博士論文

論文題目

Visible actions of reductive algebraic groups on complex algebraic varieties
（簡約代数群の複素代数多様体への可視的作用について）

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Dedicated to my grandfathers Kesao Tanaka and Morio Saeki, grandmother Iku Saeki and friend Yohei Sato with wishes for good health of my grandmother Setsuko Tanaka and greatuncle Shin-nosuke Saeki.

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Chapter 1

Introduction

We study visible actions on complex algebraic varieties, and the main result is a classification of visible actions on generalized flag varieties.

Definition 1.0.1 (Kobayashi [Ko2])**.** We say a holomorphic action of a Lie group G on a complex manifold X is strongly visible if the following two conditions are satisfied:

1. There exists a real submanifold S (called a "slice") such that

$$X' \ := \ G \cdot S \text{ is an open subset of } X.$$

2. There exists an anti-holomorphic diffeomorphism σ of X' such that

$$\begin{aligned} \sigma|_S &= \mathrm{id}_S, \\ \sigma(G \cdot x) &= G \cdot x \text{ for any } x \in X'. \end{aligned}$$

In the above setting, we say the action of G on X is S-visible. This terminology will be used also if S is just a subset of X.

Definition 1.0.2 (Kobayashi [Ko2])**.** We say a holomorphic action of a Lie group G on a complex manifold X is previsible if the condition (1) of Definition 1.0.1 is satisfied for a totally real submanifold S of X.

The notion of visible actions on complex manifolds was introduced by T. Kobayashi [Ko2] with the aim of uniform treatment of multiplicity-free representations of Lie groups.

Definition 1.0.3. We say a unitary representation V of a locally compact group G is multiplicity-free if the ring $\mathrm{End}_G(V)$ of intertwining operators on V is commutative.

There are various kinds of multiplicity-free representations (c.f. [BR, HU, Ka, VK]), and for the proof of the multiplicity-freeness property of representations, typical approaches are the following: verifying the existence of an open orbit of a Borel subgroup; using a combinatorial method (computing or estimating coefficients of the character of a representation). These two approaches work very well for (the direct sum of) finite dimensional representations, but it would be hard to apply them to the infinite dimensional representations with continuous spectra. A new approach has been introduced by Kobayashi, namely, the propagation theorem of the multiplicity-freeness property under visible actions:

Fact 1.0.4 (Kobayashi [Ko3]). *Let G be a Lie group and $\mathcal{W}$ a G-equivariant Hermitian holomorphic vector bundle on a connected complex manifold X. Let V be a unitary representation of G. If the following conditions from* (0) *to* (3) *are satisfied, then V is multiplicity-free as a representation of G.*

(0) *There exists a continuous and injective G-intertwining operator from V to the space $\mathcal{O}(X, \mathcal{W})$ of holomorphic sections of $\mathcal{W}$.*

(1) *The action of G on X is S-visible. That is, there exist a subset $S \subset X$ and an anti-holomorphic diffeomorphism σ of X' satisfying the conditions given in Definition 1.0.1. Further, there exists an automorphism $\hat{\sigma}$ of G such that $\sigma(g \cdot x) = \hat{\sigma}(g) \cdot \sigma(x)$ for any $g \in G$ and $x \in X'$.*

(2) *For any $x \in S$, the fiber $\mathcal{W}_x$ at x decomposes as the multiplicity-free sum of irreducible unitary representations of the isotropy subgroup G_x. Let $\mathcal{W}_x = \bigoplus_{1 \leq i \leq n(x)} \mathcal{W}_x^{(i)}$ denote the irreducible decomposition of $\mathcal{W}_x$.*

(3) *σ lifts to an anti-holomorphic automorphism $\tilde{\sigma}$ of $\mathcal{W}$ and satisfies $\tilde{\sigma}(\mathcal{W}_x^{(i)}) = \mathcal{W}_x^{(i)}$ for each $x \in S$ $(1 \leq i \leq n(x))$.*

The advantage of this new approach is that not only finite dimensional cases but also infinite dimensional (both discrete and continuous spectra) cases can be applied by this method. Indeed, we can see in the statement of the above theorem that we do not need to assume

G is compact, reductive,

V is of finite-dimensional, discretely decomposable, or

X is compact.

In the following, we quote a few examples of applications of Fact 1.0.4 from [Ko2]. The first example is an infinite dimensional unitary representation with only continuous spectrum.

Example 1.0.5. Let G be a semisimple Lie group and K a maximal compact subgroup of G. Then it is well-known that the space $L^2(G/K)$ of square integrable functions on the Riemannian symmetric space G/K is multiplicity-free (see [Wo] for example). We can also prove the multiplicity-freeness property by combining Fact 1.0.4 with the following facts.

- The G-action on the complexification $G_{\mathbb{C}}/K_{\mathbb{C}}$ is strongly visible by Kobayashi [Ko2].
- Let U be the complex crown of G/K, which was introduced by Akhiezer and Gindikin [AG]. Then there exists a G-embedding $L^2(G/K) \hookrightarrow \mathcal{O}(U)$ by Krötz and Stanton [KS].

