Analysis on Minimal Representations

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Lie Theory and Its Applications in Physics
25 June 2011, Varna, Bulgaria

§1 What are minimal representations?

§2 Conformal model of minimal representations

§3 L^2 model of minimal representations

§4 Deformation of Fourier transform

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Minimal representations of a reductive group G

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  (finitely many) (continuously many)
- 'attached to' minimal nilpotent orbits (orbit method)
- matrix coefficients are of bad decay

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$\uparrow$ "induction", etc.

finitely many "very small" irred. unitary reps. of reductive groups

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Observation. ϖ: minimal rep of G

$\mathrm{DIM}(\varpi)$ (Gelfand–Kirillov dimension)

$= \frac{1}{2}$ dimension of minimal nilpotent orbits

$<$ dimension of any non-trivial G-space

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Viewpoint:

Minimal representation ($\Leftarrow$ group)

$\approx$ Maximal symmetries ($\Leftarrow$ rep. space)

Geometric analysis on minimal reps

[1] Schrödinger model of minimal representations of $O(p,q)\cdots$
Memoirs of Amer. Math. Soc. (2011), no.1000, 132 pp.

[2] Algebraic analysis on minimal representations $\cdots$
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[4] Special functions associated to a fourth order differential equation $\cdots$
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[8] Generalized Fourier transforms $\mathcal{F}_{k,a}\cdots$ C.R.A.S. Paris 2009

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Collaborated with S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers and B. Ørsted

Indefinite orthogonal group $O(p+1, q+1)$

Throughout this talk, $p, q \geq 1$, $p+q$: even > 2

$$\begin{aligned} G &= O(p+1, q+1) \\ &= \{g \in GL(p+q+2, \mathbb{R}) : {}^t g \begin{pmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{pmatrix} g = \begin{pmatrix} I_{p+1} & O \\ O & -I_{q+1} \end{pmatrix} \end{aligned}$$

$\cdots$ real simple Lie group of type D

Minimal representation of $G = O(p+1, q+1)$

- $q = 1$
  highest weight module $\oplus$ lowest weight module
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  non-highest, non-spherical
  - algebraic construction (e.g. dual pair)
    (Binegar–Zierau, Howe–Tan, Huang–Zhu)
  - construction by conformal geometry (K–Ørsted)
  - L^2 construction (K–Ørsted, K–Mano)

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Two constructions of minimal reps.

1. Conformal model
   Theorem B

2. L^2 model
(Schrödinger model)
   Theorem D

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal model		
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v.s.		
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Clear Picture $\cdots$ advantage of the model

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3. Deformation of Fourier transforms (Theorems F, G, H)
   (interpolation, special functions, Dunkl operators)

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Conformal construction of minimal reps.

Idea: Composition of holomorphic functions

holomorphic $\circ$ holomorphic = holomorphic

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$\Downarrow$ taking real parts

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⟹ Try to modify the definition!

$\mathrm{Conf}(X, g) \supset \mathrm{Isom}(X, g)$

(X, g) Riemannian manifold
$\varphi \in \mathrm{Diffeo}(X)$

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Def.
φ is isometry $\iff \varphi^* g = g$
φ is conformal $\iff {}^{\exists}$positive function $C_\varphi \in C^\infty(X)$ s.t.
$\varphi^* g = C_\varphi^2\, g$

C_φ : conformal factor

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$$\mathrm{Diffeo}(X) \supset \underset{\text{Conformal group}}{\mathrm{Conf}(X, g)} \supset \underset{\text{isometry group}}{\mathrm{Isom}(X, g)}$$

$\mathrm{Conf}(X, g) \supset \mathrm{Isom}(X, g)$

(X, g) pseudo-Riemannian manifold
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$$\mathrm{Diffeo}(X) \supset \underset{\text{Conformal group}}{\mathrm{Conf}(X, g)} \supset \underset{\text{isometry group}}{\mathrm{Isom}(X, g)}$$

Harmonic ∘ conformal ≠ harmonic

Modification

$\varphi \in \mathrm{Conf}(X^n, g), \;\; \varphi^* g = C_\varphi^2 g$

Harmonic $\circ$ conformal $\neq$ harmonic

Modification

$\varphi \in \mathrm{Conf}(X^n, g),\ \ \varphi^* g = C_\varphi^2 g$

- pull-back $\rightsquigarrow$ twisted pull-back

  $f \circ \varphi \quad \rightsquigarrow \quad C_\varphi^{-\frac{n-2}{2}}\ f \circ \varphi$

  conformal factor

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- pull-back ⇝ twisted pull-back

$$f \circ \varphi \quad \rightsquigarrow \quad C_\varphi^{-\frac{n-2}{2}} \ f \circ \varphi$$

conformal factor

- $\mathcal{S}ol(\Delta_X) = \{f \in C^\infty(X) : \Delta_X f = 0\}$ (harmonic functions)

$$\rightsquigarrow \ \mathcal{S}ol(\widetilde{\Delta_X}) = \{f \in C^\infty(X) : \widetilde{\Delta_X} f = 0\}$$

$$\widetilde{\Delta_X} \quad := \quad \Delta_X \quad + \frac{n-2}{4(n-1)} \kappa$$

