INVARIANT MEASURES ON HOMOGENEOUS MANIFOLDS OF REDUCTIVE TYPE

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Abstract. We say that the homogeneous manifold G/H is of reductive type if G is a real reductive linear Lie group and if H is a connected closed subgroup which is reductive in G. Semisimple symmetric spaces (especially, Riemannian symmetric spaces and semisimple group manifolds) and semisimple orbits are of reductive type. In this paper, we give an upper estimate of the invariant measure on the homogeneous manifold G/H of reductive type. Furthermore, we also prove a comparison theorem of the measures of homogeneous submanifolds.

§1. Introduction

1.1. Consider a real reductive linear Lie group G and a maximal compact subgroup K. The Cartan decomposition $G = KAK$ is an analogue of the polar coordinate in a Euclidean space, where $A \simeq \mathbb{R}^k$ with $k := \mathbb{R}\text{-rank}\, G$. Harmonic analysis on a group manifold G and on a Riemannian symmetric space G/K relied very much on the integration formula with respect to the Cartan decomposition $G = KAK$ which was due to Harish-Chandra (e.g. [5], Chapter 1).

1.2. Semisimple symmetric spaces are a wider class of homogeneous manifolds, including both a group manifold and a Riemannian symmetric space. Later on, harmonic analysis on semisimple symmetric spaces G/H has been developed largely in the last two decades (e.g. [2], [14], [3]), in which a generalized Cartan decomposition $G = KBH$ with $B \simeq \mathbb{R}^l$ ($l := \mathbb{R}\text{-rank}\, G/H$) plays also a fundamental role; for example, each joint eigenfunction f of G-invariant differential operators on G/H has a certain asymptotic behavior of exponential growth (or decay) along "Weyl chambers" of $B \simeq \mathbb{R}^l$ if f is K-finite [13], which combined with the integration formula for $G = KBH$ enables us to classify discrete series representations for semisimple symmetric spaces ([1], [12], [2]).

1.3. Now, we consider a more general setting: We shall say that the homogeneous manifold G/H is *of reductive type* if H is a θ-stable closed subgroup of G with at most finite connected components. Semisimple symmetric spaces are of reductive type. Semisimple orbits $G/Z_G(X) \simeq \mathrm{Ad}(G) \cdot X$ are also of reductive type, where X is a semisimple element of the Lie algebra $\mathfrak{g}$ of G. Although homogeneous spaces of

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reductive type naturally arise in various fields such as the geometric construction of unitary representations in the philosophy of the Kirillov-Kostant orbit method and the study of automorphic forms, few results on L^2-harmonic analysis have been known except for semisimple symmetric spaces (see [9] and references therein). One of the difficulties is the lack of a structure theory based on some "root system". For example, there is no analogue of the Cartan decomposition $G = KBH$ (this is easily observed by the fact that $\dim G$ can be larger than $\dim K + \operatorname{rank} G + \dim H$) except for semisimple symmetric spaces (and some other few cases). In particular, we cannot expect an integration formula based on a "Cartan decomposition" for a general homogeneous space of reductive type.

1.4. In this paper, we study the invariant measure on the homogeneous manifold G/H of reductive type and give an explicit upper estimate. This will be useful especially if we want to prove that given functions on G/H belong to $L^p(G/H)$ (see Theorem 3.7). Difficulties arising from the lack of an analogue of the "Cartan decomposition" are overcome by using a natural indefinite-Riemannian metric on a certain principal fiber bundle over G/H instead of the G-invariant measure on G/H itself (see §2). Some inequality in linear algebra will be presented in §4.

Another object of our study is the comparison of the measures of G'/H' and G/H if $G'/H' \subset G/H$ is an embedding of a homogeneous manifold of reductive type (see Theorem 5.6). The idea of the proof is to employ the Riemannian geometry with non-positive sectional curvature. Our comparison theorem is basic in the study of the restriction of irreducible representations of G (realized in the space of sections of equivariant vector bundles over G/H) with respect to G'.

1.5. I have omitted in this paper all applications to representation theory and non-commutative harmonic analysis. One of the applications will be given in a subsequent paper [10] which deals with new discrete series representations for non-symmetric homogeneous manifolds. Here, combined with the knowledge of the restriction of irreducible unitary representations, the comparison theorem of the invariant measure obtained in this paper will be a main tool.

We also note that symmetric assumptions were required in the unitarization of geometric construction of Vogan-Zuckerman's derived functor modules [19] and non-zero harmonic forms associated to modular symbols defined by arithmetic quotients of Riemannian symmetric spaces [15], where it seems desirable to drop the symmetric assumptions. We hope to return these problems in future.

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§2. Invariant measure on a homogeneous manifold of reductive type.

2.1. First we review the notion of invariant connection of a reductive homogeneous manifold (cf. [6]).

Let L be a Lie group with Lie algebra $\mathfrak{l}$ and $\pi\colon P \to M$ a smooth principal L-bundle (on the right). A connection on $P \to M$ is a splitting of the tangent bundle $TP \to P$ into an L-equivariant Whitney sum $TP = Ver(P) \oplus Hor(P)$, where $Ver(P) = \operatorname{Ker}(d\pi\colon TP \to TM)$ is the tangent bundle along fibers, and $Hor(P)$ is so called a horizontal subbundle. The right action of L on P induces a vector field $\widetilde{X}$ on P for each $X \in \mathfrak{l}$ given by $\widetilde{X}_p := \frac{d}{dt}_{|t=0} p \cdot \exp(tX) \in Ver(P)_p \subset TP_p$, which is called the fundamental vector field on P. We note that $\mathfrak{l} \to Ver(P)_p, X \mapsto \widetilde{X}_p$ is a bijection for each $p \in P$. The connection form is the $\mathfrak{l}$-valued 1-form on P, denoted by $\alpha \in \mathcal{E}^1(P, \mathfrak{l})$, which is defined by the composition of

$$TP \to Ver(P),$$

the first projection of the splitting $TP = Ver(P) \oplus Hor(P)$, and

$$Ver(P) \to \mathfrak{l},$$

the inverse of $\mathfrak{l} \xrightarrow{\sim} Ver(P)_p$, $X \mapsto \widetilde{X}_p$.

2.2. In the setting of §2.1, we suppose that M is equipped with a pseudo-Riemannian metric g and that L is a real reductive Lie group. There is a non-degenerate symmetric $\operatorname{Ad}(L)$-invariant bilinear form $B_{\mathfrak{l}}$ on $\mathfrak{l}$. Then we can define naturally a pseudo-Riemannian metric $G \equiv G(\alpha, B_{\mathfrak{l}}, g)$ on P by

$$G_p(u, v) := g_{\pi(p)}(d\pi(u), d\pi(v)) + B_{\mathfrak{l}}(\alpha(u), \alpha(v)) \qquad (u, v \in TP_p). \tag{2.2.1}$$

Lemma 2.2. *The pseudo-Riemannian metric G is L-invariant.*

Proof. We write $u = u_1 + u_2 \in TP_p = Ver(P)_p \oplus Hor(P)_p$ and $X := \alpha(u) \in \mathfrak{l}$. By the definition of α, we have $u_1 = \frac{d}{dt}_{|t=0} p \cdot \exp(tX)$. Fix $l \in L$. Since $TP = Ver(P) \oplus Hor(P)$ is an L-equivariant splitting, we have $u \cdot l = u_1 \cdot l + u_2 \cdot l \in Ver(P)_{p \cdot l} \oplus Hor(P)_{p \cdot l}$. Hence we have $\alpha(u \cdot l) = \operatorname{Ad}(l)^{-1}\alpha(u)$ because

$$u_1 \cdot l = \frac{d}{dt}_{|t=0} p \cdot \exp(tX) l = \frac{d}{dt}_{|t=0} p \cdot l \exp(t \operatorname{Ad}(l)^{-1} X) = (\widetilde{\operatorname{Ad}(l)^{-1} X})_{p \cdot l}.$$

