4\pi}\,c(n-1-\nu,\delta) \int_{\mathbb{R}^{2n}} h(S)\ \mathcal{G}\left(X \mapsto |\,[X,\,S]\,|_\delta^{-n+\nu}\right)\ dS \\ &= \frac{1}{4\pi}\int_{\mathbb{R}^{2n}} h(S)\ \mu_S(n-1-\nu,\,\delta)\ dS\,, \end{aligned} \tag{4.6}$$

after one has used the equation

$$(-1)^\delta\,c(\rho,\delta)\,c(-\rho-1,\delta) = 1\ : \tag{4.7}$$

this leads to a decomposition of h into rays if coupled with the equation

$$h = \sum_{\delta=0,1} \int_{\operatorname{Re}\nu=a} h_{-\nu,\delta}\,\frac{d\nu}{i}\,, \tag{4.8}$$

in which, starting from a value of a between $-n$ and $1-n$, we can actually take $a=0$ when so desired.

The following lemma will enable us to deal with multipliers of the species which occurs consistently in the present work.

Lemma 4.1. *Let $S \in \mathbb{R}^{2n}\backslash\{0\}$. If $\varepsilon,\,\delta = 0$ or 1 and $\alpha,\,\nu \in \mathbb{C}$ satisfy the condition $-\frac{1}{2} < \operatorname{Re}\alpha < \frac{1}{2} + \operatorname{Re}\nu$, the multiplication by the function $X_* \mapsto |\,[S,\,X_*]\,|_\varepsilon^\alpha$ sends the space $C^\infty(\pi_{\nu,\delta})$ of C^∞ vectors of the representation $\pi_{\nu,\delta}$ to the space*

$L^2(\mathcal{M}_0)$.

Proof. It is no loss of generality to assume that $S = e_{n+1}$, *i.e.*, $[S,\, X_*] = x_1$. Given $f \in C^\infty(\pi_{\nu,\delta})$ extending to $\mathbb{R}^{2n}\backslash\{0\}$ as a function $f^\sharp$ of type $(-n-\nu,\, \delta)$, the function

$$k(x\,;\,\xi) = |x_1|_\varepsilon^\alpha \quad |\xi_1|_{\varepsilon+\delta \bmod 2}^{\nu-\alpha} \quad f^\sharp(x\,;\,\xi) \tag{4.9}$$

is of type $(-n, 0)$. Since the corresponding representation $\pi_{0,0}$ preserves the Hilbert space $L^2(\mathcal{M}_0)$, it suffices, in view of Remark 2.2, to check that the restriction of the function k , to $\mathcal{M}_0$ lies in the space $L^2_{\mathrm{loc}}(\mathcal{M}_0)$, which leads to the two conditions indicated.

$\square$

We now come back to a study of the bilinear operator $(f_1,\, f_2) \mapsto \mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(f_1,\, f_2)$, or of the associated triple product obtained when testing this distribution against $f \in C^\infty(\pi_{-\nu,\delta})$. Recall from the end of Section 2 that such expressions can also use as arguments objects with the proper type defined in $\mathbb{R}^{2n}\backslash\{0\}$ rather than their restrictions to $\mathcal{M}_0$, the distinction being purely notational. We shall eventually assume, but not at one stroke, that

$$f_1 = (h_1)_{\nu_1,\delta_1}\,, \quad f_2 = (h_2)_{\nu_2,\delta_2}\,, \quad f = h_{-\nu,\delta} \tag{4.10}$$

for a triple of functions $h_1,\, h_2,\, h \in \mathcal{S}(\mathbb{R}^{2n})$.

Lemma 4.2. *Assume that* $h_2 \in \mathcal{S}(\mathbb{R}^{2n})$ *and that all hypotheses of Proposition 3.1 are valid. Moreover, assume that* $\mathrm{Re}\,\nu_2 < n$ *and that*

$$\mathrm{Re}\,(\nu - \nu_1 + \nu_2) = n, \quad \mathrm{Re}\,\nu_1 > -\frac{1}{2}\,, \quad \mathrm{Re}\,\nu < \frac{1}{2}\,. \tag{4.11}$$

If $f_1 \in C^\infty(\pi_{\nu_1,\delta_1})$ *, one has in the weak sense,* i.e., *when integrated against* $f(X_*)\,dm(X_*)$ *for some* $f \in C^\infty(\pi_{-\nu,\delta})$ *,*

$$\mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(f_1,\, (h_2)_{\nu_2,\delta_2})(X_*) = \frac{1}{4\pi}\,\frac{(-1)^{\varepsilon_2}}{c(\frac{n-2+\nu-\nu_1+\nu_2}{2},\varepsilon_2)}\int_{\mathbb{R}^{2n}} h_2(S)\,dS \tag{4.12}$$

$$|\,[X_*,\,S]\,|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}} \quad \left[\theta_{\frac{n-\nu+\nu_1-\nu_2}{2},\varepsilon_2}\left(Y_* \mapsto |\,[S,\,Y_*]\,|_\varepsilon^{\frac{-n+\nu+\nu_1+\nu_2}{2}} \quad f_1(Y_*)\right)\right](X_*)\,.$$

Proof. First, we observe, as a consequence of Lemma 4.1, that, under the conditions (4.11), the multiplication by the function $Y_* \mapsto |\,[S,\,Y_*]\,|_\varepsilon^{\frac{-n+\nu+\nu_1+\nu_2}{2}}$ sends the space $C^\infty(\pi_{\nu_1,\delta_1})$ to the space $L^2(\mathcal{M}_0)$ and that the multiplication by the function $X_* \mapsto |\,[X_*,\,S]\,|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}}$ sends the space $L^2(\mathcal{M}_0)$ to the space of distributions $C^{-\infty}(\pi_{\nu,\delta})$, the topological dual of $C^\infty(\pi_{-\nu,\delta})$ (*i.e.*, the linear space of continuous linear forms on that space). On the other hand, the first condition

(4.11) gives the intertwining operator $\theta_{\frac{n-\nu+\nu_1-\nu_2}{2},\varepsilon_2}$ a meaning as a unitary operator in $L^2(\mathcal{M}_0)$, so that the right-hand side of the equation to be proved is meaningful.

If one makes there the integral kernel of the operator $\theta_{\frac{n-\nu+\nu_1-\nu_2}{2},\varepsilon_2}$ explicit, as $(-1)^{\varepsilon_2}\, c(\frac{n-2+\nu-\nu_1+\nu_2}{2},\varepsilon_2)\,|\,[Y_*,\,X_*]\,|_{\varepsilon_2}^{\frac{-n-\nu+\nu_1-\nu_2}{2}}$, then if one sets $S = sZ_*$, so that

$$|s|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}}\;|s|_{\varepsilon}^{\frac{-n+\nu+\nu_1+\nu_2}{2}}\;dS = |s|_{\delta_2}^{n-1+\nu_2}\;ds\,dm(Z_*)\,, \tag{4.13}$$

and if one uses the equation

$$(h_2)\nu_2,\delta_2(X) = \frac{1}{4\pi}\int_{-\infty}^{\infty}|s|_{\delta}^{n-1+\nu_2}\,h_2(sX)\,ds\,, \tag{4.14}$$

one transforms the right-hand side of (4.12) into the left-hand side. However, the operator on the left-hand side has been defined with the help of the desingularization of its integral kernel as done in Section 3, while on the right-hand side, the claimed unitarity of the intertwining operator into consideration is a consequence of Definition 2.2: to identify the two ways to introduce it, one must use again the connection between (2.21) and (2.22).

□

Let us rewrite (4.12), as tested against f, with

$$f(X_*) = h_{-\nu,\delta}(X_*) = \frac{1}{4\pi}\int_{-\infty}^{\infty}|t|_{\delta}^{n-1-\nu}\,h(tX_*)\,dt\,. \tag{4.15}$$

One has

$$\langle\,\mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}(f_1,\,(h_2)_{\nu_2,\delta_2})\,,\,h_{-\nu,\delta}\,\rangle = \frac{1}{(4\pi)^2}\;\frac{(-1)^{\varepsilon_2}}{c(\frac{n-2+\nu-\nu_1+\nu_2}{2},\varepsilon_2)}\int_{\mathbb{R}^{2n}}h_2(S) \tag{4.16}$$
$$\langle\,\mathcal{F}(Y\mapsto|\,[S,\,Y]\,|_{\varepsilon}^{\frac{-n+\nu+\nu_1+\nu_2}{2}}\;f_1(Y))\,,\,T\mapsto|\,[T,\,S]\,|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}}\;h(T)\,\rangle\;dS\;:$$

note that the two pairs of brackets $\langle\ ,\ \rangle$ do not denote the same pairings: on the left-hand side, it corresponds to the duality between $C^{-\infty}(\pi_{\nu,\delta})$ and $C^{\infty}(\pi_{-\nu,\delta})$; within the integrand on the right-hand side, it corresponds to the one between $\mathcal{S}'(\mathbb{R}^{2n})$ and $\mathcal{S}(\mathbb{R}^{2n})$. To prove this, we start from the right-hand side, expressing the intertwining operator there as a Fourier transformation. The function

$$T\mapsto|\,[T,\,S]\,|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}}\;\mathcal{F}(Y\mapsto|\,[S,\,Y]\,|_{\varepsilon}^{\frac{-n+\nu+\nu_1+\nu_2}{2}}\;f_1(Y))\,(T) \tag{4.17}$$

is of type (recalling (2.33))

$$(\frac{-n-\nu-\nu_1+\nu_2}{2},\,\varepsilon_1)+(-2n,\,0)+(\frac{n-\nu-\nu_1-\nu_2}{2},\,\varepsilon)+(n+\nu_1,\,\delta_1) = (-n-\nu,\,\delta)\,. \tag{4.18}$$

Set $T = tX_*$, so that $dT = |t|^{2n-1}\,dt\,dm(X_*)$: then, the right-hand side of (4.16) transforms into the left-hand side in view of (4.18) and (4.15).

As a last step, we now use the decomposition

$$(h_1)_{\nu_1,\delta_1}(Y) = \frac{2^{-\nu_1}}{4\pi}\, c(n-1+\nu_1,\delta_1) \int_{\mathbb{R}^{2n}} (\mathcal{G}h_1)(R)\, |\,[Y,\,R]\,|_{\delta_1}^{-n-\nu_1}\, dR \tag{4.19}$$

of $f_1 = (h_1)_{\nu_1,\delta_1}$, as provided by (4.4).

Proposition 4.3. *Assume that all hypotheses from Proposition 3.1 are satisfied and that, moreover,*

$$\nu+\nu_1 \neq \delta_2,\, \delta_2+2,\, \dots\,, \qquad \frac{-n+\nu+\nu_1+\nu_2}{2} \neq \varepsilon,\, \varepsilon+2,\, \dots\,, \tag{4.20}$$
$$\frac{2-n-\nu+\nu_1+\nu_2}{2} \neq \varepsilon_2+1,\, \varepsilon_2+3,\, \dots$$

and

$$\mathrm{Re}\,\nu_1 > -n\,, \quad \mathrm{Re}\,\nu_2 < n\,, \quad \mathrm{Re}\,\nu < n\,. \tag{4.21}$$

Then,

$$\langle\, \mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}((h_1)_{\nu_1,\delta_1},\,(h_2)_{\nu_2,\delta_2})\,,\,h_{-\nu,\delta}\,\rangle = \frac{(-1)^{\varepsilon_2}\,2^{-\nu_1}}{(4\pi)^3}\, \frac{c(\frac{-n+\nu+\nu_1+\nu_2}{2},\varepsilon)}{c(\frac{n-2+\nu-\nu_1+\nu_2}{2},\varepsilon_2)} \tag{4.22}$$
$$\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} (\mathcal{G}h_1)(R)\; h_2(S)\; |\,[R,\,S]\,|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}}\; dR\,dS$$
$$\int_{\mathbb{R}^2} |r|_j^{\frac{n-2-\nu+\nu_1+\nu_2}{2}}\; |s|_{\varepsilon}^{\frac{n-2-\nu-\nu_1-\nu_2}{2}}\; h(r\,R+s\,S)\; dr\,ds\,,$$

where the last integral must be understood in the distribution sense: recall that j was defined in (2.34).