Next, we give an example of a multiplicity-free representation arising from a visible action of a semisimple Lie group on a Hermitian symmetric space.

Example 1.0.6. Let G be a simple Lie group of Hermitian type, K a maximal compact subgroup and H a symmetric subgroup of G, i.e., H is an open subgroup of the τ-fixed points subgroup G^τ for an involution τ of G. Let π be a unitary highest weight representation of the scalar type of G. Then the restriction of π to H is multiplicity-free [Ko5] by Fact 1.0.4 combined with the following facts.

- π can be realized in the space $\mathcal{O}(G/K, \mathcal{L})$ of holomorphic sections of a G-equivariant holomorphic line bundle $\mathcal{L}$ on the Hermitian symmetric space G/K.
- The H-action on G/K is strongly visible by Kobayashi [Ko5] by the Cartan decomposition $G = HAK$ in the symmetric setting (see Flensted-Jensen [Fl1], Hoogenboom [Ho] and Matsuki [Ma1, Ma2]).

As the last example, we show a multiplicity-free representation of a non-reductive Lie group.

Example 1.0.7. Let G, K and π as in Example 1.0.6. Let N be a maximal unipotent subgroup of G. Then the restriction of π to N is multiplicity-free by Fact 1.0.4 combined with the facts that π can be realized in $\mathcal{O}(G/K, \mathcal{L})$ for a G-equivariant holomorphic line bundle $\mathcal{L}$ on G/K, and that the action of N on G/K is strongly visible by Kobayashi [Ko2] by the Iwasawa decomposition $G = NAK$.

As these examples show, we can obtain multiplicity-free representations from a visible action of a Lie group. Therefore it would be natural to try to find, or even classify, visible actions. In the following, we exhibit preceding results on a classification problem of visible actions. We firstly state a result on visible actions on symmetric spaces.

Fact 1.0.8 (Kobayashi [Ko5])**.** *Let (G, K) be a Hermitian symmetric pair and (G, H) a symmetric pair. Then H acts on the Hermitian symmetric space G/K strongly visibly.*

The next result concerns the visibility of linear actions. Let $G_{\mathbb{C}}$ be a connected complex reductive algebraic group and V a finite-dimensional representation of $G_{\mathbb{C}}$.

Definition 1.0.9. We say V is a linear multiplicity-free space of $G_{\mathbb{C}}$ if the space $\mathbb{C}[V]$ of polynomials on V is multiplicity-free as a representation of $G_{\mathbb{C}}$.

Fact 1.0.10 (Sasaki [Sa1, Sa4])**.** *Let V be a linear multiplicity-free space of $G_{\mathbb{C}}$. Then a compact real form U of $G_{\mathbb{C}}$ acts on V strongly visibly.*

Remark 1.0.11. We note that if U acts on a representation V of $G_{\mathbb{C}}$ strongly visibly, then V is a linear multiplicity-free space of $G_{\mathbb{C}}$ by Fact 1.0.4.

A linear multiplicity-free space is a special case of smooth affine spherical varieties. Let $G_{\mathbb{C}}$ be a connected complex reductive algebraic group and X a connected complex algebraic $G_{\mathbb{C}}$-variety.

Definition 1.0.12. We say X is a spherical variety of $G_{\mathbb{C}}$ if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit on X.

A typical example of spherical varieties is a complex symmetric space (e.g. $G_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$ and $X = \mathrm{GL}(n, \mathbb{C})/(\mathrm{GL}(m, \mathbb{C}) \times \mathrm{GL}(n-m, \mathbb{C}))$). The third result deals with visible actions on affine homogeneous spherical varieties.

Fact 1.0.13 (Sasaki [Sa2, Sa3, Sa5])**.** *Let $G_{\mathbb{C}}/H_{\mathbb{C}}$ be one of the following affine homogeneous spherical varieties:*

$$\begin{aligned}
&\mathrm{SL}(m+n, \mathbb{C})/(\mathrm{SL}(m, \mathbb{C}) \times \mathrm{SL}(n, \mathbb{C}))\ (m \neq n),\\
&\mathrm{Spin}(4n+2, \mathbb{C})/\mathrm{SL}(2n+1, \mathbb{C}),\\
&\mathrm{SL}(2n+1, \mathbb{C})/\mathrm{Sp}(n, \mathbb{C}),\\
&\mathrm{E}_6(\mathbb{C})/\mathrm{Spin}(10, \mathbb{C}),\\
&\mathrm{SO}(8, \mathbb{C})/\mathrm{G}_2(\mathbb{C}).
\end{aligned}$$

Then the action of a compact real form U of $G_{\mathbb{C}}$ on $G_{\mathbb{C}}/H_{\mathbb{C}}$ is strongly visible.