In light of $d\pi(u \cdot l) = d\pi(u) \in TM_{\pi(p \cdot l)} = TM_{\pi(p)}$ and the L-invariance of $B_{\mathfrak{l}}$, we have $G_{p \cdot l}(u \cdot l, v \cdot l) = g_{\pi(p \cdot l)}(d\pi(u \cdot l), d\pi(v \cdot l)) + B_{\mathfrak{l}}(\alpha(u \cdot l), \alpha(v \cdot l)) = G_p(u, v)$. $\square$

We write dl, $d\mu(x)$ and $d\sigma(p)$ for the measures on L, M and P which are induced from $B_{\mathfrak{l}}$, g and $G_{\alpha, B_{\mathfrak{l}}, g}$, respectively. We identify dl with a non-zero element of $\wedge^{\dim L} \mathfrak{l}^*$ by left translations. Because L is unimodular, the right translation yields the same measure. The connection form $\alpha \in \mathcal{E}^1(P, \mathfrak{l})$ induces a linear map $\widetilde{\alpha}\colon \mathfrak{l}^* \to \mathcal{E}^1(P)$, and then $\wedge^{\dim L} \mathfrak{l}^* \to \mathcal{E}^{\dim L}(P)$, which we also denote by $\widetilde{\alpha}$. Then we have

$$d\sigma(p) = \pi^*(d\mu(x))\, \widetilde{\alpha}(dl). \tag{2.2.2}$$

Let us prove that the measure $d\sigma(p)$ does not depend on the choice of the connection. To see this, it suffices to show that the restriction of $\widetilde{\alpha}(\omega) \in \mathcal{E}^1(TP)$ with respect to the vertical submanifold $\pi^{-1}(x)$ $(x \in M)$ is independent of α for each $\omega \in \mathfrak{l}^*$. Fix $p \in \pi^{-1}(x) \subset P$ and $v \in Ver(P)_p$. We find a unique element $X \in \mathfrak{l}$ such that $v = \widetilde{X}_p \in Ver(P)_p \subset TP_p$. Then we have $\langle v, \widetilde{\alpha}(\omega)_p \rangle = \langle \alpha(v), \omega \rangle = \langle X, \omega \rangle$, which shows the independence of the connection of the principal bundle.

Lemma 2.3. *With the setting in §2.2, we assume that L is compact.*
1) *For any Borel measurable function f on M, we have*

$$\int_P (\pi^* f)(p) d\sigma(p) = \mathrm{volume}(L) \int_M f(x) d\mu(x). \tag{2.3.1}$$

2) *An L-invariant measure on P satisfying (2.3.1) is unique.*

Proof. The first statement is clear from (2.2.2). Because L is compact, an L-invariant measure on P is determined by the integral of all L-invariant functions on P, namely, all the functions of the form $\pi^* f$. Hence the second statement follows. □

2.4. Now we fix a main setting throughout this paper:

Setting 2.4. *Let G be a real reductive linear Lie group, K a maximal compact subgroup, θ the corresponding Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G given by θ. We fix a non-degenerate symmetric $\mathrm{Ad}(G)$-invariant bilinear form B on $\mathfrak{g}$ such that B is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$ and that $\mathfrak{k}$ and $\mathfrak{p}$ is orthogonal. Suppose H is a θ-stable closed subgroup with finitely many connected components. Then H is also a real reductive linear Lie group and we say that the homogeneous manifold G/H is of reductive type. We write $o := eH \in G/H$. Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to B. Then we have a direct sum decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ because the restriction $B_{|\mathfrak{h}}$ is also a non-degenerate symmetric bilinear form.*

2.5. Suppose we are in the setting 2.4. We recall that G/H has a K-equivariant fiber bundle structure with the base $K/H \cap K$ and fibers $\mathfrak{q} \cap \mathfrak{p}$ (cf. [7], Lemma 2.7), by the map

$$\pi \colon K \times (\mathfrak{q} \cap \mathfrak{p}) \ni (k, X) \mapsto k \exp(X) \cdot o \in G/H.$$

That is, we have a diffeomorphism

$$G/H \simeq (K/H \cap K) \underset{\mathrm{Ad}_{|H \cap K}}{\times} (\mathfrak{q} \cap \mathfrak{p}). \tag{2.5.1}$$

We apply results in §2.2 by putting

$$P := K \times (\mathfrak{q} \cap \mathfrak{p}), \qquad M := G/H, \qquad L := H \cap K.$$

Then $L = H \cap K$ acts on P freely from the right by $(k, X) \mapsto (kh, \mathrm{Ad}(h^{-1})X)$ for $h \in H \cap K$. Then (2.5.1) means that the quotient map

$$\pi \colon P \to M$$

is a principal $H \cap K$-bundle. Let $(k, X) \in P = K \times (\mathfrak{q} \cap \mathfrak{p})$. By left translations, the tangent spaces are identified with Lie algebras as follows:

$$L_{k*} \oplus L_{X*} \colon \mathfrak{k} \oplus (\mathfrak{q} \cap \mathfrak{p}) \xrightarrow{\sim} TK_k \oplus T(\mathfrak{q} \cap \mathfrak{p})_X = TP_{(k,X)}, \tag{2.5.2}$$

$$L_{ke^X *} \colon \mathfrak{g}/\mathfrak{h} \xrightarrow{\sim} T(G/H)_{ke^X \cdot o} = TM_{\pi(k,X)}, \tag{2.5.3}$$

where $L_k : K \to K,\ x \mapsto kx,\ L_X : \mathfrak{q} \cap \mathfrak{p} \to \mathfrak{q} \cap \mathfrak{p},\ Z \mapsto X + Z$ and $L_g : G/H \to G/H,\ x \cdot o \mapsto gx \cdot o$. With these identifications, the differential $\pi_{(k,X)*} : TP_{(k,X)} \to TM_{\pi(k,X)}$ is given by (see [4], Chapter II, Theorem 1.7)

$$\begin{aligned} &\beta \equiv \beta_X : \mathfrak{k} \oplus (\mathfrak{q} \cap \mathfrak{p}) \to \mathfrak{g}/\mathfrak{h}, \\ &\qquad (Y, Z) \qquad \mapsto e^{-\operatorname{ad}(X)} Y + \frac{1 - e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)} Z \mod \mathfrak{h}. \end{aligned} \tag{2.5.4}$$

<u>**2.6.**</u> With notation in (2.5.2), we define a connection of the principal bundle $P = K \times (\mathfrak{q} \cap \mathfrak{p}) \to M = G/H$ by putting a horizontal subbundle

$$Hor(P)_{(k,X)} := L_{k*}(\mathfrak{q} \cap \mathfrak{k}) \oplus L_{X*}(\mathfrak{q} \cap \mathfrak{p}) \subset TP_{(k,X)} = TK_k \oplus T(\mathfrak{q} \cap \mathfrak{p})_X. \tag{2.6.1}$$

Lemma 2.6. *With the identification (2.5.2), the connection form $\alpha \in \mathcal{E}^1(P, \mathfrak{h} \cap \mathfrak{k})$ (see §2.1) is given by*

$$\alpha : \mathfrak{k} \oplus (\mathfrak{q} \cap \mathfrak{p}) \to \mathfrak{h} \cap \mathfrak{k}, \quad (Y, Z) \mapsto Y_+, \tag{2.6.2}$$

which is independent of $(k, X) \in P$. Here we write $Y = Y_+ + Y_-$ if $Y \in \mathfrak{k}$ according to the direct sum $\mathfrak{k} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{k})$.