Proof. First, write the equation, of immediate verification,

$$\mathcal{F}((y\,;\eta)\mapsto |-y_1|_{\delta_1}^{-n-\nu_1}\,|-\eta_1|_{\varepsilon}^{\frac{-n+\nu+\nu_1+\nu_2}{2}})(t_1,t_*\,;\,\tau_1,\tau_*) = (-1)^{\delta_1}\times \tag{4.23}$$
$$c(-n-\nu_1,\delta_1)\; c(\frac{-n+\nu+\nu_1+\nu_2}{2},\varepsilon)\; |t_1|_{\varepsilon}^{\frac{n-2-\nu-\nu_1-\nu_2}{2}}\; |\tau_1|_{\delta_1}^{n-1+\nu_1}\; \delta(t_*)\,\delta(\tau_*)\,.$$

Next, under the generic condition $[R,\,S]\neq 0$, one can find $g\in \mathrm{Sp}(n,\mathbb{R})$ such that

$$S = ge_1\,, \qquad R = [R,\,S]\,ge_{n+1}\ : \tag{4.24}$$

it follows that

(4.25)
$$\begin{aligned}
&\langle\, \mathcal{F}\,(Y \mapsto |\,[S,\,Y]\,|_{\varepsilon}^{\frac{-n+\nu+\nu_1+\nu_2}{2}}\ |\,[Y,\,R]\,|_{\delta_1}^{-n-\nu_1})\,,\, T \mapsto |\,[T,\,S]\,|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}}\ h(T)\,\rangle \\
&\qquad = (-1)^{\delta_1}\ c(-n-\nu_1,\delta_1)\ c(\frac{-n+\nu+\nu_1+\nu_2}{2},\varepsilon)\ |\,[R,\,S]\,|_{\delta_1}^{-n-\nu_1} \\
&\langle\, |t_1|_{\varepsilon}^{\frac{n-2-\nu-\nu_1-\nu_2}{2}}\ |\tau_1|_{\delta_1}^{n-1+\nu_1}\ \delta(t_*)\,\delta(\tau_*)\,,\, |\tau_1|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}}\ (h\,\circ\,g)(t_1,t_*\,;\,\tau_1,\tau_*)\,\rangle\,.
\end{aligned}$$

Since

$$(h\,\circ\,g)(t_1,0\,;\,\tau_1,0) = h\left(t_1\,S + \tau_1\,\frac{R}{[R,\,S]}\right)\,, \tag{4.26}$$

we set $\tau_1 = [R,\,S]\,r$ and, for clarity, $t_1 = s$, getting

(4.27)
$$\begin{aligned}
&\langle\, \mathcal{F}\,(Y \mapsto |\,[S,\,Y]\,|_{\varepsilon}^{\frac{-n+\nu+\nu_1+\nu_2}{2}}\ |\,[Y,\,R]\,|_{\delta_1}^{-n-\nu_1})\,,\, T \mapsto |\,[T,\,S]\,|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}}\ h(T)\,\rangle \\
&\qquad = (-1)^{\delta_1}\ c(-n-\nu_1,\delta_1)\ c(\frac{-n+\nu+\nu_1+\nu_2}{2},\varepsilon)\ |\,[R,\,S]\,|_{\varepsilon_1}^{\frac{-n-\nu-\nu_1+\nu_2}{2}} \\
&\qquad\qquad \int_{\mathbb{R}^2} |r|_j^{\frac{n-2-\nu+\nu_1+\nu_2}{2}}\ |s|_{\varepsilon}^{\frac{n-2-\nu-\nu_1-\nu_2}{2}}\ h(r\,R + s\,S)\ dr\,ds
\end{aligned}$$

as a result.

Then, using (4.16) and (4.19) together with the equation (4.7)

$$(-1)^{\delta_1}\,c(n-1+\nu_1,\delta_1)\,c(-n-\nu_1,\delta_1) = 1\,, \tag{4.28}$$

we obtain (4.22) under the conditions which made Lemma 4.2, and (4.16) as a consequence, valid. Analytic continuation is possible, the hypotheses from Proposition 3.1 giving a meaning to the left-hand side. The conditions (4.21) make it possible to extract $(h_1)_{\nu_1,\delta_1}$, $(h_2)_{\nu_2,\delta_2}$ and $h_{-\nu,\delta}$ from $h_1,\,h_2,\,h$; the first condition (4.20) gives a meaning to $|s|_{\delta_2}^{-1-\nu-\nu_1}$ as a distribution (the factor depending on r is already locally summable from the previous condition), and the other two inequalities (4.20) make up half the conditions needed in order that the ratio $\frac{c(\frac{-n+\nu+\nu_1+\nu_2}{2},\varepsilon)}{c(\frac{n-2+\nu-\nu_1+\nu_2}{2},\varepsilon_2)}$ be well-defined and nonzero while, as it turns out, the other two conditions necessary for that have already been taken care of by the assumptions of Proposition 3.1.

□

5. Some one-dimensional preparation

Let us briefly recall the spectral decomposition of the one-dimensional Euler operator in $L^2(\mathbb{R})$, with the notation of Section 2. Given a function $h_{i\lambda,\delta}$ on $\mathbb{R}^2$, homogeneous of degree $-1-i\lambda$ and with a given parity specified by the index $\delta = 0$ or 1, we set

$$h^\flat_{i\lambda,\delta}(s) = h_{i\lambda,\delta}(s,\,1) \tag{5.1}$$

so that

$$h_{i\lambda,\delta}(x,\,\xi) = |\xi|_\delta^{-1-i\lambda}\, h^\flat_{i\lambda,\delta}(\frac{x}{\xi})\,. \tag{5.2}$$

Then, every function $h \in L^2(\mathbb{R}^2)$ can be decomposed as

$$h = \sum_{\delta=0,1} \int_{-\infty}^{\infty} h_{i\lambda,\delta}\, d\lambda \tag{5.3}$$

with

$$h_{i\lambda,\delta}(x,\,\xi) = \frac{1}{2\pi} \int_0^\infty t^{i\lambda}\, h_\delta(tx,\,t\xi)\, dt\,, \tag{5.4}$$

where h_δ denotes the even, or odd, part of h, according to whether $\delta = 0$ or 1. Note that we denote here as $h^\flat_{i\lambda,\delta}$ the function denoted as $h^\flat_{\lambda,\delta}$ in [11, p. 34].

Using the equations (in which signed powers such as $|s|_\delta^\alpha$ have been defined in (2.2))

$$\frac{d}{dx}\, |x|_\delta^{-1-\nu} = -(1+\nu)\, |x|_{1-\delta}^{-\nu-2} \qquad \text{and} \qquad \frac{d}{dx} \log|x| = x^{-1}\,, \tag{5.5}$$

one obtains the well-known fact, already used in Section 3, that the function $\nu \mapsto |x|_\delta^{-1-\nu}$, a locally summable function if $\mathrm{Re}\,\nu < 0$, extends as a distribution-valued holomorphic function of ν for $\nu \neq \delta,\, \delta+2,\, \dots$.

If $|x|_{\delta_1}^{-1-\nu_1}$ and $|\xi|_{\delta_2}^{-1-\nu_2}$ make sense as distributions as just defined, the symbol $h(x,\,\xi) = |x|_{\delta_1}^{-1-\nu_1} \,\#\, |\xi|_{\delta_2}^{-1-\nu_2}$ makes sense as a tempered distribution on $\mathbb{R}^2$: in other words, the composition of the two operators, the first of which is the convolution by the inverse Fourier transform of $|\xi|_{\delta_2}^{-1-\nu_2}$, and the second is the multiplication by $|x|_{\delta_1}^{-1-\nu_1}$, is well-defined as an operator from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$. To see this, one may use as an intermediary space the space $\mathcal{O}_M$ [10, p. 101] of C^∞ functions on the line each derivative of which is bounded by some polynomial.

Under the lift from $h^\flat_{i\lambda,\delta}$ to $h_{i\lambda,\delta}$ provided by (5.2), the distribution associated to the function $|s|^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}$ is given as

$$(x,\,\xi) \mapsto |x|^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |\xi|_\delta^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}} \tag{5.6}$$

and the distribution associated to the function $\langle s\rangle^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}$ is given as

$$(x,\,\xi) \mapsto \langle x\rangle^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |\xi|_{1-\delta}^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}\,. \tag{5.7}$$

Both distributions make sense if $\frac{-1\pm(\nu_1-\nu_2)-i\lambda}{2} \neq -1, -2, \dots$, which is the case whenever $\lambda \in \mathbb{R}$ if one assumes that $|\mathrm{Re}\,(\nu_1-\nu_2)| < 1$.

We may then recall Lemma 5.1 from [11] as follows:

Lemma 5.1. *Let* $\nu_1,\, \nu_2 \in \mathbb{C}$ *and* $\delta_1,\, \delta_2 = 0$ *or* 1 *: assume that* $\nu_1 \neq \delta_1,\, \nu_2 \neq \delta_2$ *and that* $|\mathrm{Re}\,(\nu_1 \pm \nu_2)| < 1$ *which implies that* $|\mathrm{Re}\,\nu_1| < 1,\, |\mathrm{Re}\,\nu_2| < 1$ *. Let* $\delta = 0$ *or* 1 *be such that* $\delta \equiv \delta_1 + \delta_2 \mod 2$ *. Set* $h_1(x,\,\xi) = |x|_{\delta_1}^{-1-\nu_1}$ *,* $h_2(x,\,\xi) = |\xi|_{\delta_2}^{-1-\nu_2}$ *and* $h = h_1 \,\#\, h_2$ *, a tempered distribution in* $\mathbb{R}^2$ *. It admits the weak decomposition in* $\mathcal{S}'(\mathbb{R}^2)$ *given as*

$$h = \int_{-\infty}^{\infty} h_{i\lambda,\delta}\, d\lambda \tag{5.8}$$

with

$$h_{i\lambda,\delta}(x,\,\xi) = 2^{\frac{\nu_1+\nu_2-i\lambda-5}{2}}\, \pi^{\frac{\nu_1+\nu_2-i\lambda}{2}}\, \frac{\Gamma(\frac{-\nu_1+\delta_1}{2})\Gamma(\frac{-\nu_2+\delta_2}{2})}{\Gamma(\frac{\nu_1+\delta_1+1}{2})\Gamma(\frac{\nu_2+\delta_2+1}{2})} \tag{5.9}$$

$$\Bigg[i^{\delta_2-\delta}\, \frac{\Gamma(\frac{1+\nu_1-\nu_2+i\lambda}{4})\Gamma(\frac{1+\nu_1+\nu_2-i\lambda+2\delta_1}{4})\Gamma(\frac{1-\nu_1+\nu_2+i\lambda+2\delta}{4})}{\Gamma(\frac{1-\nu_1+\nu_2-i\lambda}{4})\Gamma(\frac{1-\nu_1-\nu_2+i\lambda+2\delta_1}{4})\Gamma(\frac{1+\nu_1-\nu_2-i\lambda+2\delta}{4})}\, |x|^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |\xi|_{\delta}^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}$$

$$+ i^{-\delta_2-\delta+1}\, \frac{\Gamma(\frac{3+\nu_1-\nu_2+i\lambda}{4})\Gamma(\frac{3+\nu_1+\nu_2-i\lambda-2\delta_1}{4})\Gamma(\frac{3-\nu_1+\nu_2+i\lambda-2\delta}{4})}{\Gamma(\frac{3-\nu_1+\nu_2-i\lambda}{4})\Gamma(\frac{3-\nu_1-\nu_2+i\lambda-2\delta_1}{4})\Gamma(\frac{3+\nu_1-\nu_2-i\lambda-2\delta}{4})}\, \langle x\rangle^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |\xi|_{1-\delta}^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}} \Bigg].$$

Note that the integrand, as a distribution-valued function of λ , has no singularity on the real line. Also, as a consequence of Stirling's formula, the coefficient is bounded, for large $|\lambda|$ by some power of $|\lambda|$: since our claim is that the integral decomposition (5.8) is valid in a weak sense in $\mathcal{S}'(\mathbb{R}^2)$, we may ensure convergence by means of the equation

$$|x|^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |\xi|_{\delta}^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}} = (1+\lambda^2)^{-N}\, (1+4\pi^2\mathcal{E}^2)^N \left(|x|^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |\xi|_{\delta}^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}} \right), \tag{5.10}$$

in which $2i\pi\mathcal{E} = 1 + x\,\frac{\partial}{\partial x} + \xi\,\frac{\partial}{\partial \xi}$, and of a similar one involving the second term on the right-hand side of (5.9).