Lastly we state a classification result on visible actions on generalized flag varieties of type A, which is the prototype of the main result of this paper. Let $G = \mathrm{U}(n)$ and L, H Levi subgroups of G. Kobayashi [Ko4] classified the triple (G, H, L) such that the following actions are strongly visible (we denote by $\Delta(G)$ the diagonal subgroup of $G \times G$).

$$L \curvearrowright G/H, \ H \curvearrowright G/L, \ \Delta(G) \curvearrowright (G \times G)/(H \times L).$$

In fact, all the above three actions are strongly visible if and only if at least one of those is strongly visible [Ko2]. The visibility of the three actions on generalized flag varieties was proved by giving a generalized Cartan decomposition:

Definition 1.0.14. Let G be a connected compact Lie group, T a maximal torus and H, L Levi subgroups of G, which contain T. We take a Chevalley–Weyl involution σ of G with respect to T. If the multiplication mapping

$$L \times B \times H \to G$$

is surjective for a subset B of the σ-fixed points subgroup G^σ, then we say the decomposition $G = LBH$ is a generalized Cartan decomposition.

Definition 1.0.15. An involution σ of a compact Lie group G is said to be a Chevalley–Weyl involution if there exists a maximal torus T of G such that $\sigma(t) = t^{-1}$ for any $t \in T$.

The definition of a generalized Cartan decomposition comes from that of a visible action. Let us explain. We retain the setting of Definition 1.0.14. Suppose that $G = LBH$ holds for some $B \subset G^\sigma$. Since σ acts on generalized flag varieties

$$G/H, \qquad G/L, \qquad (G \times G)/(H \times L)$$

as anti-holomorphic diffeomorphisms, we can obtain three strongly visible actions.

$$L \curvearrowright G/H, \quad H \curvearrowright G/L, \quad \Delta(G) \curvearrowright (G \times G)/(H \times L).$$

Furthermore, we can obtain three multiplicity-free theorems by using Fact 1.0.4.

$$\mathrm{ind}_H^G \chi_H|_L, \quad \mathrm{ind}_L^G \chi_L|_H, \quad \mathrm{ind}_H^G \chi_H \otimes \mathrm{ind}_L^G \chi_L.$$

Here $\mathrm{ind}_H^G \chi_H$ and $\mathrm{ind}_L^G \chi_L$ denote the holomorphically induced representations from unitary characters χ_H and χ_L of H and L, respectively. As we saw, one generalized Cartan decomposition leads us to three strongly visible actions, and three multiplicity-free theorems (Kobayashi's triunity principle [Ko1]).

As the name indicates, the decomposition $G = LBH$ can be regarded as a generalization of the Cartan decomposition. Under the assumption that both (G, H) and (G, L) are symmetric pairs, the decomposition theorem of the form $G = LBH$ or its variants has been well-established: $G = KAK$ with K compact by É. Cartan, $G = KAH$ with G, H non-compact and K compact by Flensted-Jensen [Fl1], $G = KAH$ with G compact by Hoogenboom [Ho], and the double coset decomposition $L\backslash G/H$ by Matsuki [Ma1, Ma2]. We note that in our setting the subgroups L and H of G are not necessarily symmetric.

1.1 Main result 1: Classification of visible triples

The theorem below gives a classification of generalized Cartan decompositions (Definition 1.0.14).

Theorem 1.1.1 ([Ta2, Ta3, Ta4, Ta5])**.** *Let G be a connected compact simple Lie group, T a maximal torus, Π a simple system and L_1, L_2 Levi subgroups of G, whose simple systems are given by proper subsets Π_1, Π_2 of Π. Let σ be a Chevalley–Weyl involution of G with respect to T. Then the triples (G, L_1, L_2) listed below exhaust all the triples such that the multiplication mapping*

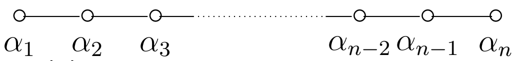
$$L_1 \times B \times L_2 \to G$$

is surjective for a subset B of G^σ.

Remark 1.1.2. For the type A simple Lie groups (or $G = \mathrm{U}(n)$), this theorem was proved by Kobayashi [Ko4].

In the following, we specify only the types of simple Lie groups G since our classification is independent of coverings, and list pairs (Π_1, Π_2) of proper subsets of Π instead of pairs (L_1, L_2) of Levi subgroups of G. Also, we put $(\Pi_j)^c := \Pi \setminus \Pi_j$ $(j = 1, 2)$.

Classification for type $\mathbf{A}_n$ [Ko4]



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j\}$.

Non-Hermitian type:

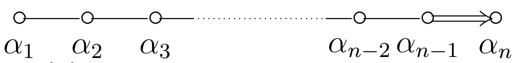
I. $(\Pi_1)^c = \{\alpha_i, \alpha_j\}$, $(\Pi_2)^c = \{\alpha_k\}$, $\min\limits_{p=i,j}\{p, n+1-p\} = 1$ or $i = j \pm 1$.