Proof. The fundamental vector field $\widetilde{Y_+}$ on P induced by $Y_+ \in \mathfrak{h} \cap \mathfrak{k}$ is given by

$$\begin{aligned} \widetilde{Y_+}_{(k,X)} &= \frac{d}{dt}|_{t=0}(k \exp(tY_+), \operatorname{Ad}(\exp(-tY_+))X) \\ &= (L_{k*} \oplus L_{X*})(Y_+, [X, Y_+]) \quad \in Ver(P)_{(k,X)}, \end{aligned} \tag{2.6.3}$$

for $(k, X) \in K \times (\mathfrak{q} \cap \mathfrak{p}) = P$. Because $[X, Y_+] \in [\mathfrak{q} \cap \mathfrak{p}, \mathfrak{h} \cap \mathfrak{k}] \subset \mathfrak{q} \cap \mathfrak{p}$, we have $(L_{k*} \oplus L_{X*})(Y_-, Z - [X, Y_+]) \in Hor(P)_{(k,X)}$ by (2.6.1). Therefore, we have

$$\begin{aligned} (L_{k*} \oplus L_{X*})(Y, Z) &= (L_{k*} \oplus L_{X*})(Y_+, [X, Y_+]) + (L_{k*} \oplus L_{X*})(Y_-, Z - [X, Y_+]) \\ &= \widetilde{Y_+}_{(k,X)} + (L_{k*} \oplus L_{X*})(Y_-, Z - [X, Y_+]) \\ &\in Ver(P)_{(k,X)} \oplus Hor(P)_{(k,X)}. \end{aligned}$$

Hence the vertical projection $TP_{(k,X)} \to Ver(P)_{(k,X)}$ is given by

$$(L_{k*} \oplus L_{X*})(Y, Z) \mapsto \widetilde{Y_+}_{(k,X)}.$$

By the definition of α, this formula implies $\alpha(Y, Z) = Y_+$ via the identification (2.5.2). $\square$

<u>**2.7.**</u> Let us fix $X \in \mathfrak{q} \cap \mathfrak{p}$. Combining two maps (2.5.4) and (2.6.2), we define a linear map $\varphi \equiv \varphi_X = \beta_X \oplus \alpha : \mathfrak{k} \oplus (\mathfrak{q} \cap \mathfrak{p}) \to (\mathfrak{g}/\mathfrak{h}) \oplus (\mathfrak{h} \cap \mathfrak{k})$ by

$$(Y, Z) = (Y_+ + Y_-, Z) \mapsto (e^{-\operatorname{ad}(X)} Y + \frac{1 - e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)} Z \mod \mathfrak{h}, Y_+). \tag{2.7.1}$$

We write

$$N_H := \dim(\mathfrak{k} \oplus (\mathfrak{q} \cap \mathfrak{p})) = \dim((\mathfrak{g}/\mathfrak{h}) \oplus (\mathfrak{h} \cap \mathfrak{k})) = \dim \mathfrak{g} - \dim(\mathfrak{h} \cap \mathfrak{p}). \tag{2.7.2}$$

The bilinear form B (see §2.4) induces non-degenerate symmetric bilinear forms on $\mathfrak{k}$, $\mathfrak{q}\cap\mathfrak{p}$, $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{h}\cap\mathfrak{k}$, which we denote by $B_{\mathfrak{k}}$, $B_{\mathfrak{q}\cap\mathfrak{p}}$, $B_{\mathfrak{g}/\mathfrak{h}}$ and $B_{\mathfrak{h}\cap\mathfrak{k}}$, respectively. In particular, they define invariant (pseudo-)Riemannian structures on K, $\mathfrak{q} \cap \mathfrak{p}$, G/H and $H \cap K$, respectively, by left translations. We denote the corresponding invariant measures by dk, dX, $d\mu$ and dh, respectively. Note that $B_{\mathfrak{q}\cap\mathfrak{p}}$ is positive definite and dX is the Lebesgue measure. The induced pseudo-Riemannian metric on the principal bundle $P = K \times (\mathfrak{q} \cap \mathfrak{p})$ from those on the base space G/H and the fiber $H \cap K$ is given by using the identification (2.5.2) as follows:

$$\begin{aligned} &G_{k,X}(Y_1 + Z_1, Y_2 + Z_2) \\ &= B_{\mathfrak{g}/\mathfrak{h}}(\beta_X(Y_1, Z_1), \beta_X(Y_2, Z_2)) + B_{\mathfrak{h}\cap\mathfrak{k}}(\alpha(Y_1, Z_1), \alpha(Y_2, Z_2)), \end{aligned} \tag{2.7.3}$$

for $(k, X) \in K \times (\mathfrak{q} \cap \mathfrak{p}) = P$ and for $Y_i + Z_i \in \mathfrak{k} \oplus (\mathfrak{q} \cap \mathfrak{p})$ $(i = 1, 2)$. Now, we are ready to describe explicitly the measure $d\sigma$ on $P \simeq K \times (\mathfrak{q} \cap \mathfrak{p})$ induced from the indefinite-Riemannian metric defined in §2.2. We fix an orthonormal base $\{u_j\}$ of $\mathfrak{k} \oplus (\mathfrak{q} \cap \mathfrak{p})$ with respect to the symmetric (indefinite) bilinear form $B_{\mathfrak{k}} \oplus B_{\mathfrak{q}\cap\mathfrak{p}}$ such that u_j belongs to one of $\mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{q} \cap \mathfrak{k}$ or $\mathfrak{q} \cap \mathfrak{p}$. We define a non-negative function $\delta\colon \mathfrak{q} \cap \mathfrak{p} \to \mathbb{R}$ by

$$\delta(X) := \left|\det\left((B_{\mathfrak{g}/\mathfrak{h}} \oplus B_{\mathfrak{h}\cap\mathfrak{k}})(\varphi_X(u_i), \varphi_X(u_j))\right)_{1\le i,j\le N_H}\right|^{\frac{1}{2}}. \tag{2.7.4}$$

In light of (2.7.3), we have the following Proposition:

Proposition 2.7. *With notation as above, the measure on the principal bundle $P = K \times (\mathfrak{q} \cap \mathfrak{p})$ is given by $d\sigma(k, X) = \delta(X)dkdX$.*

§3. Upper estimate of the invariant measure.

3.1. We retain the setting §2.4. We give an upper estimate of the natural measure on $P = K \times (\mathfrak{q} \cap \mathfrak{p})$ by using the function $\delta(X)$ defined in (2.7.4).

3.2. Let $\mathfrak{a}_{\mathfrak{p}}$ be a maximally abelian subspace of $\mathfrak{p}$. For $\mathfrak{a}_{\mathfrak{p}}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_{\mathfrak{p}}, \mathbb{R})$, we define $\mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; \lambda) := \{X \in \mathfrak{g} : [H, X] = \lambda(H)X \text{ for } H \in \mathfrak{a}_{\mathfrak{p}}\}$. The finite set $\Sigma(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}}) := \{\lambda \in \mathfrak{a}_{\mathfrak{p}}^* \setminus \{0\} : \mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; \lambda) \neq 0\}$ is said to be the restricted root system of the pair $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$. Let $\bar{B}$ be a positive definite bilinear form on $\mathfrak{g}$ given by $\bar{B}(X, Y) := -B(X, \theta Y)$. $\bar{B}$ coincides with B on $\mathfrak{p}$. Denote by $\|\cdot\|$ the norm on $\mathfrak{g}$ defined by $\bar{B}$. For a subspace V of $\mathfrak{g}$, we write $\|\cdot\|_{V^*}$ and $\|\cdot\|_{\mathfrak{g}/V}$ for the induced norms on V^* and $\mathfrak{g}/V$, respectively. We put

$$\Lambda \equiv \Lambda(G) := \max\{\|\lambda\|_{\mathfrak{a}_{\mathfrak{p}}^*} : \lambda \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})\}. \tag{3.2.1}$$

With the notation in §2.4, we write the projection

$$\mathfrak{g} \to \mathfrak{q}, Y \mapsto Y_-$$

with respect to the direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Then $\|Y\|^2_{\mathfrak{g}/\mathfrak{h}} = \|Y_-\|^2$ by definition. We note that $Y \mapsto Y_-$ induces the projection $\mathfrak{p} \to \mathfrak{q} \cap \mathfrak{p}$. We regard $\mathfrak{a}^*_{\mathfrak{p}} \subset \mathfrak{p}^* \simeq \mathfrak{p}$ by the bilinear form $\bar{B}_{\mathfrak{p}\times\mathfrak{p}}$ and write $\mathfrak{p}^* \to (\mathfrak{q} \cap \mathfrak{p})^*, \lambda \mapsto \lambda_-$ for the projection. We define

$$\text{(3.2.2)} \quad \Lambda_H \equiv \Lambda_H(G) := \max\{\| (\mathrm{Ad}^*(k)\lambda)_- \|_{(\mathfrak{q}\cap\mathfrak{p})^*} : k \in K, \lambda \in \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})\},$$
$$\text{(3.2.3)} \quad \nu_{G/H} \equiv \nu_H(G) := N_H \Lambda_H, \qquad \nu \equiv \nu(G) := \Lambda \cdot \dim \mathfrak{g}.$$

Here we recall $N_H = \dim \mathfrak{g} - \dim(\mathfrak{h} \cap \mathfrak{p})$.