We now need to consider the case of two symbols $|x|_{\delta_1}^{-n-\nu_1}$ and $|\xi|_{\delta_2}^{-n-\nu_2}$, in which $n = 1, 2, \dots$ is given, the same in both functions. The reason is that, even though the proof of the main theorem depends on the decomposition of symbols into homogeneous hyperplane waves, which are essentially one-dimensional objects, the spectral decomposition of the Euler operator in $L^2(\mathbb{R}^{2n})$ demands that we consider decompositions of the same species as (5.3) in which, however, the degrees of homogeneity of the functions in the decomposition lie on the complex line with real part $-n$ rather than -1.

Let Q and P be the basic infinitesimal operators of Heisenberg's representation, where Q is the operator of multiplication by the variable x on the real line, and $P = \frac{1}{2i\pi}\frac{d}{dx}$. Then, in the one-dimensional Weyl calculus, one has the commutation relations

$$[\,Q\,,\,\mathrm{Op}(h)\,] = -\frac{1}{2i\pi}\,\mathrm{Op}(\frac{\partial h}{\partial \xi})\,, \qquad [\,P\,,\,\mathrm{Op}(h)\,] = \frac{1}{2i\pi}\,\mathrm{Op}(\frac{\partial h}{\partial x})\,. \tag{5.11}$$

Also, $P\,\mathrm{Op}(h) = \mathrm{Op}(\xi\,h + \frac{1}{4i\pi}\frac{\partial h}{\partial x})$. If h_1 (*resp.* h_2) is a tempered distribution depending only on x (*resp.* ξ), and if one sets $A_1 = \mathrm{Op}(h_1)$, $A_2 = \mathrm{Op}(h_2)$, one has (using the facts that A_1 commutes with Q, A_2 commutes with P and the Heisenberg relation $[P,\,Q] = \frac{1}{2i\pi}$)

$$[P,\,A_1]\,[Q,\,A_2] = P\,[Q,\,A_1A_2] - [Q,\,A_1A_2P] - \frac{1}{2i\pi}\,A_1A_2\ : \tag{5.12}$$

it follows that if $h = h_1 \,\#\, h_2$, the symbol of the operator $[P,\,\mathrm{Op}(h_1)]\,[Q,\,\mathrm{Op}(h_2)]$ is the function

(5.13)
$$\left(\xi + \frac{1}{4i\pi}\,\frac{\partial}{\partial x}\right)\left(-\frac{1}{2i\pi}\,\frac{\partial h}{\partial \xi}\right) + \frac{1}{2i\pi}\,\frac{\partial}{\partial \xi}\,\left(\xi\,h - \frac{1}{4i\pi}\,\frac{\partial h}{\partial x}\right) - \frac{1}{2i\pi}\,h = \frac{1}{4\pi^2}\,\frac{\partial^2 h}{\partial x\,\partial \xi}\,.$$

In other words, under the present assumptions,

$$\frac{\partial h_1}{\partial x}\,\#\,\frac{\partial h_2}{\partial \xi} = \frac{\partial^2 h}{\partial x\,\partial \xi}\,. \tag{5.14}$$

Introduce, for $k = 0, 1, \dots$ and $a \in \mathbb{C}$, the Pochhammer symbols $(a)_k = a(a{+}1)\dots(a{+}k{-}1)$, and extend the definition of $|s|_\delta^\alpha$ beyond the case when $\delta = 0$ or 1, setting $|s|_p^\alpha = |s|_{p \bmod 2}^\alpha$. With the same assumptions about $\nu_1,\,\nu_2,\,\delta_1,\,\delta_2$ as in Lemma 3.1, one has for $n = 1, 2, \dots$ (using (5.5) the equation

$$\begin{aligned}
(5.15)\quad & (1+\nu_1)_{n-1}\,(1+\nu_2)_{n-1}\ |x|_{n-1-\delta_1}^{-n-\nu_1}\,\#\,|\xi|_{n-1-\delta_2}^{-n-\nu_2} \\
&= \int_{-\infty}^{\infty} (\frac{1+\nu_1-\nu_2+i\lambda}{2})_{n-1}\,(\frac{1-\nu_1+\nu_2+i\lambda}{2})_{n-1} \\
&\qquad 2^{\frac{\nu_1+\nu_2-i\lambda-5}{2}}\ \pi^{\frac{\nu_1+\nu_2-i\lambda}{2}}\ \frac{\Gamma(\frac{-\nu_1+\delta_1}{2})\Gamma(\frac{-\nu_2+\delta_2}{2})}{\Gamma(\frac{\nu_1+\delta_1+1}{2})\Gamma(\frac{\nu_2+\delta_2+1}{2})} \\
&\Bigg[\, i^{\delta_2-\delta}\ \frac{\Gamma(\frac{1+\nu_1-\nu_2+i\lambda}{4})\Gamma(\frac{1+\nu_1+\nu_2-i\lambda+2\delta_1}{4})\Gamma(\frac{1-\nu_1+\nu_2+i\lambda+2\delta}{4})}{\Gamma(\frac{1-\nu_1+\nu_2-i\lambda}{4})\Gamma(\frac{1-\nu_1-\nu_2+i\lambda+2\delta_1}{4})\Gamma(\frac{1+\nu_1-\nu_2-i\lambda+2\delta}{4})}\ |x|_{n-1}^{\frac{1-2n-\nu_1+\nu_2-i\lambda}{2}}\ \ |\xi|_{n-1+\delta}^{\frac{1-2n+\nu_1-\nu_2-i\lambda}{2}} \\
&+ i^{-\delta_2-\delta+1}\ \frac{\Gamma(\frac{3+\nu_1-\nu_2+i\lambda}{4})\Gamma(\frac{3+\nu_1+\nu_2-i\lambda-2\delta_1}{4})\Gamma(\frac{3-\nu_1+\nu_2+i\lambda-2\delta}{4})}{\Gamma(\frac{3-\nu_1+\nu_2-i\lambda}{4})\Gamma(\frac{3-\nu_1-\nu_2+i\lambda-2\delta_1}{4})\Gamma(\frac{3+\nu_1-\nu_2-i\lambda-2\delta}{4})}\ |x|_n^{\frac{1-2n-\nu_1+\nu_2-i\lambda}{2}}\ \ |\xi|_{n-\delta}^{\frac{1-2n+\nu_1-\nu_2-i\lambda}{2}}\ \Bigg] \\
&\qquad d\lambda\,.
\end{aligned}$$

Note that the degree of homogeneity of each of the two terms under the integral sign is $1-2n-i\lambda$, not $-n-i\lambda$ as we would wish it to be: we must thus perform a deformation of contour. We substitute $z \in \mathbb{C}$ for $i\lambda$ and we must move z from the pure imaginary line to the line with real part $1-n$. There is no convergence problem at infinity in the process, in view of (5.10). We must then chase for possible poles, setting $\mu = \frac{\nu_1-\nu_2+z}{2}$ and $\mu' = \frac{\nu_1-\nu_2-z}{2}$. The only singularities can arise from the factors depending on x or ξ, or from the first and third Gamma functions in the numerator of each of the two major coefficients. We make a group of each of the expressions

$$\begin{aligned}
&(\frac{1}{2}+\mu)_{n-1}\ \Gamma(\frac{1}{4}+\frac{\mu}{2})\ |x|_{n-1}^{\frac{1}{2}-n-\mu}\,,\\
&(\frac{1}{2}+\mu)_{n-1}\ \Gamma(\frac{3}{4}+\frac{\mu}{2})\ |x|_{n}^{\frac{1}{2}-n-\mu}\,,\\
&(\frac{1}{2}-\mu')_{n-1}\ \Gamma(\frac{1}{4}+\frac{\delta}{2}-\frac{\mu'}{2})\ |\xi|_{n-1+\delta}^{\frac{1}{2}-n+\mu'}\,,\\
&(\frac{1}{2}-\mu')_{n-1}\ \Gamma(\frac{3}{4}-\frac{\delta}{2}-\frac{\mu'}{2})\ |\xi|_{n-\delta}^{\frac{1}{2}-n+\mu'}\,. \qquad (5.16)
\end{aligned}$$

We now show that each of the four functions under consideration remains a holomorphic function of z in a neighbourhood of the closed strip $1-n \le \operatorname{Re} z \le 0$. First we show that the Gamma factor and the distribution (in x or ξ) on any of the four lines have disjoint sets of singularities as functions of z. This is a consequence of the fact, noted just after (5.5), that $|x|_\delta^{-\alpha}$ a well-defined distribution in x provided that $\alpha \neq \delta+1, \delta+3, \ldots$. For, as a consequence, the singularities of the factor depending on x or ξ on the four lines are reached when $\mu \in \frac{1}{2}+2\mathbb{N}$, *resp.* $\mu \in \frac{3}{2}+2\mathbb{N}$, *resp.* $\mu \in -\delta-\frac{1}{2}-2\mathbb{N}$, *resp.* $\mu \in \delta-\frac{3}{2}+2\mathbb{N}$, while the singularities of the corresponding Gamma factors are reached when $\mu \in -\frac{1}{2}-2\mathbb{N}$, *resp.* $\mu \in -\frac{3}{2}-2\mathbb{N}$, *resp.* $\mu \in -\delta+\frac{1}{2}+2\mathbb{N}$, *resp.* $\mu \in -\delta+\frac{3}{2}+2\mathbb{N}$.

Since the two sets of singularities under consideration are disjoint, what remains to be proved is that each of the eight expressions

$$\begin{aligned}
&(\frac{1}{2}+\mu)_{n-1}\ \Gamma(\frac{1}{4}+\frac{\mu}{2})\,, && (\frac{1}{2}+\mu)_{n-1}\ |x|_{n-1}^{\frac{1}{2}-n-\mu}\,,\\
&(\frac{1}{2}+\mu)_{n-1}\ \Gamma(\frac{3}{4}+\frac{\mu}{2})\,, && (\frac{1}{2}+\mu)_{n-1}\ |x|_{n}^{\frac{1}{2}-n-\mu}\,,\\
&(\frac{1}{2}-\mu')_{n-1}\ \Gamma(\frac{1}{4}+\frac{\delta}{2}-\frac{\mu'}{2})\,, && (\frac{1}{2}-\mu')_{n-1}\ |\xi|_{n-1+\delta}^{\frac{1}{2}-n+\mu'}\,,\\
&(\frac{1}{2}-\mu')_{n-1}\ \Gamma(\frac{3}{4}-\frac{\delta}{2}-\frac{\mu'}{2})\,, && (\frac{1}{2}-\mu')_{n-1}\ |\xi|_{n-\delta}^{\frac{1}{2}-n+\mu'}\,. \qquad (5.17)
\end{aligned}$$

is regular for z lying in the strip $1-n \le \operatorname{Re} z \le 0$. So far as the distribution on the right of each line is concerned, we write it as $(-1)^{n-1}$ times the $(\frac{d}{dx})^{n-1}$, or $(\frac{d}{d\xi})^{n-1}$-derivative of the distribution $|x|^{-\frac{1}{2}-\mu}$, *resp.* $\langle x\rangle^{-\frac{1}{2}-\mu}$, *resp.*