II. $(\Pi_1)^c = \{\alpha_i, \alpha_j\}$, $(\Pi_2)^c = \{\alpha_k\}$, $\min\{k, n+1-k\} = 2$.

III. $(\Pi_1)^c = \{\alpha_l\}$, Π_2: arbitrary, $l = 1$ or n.

Here i, j, k satisfy $1 \leq i, j, k \leq n$.

Classification for type $\mathbf{B}_n$



Hermitian type:

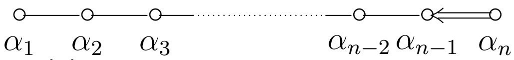
I. $(\Pi_1)^c = \{\alpha_1\}$, $(\Pi_2)^c = \{\alpha_1\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_n\}$, $(\Pi_2)^c = \{\alpha_n\}$.

II. $(\Pi_1)^c = \{\alpha_1\}$, $(\Pi_2)^c = \{\alpha_i\}$, $2 \leq i \leq n$.

Classification for type $\mathbf{C}_n$



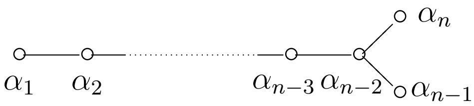
Hermitian type:

I. $(\Pi_1)^c = \{\alpha_n\}$, $(\Pi_2)^c = \{\alpha_n\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_1\}$, $(\Pi_2)^c = \{\alpha_i\}$, $1 \leq i \leq n$.

Classification for type $\mathbf{D}_n$



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j\}$, $i, j \in \{1, n-1, n\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_1\}$, $(\Pi_2)^c = \{\alpha_j\}$, $j \neq 1, n-1, n$.

II. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j\}$, $i \in \{n-1, n\},\ j \in \{2, 3\}$.

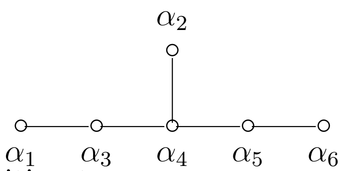
III. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j, \alpha_k\}$, $i \in \{n-1, n\},\ j, k \in \{1, n-1, n\}$.

IV. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_1, \alpha_2\}$, $i \in \{n-1, n\}$.

V. $(\Pi_1)^c = \{\alpha_1\}$, $(\Pi_2)^c = \{\alpha_j, \alpha_k\}$, j or $k \in \{n-1, n\}$.

VI. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_2, \alpha_j\}$, $n = 4,\ (i, j) = (3, 4)$ or $(4, 3)$.

Classification for type $\mathbf{E}_6$



Hermitian type:

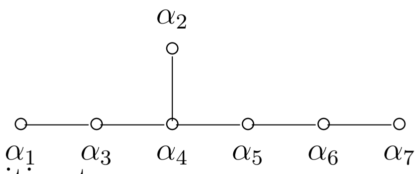
I. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j\}$, $i, j \in \{1, 6\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_1, \alpha_6\}$, $i = 1$ or 6.

II. $(\Pi_1)^c = \{\alpha_i\}$, $(\Pi_2)^c = \{\alpha_j\}$, $i = 1$ or $6,\ j \neq 1, 4, 6$.

Classification for type $\mathbf{E}_7$



Hermitian type:

I. $(\Pi_1)^c = \{\alpha_7\}$, $(\Pi_2)^c = \{\alpha_7\}$.

Non-Hermitian type:

I. $(\Pi_1)^c = \{\alpha_7\}$, $(\Pi_2)^c = \{\alpha_i\}$, $i = 1$ or 2.

Classification for type $\mathbf{E}_8$, $\mathbf{F}_4$, $\mathbf{G}_2$

There is no pair (Π_1, Π_2) of proper subsets of Π such that $G = L_1 G^\sigma L_2$ holds.

For the proof of sufficiency of Theorem 1.1.1, we use the herringbone stitch method introduced by Kobayashi [Ko4], which reduces unknown decompositions to the known decomposition in the symmetric case. This method enables us to obtain a generalized Cartan decomposition $G = L_1 B L_2$ with $B \subset G^\sigma$ (Definition 1.0.14). For the proof of necessity in the classical case, we prove that $G \neq L_1 G^\sigma L_2$ for any pair (Π_1, Π_2) which is not in the list in Theorem 1.1.1 by using invariant theory for quivers associated to Levi subgroups. For the proof in the exceptional case, we use Fact 1.0.4 and Stembridge's classification of multiplicity-free tensor product representations ([St2]).

1.2 Main result 2: Classification of visible actions on generalized flag varieties

As we explained before, one generalized Cartan decomposition (Definition 1.0.14) leads us to three strongly visible actions. The following corollary shows that the converse is also true in our setting. Therefore we can obtain a classification of visible actions on generalized flag varieties from Theorem 1.1.1.