Lemma 3.3. *For $X, Y \in \mathfrak{g}$.*
1) *$\Lambda_H \leq \Lambda$ and $\nu_{G/H} \leq \nu$.*
2) *For any $X \in \mathfrak{q} \cap \mathfrak{p}$, and any eigenvalue μ of $\mathrm{ad}(X)$ on $\mathfrak{g}$ we have $|\mu| \leq \Lambda_H \|X\|$.*
3) *For any $X \in \mathfrak{q} \cap \mathfrak{p}$ and $Y \in \mathfrak{g}$, $\| \exp(-\mathrm{ad}(X))Y\| \leq \exp(\Lambda_H\|X\|)\|Y\|$.*
4) *For any $X \in \mathfrak{q} \cap \mathfrak{p}$ and $Y \in \mathfrak{g}$, $\|\frac{1-\exp(-\mathrm{ad}(X))}{\mathrm{ad}(X)} Y\| \leq \exp(\Lambda_H\|X\|)\|Y\|$.*

Proof. 1) Clear.
2) We take $k \in K$ such that $X' := \mathrm{Ad}(k)X \in \mathfrak{a}_{\mathfrak{p}}$. If $Y \in \mathfrak{g}(\mathfrak{a}_{\mathfrak{p}}; \lambda)$ and if we put $Z := \mathrm{Ad}(k^{-1})Y$ then $\mathrm{ad}(X)Z = \mathrm{Ad}(k^{-1})\,(\mathrm{ad}(X')Y) = \lambda(X')Z$. Thus the set of eigenvalues of $\mathrm{ad}(X)$ is given by $\{\lambda(\mathrm{Ad}(k)X) : \lambda \in \Sigma(\mathfrak{g}; \mathfrak{a}_{\mathfrak{p}}) \cup \{0\}\}$. Then we have

$$|\lambda(\mathrm{Ad}(k)X)| = |(\mathrm{Ad}^*(k^{-1})\lambda)(X)| = |(\mathrm{Ad}^*(k^{-1})\lambda)_-(X)| \leq \Lambda_H\|X\|,$$

where the last inequality is from the definition (3.2.1). Hence we have proved (2).
3) $\mathrm{ad}(X)$ is a symmetric operator on $\mathfrak{g}$ with respect to the positive definite form $\bar{B}$. Let $\mathrm{Ker}(\mathrm{ad}(X) - \mu)$ be the eigenspace of $\mathrm{ad}(X)$ with eigenvalue $\mu \in \mathbb{R}$. We take an orthonormal basis $Y_{\mu,i}$ of $\mathfrak{g}$ with respect to $\bar{B}$ such that $Y_{\mu,i}$ belongs to the eigenspace $\mathrm{Ker}(\mathrm{ad}(X) - \mu)$ $(1 \leq i \leq \dim \mathrm{Ker}(\mathrm{ad}(X) - \mu))$. We write $Y = \sum_{\mu,i} a_{\mu,i} Y_{\mu,i}$. Then we have $\mathrm{ad}(X)Y = \sum_{\mu,i} a_{\mu,i}\mu Y_{\mu,i}$ and therefore

$$\|e^{-\mathrm{ad}(X)}Y\|^2 = \sum_{\mu,i} |a_{\mu,i}|^2 e^{-2\mu} \leq e^{2\Lambda_H\|X\|} \sum_{\mu,i} |a_{\mu,i}|^2 = e^{2\Lambda_H\|X\|}\|Y\|^2.$$

4) The proof is similar to (3) because $|\frac{1-\exp(-\mu)}{\mu}| \leq \exp|\mu|$ for $\mu \in \mathbb{R}$. ☐

<u>**3.4.**</u> As is similar to B in §2.7, $\bar{B}$ induces non-degenerate symmetric bilinear forms on $\mathfrak{k}$, $\mathfrak{q}\cap\mathfrak{p}$, $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{h}\cap\mathfrak{k}$, which we denote by $\bar{B}_{\mathfrak{k}}$, $\bar{B}_{\mathfrak{q}\cap\mathfrak{p}}$, $\bar{B}_{\mathfrak{g}/\mathfrak{h}}$ and $\bar{B}_{\mathfrak{h}\cap\mathfrak{k}}$, respectively.

Lemma 3.4. *Let u_j $(1 \leq j \leq N_H)$ be as in §2.7. Then we have*

$$(\bar{B}_{\mathfrak{g}/\mathfrak{h}} \oplus \bar{B}_{\mathfrak{h}\cap\mathfrak{k}})(\varphi_X(u_j), \varphi_X(u_j)) \leq 2\exp(\Lambda_H\|X\|), \quad 1 \leq j \leq N_H.$$

Proof. First we note that $\|u_j\| = 1$ because $B(u_j, u_j) = \pm 1$ and because u_j belongs to $\mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{q} \cap \mathfrak{k}$ or $\mathfrak{q} \cap \mathfrak{p}$. We divide into three cases:
1) If $u_j \in \mathfrak{h} \cap \mathfrak{k}$ then $\varphi_X(u_j) = (e^{-\operatorname{ad}(X)} u_j, u_j) \in (\mathfrak{g}/\mathfrak{h}) \oplus (\mathfrak{h} \cap \mathfrak{k})$. Hence we have

$$\begin{aligned}
(\bar{B}_{\mathfrak{g}/\mathfrak{h}} \oplus \bar{B}_{\mathfrak{h}\cap\mathfrak{k}})(\varphi_X(u_j), \varphi_X(u_j)) &= \left\| (e^{-\operatorname{ad}(X)} u_j)_- \right\|^2 + \|u_j\|^2 \\
&\le \left\| e^{-\operatorname{ad}(X)} u_j \right\|^2 + \|u_j\|^2 \\
&\le (e^{2\Lambda_H \|X\|} + 1)\|u_j\|^2 \\
&= e^{2\Lambda_H \|X\|} + 1 \\
&\le 2e^{2\Lambda_H \|X\|}.
\end{aligned}$$

Here we have used Lemma 3.3 (3).
2) If $u_j \in \mathfrak{q} \cap \mathfrak{k}$ then $\varphi_X(u_j) = (e^{-\operatorname{ad}(X)} u_j, 0)$. As in the above proof, we have $(\bar{B}_{\mathfrak{g}/\mathfrak{h}} \oplus \bar{B}_{\mathfrak{h}\cap\mathfrak{k}})(\varphi_X(u_j), \varphi_X(u_j)) \le e^{2\Lambda_H \|X\|}$.
3) If $u_j \in \mathfrak{q} \cap \mathfrak{p}$ then $\varphi_X(u_j) = (\frac{1 - e^{-\operatorname{ad}(X)}}{\operatorname{ad}(X)} u_j, 0)$. By using Lemma 3.3 (4), we have $(\bar{B}_{\mathfrak{g}/\mathfrak{h}} \oplus \bar{B}_{\mathfrak{h}\cap\mathfrak{k}})(\varphi_X(u_j), \varphi_X(u_j)) \le e^{2\Lambda_H \|X\|}$.
Therefore, we have proved Lemma. □

Lemma 3.5. *Suppose $v_i \in (\mathfrak{g}/\mathfrak{h}) \oplus (\mathfrak{h} \cap \mathfrak{k})$ for $1 \le i \le N_H$. Then we have*

$$\left| \det \left((B_{\mathfrak{g}/\mathfrak{h}} \oplus B_{\mathfrak{h}\cap\mathfrak{k}})(v_i, v_j) \right)_{1 \le i,j \le N_H} \right| \le \det \left((\bar{B}_{\mathfrak{g}/\mathfrak{h}} \oplus \bar{B}_{\mathfrak{h}\cap\mathfrak{k}})(v_i, v_j) \right)_{1 \le i,j \le N_H}.$$

We shall postpone the proof of this lemma to the next section (see §4.1).