$|\xi|_{\delta}^{\frac{1}{2}+\mu'}$, *resp.* $|\xi|_{1-\delta}^{-\frac{1}{2}+\mu'}$. Now, the condition $\operatorname{Re} z \leq 0$, together with the assumption $|\operatorname{Re}(\nu_1 - \nu_2)| < 1$, implies that $\operatorname{Re}\mu < \frac{1}{2}$ and $\operatorname{Re}\mu' > -\frac{1}{2}$, which gives the four distributions under consideration a meaning as a locally summable function. So far as the Gamma factors are concerned, every other term in the product

$$(\frac{1}{2}+\mu)_{n-1} = (\frac{1}{2}+\mu)\,(\frac{3}{2}+\mu)\dots(n-\frac{1}{2}+\mu) \quad \text{or}$$

$$(\frac{1}{2}-\mu')_{n-1} = (\frac{1}{2}-\mu')\,(\frac{3}{2}-\mu')\dots(n-\frac{1}{2}-\mu') \tag{5.18}$$

will help in killing the relevant poles of the corresponding Gamma factor. Indeed, with $p = 1, 2, \dots$, each of the two expressions $(\frac{1}{2}+\mu)_{2p-1}\,\Gamma(\frac{1}{4}+\frac{\mu}{2})$ and $(\frac{1}{2}+\mu)_{2p-2}\,\Gamma(\frac{1}{4}+\frac{\mu}{2})$ is the product of a polynomial in μ by the function $\Gamma(p+\frac{1}{4}+\frac{\mu}{2})$, while each of the two expressions $(\frac{1}{2}+\mu)_{2p-1}\,\Gamma(\frac{3}{4}+\frac{\mu}{2})$ and $(\frac{1}{2}+\mu)_{2p-2}\,\Gamma(\frac{3}{4}+\frac{\mu}{2})$ is the product of a polynomial in μ by the function $\Gamma(p-\frac{1}{4}+\frac{\mu}{2})$. The last two expressions to be analyzed are $(\frac{1}{2}-\mu')_{n-1}\,\Gamma(\frac{1}{4}-\frac{\mu'}{2})$ and $(\frac{1}{2}-\mu')_{n-1}\,\Gamma(\frac{3}{4}-\frac{\mu'}{2})$. We use this time the inequality $\operatorname{Re}\mu' < \frac{n}{2}$ and observe that each of the two expressions $(\frac{1}{2}-\mu')_{2p-1}\,\Gamma(\frac{1}{4}-\frac{\mu'}{2})$ and $(\frac{1}{2}-\mu')_{2p-2}\,\Gamma(\frac{1}{4}-\frac{\mu'}{2})$ is the product of a polynomial by $\Gamma(p+\frac{1}{4}-\frac{\mu'}{2})$, while each of the two expressions $(\frac{1}{2}-\mu')_{2p-1}\,\Gamma(\frac{3}{4}-\frac{\mu'}{2})$ and $(\frac{1}{2}-\mu')_{2p-2}\,\Gamma(\frac{3}{4}-\frac{\mu'}{2})$ is the product of a polynomial by $\Gamma(p-\frac{1}{4}-\frac{\mu'}{2})$.

Performing the change of contour which was the aim of the lengthy preparation just made, we finally obtain the following.

Lemma 5.2. *Let* $\nu_1, \nu_2 \in \mathbb{C}$ *and* $\delta_1, \delta_2 = 0$ *or* 1 *: assume that* $\nu_1 \neq \delta_1, \nu_2 \neq \delta_2$ *and that* $|\operatorname{Re}(\nu_1 \pm \nu_2)| < 1$ *. Let* $n = 1, 2, \dots$ *, and let* $\delta, \delta_1', \delta_2'$ *be the numbers, all equal to* 0 *or* 1 *, characterized by the congruences mod* 2

$$\delta \equiv \delta_1 + \delta_2\,, \qquad \delta_1' \equiv n - 1 - \delta_1\,, \quad \delta_2' \equiv n - 1 - \delta_2\,. \tag{5.19}$$

Set $h_1(x,\xi) = |x|_{\delta_1}^{-n-\nu_1}$ *,* $h_2(x,\xi) = |\xi|_{\delta_2}^{-n-\nu_2}$ *and let* $h = h_1 \# h_2$ *, a tempered distribution in* $\mathbb{R}^2$ *. It admits the weak decomposition in* $\mathcal{S}'(\mathbb{R}^2)$ *given as*

$$h = \int_{-\infty}^{\infty} h_{i\lambda,\delta}^{(n)}\, d\lambda \tag{5.20}$$

with

(5.21)
$$h^{(n)}_{i\lambda,\delta}(x,\,\xi) = (1{+}\nu_1)^{-1}_{n-1}\,(1{+}\nu_2)^{-1}_{n-1}\,(\frac{2-n+\nu_1-\nu_2+i\lambda}{2})_{n-1}\,(\frac{2-n-\nu_1+\nu_2+i\lambda}{2})_{n-1}$$
$$\times\; 2^{\frac{\nu_1+\nu_2-i\lambda+n-6}{2}}\,\pi^{\frac{n-1+\nu_1+\nu_2-i\lambda}{2}}\,\frac{\Gamma(\frac{-\nu_1+\delta_1'}{2})\Gamma(\frac{-\nu_2+\delta_2'}{2})}{\Gamma(\frac{\nu_1+\delta_1'+1}{2})\Gamma(\frac{\nu_2+\delta_2'+1}{2})}$$
$$\times\Big[\, i^{\delta_2-\delta}\,\frac{\Gamma(\frac{2-n+\nu_1-\nu_2+i\lambda}{4})\Gamma(\frac{n+\nu_1+\nu_2-i\lambda+2\delta_1'}{4})\Gamma(\frac{2-n-\nu_1+\nu_2+i\lambda+2\delta}{4})}{\Gamma(\frac{n-\nu_1+\nu_2-i\lambda}{4})\Gamma(\frac{2-n-\nu_1-\nu_2+i\lambda+2\delta_1'}{4})\Gamma(\frac{n+\nu_1-\nu_2-i\lambda+2\delta}{4})}$$
$$\times\; |x|_{n-1}^{\frac{-n-\nu_1+\nu_2-i\lambda}{2}}\,|\xi|_{n-1-\delta}^{\frac{-n+\nu_1-\nu_2-i\lambda}{2}}$$
$$+\, i^{-\delta_2-\delta+1}\,\frac{\Gamma(\frac{4-n+\nu_1-\nu_2+i\lambda}{4})\Gamma(\frac{n+2+\nu_1+\nu_2-i\lambda-2\delta_1'}{4})\Gamma(\frac{4-n-\nu_1+\nu_2+i\lambda-2\delta}{4})}{\Gamma(\frac{n+2-\nu_1+\nu_2-i\lambda}{4})\Gamma(\frac{4-n-\nu_1-\nu_2+i\lambda-2\delta_1'}{4})\Gamma(\frac{n+2+\nu_1-\nu_2-i\lambda-2\delta}{4})}$$
$$\times\; |x|_{n}^{\frac{-n-\nu_1+\nu_2-i\lambda}{2}}\,|\xi|_{n-\delta}^{\frac{-n+\nu_1-\nu_2-i\lambda}{2}}\,\Big],$$

where we recall our convention that $|s|^{\alpha}_{p} = |s|^{\alpha}_{p'}$ *with* $p'=0$ *or* 1 *and* $p \equiv p'$ *mod* 2.

In the proof of Lemma 5.2, we have avoided moving ν_1 and ν_2, which would have complicated the pole chasing even more. It is, however, necessary to check that analytic continuation with respect to ν_1 and ν_2 is possible up to some point, in the sense of the following lemma.

Lemma 5.3. *Set* $\nu_1' = n-1+\nu_1$, $\nu_2' = n-1+\nu_2$, *so that* $|x|^{-1-\nu_1'}_{\delta_1} = |x|^{-n-\nu_1}_{\delta_1}$ *and* $|\xi|^{-1-\nu_2'}_{\delta_2} = |\xi|^{-n-\nu_2}_{\delta_2}$. *To obtain the term* $h^{(n)}_{i\lambda,\delta}$ *from the decomposition (5.20) of* $h_1 \,\#\, h_2$ *(same notation as in Lemma 5.2), it suffices to perform the substitutions* $\nu_1 \mapsto \nu_1'$, $\nu_2 \mapsto \nu_2'$ *and* $i\lambda \mapsto \nu' = i\lambda + n - 1$ *on the right-hand side of (5.9).*

Proof. The proof, based on the duplication formula and on the formula of complements for the Gamma function, is perfectly ugly, though one can take solace in the fact that it offers a means of verification. Starting from the right-hand side of (5.9) and making the substitution $(\nu_1,\,\nu_2,\,i\lambda) \mapsto (\nu_1',\,\nu_2',\,i\lambda+n-1)$, we want to show that we just obtain the right-hand side of (5.21). We shall limit ourselves to the case when n is odd. One has

$$(1+\nu_1)^{-1}_{n-1} = \frac{\Gamma(1-n-\nu_1)}{\Gamma(-\nu_1)} = 2^{1-n}\,\frac{\Gamma(\frac{1-n-\nu_1+\delta_1'}{2})\,\Gamma(\frac{2-n-\nu_1-\delta_1'}{2})}{\Gamma(\frac{-\nu_1+\delta_1'}{2})\,\Gamma(\frac{1-\nu_1-\delta_1'}{2})}, \tag{5.22}$$

so that
(5.23)
$$(1+\nu_1)_{n-1}^{-1}\,\frac{\Gamma(\frac{-\nu_1+\delta_1'}{2})}{\Gamma(\frac{1+\nu_1+\delta_1'}{2})}\,2^{1-n}\,\frac{\Gamma(\frac{1-n-\nu_1+\delta_1'}{2})\,\Gamma(\frac{2-n-\nu_1-\delta_1'}{2})}{\Gamma(\frac{1+\nu_1+\delta_1'}{2})\,\Gamma(\frac{1-\nu_1-\delta_1'}{2})}=2^{1-n}\,\frac{\Gamma(\frac{1-n-\nu_1+\delta_1'}{2})}{\Gamma(\frac{n+\nu_1+\delta_1'}{2})}\,,$$

2^{1-n} times the corresponding coefficient $\frac{\Gamma(\frac{-\nu_1'+\delta_1}{2})}{\Gamma(\frac{1+\nu_1'+\delta_1}{2})}$ arising after the shift $\nu_1\mapsto\nu_1'$ from a factor in (5.9). The same goes so far as the comparable coefficient depending on ν_2 is concerned. The powers of 2 and π, as well as the Gamma factors in the middle of the coefficients we are interested in, transform in an immediately satisfactory way. The remaining headache arises from the coefficient, obtained from (5.9) and the required shift,

$$B\colon=\frac{\Gamma(\frac{n+\nu_1-\nu_2+i\lambda}{4})\,\Gamma(\frac{n-\nu_1+\nu_2+i\lambda+2\delta}{4})}{\Gamma(\frac{2-n-\nu_1+\nu_2-i\lambda}{4})\,\Gamma(\frac{2-n+\nu_1-\nu_2-i\lambda+2\delta}{4})}\ : \tag{5.24}$$

multiplying by $\Gamma(\frac{4-n-\nu_1+\nu_2-i\lambda}{4})\,\Gamma(\frac{4-n+\nu_1-\nu_2-i\lambda-2\delta}{4})$ up and down, using the formula of complements upstairs and the duplication formula downstairs, we obtain

$$\begin{aligned}B=\frac{\pi}{2^{n+i\lambda}}&\left[\sin\pi(\frac{n+\nu_1-\nu_2+i\lambda}{4})\,\sin\pi(\frac{n-\nu_1+\nu_2+i\lambda+2\delta}{4})\right]^{-1}\\&\times\left[\Gamma(\frac{2-n-\nu_1+\nu_2-i\lambda}{2})\,\Gamma(\frac{2-n+\nu_1-\nu_2-i\lambda}{2})\right]^{-1}.\end{aligned}\tag{5.25}$$