Yamabe operator — Laplacian — scalar curvature

Distinguished rep. of conformal groups

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$\Downarrow$ Modification

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$\Downarrow$ Modification

Theorem A ([K–Ørsted 03]) (X^n, g): Riemannian mfd

$\Longrightarrow \mathrm{Conf}(X, g)$ acts on $\mathcal{S}ol(\widetilde{\Delta_X})$ by $f \mapsto C_\varphi^{-\frac{n-2}{2}} f \circ \varphi$

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Point $\widetilde{\Delta_X} = \Delta_X + \frac{n-2}{4(n-1)}\kappa$

$\widetilde{\Delta_X}$ is not invariant by $\mathrm{Conf}(X, g)$.

But $\mathcal{S}ol(\widetilde{\Delta_X})$ is invariant by $\mathrm{Conf}(X, g)$.

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$\mathrm{Diffeo}(X) \supset \mathrm{Conf}(X,g) \supset \mathrm{Isom}(X,g)$

Conformal group — isometry group

Application of Theorem A

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Theorem B ([7, Part I]) $\widetilde{\Delta_X} = \Delta_{S^p} - \Delta_{S^q} + \text{const.}$

0) $\mathrm{Conf}(X, g) \simeq O(p+1, q+1)$

1) $\mathcal{S}ol(\widetilde{\Delta_X}) \neq \{0\} \iff p+q$ even

2) If $p+q$ is even and > 2, then
$\mathrm{Conf}(X, g) \curvearrowright \mathcal{S}ol(\widetilde{\Delta_X})$ is irreducible,
and for $p+q > 6$ it is a minimal rep of $O(p+1, q+1)$.

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↑

$^{\exists}$a $\mathrm{Conf}(X,g)$-invariant inner product, and take the Hilbert completion

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1. Conformal construction		
Theorem B	Clear	?
v.s.		
2. L^2 construction		
(Schrödinger model) Theorem D	?	Clear

Clear $\cdots$ advantage of the model

Hilbert structure on $\mathcal{S}ol(\widetilde{\Delta})$

Three methods:

1. Parseval-type formula [7, Part II]
2. Using Green function [7, Part III]
3. 'Intrinsic formula'

Hilbert structure on $Sol(\widetilde{\Delta})$

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Flat model

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Functoriality of Theorem A

$$\begin{array}{ccc} \mathcal{S}ol(\widetilde{\Delta}_{S^p \times S^q}) & \subset & \mathcal{S}ol(\widetilde{\Delta}_{\mathbb{R}^{p,q}}) \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{Conf}(S^p \times S^q) & \hookrightarrow & \mathrm{Conf}(\mathbb{R}^{p,q}) \end{array}$$

Conservative quantity for ultra-hyperbolic eqn.

$$\mathbb{R}^{p,q} = \mathbb{R}^{p+q},\ ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$$

$$\widetilde{\Delta}_{\mathbb{R}^{p,q}} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \equiv \square_{p,q}$$

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Problem Find an 'intrinsic' inner product on (a 'large' subspace of) $\mathcal{S}ol(\Box_{p,q})$ if exists.

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Unitarization of subrep (representation theory)

$\Longleftrightarrow$

Conservative quantity (differential eqn)

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$q = 1$ wave operator

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energy $\cdots$ conservative quantity for wave equations
w.r.t. time translation $\mathbb{R}$

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$\Downarrow$

? $\cdots$ conservative quantity for ultra-hyperbolic eqs
w.r.t. conformal group $O(p+1, q+1)$

Conservative quantity for $\square_{p,q} f = 0$

Fix $\alpha \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

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For $f \in \mathcal{S}ol(\square_{p,q})$

$$(f, f) := \int_\alpha Q_\alpha f \quad \text{(to be defined soon)} \qquad \cdots\cdots ①$$

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Theorem C ([7, Part III]+K– 2011)

1) ① is independent of hyperplane α.

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Theorem C ([7, Part III]+K– 2011)

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$$O(p, q) \quad \curvearrowright \quad \mathbb{R}^{p,q} \quad \text{(linear)}$$

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~~$O(p,q)$~~ $O(p+1, q+1)$ $\curvearrowright$ $\mathbb{R}^{p,q}$ ~~(linear)~~ (Möbius transform)

Construction of $Q_\alpha f$

$$\mathbb{R}^{p,q} = (\mathbb{R}^{p+q}, ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2)$$

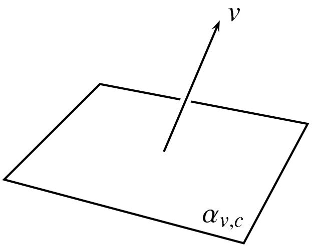
Fix $v \in \mathbb{R}^{p,q}$ s.t. $(v,v)_{\mathbb{R}^{p,q}} = \pm 1$