Lemma 3.6. *Retaining the notation, we have*

$$\left| \det \left((\bar{B}_{\mathfrak{g}/\mathfrak{h}} \oplus \bar{B}_{\mathfrak{h}\cap\mathfrak{k}})(\varphi_X(u_i), \varphi_X(u_j)) \right)_{1 \le i,j \le N_H} \right| \le 2^{N_H} \exp(\nu_{G/H} \|X\|). \tag{3.6.1}$$

Proof. We recall an elementary fact that if A is a positive semi-definite symmetric $n \times n$ matrix and if $V := \left(\vec{v_1}, \dots, \vec{v_n} \right) \in M(n, \mathbb{R})$ with $\vec{v_j} \in \mathbb{R}^n$ $(1 \le j \le n)$, then $\det {}^t V A V \le \prod_{j=1}^n (A\vec{v_j}, \vec{v_j})$. In fact, the volume of the parallelotope spanned by $\sqrt{A}\vec{v_1}, \dots, \sqrt{A}\vec{v_n}$ is less than $\prod_{j=1}^n \|\sqrt{A}\vec{v_j}\|$. This implies that

$$\det({}^t V A V) = \det(\sqrt{A} V)^2 \le \prod_{j=1}^n \|\sqrt{A}\vec{v_j}\|^2 = \prod_{j=1}^n (\vec{v_j}, A\vec{v_j}).$$

Since $\bar{B}_{\mathfrak{g}/\mathfrak{h}} \oplus \bar{B}_{\mathfrak{h}\cap\mathfrak{k}}$ is a positive definite symmetric bilinear form, we have

$$\begin{aligned}
(\text{the left side of (3.6.1)}) &\le \prod_{j=1}^{N_H} \left(\bar{B}_{\mathfrak{g}/\mathfrak{h}} \oplus \bar{B}_{\mathfrak{h}\cap\mathfrak{k}})(\varphi_X(u_j), \varphi_X(u_j) \right) \\
&\le (2 \exp(\Lambda_H \|X\|))^{N_H} = (\text{the right side of (3.6.1)}),
\end{aligned}$$

where the second inequality follows from the previous lemma. □

Theorem 3.7. *Let G/H be a homogeneous manifold of reductive type and $d\mu(x)$ a G-invariant measure. Then there exists a non-negative function $\delta\colon \mathfrak{q}\cap\mathfrak{p} \to \mathbb{R}$ such that*

$$\int_{G/H} f(x)d\mu(x) = \int_K \int_{\mathfrak{q}\cap\mathfrak{p}} f(ke^X \cdot o)\,\delta(X)\,dkdX \qquad \text{for any } f \in C_c(G/H).$$

Furthermore, there exist constants $\nu_{G/H} > 0$ and $C > 0$ such that

$$\delta(X) \le C\exp(\nu_{G/H}\|X\|) \qquad \text{for any } X \in \mathfrak{q}\cap\mathfrak{p}.$$

The precise formula of δ is given in (2.7.4).

We note that with the normalization of $\|\cdot\|$ induced from a fixed norm on $\mathfrak{p}$, there exists a constant $\nu \equiv \nu(G)$ such that $\nu_{G/H} \le \nu$ for any θ-stable closed subgroup H with finitely many components.

Proof. The first statement follows from Lemma 2.3 and Proposition 2.7. The upper estimate of $\delta(X)$ is deduced from (2.7.4), Lemma 3.5 and Lemma 3.6. The last statement was proved in Lemma 3.3 with the constant $\nu \equiv \nu(G)$ given in (3.3.3). □

<u>**3.8.**</u> Given $\xi \in \mathbb{R}$, we introduce a subspace of continuous functions

$$C(G/H;\xi) := \{f \in C(G/H) : \sup_{k\in K}\ \sup_{X\in\mathfrak{p}\cap\mathfrak{q}} f(k\exp X)\exp(\xi\|X\|) < \infty\}.$$

We note that $C(G/H;\xi) \subset C(G/H;\xi')$ if $\xi > \xi'$.

Corollary 3.9. *Let G/H be a homogeneous manifold of reductive type and $d\mu$ a G-invariant measure. If $1 \le p \le \infty$ then we have*

$$C(G/H;\xi) \subset L^p(G/H;d\mu) \qquad \text{for any } \xi > \frac{\nu_{G/H}}{p}.$$

Proof. If $p = \infty$ then any function belonging to $C(G/H;\xi)$ ($\xi > 0$) is bounded by definition and there is nothing to prove. We assume $1 \le p < \infty$. Suppose $f \in C(G/H;\xi)$ for $\xi > \frac{\nu_{G/H}}{p}$. Then we find a constant $C' > 0$ such that

$$|f(k\exp(X))| \le C'\exp(-\xi\|X\|).$$

It follows from Theorem 3.7 that

$$\begin{aligned}\int_{G/H} |f(x)|^p d\mu(x) &= \int_K \int_{\mathfrak{q}\cap\mathfrak{p}} |f(k\exp(X)\cdot o)|^p \delta(X)\,dkdX \\ &\le (C')^p C \int_K \int_{\mathfrak{q}\cap\mathfrak{p}} \exp((\nu_{G/H} - p\xi)\|X\|)\,dkdX < \infty.\end{aligned}$$

□

§4. Some elementary linear algebra.

<u>**4.1.**</u> The purpose of this section is to prove an inequality of the determinants of symmetric matrices:

Lemma 4.1. *Suppose $A, B \in M(n, \mathbb{R})$ are symmetric and positive semi-definite matrices. Let $\tau \in GL(n, \mathbb{R})$ satisfy $\tau^2 = 1$ and ${}^t\tau B \tau = B$. Then we have*

$$|\det(A + B\tau)| \leq \det(A + B).$$

Admitting Lemma 4.1, we complete the proof of Lemma 3.5.

Proof of Lemma 3.5. We fix a base $\{e_1, \dots, e_n\}$ ($n := N_H$) of $(\mathfrak{g}/\mathfrak{h}) \oplus (\mathfrak{h} \cap \mathfrak{k})$ so that each e_j belongs to one of $\mathfrak{k}/(\mathfrak{h} \cap \mathfrak{k})$, $\mathfrak{p}/(\mathfrak{h} \cap \mathfrak{p})$ or $\mathfrak{h} \cap \mathfrak{k}$. We put $A := \bar{B}_{\mathfrak{h} \cap \mathfrak{k}}$ and $B := \bar{B}_{\mathfrak{g}/\mathfrak{h}}$, which are identified with positive semi-definite symmetric matrices via the above base. Because $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is orthogonal with respect to the bilinear form B on $\mathfrak{g}$, we have $\theta \in O(B)$ by Lemma 4.2 (see below). Then the induced bilinear form $\bar{B}_{\mathfrak{g}/\mathfrak{h}}$ on $\mathfrak{g}/\mathfrak{h}$ also satisfies ${}^t\theta \bar{B}_{\mathfrak{g}/\mathfrak{h}} \theta = \bar{B}_{\mathfrak{g}/\mathfrak{h}}$ where θ is regarded as an involution on $(\mathfrak{g}/\mathfrak{h}) \oplus (\mathfrak{h} \cap \mathfrak{k}) \simeq \mathbb{R}^n$. Since $\bar{B}_{\mathfrak{h} \cap \mathfrak{k}} = -B_{\mathfrak{h} \cap \mathfrak{k}}$ and $\bar{B}_{\mathfrak{g}/\mathfrak{h}} \theta = -B_{\mathfrak{g}/\mathfrak{h}}$, we have from Lemma 4.1 that

$$|\det\left(B_{\mathfrak{h} \cap \mathfrak{k}} + B_{\mathfrak{g}/\mathfrak{h}}\right)| = |\det\left(\bar{B}_{\mathfrak{h} \cap \mathfrak{k}} + \bar{B}_{\mathfrak{g}/\mathfrak{h}} \theta\right)| \leq \det\left(\bar{B}_{\mathfrak{h} \cap \mathfrak{k}} + \bar{B}_{\mathfrak{g}/\mathfrak{h}}\right).$$

Hence we have Lemma 3.5. $\square$

The proof of Lemma 4.1 is elementary, but we shall give the proof for the sake of completeness.