This must be compared to the similar coefficient from (5.21), which must be accompanied, as a factor, by the product of the two remaining Pochhammer symbols. This is

$$A\colon=\frac{\Gamma(\frac{n-\nu_1+\nu_2-i\lambda}{2})}{\Gamma(\frac{2-n-\nu_1+\nu_2-i\lambda}{2})}\,\frac{\Gamma(\frac{n+\nu_1-\nu_2-i\lambda}{2})}{\Gamma(\frac{2-n+\nu_1-\nu_2-i\lambda}{2})} \tag{5.26}$$

$$\times\ \frac{\Gamma(\frac{2-n+\nu_1-\nu_2+i\lambda}{4})\,\Gamma(\frac{2-n-\nu_1+\nu_2+i\lambda+2\delta}{4})}{\Gamma(\frac{n-\nu_1+\nu_2-i\lambda}{4})\,\Gamma(\frac{n+\nu_1-\nu_2-i\lambda+2\delta}{4})}\ : \tag{5.27}$$

If we multiply the product of fractions on the second line, up and down, by $\Gamma(\frac{2+n-\nu_1+\nu_2-i\lambda}{4})\,\Gamma(\frac{2+n+\nu_1-\nu_2-i\lambda-2\delta}{4})$, if we apply again the formula of complements upstairs and the duplication formula downstairs, it becomes

(5.28)
$$\begin{aligned}\frac{\pi}{2^{2-n+i\lambda}}&\left[\sin\pi(\frac{2-n+\nu_1-\nu_2+i\lambda}{4})\,\sin\pi(\frac{2-n-\nu_1+\nu_2+i\lambda+2\delta}{4})\right]^{-1}\\&\times\left[\Gamma(\frac{n-\nu_1+\nu_2-i\lambda}{2})\,\Gamma(\frac{n+\nu_1-\nu_2-i\lambda}{2})\right]^{-1}.\end{aligned}$$

It follows that $A=2^{2n-2}\,B$, which completes our verification, in the case when n is odd, so far as the coefficient of the first term on the right-hand side of (5.9) or (5.21) is concerned. We shall not write down everything in the case when (still with n odd) the coefficient of the second term is concerned. The trick is, this time, to multiply the fraction B' which takes the place of B, up and down, by $\Gamma(\frac{2-n-\nu_1+\nu_2-i\lambda}{4})\,\Gamma(\frac{2-n+\nu_1-\nu_2-i\lambda+2\delta}{4})$; next, the fraction on the second line of the expression A' which takes the place of A is to be multiplied, up and down, by $\Gamma(\frac{n-\nu_1+\nu_2-i\lambda}{4})\,\Gamma(\frac{n+\nu_1-\nu_2-i\lambda+2\delta}{4})$: again, we find that $A' = 2^{2n-2}B'$. The lemma is thus proved in the case when n is odd. The proof is of course similar in the case when it is even: only, one should not forget that, in this case, $\delta_1' = 1-\delta_1$ and $\delta_2' = 1-\delta_2$. Also, the right-hand side of (5.9) will yield, after transformation, the two terms on the right-hand side of (5.21) in reverse order.

$\square$

Making all Gamma factors apparent has been necessary for the discussion of the change of complex contour. Using the shorthand provided by (3.32), *i.e.*, making the substitution

$$\frac{\Gamma(\frac{\rho+1+\delta}{2})}{\Gamma(\frac{-\rho+\delta}{2})} = i^{\delta}\,\pi^{\rho+\frac{1}{2}}\,c(\rho,\,\delta)\,, \tag{5.29}$$

one obtains the following.

Proposition 5.4. *Under the assumptions of Lemma 5.2, one has*

$$\begin{aligned} h^{(n)}_{i\lambda,\delta}(x,\,\xi) &= C_0(\nu_1,\,\nu_2,\,i\lambda\,;\,\delta_1,\,\delta_2,\,\delta)\;|x|^{\frac{-n-\nu_1+\nu_2-i\lambda}{2}}\;|\xi|_{\delta}^{\frac{-n+\nu_1-\nu_2-i\lambda}{2}} \\ &\quad + C_1(\nu_1,\,\nu_2,\,i\lambda\,;\,\delta_1,\,\delta_2,\,\delta)\;\langle x\rangle^{\frac{-n-\nu_1+\nu_2-i\lambda}{2}}\;|\xi|_{1-\delta}^{\frac{-n+\nu_1-\nu_2-i\lambda}{2}}\,, \end{aligned} \tag{5.30}$$

with

(5.31)

$$C_0(\nu_1,\,\nu_2,\,i\lambda\,;\,\delta_1,\,\delta_2,\,\delta) = 2^{\frac{\nu_1+\nu_2-i\lambda+n-6}{2}}\;\pi^{-1}\;(-1)^{\delta}\;c(-n-\nu_1,\delta_1)\;\;c(-n-\nu_2,\delta_2)$$
$$c(\frac{n-2+\nu_1-\nu_2+i\lambda}{2},0)\;\;c(\frac{n-2+\nu_1+\nu_2-i\lambda}{2},\delta_1)\;\;c(\frac{n-2-\nu_1+\nu_2+i\lambda}{2},\delta)$$

and

(5.32)

$$C_1(\nu_1,\,\nu_2,\,i\lambda\,;\,\delta_1,\,\delta_2,\,\delta) = 2^{\frac{\nu_1+\nu_2-i\lambda+n-6}{2}}\;\pi^{-1}\;(-1)^{\delta}\;c(-n-\nu_1,\delta_1)\;\;c(-n-\nu_2,\delta_2)$$
$$c(\frac{n-2+\nu_1-\nu_2+i\lambda}{2},1)\;\;c(\frac{n-2+\nu_1+\nu_2-i\lambda}{2},1-\delta_1)\;\;c(\frac{n-2-\nu_1+\nu_2+i\lambda}{2},1-\delta)\,.$$

In view of the proof of the main theorem in next section, and as a final topic in this very computational section, we compute the $\mathcal{G}$-transform (4.1) of the symbol $|x_1|_{\delta_1}^{-n-\nu_1}\,\#\,|\xi_1|_{\delta_2}^{-n+\nu_2}$, considered as a distribution in $\mathbb{R}^{2n}$: we still set $x = (x_1, x_*)$, $\xi = (\xi_1, \xi_*)$. The change $\nu_2 \mapsto -\nu_2$ is needed for the application in next section: at the same time, we change the variable of integration λ to $-\lambda$

below so as to decompose the result as an integral superposition of distributions of type $(-n-i\lambda,\,\delta)$; we denote as $k^{(n)}_{-i\lambda,\delta}$ the function obtained from $h^{(n)}_{i\lambda,\delta}$ after these two sign changes.

Proposition 5.5. *Assume that $\nu_1 \neq \delta_1$, $-\nu_2 \neq \delta_2$ and $|\mathrm{Re}\,(\nu_1 \pm \nu_2)| < 1$. One has the weak decomposition in $\mathcal{S}'(\mathbb{R}^{2n})$, given by the equation*

$$\left[\mathcal{G}\,\left(Y \mapsto |y_1|_{\delta_1}^{-n-\nu_1} \,\#\, |\eta_1|_{\delta_2}^{-n+\nu_2}\right)\right](x,\,\xi) = \int_{-\infty}^{\infty} (\mathcal{G}\,k^{(n)}_{-i\lambda,\delta})(x,\,\xi)\,d\lambda \tag{5.33}$$

with

$$\begin{aligned}(\mathcal{G}\,k^{(n)}_{-i\lambda,\delta})(x,\,\xi) &= B_0(\nu_1,\,\nu_2,\,i\lambda\,;\,\delta_1,\,\delta_2,\,\delta)\;|x_1|_{\delta}^{\frac{n-2-\nu_1-\nu_2-i\lambda}{2}}\;|\xi_1|^{\frac{n-2+\nu_1+\nu_2-i\lambda}{2}}\;\delta(x_*)\,\delta(\xi_*)\\ &+ B_1(\nu_1,\,\nu_2,\,i\lambda\,;\,\delta_1,\,\delta_2,\,\delta)\;|x_1|_{1-\delta}^{\frac{n-2-\nu_1-\nu_2-i\lambda}{2}}\;\langle\xi_1\rangle^{\frac{n-2+\nu_1+\nu_2-i\lambda}{2}}\;\delta(x_*)\,\delta(\xi_*)\,,\end{aligned} \tag{5.34}$$

where

$$\begin{aligned}B_0(\nu_1,\,\nu_2,\,i\lambda\,;\,\delta_1,\,\delta_2,\,\delta) &= 2^{\frac{\nu_1-\nu_2-i\lambda+n-6}{2}}\;\pi^{-1}\\ &\times\; c(-n-\nu_1,\delta_1)\;\;c(-n+\nu_2,\delta_2)\;\;c(\frac{n-2+\nu_1-\nu_2+i\lambda}{2},\delta_1)\end{aligned} \tag{5.35}$$

and

$$\begin{aligned}B_1(\nu_1,\,\nu_2,\,i\lambda\,;\,\delta_1,\,\delta_2,\,\delta) &= -2^{\frac{\nu_1-\nu_2-i\lambda+n-6}{2}}\;\pi^{-1}\\ &\times\; c(-n-\nu_1,\delta_1)\;\;c(-n+\nu_2,\delta_2)\;\;c(\frac{n-2+\nu_1-\nu_2+i\lambda}{2},1-\delta_1)\,.\end{aligned} \tag{5.36}$$

Proof. This is a consequence of the preceding proposition, together with the equation

$$\begin{aligned}&(\mathcal{G}\,(Y \mapsto |y_1|^{\alpha}_{\omega_1}\,|\xi_1|^{\beta}_{\omega_2}))(x,\,\xi)\\ &\quad= 2^{-n-\alpha-\beta}\,(-1)^{\omega_2}\;c(\alpha,\,\omega_1)\;c(\beta,\,\omega_2)\;|x_1|^{-1-\beta}_{\omega_2}\,|\xi_1|^{-1-\alpha}_{\omega_1}\;\delta(x_*)\,\delta(\xi_*)\,.\end{aligned} \tag{5.37}$$

A simplification occurs from the use of the equations (4.7)

$$\begin{aligned}c(\frac{n-2+\nu_1+\nu_2-i\lambda}{2},0)\;c(\frac{-n-\nu_1-\nu_2+i\lambda}{2},0) &= 1\,,\\ c(\frac{n-2+\nu_1+\nu_2-i\lambda}{2},\delta)\;c(\frac{-n-\nu_1-\nu_2+i\lambda}{2},\delta) &= (-1)^{\delta}\,,\\ c(\frac{n-2+\nu_1+\nu_2-i\lambda}{2},1)\;c(\frac{-n-\nu_1-\nu_2+i\lambda}{2},1) &= -1\,,\\ c(\frac{n-2+\nu_1+\nu_2-i\lambda}{2},1-\delta)\;c(\frac{-n-\nu_1-\nu_2+i\lambda}{2},1-\delta) &= (-1)^{1-\delta}\,.\end{aligned} \tag{5.38}$$

$\square$

6. Another composition of Weyl symbols

Theorem 6.1. *Given δ_1, δ_2 and $\delta = 0$ or 1 with $\delta \equiv \delta_1 + \delta_2 \mod 2$, and $j = 0$ or 1, define $\varepsilon_1, \varepsilon_2, \varepsilon$ by means of (2.34), and set, for real $\lambda_1, \lambda_2, \lambda$,*

$$
\begin{aligned}
(6.1) \quad a^{(j)}_{\delta_1,\delta_2;\delta}(i\lambda_1, i\lambda_2;\, i\lambda) = 2^{\frac{n-6+i(\lambda_1+\lambda_2-\lambda)}{2}}\, i^{\varepsilon-\varepsilon_1-\varepsilon_2}\, \pi^{\frac{3(1-n)-2+i(\lambda_1+\lambda_2-\lambda)}{2}} \times \\
\frac{\Gamma(\frac{n+i(\lambda_1-\lambda_2+\lambda)+2\varepsilon_1}{2})}{\Gamma(\frac{2-n-i(\lambda_1-\lambda_2+\lambda)+2\varepsilon_1}{2})}\, \frac{\Gamma(\frac{n+i(-\lambda_1+\lambda_2+\lambda)+2\varepsilon_2}{2})}{\Gamma(\frac{2-n-i(-\lambda_1+\lambda_2+\lambda)+2\varepsilon_2}{2})}\, \frac{\Gamma(\frac{n-i(\lambda_1+\lambda_2+\lambda)+2\varepsilon}{2})}{\Gamma(\frac{2-n+i(\lambda_1+\lambda_2+\lambda)+2\varepsilon}{2})}\,.
\end{aligned}
$$