Corollary 1.2.1 ([Ta1]). *We retain the setting of Theorem 1.1.1. We denote by $G_{\mathbb{C}}$ and $(L_j)_{\mathbb{C}}$ the complexifications of G and L_j, respectively ($j = 1, 2$). We let P_j be a parabolic subgroup of $G_{\mathbb{C}}$ with Levi subgroup $(L_j)_{\mathbb{C}}$, and put $\mathcal{P}_j = G_{\mathbb{C}}/P_j$ ($j = 1, 2$). Then the following eleven conditions are equivalent.*

(i) The multiplication mapping $L_1 \times G^\sigma \times L_2 \to G$ is surjective.

(ii) The natural action $L_1 \curvearrowright \mathcal{P}_2$ is strongly visible.

(iii) The natural action $L_2 \curvearrowright \mathcal{P}_1$ is strongly visible.

(iv) The diagonal action $\Delta(G) \curvearrowright \mathcal{P}_1 \times \mathcal{P}_2$ is strongly visible.

(v) Any irreducible representation of G, which belongs to $\mathcal{P}_2$-series is multiplicity-free when restricted to L_1.

(vi) Any irreducible representation of G, which belongs to $\mathcal{P}_1$-series is multiplicity-free when restricted to L_2.

(vii) The tensor product of arbitrary two irreducible representations π_1 and π_2 of G, which belong to $\mathcal{P}_1$ and $\mathcal{P}_2$-series, respectively, is multiplicity-free.

(viii) $\mathcal{P}_2$ is a spherical variety of $(L_1)_{\mathbb{C}}$.

(ix) $\mathcal{P}_1$ is a spherical variety of $(L_2)_{\mathbb{C}}$.

(x) $\mathcal{P}_1 \times \mathcal{P}_2$ is a spherical variety of $\Delta(G_{\mathbb{C}})$.

(xi) The pair (Π_1, Π_2) is one of the entries listed in Theorem 1.1.1 up to switch of the factors.

Here an irreducible representation of G is in $\mathcal{P}_j$-series if it is a holomorphically induced representation from a unitary character of the Levi subgroup L_j $(j = 1, 2)$.

Proof. [*] We prove that Theorem 1.1.1 implies this corollary. The strategy of the proof is summarized in the below diagram.

$$
\begin{array}{ccccccl}
 & & (\text{vii}) & & & & \cdots\cdots\ \text{multiplicity-free}\\
 & & \Updownarrow & & & & \\
 & & (\text{xi}) & & & & \cdots\cdots\ \text{classification of } (L_1, L_2)\\
 & & \Updownarrow & & & & \\
 & & (\text{i}) & & & & \cdots\cdots\ \text{Cartan decomposition}\\
 & \swarrow & \Downarrow & \searrow & & & \\
(\text{ii}) & & (\text{iv}) & & (\text{iii}) & & \cdots\cdots\ \text{visible action}\\
\Downarrow & & \Downarrow & & \Downarrow & & \\
(\text{v}) & \Leftrightarrow & (\text{vii}) & \Leftrightarrow & (\text{vi}) & & \cdots\cdots\ \text{multiplicity-free}\\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \\
(\text{viii}) & & (\text{x}) & & (\text{ix}) & & \cdots\cdots\ \text{spherical action.}
\end{array}
$$

The implication (vii) $\Rightarrow$ (xi) can be verified by comparing Stembridge's classification [St2] with Theorem 1.1.1. The converse implication (xi) $\Rightarrow$ (vii) follows from Fact 1.0.4. The equivalence (xi) $\Leftrightarrow$ (i) is Theorem 1.1.1. The implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) are the triunity of visibility ([Ko1]). Each of the three implications (ii) $\Rightarrow$ (v), (iii) $\Rightarrow$ (vi) and (iv) $\Rightarrow$ (vii) is followed by Fact 1.0.4. As in the proof of [Ko2, Corollary 15], we see that a result of Vinberg and Kimel'fel'd [VK, Corollary 1] implies the three equivalences (v) $\Leftrightarrow$ (viii), (vi) $\Leftrightarrow$ (ix) and (vii) $\Leftrightarrow$ (x). The equivalence (v) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (vi) on the multiplicity-freeness property of representations follows from a result of Stembridge [St2, Corollary 2.5]. This completes the proof of the corollary. □

Remark 1.2.2. For the type A simple Lie groups (or $G = \mathrm{U}(n)$), this corollary was proved by Kobayashi [Ko2].

Remark 1.2.3. Littelmann [Li] classified for any simple algebraic group G over any algebraically closed field of characteristic zero, all the pairs of maximal parabolic subgroups P_ω and $P_{\omega'}$ corresponding to fundamental weights ω and ω', respectively, such that the

[*] This proof for Corollary 1.2.1 is quoted from [Ta1].

tensor product representation $V_{n\omega} \otimes V_{m\omega'}$ decomposes multiplicity-freely for arbitrary non-negative integers n and m. Moreover, he found the branching rules of $V_{n\omega} \otimes V_{m\omega'}$ and the restriction of $V_{n\omega}$ to the maximal Levi subgroup $L_{\omega'}$ of $P_{\omega'}$ for any pair (ω, ω') that admits a $\Delta(G)$-spherical action on $G/P_\omega \times G/P_{\omega'}$.

