Visible Actions and Multiplicity-free Representations

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ABSTRACT

In the lecture course, I will explain an application of complex geometry to representation theory of Lie groups with emphasis on multiplicity-free representations.

The property "multiplicity-freeness" (each building block is used no more than once) is sometimes hidden as one of the most fundamental structures in classical mathematical methods such as the Taylor series, the Fourier transform, and the theory of spherical harmonics.

Historically, there have been various methods in (finite-dimensional) representation theory from combinatorics to algebraic representation theory which verifies the multiplicity-free property.

In the lecture course, I will explain yet another new and simple principle that gives rise to multiplicity-free representations not only for finite-dimensional cases but also for infinite-dimensional cases.

I plan to begin with a number of elementary examples of multiplicity-free representations. Then I will introduce a theory of "visible actions" on complex manifolds, and a theory of reproducing kernels for unitary representations which are realized in the space of sections for holomorphic bundles. With these preparation, I will explain a *propagation theorem* of multiplicity-free property from fibers to sections under visible actions.

It turns out that the propagation theorem is an effective method that makes new examples of multiplicity-free representations out of old, and big out of small.

Then systematic and synthetic applications of this idea will be illustrated in branching problems (the description of "breaking symmetries"). Examples include both finite and infinite dimensional cases.

In the lecture course, we assume basic knowledge on Lie groups, Lie algebras and homogeneous spaces, differential manifolds (e.g. fiber bundles), Hilbert spaces, and some elementary theory of finite-dimensional representations of compact Lie groups (e.g. Knapp's textbook [1]).

PROGRAM OF THE COURSE

I. EXAMPLES OF MULTIPLICITY-FREE REPRESENTATIONS

1. Taylor series, Fourier series, Peter-Weyl theory, spherical harmonics
2. Finite dimensional representations of $GL(n)$
3. Clebsh–Gordan formula, Pieri's law, Stembridge's classification
4. Branching laws and tensor products, GL-GL duality

II. INFINITE DIMENSIONAL EXAMPLES

1. Fourier analysis
2. Unitary representations
3. Harmonic analysis on Riemannian symmetric spaces

III. PROPAGATION THEOREM OF MULTIPLICITY-FREE REPRESENTATIONS

1. Borel-Weil theory
2. Reproducing kernels

IV. VISIBLE ACTIONS ON COMPLEX MANIFOLDS

1. Definition
2. Various examples of visible actions
3. Classification theory

V. APPLICATIONS OF VISIBLE ACTIONS TO MULTIPLICITY-FREE REPRESENTATIONS

1. Homogeneous complex manifolds
2. Triunity of visible actions
3. Verma modules and symmetry breaking operators

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