<u>4.2.</u> Let V be a finite dimensional vector space over $\mathbb{R}$ and T a symmetric bilinear form on V which is possibly degenerate. We define the orthogonal group with respect to T by

$$O(T) := \{g \in GL_{\mathbb{R}}(V) : T(u, v) = T(gu, gv) \text{ for any } u, v \in V\}.$$

Suppose $\sigma \in GL_{\mathbb{R}}(V)$ is an involution, that is, $\sigma^2 = 1$. Then we have a direct sum decomposition $V = V_+ + V_-$, where V_+ and V_- are the eigenspaces for $+1$ and -1 eigenvalues of σ, respectively.

Lemma 4.2. *$\sigma \in O(T)$ if and only if V_+ is orthogonal to V_- with respect to T.*

Proof. If $\sigma \in O(T)$, then $T(u + \sigma u, v - \sigma v) = (T(u, v) - T(\sigma u, \sigma v)) + (T(\sigma u, v) - T(u, \sigma v)) = 0$ for $u, v \in V$. Because $V_\pm = \{u \pm \sigma u : u \in V\}$, we have $V_+ \perp V_-$. Conversely, assume $V_+ \perp V_-$. Let $u = u_+ + u_-$, $v = v_+ + v_-$ be arbitrary elements of $V = V_+ + V_-$. Then $T(\sigma u, \sigma v) = T(u_+ - u_-, v_+ - v_-) = T(u_+, v_+) + T(u_+, u_-) = T(u, v)$, which shows $\sigma \in O(T)$. $\square$

Definition 4.3. Under the equivalent conditions in Lemma 4.2, we define a bilinear form by

$$T_\sigma(u, v) := T(\sigma u, v) = T(u, \sigma v), \quad \text{for } u, v \in V.$$

<u>4.4.</u> Suppose $g \in GL_{\mathbb{R}}(V)$ and $\sigma \in GL_{\mathbb{R}}(V)$. We define a bilinear form by $T^g := {}^t g T g$, that is, $T^g(u, v) = T(gu, gv)$ $(u, v \in V)$. Let $\sigma^g := g^{-1} \sigma g$. Then the following is obvious from definition:

Lemma 4.4. *$\sigma \in O(T)$ if and only if $\sigma^g \in O(T^g)$.*

<u>4.5.</u> We fix a base of V and identify $\mathrm{End}(V)$ with $M(n,\mathbb{R})$ where $n = \dim_{\mathbb{R}} V$. We also identify an $n \times n$ symmetric matrix T with a symmetric bilinear form on V and write $O(T)$ for the orthogonal group with respect to T as in §4.2. $O(I_n)$ will be denoted by $O(n)$ as usual. Denote by E_{ij} the matrix unit. Fix $m\,(\leq n)$ and we put $T_m := \sum_{i=1}^m E_{ii} \in M(n,\mathbb{R})$.

Lemma 4.5. *Suppose $\sigma = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$, where $P \in M(m,\mathbb{R}), Q \in M(m,n-m;\mathbb{R}), R \in M(n-m,m;\mathbb{R})$ and $S \in M(n-m,\mathbb{R})$.*
1) $\sigma \in O(T_m)$ if and only if $P \in O(m)$ and $Q = 0$. Then, $T_m\sigma = \begin{pmatrix} P & O \\ O & O \end{pmatrix}$.
2) If $\sigma \in O(T_m)$ and if $\sigma^2 = 1$, then there exist $k \in O(m)$ and $\varepsilon_j = \pm 1$ $(1 \leq j \leq m)$ such that $P = k\,\mathrm{diag}(\varepsilon_1,\ldots,\varepsilon_m)k^{-1}$.

Proof. 1) Since ${}^t\sigma T_m \sigma = \begin{pmatrix} {}^tPP & {}^tPQ \\ {}^tQP & {}^tQQ \end{pmatrix}$, we have ${}^t\sigma T_m\sigma = T_m$ if and only if $P \in O(m)$ and $Q = 0$. The last statement follows from the formula $T_m\sigma = \begin{pmatrix} P & Q \\ O & O \end{pmatrix}$.
2) Since $\sigma \in O(T_m)$, σ is of the form $\begin{pmatrix} P & O \\ R & S \end{pmatrix}$ with $P \in O(m)$ from (1). Since $\sigma^2 = 1$, $P \in O(m)$ satisfies $P^2 = I_m$, whence the second statement. $\square$

<u>4.6.</u> Given $C_j > 0$ for $1 \leq j \leq m\,(\leq n)$, we define a symmetric matrix by $T := \mathrm{diag}(C_1,\ldots,C_m,0,\ldots,0) \in M(n,\mathbb{R})$.

Lemma 4.6. *Suppose $\sigma \in O(T)$ satisfies $\sigma^2 = 1$. Then there exist $k \in O(m)$ and $\varepsilon_j = \pm 1$ $(1 \leq j \leq m)$ such that*

$$T\sigma = g\begin{pmatrix} k\varepsilon k^{-1} & O \\ O & O \end{pmatrix} g,$$

where $\varepsilon := \mathrm{diag}(\varepsilon_1,\ldots,\varepsilon_m) \in O(m)$ and $g := \mathrm{diag}(\sqrt{C_1},\ldots,\sqrt{C_m},1,\ldots,1) \in M(n,\mathbb{R})$.

Proof. From Lemma 4.4 we have $\sigma^{g^{-1}} \in O(T^{g^{-1}})$. We note that

$$T^{g^{-1}} = {}^tg^{-1}Tg^{-1} = \sum_{i=1}^m E_{ii} \equiv T_m.$$

Then we can find $k \in O(m)$ and $\varepsilon_j = \pm 1$ $(1 \leq j \leq m)$ such that $T^{g^{-1}}\sigma^{g^{-1}} = \begin{pmatrix} k\varepsilon k^{-1} & O \\ O & O \end{pmatrix}$ by Lemma 4.5. The left side equals ${}^tg^{-1}T\sigma g^{-1}$, whence Lemma. $\square$

Lemma 4.7. *Let $u_1, \dots, u_n \in \mathbb{R}^n$ and $\| \cdot \|$ be the Euclidean norm in $\mathbb{R}^n$. Then*

$$|\det(u_1, \dots, u_n)| \leq \prod_{j=1}^{n} \|u_j\|.$$

Proof. Clear from the Gram-Schmidt orthogonalization. □

4.8. *Proof of Lemma 4.1* First we assume $\det(A+B) = 0$. Then we find a non-zero vector $u \in \mathbb{R}^n$ such that ${}^t u(A+B)u = 0$. Because both A and B are positive semi-definite, we have ${}^t uAu = {}^t uBu = 0$. By using the positive semi-definiteness again, we have $Au = Bu = 0$. Therefore $(A + B\tau)u = (A + {}^t\tau B)u = 0$. Hence $\det(A + B\tau) = 0$, proving $|\det(A + B\tau)| \leq \det(A + B)$.