Remark 1.2.4. Stembridge [St2] gave a complete list of a pair (μ, ν) of highest weights such that the corresponding tensor product representation $V_\mu \otimes V_\nu$ is multiplicity-free for any complex simple Lie algebra $\mathfrak{g}$. His method was combinatorial and not based on the Borel–Weil realization of irreducible representations. He also classified all the pairs $(\mu, \mathfrak{l})$ of highest weights μ and Levi subalgebras $\mathfrak{l}$ of $\mathfrak{g}$ with the restrictions $V_\mu|_{\mathfrak{l}}$ multiplicity-free.

1.3 Main result 3: Seeds and visible actions for the orthogonal group

As we mentioned in Remark 1.2.4, finite dimensional multiplicity-free tensor product representations were classified by Stembridge [St2]. By using the notion of visible actions on complex manifolds, we would be able to, and indeed can in the types A, B and D cases, understand his classification more deeply. By Fact 1.0.4, we can reduce complicated multiplicity-free theorems to a pair of data:

visible actions on complex manifolds, and

much simpler multiplicity-free representations (*seeds* of multiplicity-free representations).

For the type A simple Lie groups, Kobayashi found the following *seeds* of multiplicity-free representations that combined with visible actions can produce all the cases of the pair of two representations (V_1, V_2) of $\mathrm{U}(n)$ such that $V_1 \otimes V_2$ is multiplicity-free [Ko1].

- One-dimensional representations.
- $(\mathrm{U}(n) \downarrow \mathbb{T}^n)$ The restriction of an alternating tensor product representation $\Lambda^k(\mathbb{C}^n)$.
- $(\mathrm{U}(n) \downarrow \mathbb{T}^n)$ The restriction of a symmetric tensor product representation $S^k(\mathbb{C}^n)$.
- $(\mathrm{U}(n) \downarrow \mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{U}(n_3))$ The restriction of an irreducible representation $V_{2\omega_k}$ $(n = n_1 + n_2 + n_3)$.

Here V_λ denotes an irreducible representation of $\mathrm{U}(n)$ with highest weight λ and $\{\omega_k\}_{1\le k\le n-1}$ is the set of fundamental weights of $\mathrm{U}(n)$. On the other hand, he classified in [Ko4] visible actions on generalized flag varieties of type A as listed in Theorem 1.1.1. By combining the above *seeds* of multiplicity-free representations with the visible actions and using his triunity principle, Kobayashi constructed all the multiplicity-free tensor product representations of $\mathrm{U}(n)$ [Ko1]. In this paper we construct all the multiplicity-free tensor product representations for $\mathrm{SO}(N)$ and its covering group $\mathrm{Spin}(N)$ by following Kobayashi's argument for $\mathrm{U}(n)$. In our case, visible actions come from triples (G, L_1, L_2) for $G = \mathrm{Spin}(N)$ listed in Theorem 1.1.1 as in the case of the type A groups. On the other hand, seeds of multiplicity-free tensor product representations arise only from one-dimensional representations, alternating tensor product representations and spin representations. These are exhibited in Proposition 1.3.1. We can see how to combine those visible actions and seeds to obtain multiplicity-free tensor product representations in Theorem 1.3.2.

We denote by $\Pi = \{\alpha_i\}_{1\leq i\leq [N/2]}$ (see Theorem 1.1.1 for the labeling of the Dynkin diagrams) a simple system of the root system of $G = \mathrm{Spin}(N)$ with respect to its maximal torus T, and by $\{H_i\}_{1\leq i\leq [N/2]}$ the dual basis of Π. We define a subgroup M of $\mathrm{Spin}(2n+1)$ as follows.

$$M := \left\{\exp\left(\sqrt{-1}m\pi H_1\right)\right\}_{1\leq m\leq 4} \cdot \mathrm{Spin}(2n-1), \tag{1.3.1}$$

where exp denotes the exponential mapping, and the simple system of $\mathrm{Spin}(2n-1)$ is given by $\{\alpha_k \in \Pi : 2 \leq k \leq n\}$.

Proposition 1.3.1. *We denote by* $\mathbf{1}$, $\mathbb{C}^N$ *and* Spin_N *for the one-dimensional trivial representation, the natural representation and the spin representation of* $\mathrm{Spin}(N)$, *respectively. Then the following hold.*

(1) *One-dimensional representations are multiplicity-free.*

(2) $\mathbf{1}$, $\mathbb{C}^N$ *and* Spin_N *are multiplicity-free as representations of a maximal torus* T *of* $\mathrm{Spin}(N)$.