Given two symbols h_1 and h_2 in the space $\mathcal{S}(\mathbb{R}^{2n})$, one has, in the weak sense in $\mathcal{S}'(\mathbb{R}^{2n})$,

$$
(6.2) \qquad h_1 \# h_2 = \int_{-\infty}^{\infty} (h_1 \# h_2)_{i\lambda}\, d\lambda\,,
$$

with

$$
\begin{aligned}
(6.3) \quad (h_1 \# h_2)_{i\lambda} = \sum_{\delta_1=0,\,1}\,\sum_{\delta_2=0,\,1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \\
\sum_{j=0,1} a^{(j)}_{\delta_1,\delta_2;\delta}(i\lambda_1, i\lambda_2;\, i\lambda)\, \mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{i\lambda_1,i\lambda_2;\,i\lambda}\left((h_1)_{i\lambda_1,\delta_1}\,,\,(h_2)_{i\lambda_2,\delta_2}\right)\, d\lambda_1\, d\lambda_2\,,
\end{aligned}
$$

where $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{i\lambda_1,i\lambda_2;\,i\lambda}$ is the bilinear operator from $C^\infty(\pi_{i\lambda_1,\delta_1}) \times C^\infty(\pi_{i\lambda_2,\delta_2})$ to $C^{-\infty}(\pi_{i\lambda,\delta})$ formally introduced in (2.38) and discussed in Section 3.

Proof. One has $h_1 \# h_2 = \mathcal{G}(h_1 \# \mathcal{G}h_2)$, as it follows from the interpretation of the transformation $\mathcal{G}$ of symbols recalled in the beginning of Section 4. Next, we decompose h_1 into hyperplane waves with the help of (4.4), and h_2 into rays with the help of (4.6), recalling that one can move the line of integration up to the spectral line and writing

$$
(6.4) \quad h_1 = \sum_{\delta_1=0,\,1} \int_{-\infty}^{\infty} (h_1)_{i\lambda_1,\delta_1}\, d\lambda_1\,, \qquad \mathcal{G}h_2 = \sum_{\delta_2=0,\,1} \int_{-\infty}^{\infty} (\mathcal{G}h_2)_{-i\lambda_2,\delta_2}\, d\lambda_2\,,
$$

with

$$
(6.5) \quad
\begin{aligned}
(h_1)_{i\lambda_1,\delta_1}(X) &= \frac{2^{-i\lambda_1}}{4\pi}\, c(n-1+i\lambda_1, \delta_1) \int_{\mathbb{R}^{2n}} (\mathcal{G}h_1)(R)\, |\, [X,\, R]\, |^{-n-i\lambda_1}_{\delta_1}\ dR\,, \\
(\mathcal{G}h_2)_{-i\lambda_2,\delta_2}(X) &= \frac{2^{i\lambda_2}}{4\pi}\, c(n-1-i\lambda_2, \delta_2) \int_{\mathbb{R}^{2n}} h_2(S)\, |\, [X,\, S]\, |^{-n+i\lambda_2}_{\delta_2}\ dS\ :
\end{aligned}
$$

recall that the product $c(n-1+\nu_1, \delta_1)\, |\, [X,\, R]\, |^{-n-\nu_1}_{\delta_1}$, can be continued analytically with respect to ν_1, as a distribution in X. Then,

$$
(6.6) \qquad (h_1 \# h_2)(X) = \sum_{\delta_1=0,\,1}\,\sum_{\delta_2=0,\,1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^{\delta_1,\delta_2}_{i\lambda_1,i\lambda_2}(X)\, d\lambda_1\, d\lambda_2
$$

with

$$(6.7)\quad F^{\delta_1,\delta_2}_{\nu_1,\nu_2}(X)=\frac{2^{-\nu_1+\nu_2}}{(4\pi)^2}\ c(n-1+\nu_1,\delta_1)\ c(n-1-\nu_2,\delta_2)$$
$$\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}}(\mathcal{G}h_1)(R)\ h_2(S)\ (\mathcal{G}\ \left(\,|\,[X,\,R]\,|_{\delta_1}^{-n-\nu_1}\,\#\,|\,[X,\,S]\,|_{\delta_2}^{-n+\nu_2}\,\right))\ dR\,dS\,,$$

the two signed powers under the sharp product of which appears under the integral sign being regarded as functions of X. Actually, so as to obtain the last equation, we have changed the order of the bilinear operation $\#$ and of the integration with respect to $dR\,dS$. Though not completely trivial, the justification is fully similar to that, based on the consideration of the domains of powers of the harmonic oscillator, which occurred, in the one-dimensional case, in [12, p. 209]: we shall not reproduce it here.

Generically, one has $[R,\,S]\neq 0$ and, as noticed in (4.24), there exists $g\in \mathrm{Sp}(n,\mathbb{R})$ such that

$$(6.8)\qquad g^{-1}S=e_1\,,\qquad g^{-1}R=[R,\,S]\,e_{n+1}$$

in terms of the canonical basis of $\mathbb{R}^n\times\mathbb{R}^n$. Then, using the covariance of the Weyl calculus, and the fact that the transformation $\mathcal{G}$ commutes with symplectic changes of coordinates, we obtain

(6.9)
$$F^{\delta_1,\delta_2}_{\nu_1,\nu_2}(X)=\frac{2^{-\nu_1+\nu_2}}{(4\pi)^2}\ c(n-1+\nu_1,\delta_1)\ c(n-1-\nu_2,\delta_2)\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}}(\mathcal{G}h_1)(R)\,h_2(S)$$
$$|\,[S,\,R]\,|_{\delta_1}^{-n-\nu_1}\ \mathcal{G}\,(\,Y\mapsto |y_1|_{\delta_1}^{-n-\nu_1}\,\#\,|\eta_1|_{\delta_2}^{-n+\nu_2}\,)\,(g^{-1}X)\ dR\,dS\,.$$

The function $F^{\delta_1,\delta_2}_{\nu_1,\nu_2}$ can then be made explicit, starting from (6.9), with the help of Proposition 5.5. Rewrite the result of this proposition, tested against $h\in\mathcal{S}(\mathbb{R}^{2n})$, as

$$(6.10)\quad \langle\,\mathcal{G}\,(\,Y\mapsto |y_1|_{\delta_1}^{-n-\nu_1}\,\#\,|\eta_1|_{\delta_2}^{-n+\nu_2}\,)\,,\,h\,\rangle=\int_{-\infty}^{\infty}d\lambda\int_{\mathbb{R}^2}h(s\,e_1+r\,e_{n+1})$$
$$\Big[\,B_0(\nu_1,\,\nu_2,\,i\lambda\,;\,\delta_1,\,\delta_2,\,\delta)\ |r|^{\frac{n-2+\nu_1+\nu_2-i\lambda}{2}}\ |s|_{\delta}^{\frac{n-2-\nu_1-\nu_2-i\lambda}{2}}$$
$$+\,B_1(\nu_1,\,\nu_2,\,i\lambda\,;\,\delta_1,\,\delta_2,\,\delta)\,\langle r_1\rangle^{\frac{n-2+\nu_1+\nu_2-i\lambda}{2}}\ |s|_{1-\delta}^{\frac{n-2-\nu_1-\nu_2-i\lambda}{2}}\ \Big]\ dr\,ds\,.$$

Then,

$$(6.11)\quad \langle\, (\mathcal{G}\,(\,Y \mapsto |y_1|_{\delta_1}^{-n-\nu_1} \,\#\, |\eta_1|_{\delta_2}^{-n+\nu_2}\,))\,\circ\, g^{-1}\,,\, h\,\rangle = \int_{-\infty}^{\infty} d\lambda \int_{\mathbb{R}^2} h(s\,S + r\,R)$$

$$\Big[\, B_0(\nu_1,\, \nu_2,\, i\lambda\,;\, \delta_1,\, \delta_2,\, \delta)\,|\,[R,\, S]\,|^{\frac{n+\nu_1+\nu_2-i\lambda}{2}}\ |r|^{\frac{n-2+\nu_1+\nu_2-i\lambda}{2}}\ |s|_{\delta}^{\frac{n-2-\nu_1-\nu_2-i\lambda}{2}}$$

$$+B_1(\nu_1,\, \nu_2,\, i\lambda\,;\, \delta_1,\, \delta_2,\, \delta)\,\langle\,[R,\, S]\,\rangle^{\frac{n+\nu_1+\nu_2-i\lambda}{2}}\ \langle r\rangle^{\frac{n-2+\nu_1+\nu_2-i\lambda}{2}}\ |s|_{1-\delta}^{\frac{n-2-\nu_1-\nu_2-i\lambda}{2}}\,\Big]\ dr\,ds\,,$$

as seen after one has used (6.8) and the change of variable $r \mapsto [R,\, S]\,r$, and

$$(6.12)\qquad F^{\delta_1,\delta_2}_{i\lambda_1,i\lambda_2} = \int_{-\infty}^{\infty} F^{\delta_1,\delta_2}_{i\lambda_1,i\lambda_2;\,i\lambda}\, d\lambda$$

with

$$(6.13)\quad \langle\, F^{\delta_1,\delta_2}_{i\lambda_1,i\lambda_2;\,i\lambda}\,,\, h\,\rangle$$

$$= (-1)^{\delta_1}\,\frac{2^{i(-\lambda_1+\lambda_2)}}{(4\pi)^2}\,c(n-1+i\lambda_1,\,\delta_1)\,c(n-1-i\lambda_2,\,\delta_2)$$

$$\int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} (\mathcal{G}h_1)(R)\,h_2(S)\,dR\,dS \int_{\mathbb{R}^2} h(r\,R + s\,S)$$

$$\Big[\, B_0(i\lambda_1,\, i\lambda_2,\, i\lambda\,;\, \delta_1,\, \delta_2,\, \delta)\,|\,[R,\, S]\,|_{\delta_1}^{\frac{-n+i(-\lambda_1+\lambda_2-\lambda)}{2}}\ |r|^{\frac{n-2+i(\lambda_1+\lambda_2-\lambda)}{2}}\ |s|_{\delta}^{\frac{n-2+i(-\lambda_1-\lambda_2-\lambda)}{2}}$$

$$+B_1(i\lambda_1,\, i\lambda_2,\, i\lambda\,;\, \delta_1,\, \delta_2,\, \delta)\,|\,[R,\, S]\,|_{1-\delta_1}^{\frac{-n+i(-\lambda_1+\lambda_2+\lambda)}{2}}\ \langle r\rangle^{\frac{n-2+i(\lambda_1+\lambda_2-\lambda)}{2}}\ |s|_{1-\delta}^{\frac{n-2+i(-\lambda_1-\lambda_2-\lambda)}{2}}\,\Big]\ dr\,ds\,.$$