Second we assume $\det(A+B) \neq 0$. Then $A+B$ is positive definite, and we find $g \in GL(n, \mathbb{R})$ such that ${}^t g(A+B)g = I_n$. Since ${}^t gAg$ is positive semi-definite, we find $k \in O(n)$ such that ${}^t k({}^t gAg)k = \operatorname{diag}(1, \dots, 1, a_1, \dots, a_m)$ where $a_1, \dots, a_m$ are non-negative numbers other than 1. We put $h := gk$. Then ${}^t hBh = {}^t h(A+B)h - {}^t hAh = \operatorname{diag}(0, \dots, 0, 1-a_1, \dots, 1-a_m)$. Since B is positive semi-definite, we have $a_j \leq 1$ for all j with $1 \leq j \leq m$. Together with the previous condition $0 \leq a_j \neq 1$, we have

$$0 \leq a_j < 1 \quad (1 \leq j \leq m).$$

We put

$$\begin{aligned}
b_j &:= 1 - a_j & &(1 \leq j \leq m),\\
b &:= \operatorname{diag}(1, \dots, 1, \sqrt{b_1}, \dots, \sqrt{b_m}) & &\in M(n, \mathbb{R}),\\
\sigma &:= h^{-1}\tau h & &\in GL(n, \mathbb{R}),\\
T &:= {}^t hBh = \operatorname{diag}(0, \dots, 0, b_1, \dots, b_m) & &\in M(n, \mathbb{R}).
\end{aligned}$$

Then a simple computation yields $\sigma^2 = 1$ and ${}^t\sigma T\sigma = T$. From Lemma 4.6, there exist $k \in O(m)$ and $\varepsilon_j = \pm 1$ $(1 \leq j \leq m)$ such that

$$T\sigma = {}^t hBh\sigma = b \begin{pmatrix} O & O \\ O & k\varepsilon k^{-1} \end{pmatrix} b,$$

where $\varepsilon := \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_m) \in O(m)$. Then

$$\begin{aligned}
&\det({}^t hAh + {}^t hBh\sigma)\\
&= \det\left(\operatorname{diag}(1, \dots, 1, a_1, \dots, a_m) + b \begin{pmatrix} O & O \\ O & k\varepsilon k^{-1} \end{pmatrix} b\right)\\
&= b_1 \cdots b_m \det\left(\operatorname{diag}(\frac{a_1}{b_1}, \dots, \frac{a_m}{b_m}) + k\varepsilon k^{-1}\right)\\
&= b_1 \cdots b_m \det\left(\operatorname{diag}(\frac{a_1}{b_1}, \dots, \frac{a_m}{b_m})k + k\varepsilon\right)\\
&= \det(\operatorname{diag}(a_1, \dots, a_m)k + \operatorname{diag}(b_1, \dots, b_m)k\varepsilon)\\
&= \det(((a_i + b_i\varepsilon_j)k_{ij}))_{1 \leq i,j \leq m}.
\end{aligned}$$

Here we write $k = (k_{ij})_{1\le i,j\le m}$. In view of $|a_i + b_i\varepsilon_j| \le 1$ for any i, j and $k \in O(m)$, the column vector satisfies

$$\|{}^t((a_1 + b_1\varepsilon_j)k_{1j}, \dots, (a_m + b_m\varepsilon_j)k_{mj})\| \le 1,$$

for any $1 \le j \le m$. Hence $|\det(((a_i + b_i\varepsilon_j)k_{ij})_{1\le i,j\le m})| \le 1$ by Lemma 4.7. In light of ${}^th(A+B\tau)h = \det({}^thAh + {}^thBh\sigma)$ and $(\det h)^2 = (\det g)^2 = \det(A+B)^{-1}$, we have

$$|\det(A + B\tau)| = |\det({}^th(A + B\tau)h)|\ \det(h)^{-2} \le 1 \cdot \det(A + B) = \det(A + B),$$

proving Lemma 4.1. $\square$

§5. Estimates on functions on homogeneous submanifolds.

5.1. In this section we give an estimate of the asymptotic behavior of a function on G/H when restricted to a homogeneous submanifold G'/H'. This result plays a fundamental role of harmonic analysis on the orbit space defined by two involutions ([10]).

The key idea is to reduce the problem to the Riemannian symmetric space $K\backslash G$ instead of a non-Riemannian manifold G/H and then to employ an estimate in Riemannian manifolds with non-positive sectional curvatures.

5.2. Let W be a finite dimensional vector space over $\mathbb{R}$ with an inner product. Given non-zero vector subspaces V_1 and V_2, we define the angle of V_1 and V_2 by

$$\psi(V_1, V_2) := \min\{\text{the angle of } l_1 \text{ and } l_2\} \tag{5.2.1}$$

Here, the minimum is taken over all (l_1, l_2) such that l_j are lines in V_j going through the origin for $j = 1, 2$. We note that $0 \le \psi(V_1, V_2) \le \frac{\pi}{2}$ and that

$$\psi(V_1, V_2) = 0 \ \text{ if and only if } \ V_1 \cap V_2 \ne \{0\}. \tag{5.2.2}$$

5.3. Suppose we are in the setting (2.4). Let G/H be a homogeneous manifold of reductive type. It is well known that the mappings

$$(\mathfrak{q} \cap \mathfrak{p}) \oplus (\mathfrak{h} \cap \mathfrak{p}) \to K\backslash G, \quad (X, Y) \mapsto K \exp X \exp Y, \tag{5.3.1}$$

$$\mathfrak{p} \to K\backslash G, \qquad Z \quad \mapsto K \exp Z \tag{5.3.2}$$

are surjective diffeomorphisms. In particular, a smooth map $\gamma\colon \mathfrak{p} \to \mathfrak{q}\cap\mathfrak{p}$ is uniquely defined with the relation

$$\exp Z \in K \exp(\gamma(Z)) \exp(\mathfrak{h} \cap \mathfrak{p}). \tag{5.3.3}$$

5.4. Here is a crucial step in L^p-estimates of the function restricted to a submanifold.

Lemma 5.4. *Let V be a non-zero subspace of $\mathfrak{p}$. Then we have:*

$$\|Z\| \geq \|\gamma(Z)\| \geq \sin\psi(V, \mathfrak{h}\cap\mathfrak{p})\,\|Z\| \quad \textit{for any } Z \in V.$$

Proof. The $\mathrm{Ad}(G)$-invariant symmetric bilinear form B on $\mathfrak{g}$ induces a norm $\|\cdot\|$ on $\mathfrak{p}$ and a Riemannian metric on the right coset space $K\backslash G$ by the right translation of G. Then $K\backslash G$ is a simply connected Riemannian symmetric space with non-positive sectional curvatures. We write $\exp Z = k\exp(\gamma(Z))\exp Y$ with the uniquely determined elements $k \in K$ and $Y \in \mathfrak{h}\cap\mathfrak{p}$ according to(5.3.3). Let $O := K\cdot e \in K\backslash G$ and we write $A := O\cdot\exp Y$ and $B := O\cdot\exp Z \in K\backslash G$. The curves

$$\begin{aligned}
&\{O\cdot\exp(tZ) : 0 \leq t \leq 1\},\\
&\{O\cdot\exp(tY) : 0 \leq t \leq 1\},\\
&\{O\cdot\exp(t\,\gamma(Z))\exp Y : 0 \leq t \leq 1\}
\end{aligned}$$

are the geodesics forming the sides of the triangle, namely, OB, OA and AB respectively. We write $|OB|$, $|OA|$ and $|AB|$ for the geodesic distances, which equal $\|Z\|$, $\|Y\|$ and $\|\gamma(Z)\|$, respectively. Denote by ψ_0 the angle of OA and OB. It follows from the definition (5.2.1) that

$$\psi_0 \geq \psi(V, \mathfrak{h}\cap\mathfrak{p})$$

because $Z \in V$ and $Y \in \mathfrak{h}\cap\mathfrak{p}$. Then we have (cf. [4], Chapter I, Corollary 13.2)

$$|AB|^2 \geq |OA|^2 + |OB|^2 - 2|OA||OB|\cos\psi_0 = (|OA| - |OB|\cos\psi_0)^2 + |OB|^2\sin^2\psi_0.$$