(3) $\Lambda^i(\mathbb{C}^N)$ *is multiplicity-free as a representation of a maximal Levi subgroup* $\mathrm{U}(j) \times \mathrm{SO}(N-2j)$ *of* $\mathrm{SO}(N)$ (*when* N *is even and* $i = N/2$, *we replace* $\Lambda^{N/2}(\mathbb{C}^N)$ *by its* $\mathrm{SO}(N)$*-irreducible constituent whose highest weight is* $2\omega_{N/2-1}$ *or* $2\omega_{N/2}$) *if the following condition* (3-1) *or* (3-2) *is satisfied* ($1 \leq i, j \leq [N/2]$).

(3-1) N *is odd.*

(3-2) N *is even, and* i, j *satisfy* $i + j \leq N/2$, $j = N/2$ *or* $i = N/2$.

(4) Spin_N *is multiplicity-free as a representation of* M, *where* N *is odd and* M *as in* (1.3.1).

The theorem below gives a geometric construction of all the multiplicity-free tensor product representations for the orthogonal group. For a realization of irreducible representations of a compact Lie group, we use the Borel–Weil theory. Namely, we realize an irreducible representation of a compact Lie group G as the space $\mathcal{O}(G/L, \mathcal{W})$ of holomorphic sections of a vector bundle $\mathcal{W}$ on a generalized flag variety G/L, which is associated to an irreducible representation W of a Levi subgroup L of G.

Theorem 1.3.2. *We let* $G = \mathrm{Spin}(N)$. *For any two irreducible representations* V_{λ_1} *and* V_{λ_2} *of* G *such that* $V_{\lambda_1} \otimes V_{\lambda_2}$ *is multiplicity-free, there exists a pair of*

- *a generalized flag variety* $(G \times G)/(L_1 \times L_2)$ *with a strongly visible* $\Delta(G)$*-action, and*
- *irreducible representations* (*a seed given in* Proposition 1.3.1) W_1 *and* W_2 *of* L_1 *and* L_2, *respectively,*

such that $V_{\lambda_k} \simeq \mathcal{O}(G/L_k, \mathcal{W}_k)$ *as* G*-modules* ($k = 1, 2$).

The correspondence between the data (L_k, W_k) *of visible actions and seeds, and the highest weights* λ_k *of* V_{λ_k} ($k = 1, 2$) *is given as in* Tables 1.3.1–1.3.4 *below. In the tables,* $\mathbb{C}_\lambda$ *denotes a one-dimensional representation with weight* λ, T *a maximal torus of* G *and* L_λ *a Levi subgroup of* G, *whose simple system is given by* $\{\alpha_l \in \Pi : \langle \lambda, \check{\alpha}_l\rangle = 0\}$ *where* $\check{\alpha}_l$ *is the coroot of* α_l ($1 \leq l \leq [N/2]$).

Table 1.3.1: Line bundle type

L_1	L_2	W_1	W_2	N	λ_1	λ_2
L_{λ_1}	L_{λ_2}	$\mathbb{C}_{\lambda_1}$	$\mathbb{C}_{\lambda_2}$	$2n+1$	$s\omega_1$	$t\omega_j$
					$s\omega_n$	$t\omega_n$
				$2n$	$s\omega_1$	$t\omega_j + u\omega_{n-\delta}$
					$s\omega_{n-\delta}$	$t\omega_3$, $t\omega_1 + u\omega_2$, $t\omega_1 + u\omega_{n-\delta'}$ or $t\omega_{n-1} + u\omega_n$
				8	$s\omega_{5-\epsilon}$	$t\omega_2 + u\omega_{2+\epsilon}$

$1 \leq j \leq n$, $s, t, u \in \mathbb{N}$, $\delta = 0$ or 1, $\delta' = 0$ or 1 and $\epsilon = 1$ or 2.

Table 1.3.2: Weight multiplicity-free type

L_1	L_2	W_1	W_2	N	λ_1	λ_2
G	T	V_{λ_1}	$\mathbb{C}_{\lambda_2}$	$2n+1$	0, ω_1 or ω_n	arbitrary
				$2n$	0, ω_1, ω_{n-1} or ω_n	arbitrary

Table 1.3.3: Alternating tensor product type

L_1	L_2	W_1	W_2	N	λ_1	λ_2	Condition
G	L_{λ_2}	V_{λ_1}	$\mathbb{C}_{\lambda_2}$	$2n+1$	ω_i or $2\omega_n$	$t\omega_j$	
				$2n$	ω_i	$t\omega_j$	$i + j \leq n$
					ω_i	$t\omega_{n-\delta}$	
					$2\omega_{n-\delta}$	$t\omega_j$	

$1 \leq i, j \leq n$, $t \in \mathbb{N}$ and $\delta = 0$ or 1.