Finally, making the coefficients B_0 and B_1 explicit with the help of Proposition 5.5 and using (4.7) again,

$$(6.14)\quad \frac{1}{4\pi}\,\langle\, F^{\delta_1,\delta_2}_{i\lambda_1,i\lambda_2;\,i\lambda}\,,\, h\,\rangle$$

$$= \frac{(-1)^{\delta_2}\,2^{\frac{n-2+i(-\lambda_1+\lambda_2-\lambda)}{2}}}{(4\pi)^4} \int_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} (\mathcal{G}h_1)(R)\,h_2(S)\,dR\,dS \int_{\mathbb{R}^2} h(r\,R + s\,S)$$

$$\Big[\, c(\tfrac{n-2+i(\lambda_1-\lambda_2+\lambda)}{2},\delta_1)\,|\,[R,\, S]\,|_{\delta_1}^{\frac{-n+i(-\lambda_1+\lambda_2-\lambda)}{2}}\ |r|^{\frac{n-2+i(\lambda_1+\lambda_2-\lambda)}{2}}\ |s|_{\delta}^{\frac{n-2+i(-\lambda_1-\lambda_2-\lambda)}{2}}$$

$$-c(\tfrac{n-2+i(\lambda_1-\lambda_2+\lambda)}{2},1-\delta_1)\,|\,[R,\, S]\,|_{1-\delta_1}^{\frac{-n+i(-\lambda_1+\lambda_2+\lambda)}{2}}\ \langle r\rangle^{\frac{n-2+i(\lambda_1+\lambda_2-\lambda)}{2}}\ |s|_{1-\delta}^{\frac{n-2+i(-\lambda_1-\lambda_2-\lambda)}{2}}\,\Big]\ dr\,ds\,.$$

The distribution $F^{\delta_1,\delta_2}_{i\lambda_1,i\lambda_2;\,i\lambda} \in \mathcal{S}'(\mathbb{R}^{2n})$ is of type $(-n-i\lambda,\,\delta)$. Now, given any element $\mathfrak{S}$ of $C^{-\infty}(\pi_{i\lambda,\delta})$ extended as a distribution in $\mathbb{R}^{2n}$ of type $(-n-i\lambda,\,\delta)$

with the same name, and any function $h \in \mathcal{S}(\mathbb{R}^{2n})$, one has the equation

$$\langle \mathfrak{S}\,,\, h \rangle_{\mathcal{S}'(\mathbb{R}^{2n})\times\mathcal{S}(\mathbb{R}^{2n})} = 4\pi\, \langle \mathfrak{S}\,,\, h_{-i\lambda,\delta} \rangle_{C^{-\infty}(\pi_{i\lambda,\delta})\times C^{\infty}(\pi_{-i\lambda,\delta})} \tag{6.15}$$

linking the two kinds of pairings. Starting from the case when $\mathfrak{S}$ is a function, one obtains (6.15) from the equation $\mathfrak{S}(tX_*) = |t|_\delta^{-n-i\lambda}\, \mathfrak{S}(X_*)$ and (2.4) or, if preferred, from a polarization of (2.13). The left-hand side of (6.14) can thus also be regarded as being $\langle F^{\delta_1,\delta_2}_{i\lambda_1,i\lambda_2;\,i\lambda}\,,\, h_{-i\lambda,\delta} \rangle$, the pairing now denoting that between $C^{-\infty}(\pi_{i\lambda,\delta})$ and $C^{\infty}(\pi_{-i\lambda,\delta})$. The comparison with (4.22) is now easy.

With another look at (2.34), one sees that $\mathbf{J}^{\varepsilon_1,\varepsilon_2;\,\varepsilon}_{\nu_1,\nu_2;\,\nu}$ coincides with $\mathbf{J}^{\delta_1,\delta_2;\,\delta}_{\nu_1,\nu_2;\,\nu}$ when $j=0$, and with $\mathbf{J}^{1-\delta_1,1-\delta_2;\,1-\delta}_{\nu_1,\nu_2;\,\nu}$ when $j=1$. Then, the first or second term on the right-hand side of (6.14) is a multiple of the right-hand side of (4.22) taken with $j=0$ or 1, as it follows from a comparison of the exponents and subscripts in (4.22) and in each of the two terms of (6.14) of the signed powers of $[R,\,S]$, r and s. The coefficient by which one must multiply the expression on right-hand side of (4.22) to obtain the corresponding term in right-hand side of (6.14) is

$$\frac{1}{4\pi}\, 2^{\frac{n-2+i(\lambda_1+\lambda_2-\lambda)}{2}}\; c(\frac{n-2+i(\lambda_1-\lambda_2+\lambda)}{2},\, \varepsilon_1) \;\times\; \frac{c(\frac{n-2+i(-\lambda_1+\lambda_2+\lambda)}{2},\varepsilon_2)}{c(\frac{-n+i(\lambda_1+\lambda_2+\lambda)}{2},\varepsilon)}\,. \tag{6.16}$$

Expanding, we can write this as

$$\begin{aligned} &2^{\frac{n-6+i(\lambda_1+\lambda_2-\lambda)}{2}}\; i^{\varepsilon-\varepsilon_1-\varepsilon_2}\; \pi^{\frac{3(1-n)-2+i(\lambda_1+\lambda_2-\lambda)}{2}} \times \\ &\frac{\Gamma(\frac{n+i(\lambda_1-\lambda_2+\lambda)+2\varepsilon_1}{2})}{\Gamma(\frac{2-n-i(\lambda_1-\lambda_2+\lambda)+2\varepsilon_1}{2})}\; \frac{\Gamma(\frac{n+i(-\lambda_1+\lambda_2+\lambda)+2\varepsilon_2}{2})}{\Gamma(\frac{2-n-i(-\lambda_1+\lambda_2+\lambda)+2\varepsilon_2}{2})}\; \frac{\Gamma(\frac{n-i(\lambda_1+\lambda_2+\lambda)+2\varepsilon}{2})}{\Gamma(\frac{2-n+i(\lambda_1+\lambda_2+\lambda)+2\varepsilon}{2})}\,. \end{aligned} \tag{6.17}$$

This concludes the proof of Theorem 6.1.

□

As an example, let us consider the harmonic oscillator $L = \mathrm{Op}(\pi\,\ell)$ with $\ell(x,\,\xi) = |x|^2 + |\xi|^2$, and sharp products of fractional powers of ℓ.

Proposition 6.2. *Let $\nu_1,\,\nu_2 \in \mathbb{C}$ satisfy the conditions $-n < \mathrm{Re}\,\nu_1 < n$, $-n < \mathrm{Re}\,\nu_2 < n$. Then, the decomposition into homogeneous components $h_{i\lambda}$ of the symbol $h = \ell^{\frac{-n-\nu_1}{2}} \# \ell^{\frac{-n-\nu_2}{2}}$ is given by the equation*

$$\begin{aligned} h_{i\lambda} = \frac{1}{4}\, (2\pi)^{\frac{n-2+\nu_1+\nu_2-i\lambda}{2}}\; \ell^{\frac{-n-i\lambda}{2}} \\ \times\, \frac{\Gamma(\frac{n+\nu_1+\nu_2-i\lambda}{4})\,\Gamma(\frac{n+\nu_1-\nu_2+i\lambda}{4})\,\Gamma(\frac{n-\nu_1+\nu_2+i\lambda}{4})\,\Gamma(\frac{n-\nu_1-\nu_2-i\lambda}{4})}{\Gamma(\frac{n+\nu_1}{2})\,\Gamma(\frac{n+\nu_2}{2})\,\Gamma(\frac{n-i\lambda}{2})}\,. \end{aligned} \tag{6.18}$$

Proof. It is identical to that of the one-dimensional case, as treated in [12, p. 214]. Only, one starts this time from the equation

$$\mathrm{Op}(e^{-2\pi s\ell}) = (1-s^2)^{-\frac{n}{2}} \left(\frac{1-s}{1+s}\right)^L \tag{6.19}$$

(same reference as in the one-dimensional case), leading rapidly to the equation

$$h = \frac{(2\pi)^{\frac{\nu_1+\nu_2+2n}{2}}}{\Gamma(\frac{n+\nu_1}{2})\,\Gamma(\frac{n+\nu_2}{2})} \int_0^\infty \int_0^\infty s_1^{\frac{n+\nu_1-2}{2}}\, s_2^{\frac{n+\nu_2-2}{2}}\, e^{-2\pi\frac{s_1+s_2}{1+s_1s_2}\,\ell}\, \frac{ds_1\,ds_2}{(1+s_1s_2)^n}\,, \tag{6.20}$$

then

$$\begin{aligned} h_{i\lambda} = {} & \frac{1}{2}\,(2\pi)^{\frac{\nu_1+\nu_2+n-2-i\lambda}{2}}\, \frac{\Gamma(\frac{n+i\lambda}{2})}{\Gamma(\frac{n+\nu_1}{2})\,\Gamma(\frac{n+\nu_2}{2})}\, \ell^{\frac{-n-i\lambda}{2}} \\ & \times \int_0^\infty \int_0^\infty s_1^{\frac{n+\nu_1-2}{2}}\, s_2^{\frac{n+\nu_2-2}{2}}\, (s_1+s_2)^{\frac{-n-i\lambda}{2}}\, (1+s_1s_2)^{\frac{-n+i\lambda}{2}}\, ds_1\,ds_2\,, \end{aligned} \tag{6.21}$$

from which it is easy to conclude.

Let us observe that, if not dealing with differential operators (*i.e.*, when $\frac{-n-\nu_1}{2}$ and $\frac{-n-\nu_2}{2}$ are not both non-negative integers), Moyal's expansion (1.11) would lead in this example to a sum of terms with increasing singularities at 0, without significance, even asymptotic, as a distribution in $\mathbb{R}^{2n}$: however, let us hasten to say that microlocal analysis does not attach much significance to *points* of the phase space.

□

As a comment, let us express our conviction that the new composition formula has at best limited interest so far as applications of pseudodifferential analysis to partial differential equations are concerned. This is not to mean that symplectic covariance does not play any role in P.D.E.'s: only, its role is essentially subordinate to that of the covariance under translations. It would be more correct to say that, in the more technical classes of symbols used in pseudodifferential analysis, it is rather the notion of uniformity under actions of conjugates of the group of translations under local families of symplectic transformations that is important. Here, our tilt is entirely towards the symplectic action, to the point that we have completely forgotten about the action of translations.

On the other hand, automorphic pseudodifferential analysis calls for the present point of view, as experienced in the one-dimensional case: automorphic symbols are much too singular to be even remotely reminiscent of symbols in any of the classes developed for P.D.E. applications. This does not imply that, to obtain the sharp composition of two automorphic symbols, it suffices to *apply* the present formula. Rather, the specific formula developed in this case, which has many special features inherent in the theory of modular forms, is based on the same principles (coupling symplectic covariance with the decomposition of automorphic symbols into their homogeneous components of a definite parity) as the ones which made the formula discussed here a natural one.

7. Irreducibility of the decomposition of $L^2(\mathbb{R}^{2n})$

We prove here the irreducibility of most unitary representations appearing in the spectral decomposition of Proposition 2.1 . In the last decades, general irreducibility results such as Kostant's irreducibility theorem for spherical (minimal) principal series representations [6] and Vogan–Wallach's irreducibility theorem for generic parameters [14] have been developed. Also, many specific cases have been studied in detail by R. Howe, E.-T. Tan, S.-T. Lee, S. Sahi, etc by algebraic and combinatorial methods. However, to the best of our knowledge, neither the general theory nor the known special results contain Theorem 7.3 below, the proof of which is based on the extension of the idea of branching laws to non-compact subgroups [5] and on properties of the Weyl calculus in $\mathbb{R}^{n-1}$.

Lemma 7.1. *Let $\mathcal{M}_0^{\text{vect}} = \{S = (s_1, s_* \,;\, 0, \sigma_*)\}$ denote the linear space of translations of the affine hyperplane $\mathcal{M}_0$. Given $S \in \mathcal{M}_0$, define the linear automorphism $\mathcal{T}_S$ of $\mathbb{R}^{2n}$ by the equation*

$$\mathcal{T}_S\, X = X + [S,\, X]\, e_1 + [e_1,\, X]\, S\,. \tag{7.1}$$

For every $S \in \mathcal{M}_0^{\text{vect}}$, $\mathcal{T}_S$ is a symplectic transformation of $\mathbb{R}^{2n}$ preserving $\mathcal{M}_0$. The group of all such symplectic transformations is generated by the group N of transformations $\mathcal{T}_S,\, S \in \mathcal{M}_0^{\text{vect}}$, together with the group M of transformations $(x_1, x_* \,;\, \xi_1, \xi_*) \mapsto (x_1, y_* \,;\, \xi_1, \eta_*)$, where the map $(x_* \,;\, \xi_*) \mapsto (y_* \,;\, \eta_*)$ is a symplectic transformation in the $2n-2$ variables involved; the latter normalizes the first within $\mathrm{Sp}(n, \mathbb{R})$.