Hence $|AB| \geq |OB|\sin\psi_0$. This means that

$$\|\gamma(Z)\| = |AB| \geq |OB|\sin\psi_0 = \|Z\|\sin\psi_0 \geq \sin\psi(V, \mathfrak{h}\cap\mathfrak{p})\,\|Z\|.$$

Similarly, we have $|OB| \geq |AB|\sin\angle OAB$. We translate $A = O\cdot\exp(Y)$ into O by the right action of $\exp(-Y)$. Put $A' := O\cdot\exp(-Y)$ and $B' := B\cdot\exp(-Y) = O\cdot\exp(\gamma(Z))$. Then $\angle OAB = \angle A'OB'$, which equals $\angle OAB = \frac{\pi}{2}$ because $Y \in \mathfrak{h}\cap\mathfrak{p}$ and $\gamma(Z) \in \mathfrak{q}\cap\mathfrak{p}$ and because $(\mathfrak{h}\cap\mathfrak{p}) \perp (\mathfrak{q}\cap\mathfrak{p})$. Therefore we have $|OB| \geq |AB|$, namely, $\|Z\| \geq \|\gamma(Z)\|$. This completes the proof. □

5.5. Suppose both H and G' are θ-stable closed subgroups of G with finitely many connected components. We put $H' := G'\cap H$. Then $G'/H' \subset G/H$ are homogeneous manifolds of reductive type. Let $\mathfrak{g}'$ and $\mathfrak{h}'$ be the Lie algebras of G' and H', respectively. We put $\mathfrak{p}' := \mathfrak{g}'\cap\mathfrak{p}$, and define $\mathfrak{q}'$ to be the orthogonal complement of $\mathfrak{h}'$ in $\mathfrak{g}'$ with respect to the non-degenerate symmetric bilinear form $B_{\mathfrak{g}'}$. Then we have direct sum decompositions $\mathfrak{g}' = \mathfrak{h}'\oplus\mathfrak{q}'$ and $\mathfrak{p}' = (\mathfrak{h}'\cap\mathfrak{p}')\oplus(\mathfrak{q}'\cap\mathfrak{p}')$. We define

$$b(G'/H'; G/H) := \sin\psi(\mathfrak{q}'\cap\mathfrak{p}', \mathfrak{h}\cap\mathfrak{p}). \tag{5.5.1}$$

Lemma 5.5. *If G'/H' is noncompact then $b(G'/H';G/H) > 0$.*

Proof. It suffices to show $(\mathfrak{q}' \cap \mathfrak{p}') \cap (\mathfrak{h} \cap \mathfrak{p}) = \{0\}$ by (5.2.2). We have

$$(\mathfrak{q}' \cap \mathfrak{p}') \cap (\mathfrak{h} \cap \mathfrak{p}) = (\mathfrak{q}' \cap \mathfrak{p}') \cap (\mathfrak{h} \cap \mathfrak{p}') = (\mathfrak{q}' \cap \mathfrak{p}') \cap (\mathfrak{h}' \cap \mathfrak{p}') \subset \mathfrak{q}' \cap \mathfrak{h}' = \{0\}.$$

Hence lemma. $\square$

5.6. Now we are ready to give an estimate of the decay of a function on G/H restricted to a submanifold G'/H'.

Theorem 5.6. *Suppose that G is a real reductive linear Lie group with a Cartan involution θ and that both H and G' are θ-stable closed subgroups of G with finitely many connected components. Let $H' := H \cap G'$. We write $\iota\colon G'/H' \hookrightarrow G/H$ for the natural embedding, and $\iota^*\colon C(G/H) \to C(G'/H')$ for the pullback of functions. Let $b := b(G'/H';G/H)$ be a constant given in (5.5.1). Then*

$$\iota^* C(G/H;\xi) \subset C(G'/H';b\,\xi) \qquad \text{for any } \xi \geq 0.$$

Proof. Let $f \in C(G/H;\xi)$. It follows from definition that there exists $C > 0$ such that

$$|f(k \exp Z \cdot o)| \leq C \exp(-\xi \|Z\|) \quad \text{for any } k \in K \text{ and } Z \in \mathfrak{q} \cap \mathfrak{p},$$

where $o = eH \in G/H$. Let $k' \in K'$, $Z' \in \mathfrak{q}' \cap \mathfrak{p}'$. According to the decomposition (5.3.3), we find $k'' \in K$ and $Y \in \mathfrak{h} \cap \mathfrak{p}$ such that $\exp Z' = k'' \exp(\gamma(Z')) \exp Y$. It follows from Lemma 5.4 that $\|\gamma(Z')\| \geq b\|Z'\|$. Therefore

$$\begin{aligned}
|f(k' \exp Z' \cdot o)| &= |f(k'k'' \exp(\gamma(Z')) \exp Y \cdot o)| \\
&= |f(k'k'' \exp(\gamma(Z')) \cdot o)| \\
&\leq C \exp(-\xi\ \|\gamma(Z')\|) \\
&\leq C \exp(-\xi\ b\|Z'\|).
\end{aligned}$$

This shows that $\iota^* f \in C(G'/H';b\xi)$. $\square$

5.7. The following Corollary will play a basic role in the forthcoming paper on harmonic analysis of the double coset space associated to two involutive automorphisms, of which the geometry has been studied recently by T. Matsuki.

Corollary 5.7. *Retain the setting of Theorem 5.6. Let $d\mu$ be a G'-invariant measure on G'/H'. Let $\nu_{G'/H'}$ and $b(G'/H';G/H)$ be the constants given in (3.2.3) and in (5.5.1), respectively. Then for any $1 \leq p \leq \infty$, we have*

$$\iota^* C(G/H;\xi) \subset L^p(G'/H';d\mu) \qquad \text{for any } \xi > \frac{\nu_{G'/H'}}{b(G'/H';G/H)\ p}.$$

Proof. Corollary is immediate from Corollary 3.9 and Theorem 5.6. $\square$

Remark 5.8. Neither Theorem 5.6 nor Corollary 5.7 is true in general if we assume only G', H and $H' := G' \cap H$ are reductive in G. For example, let $G := SL(2,\mathbb{R})$, $a(t) := \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(-t) \end{pmatrix} \in G$, $H := \{a(t) : t \in \mathbb{R}\} \subset G$, $n := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$, and $G' := nHn^{-1}$. Then H, G' and $H' := G' \cap H = \{e\}$ are reductive in G. Then

$$\lim_{t\to-\infty} na(t)n^{-1}H = n \lim_{t\to-\infty} a(t)n^{-1}a(-t)H = nH \quad \text{in } G/H.$$

The natural imbedding $\iota\colon G' \simeq G'/H' \to G/H$ is given by $na(t)n^{-1} \mapsto na(t)n^{-1}H$. Thus we have

$$\lim_{t\to-\infty} \iota^* f(a(t)) = f(nH),$$

for $f \in C(G/H)$. In particular, if $f(nH) \neq 0$, then $\lim_{t\to-\infty} \iota^* f(a(t)) \neq 0$. Therefore, $\iota^* f \notin C(G'/H';\xi')$ for $\xi' > 0$ and $\iota^* f \notin L^p(G'/H';d\mu)$ for $1 \leq p < \infty$. Hence,

$$\begin{aligned} \iota^* C(G/H;\xi) &\not\subset C(G'/H';c\xi) && \text{for any } \xi > 0 \text{ and for any } c > 0, \\ \iota^* C(G/H;\xi) &\not\subset L^p(G'/H';d\mu) && \text{for any } \xi > 0 \text{ and for any } 1 \leq p < \infty. \end{aligned}$$

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