Table 1.3.4: Spin type

L_1	L_2	W_1	W_2	N	λ_1	λ_2
L_{λ_1}	L_{ω_j}	$\mathbb{C}_{\lambda_1}$	$\mathbb{C}_{(1/2+t)\omega_j} \boxtimes \mathrm{Spin}_{N-2j}$	$2n+1$	$s\omega_1$	$\omega_n + t\omega_j$

$1 \leq j \leq n-1$ and $s, t \in \mathbb{N}$.

By virtue of Fact 1.0.4 and the triunity principle [Ko1], we obtain the following corollary. This corollary was proved by Stembridge [St2] by a combinatorial method.

Corollary 1.3.3. *We retain the notation of Theorem 1.3.2. For the data* $(L_1, L_2, N, \lambda_1, \lambda_2)$ *of each row in* Tables 1.3.1–1.3.4, *the representations* V_{λ_1} *and* V_{λ_2} *of* G *decompose multiplicity-freely when restricted to the subgroups* L_2 *and* L_1 *of* G*, respectively.*

So far we have considered visible actions of Levi subgroups on generalized flag varieties. For a general spherical variety, we have the following result on the visibility of actions of compact Lie groups. Let U be a compact real form of a connected complex reductive algebraic group $G_{\mathbb{C}}$, and X a $G_{\mathbb{C}}$-spherical variety. We denote by θ the Cartan involution of $G_{\mathbb{C}}$, which corresponds to U, and by ν a Chevalley–Weyl involution of $G_{\mathbb{C}}$ (i.e., ν is an involution of $G_{\mathbb{C}}$, which satisfies $\nu(t) = t^{-1}$ for any element $t \in T_{\mathbb{C}}$ for some maximal torus $T_{\mathbb{C}}$), which preserves U. We put $\iota = \theta \circ \nu$.

Theorem 1.3.4. *Assume that there exists a real structure μ on a $G_{\mathbb{C}}$-spherical variety X compatible with ι and that the μ-fixed points subset X^{μ} is non-empty. Then a compact real form U acts on X strongly visibly.*

Here by a real structure on a complex manifold Z we mean an anti-holomorphic involution $\eta : Z \to Z$ [Ak, AC]. Also for a real structure η on a complex manifold Z with an action of a group K, we say η is compatible with an automorphism ϕ of K if η satisfies $\eta(kz) = \phi(k)\eta(z)$ for any $k \in K$ and $z \in Z$. Combining Theorem 1.3.4 with Akhiezer's result [Ak] on the existence of compatible real structures on Stein manifolds, we obtain

Corollary 1.3.5. *Let $(G_{\mathbb{C}}, V)$ be a linear multiplicity-free space. Then a compact real form U acts on V strongly visibly.*

Corollary 1.3.6. *Let X be a smooth affine $G_{\mathbb{C}}$-spherical variety. Then a compact real form U acts on X strongly visibly.*

Here a typical example of smooth affine spherical varieties is a complex symmetric space. On the other hand, we have the principal affine space $G_{\mathbb{C}}/N$ (N is a maximal unipotent subgroup) as an example of non-affine smooth spherical varieties. We remark that Corollary 1.3.5 was earlier proved by Sasaki (Fact 1.0.10) by constructing slices explicitly. By combining Theorem 1.3.4 with Akhiezer and Cupit-Foutou's result [AC], we also have

Corollary 1.3.7. *Let X be a $G_{\mathbb{C}}$-wonderful variety. Then a compact real form U acts on X strongly visibly.*

Definition 1.3.8. A $G_{\mathbb{C}}$-variety X is said to be wonderful if

- X is smooth and projective,
- $G_{\mathbb{C}}$ has an open orbit on X, whose complement is a union of finitely many smooth prime divisors X_i $(i \in I)$ with normal crossings, and
- the closure of any $G_{\mathbb{C}}$-orbit on X is given as a partial intersection of X_i $(i \in I)$.

To prove the visibility of actions of non-compact reductive groups on complex manifolds, we use the following extension of a result of Matsuki [Ma1, Ma2]. Let L and H be reductive subgroups of a connected real semisimple algebraic group G such that both $G_{\mathbb{C}}/L_{\mathbb{C}}$ and $G_{\mathbb{C}}/H_{\mathbb{C}}$ are $G_{\mathbb{C}}$-spherical varieties.

Theorem 1.3.9. *There exist finitely many abelian subspaces $\mathfrak{j}_i$ of $\mathfrak{g}$ and elements x_i of G $(i = 1, \ldots, m)$ such that $\bigcup_{i=1}^{m} LC_iH$ contains an open dense subset of G, where $C_i = \exp(\mathfrak{j}_i)x_i$.*

We use this decomposition to show the previsibility of actions of non-compact reductive groups.

Theorem 1.3.10. *Let X be a $G_{\mathbb{C}}$-spherical variety and G a real form of inner type of $G_{\mathbb{C}}$. Then G acts on X previsibly.*

Here a real reductive Lie group is said to be of inner type if its Lie algebra has a compact Cartan subalgebra.

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