Proof. That $[\mathcal{T}_S\, X,\, \mathcal{T}_S\, Y] = [X,\, Y]$ for every pair X, Y is an immediate consequence of the relations $[e_1,\, e_1] = [e_1,\, S] = [S,\, S] = 0$. That the group MN generates the stabilizer of $\mathcal{M}_0$ is a consequence of the observation following (2.28). □

Equation (2.8) reduces when $g \in MN$ to

$$(\pi_{\nu,\delta}(g)\, f)(X) = f(g^{-1}X)\,, \qquad X \in \mathcal{M}_0\,. \tag{7.2}$$

If one sets $S_{**} = (s_* \,;\, \sigma_*)$, $X_{**} = (x_* \,;\, \xi_*)$, the transformation $\mathcal{T}_{-S}$ expresses itself when considered on $\mathcal{M}_0$ as

$$\mathcal{T}_{-S}(x_1, x_* \,;\, 1, \xi_*) = (x_1 - 2s_1 + [S_{**},\, X_{**}],\, x_* - s_* \,;\, 1, \xi_* - \sigma_*)\ : \tag{7.3}$$

it follows in particular that, given $(i\lambda,\, \delta) \in i\mathbb{R} \times \{0,\, 1\}$, all transformations $\pi_{i\lambda,\delta}(g)$ with $g \in MN$, when regarded as unitary transformations of $L^2(\mathcal{M}_0)$, commute with the differential operator $\frac{1}{2i\pi}\,\frac{\partial}{\partial x_1}$.

Let us first decompose the restriction of the representation $\pi_{i\lambda,\delta}$ to MN: from what has just been said, it can be analyzed when coupled with the spectral decomposition of the operator $\frac{1}{2i\pi}\,\frac{\partial}{\partial x_1}$, in other words when fixing the first variable t in the partial Fourier transform $\mathcal{F}_1 f$ of $f \in L^2(\mathcal{M}_0)$, as already done in Section 2. From (7.2), one has if $n \geq 2$ the identity
(7.4)

$$(\mathcal{F}_1\,(\pi_{i\lambda,\delta}(\mathcal{T}_S)\,f))(t, x_*\,;\,\xi_*) = e^{-2i\pi t\,(2s_1 - [S_{**},\,X_{**}])}\,(\mathcal{F}_1 f)(t, x_* - s_*\,;\,\xi_* - \sigma_*)\,,$$

a group of transformations in which we may regard $t \neq 0$ as a parameter by specializing to $s_1 = 0$, getting a projective representation $\pi^{(t)}_{i\lambda,\delta}$ of $\mathbb{R}^{2n-2}$, actually independent of $(i\lambda, \delta)$, as a result; the same is true when considering transformations $\mathcal{F}_1\,(\pi_{i\lambda,\delta}(g))\,\mathcal{F}_1^{-1}$ with $g \in M$.

Lemma 7.2. *Assume that* $n \geq 2$ *. For fixed* $t \neq 0$ *, the linear space of bounded operators in* $L^2(\mathbb{R}^{2n-2})$ *which commute with all transformations* $\mathcal{F}_1\,(\pi^{(t)}_{i\lambda,\delta}(g))\,\mathcal{F}_1^{-1}$ *with* $g \in MN$ *is generated by the identity and the transformation* $\mathcal{F}_1\,\Sigma_t\,\mathcal{F}_1^{-1}$ *characterized by the equation*

$$(\mathcal{F}_1\,\Sigma_t\,f)(t,\,X_{**}) = |t|^{n-1}\int_{\mathbb{R}^{2n-2}} e^{-2i\pi t\,[X_{**},\,Y_{**}]}\,(\mathcal{F}_1\,f)(t,\,Y_{**})\,dY_{**}\,. \tag{7.5}$$

Proof. First assume that $t = 2$. Looking at (7.4), one sees that the linear space of infinitesimal operators of the representation of N under consideration is generated by the following operators, where $j,\,k \geq 2$: (i) the operators $\xi_j + \frac{1}{4i\pi}\,\frac{\partial}{\partial x_j}$, where ξ_j denotes the operator of multiplication by ξ_j ; (ii) the operators $x_k - \frac{1}{4i\pi}\,\frac{\partial}{\partial \xi_k}$. From (1.11), these are just the operators $h \mapsto \xi_j \,\#\, h$ and $h \mapsto x_k \,\#\, h$. Taking advantage of the Weyl calculus in $\mathbb{R}^{n-1}$, set

$$\varpi^{(2)}(g)\,\mathrm{Op}(h) = \mathrm{Op}\,(\mathcal{F}_1\,(\pi^{(2)}_{i\lambda,\delta}(g))\,\mathcal{F}_1^{-1}\,h)\,, \qquad g \in MN\,, \tag{7.6}$$

defining in this way a unitary representation $\varpi^{(2)}$ of MN in the space of Hilbert-Schmidt operators in $L^2(\mathbb{R}^{n-1})$. From what has just been seen, the image $\varpi^{(2)}(N)$ consists of the automorphisms

$$A \mapsto \exp\,(2i\pi\,(\langle \eta,\,Q\rangle - \langle y,\,P\rangle))\;A \tag{7.7}$$

(where the first factor was defined in the introduction). On the other hand, in view of (1.12), the image under $\varpi^{(2)}$ of M consists of the maps $A \mapsto U\;A\;U^{-1}$ with U in the image of the metaplectic representation. Since the Heisenberg representation in $L^2(\mathbb{R}^{n-1})$ is irreducible, while that of the metaplectic representation decomposes into its restrictions to spaces of functions with a given parity, it follows that the commutant of the representation $\varpi^{(2)}$ of MN is the linear space generated by the identity together with the automorphism $A \mapsto A\,\mathrm{Ch}$, where Ch is the parity map $u \mapsto \check{u}$, of the space of Hilbert-Schmidt operators in $L^2(\mathbb{R}^{n-1})$. Going back to symbols and using what immediately follows (4.1), one obtains the case $t = 2$ of Lemma 7.2, from which one obtains the general case by a simple rescaling of coordinates of S.

□

Consider now any bounded operator $\mathcal{K}$ in the commutant of the representation $\pi_{i\lambda,\delta}$. Restricting the representation to MN, it follows from Lemma 7.2 that the operator $\mathcal{F}_1\,\mathcal{K}\,\mathcal{F}_1^{-1}$ is a linear combination, with coefficients depending on t (the variable used in the definition of the partial Fourier transform), of the operators I and $\mathcal{F}_1\,\Sigma_t\,\mathcal{F}_1^{-1}$. Introduce the group A of symplectic transformations of $\mathbb{R}^{2n}$ defined as

$$g_a\colon (x,\,\xi) \mapsto (ax,\,a^{-1}\xi)\,,\qquad a > 0\,. \tag{7.8}$$

From (2.8), one has

$$(\pi_{i\lambda,\delta}(g_a)\,f)(x_1, x_*\,;\,1,\xi_*) = a^{-n-i\lambda}\,f(a^{-2}x_1,\,a^{-2}x_*\,;\,1,\xi_*)\,. \tag{7.9}$$

Then, the operator $\mathcal{K}$ must also commute with the Euler operator $\sum_{j\geq 1} x_j\,\frac{\partial}{\partial x_j}$, and the operator $\mathcal{F}_1\,\mathcal{K}\,\mathcal{F}_1^{-1}$ must commute with the operator $-t\,\frac{\partial}{\partial t}+\sum_{j\geq 2} x_j\,\frac{\partial}{\partial x_j}$: after a change of variables in (7.5), it follows that the above-referred coefficients depend only on $\operatorname{sign} t$.

Theorem 7.3. *Given any $n \geq 1$, and any pair $(i\lambda,\,\delta) \in i\mathbb{R} \times \{0,\,1\}$ such that $(i\lambda,\,\delta) \neq (0,\,1)$ and $(i\lambda,\,\delta) \neq (0,\,0)$, the representation $\pi_{i\lambda,\delta}$ is irreducible; if $(i\lambda,\,\delta) = (0,\,1)$, it decomposes as the direct sum of two irreducible representations, and such is the case if $(i\lambda,\,\delta) = (0,\,0)$ and $n \geq 2$.*

Proof. We may assume that $n \geq 2$, since the one-dimensional case is classical [2]. From the considerations that precede in this section, any operator commuting with the representation $\pi_{i\lambda,\delta}$ must lie in the algebra generated by the following two involutions: (i) the transformation Σ defined by

$$(\mathcal{F}_1\,\Sigma\,f)(t,\,X_{**}) = |t|^{n-1}\int_{\mathbb{R}^{2n-2}} e^{-2i\pi t\,[X_{**},\,Y_{**}]}\,(\mathcal{F}_1\,f)(t,\,Y_{**})\,dY_{**}\,; \tag{7.10}$$

(ii) the transformation $\Psi = \operatorname{sign}\!\left(\frac{1}{2i\pi}\,\frac{\partial}{\partial x_1}\right)$ defined by

$$(\mathcal{F}_1\,(\Psi f))(t,\,X_{**}) = (\operatorname{sign} t)\,(\mathcal{F}_1\,f)(t,\,X_{**})\,. \tag{7.11}$$

Looking at (2.26), one may note that $\Sigma = \theta_{0,\,0}$ and that the composition $\Sigma\,\Psi = \Psi\,\Sigma$ coincides with the intertwining operator $\theta_{0,\,1}$. Now, $\theta_{0,1}$ is a non-trivial (*i.e.*, distinct from a scalar) intertwining operator of the representation $\pi_{0,1}$ with itself, and $\theta_{0,0}$ is an intertwining operator of the representation $\pi_{0,0}$ with itself, non-trivial as soon as $n \geq 2$.

What remains to be seen, fixing $n \geq 2$, is that the operator $\theta_{0,\,1}$ cannot commute with the representation $\pi_{i\lambda,\delta}$ unless $(i\lambda, \delta) = (0,\,1)$ and that the operator $\theta_{0,\,0}$ cannot commute with the representation $\pi_{i\lambda,\delta}$ unless $(i\lambda, \delta) = (0,\,0)$, finally that Ψ can never (if $n \geq 2$) commute with a representation $\pi_{i\lambda,\delta}$. Given $(i\lambda, \delta)$, set

$$\Theta_j = \theta_{i\lambda,\delta}\,\theta_{0,\,j} \tag{7.12}$$

so that, from (2.27),

$$(\mathcal{F}_1\,\Theta_j\,f)(t,\,X_{**}) = |t|_{j-\delta}^{-i\lambda}\,(\mathcal{F}_1 f)(t,\,X_{**})\,. \tag{7.13}$$

If $\theta_{0,j}$ happens to be an intertwining operator from the representation $\pi_{i\lambda,\delta}$ to itself, the operator Θ_j is an intertwining operator from $\pi_{i\lambda,\delta}$ to $\pi_{-i\lambda,\delta}$. This operator, in its realization on $L^2(\mathcal{M}_0)$, has an integral kernel which, evaluated at some pair $((x_1,\,X_{**}),\,(y_1,\,Y_{**}))$, is the product of some distribution in $x_1 - y_1$ by $\delta(X_{**} - Y_{**})$: as $n \geq 2$, it is obvious that such an integral kernel, unless it is that of a scalar operator, cannot satisfy the covariance property that would make it an intertwining operator between two representations of the species under consideration. The same applies to the operator Ψ.

□

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(TK) Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo, 153-8914 Japan,

(BØ) Matematisk Institut, Byg. 430, Ny Munkegade, 8000 Aarhus C, Denmark,

(MP, AU) Laboratoire de Mathématiques, CNRS FRE 3111, Université de Reims, Moulin de la Housse, B.P.1039, F-51687 Reims Cedex 2, France.

E-mail address: toshi@ms.u-tokyo.ac.jp, orsted@imf.au.dk, pevzner@univ-reims.fr, andre.unterberger@univ-reims.fr