$c \in \mathbb{R}$

$\Downarrow$

$$\mathbb{R}^{p,q} \supset \alpha \equiv \alpha_{v,c} := \{x \in \mathbb{R}^{p+q} : (x,v)_{\mathbb{R}^{p,q}} = c\}$$

non-characteristic hyperplane



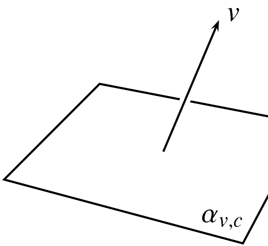
Construction of $Q_\alpha f$

Point: $f = f_+ + f_-$ (idea: Sato's hyperfunction)

Construction of $Q_\alpha f$

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at ∞

Point: $f = f_+ + f_-$ (idea: Sato's hyperfunction)



$(x, v)_{\mathbb{R}^{p,q}} = c$

Construction of $Q_\alpha f$

For $\alpha = \alpha_{v,c}$, $f \in C^\infty(\mathbb{R}^{p,q})$ with some decay at ∞

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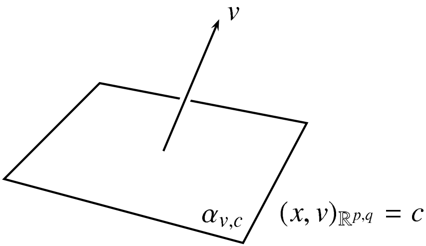
$f_\pm$ extends holomorphically to the direction $\pm\sqrt{-1}v$



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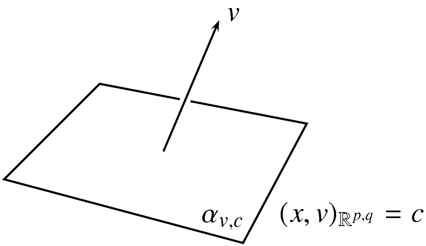
$$f_\pm(x; v) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mp f(x - tv)}{t \pm i0} dt$$



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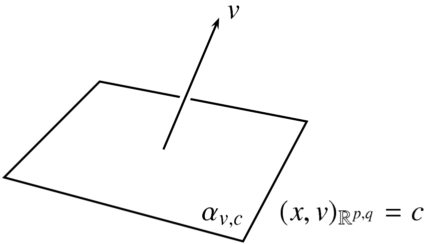
$f'_\pm \cdots$ normal derivative of $f_\pm$ w.r.t. v



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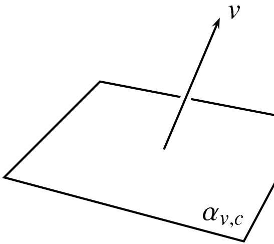
$$Q_\alpha f := \frac{1}{i}\left(f_+\overline{f'_+} - f_-\overline{f'_-}\right)$$



$(x, v)_{\mathbb{R}^{p,q}} = c$

Conservative quantity for $\square_{p,q} f = 0$

Fix $\alpha = \alpha_{v,c} \subset \mathbb{R}^{p+q}$ non-degenerate hyperplane

For $f \in \mathcal{S}ol(\square_{p,q})$

$$(f, f) := \int_{\alpha} Q_{\alpha} f \qquad \cdots\cdots ①$$

Theorem C

1) ① is independent of hyperplane α.
2) ① gives the unique inner product (up to scalar) which is invariant under $O(p+1, q+1)$.

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Theorem C

1) ① is independent of hyperplane α.
2) ① gives the unique inner product (up to scalar) which is invariant under $O(p+1, q+1)$.

Theorem C is non-trivial even for $q = 1$ (wave equation)

In space-time $\mathbb{R}^{p+1} = \mathbb{R}^{p}_{x} \times \mathbb{R}_{t}$,

average in space (i.e. time t = constant)

= average in (any hyperplane in space) $\times \mathbb{R}_t$ (time)

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction		
Theorems A, B	Clear	?
v.s.		
2.		
?	?	?

Clear · · · advantage of the model

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	conservative quantity Theorem C
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Two constructions of minimal reps.

	Group action	Hilbert structure
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2. L^2 construction (Schrödinger model) Theorem D	?	Clear

Clear $\cdots$ advantage of the model

Conformal model $\Longrightarrow$ L^2-model

$$\square_{p,q} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}$$

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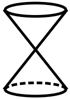
$=$  (figure for $(p, q) = (2, 1)$)

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$\overline{\cdot}$ denotes the Hilbert completion w.r.t. the invariant inner product.

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conformal model — L^2-model

Two constructions of minimal reps.

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Clear $\cdots$ advantage of the model

L^2-model of minimal reps.

Theorem D $p+q>2$, even. $\overline{Sol(\square_{p,q})} \xrightarrow{\sim} L^2(\Xi)$

conformal model L^2-model

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What is Ξ?

Geometric quantization of minimal nilpotent orbit

$\mathfrak{g}^* \supset \mathcal{O} = \mathrm{Ad}^*(G)\lambda$ coadjoint orbit

$\rightsquigarrow$? "geometric quantization"

$\widehat{G} \ni \pi$ irred. unitary rep of G

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Assume Ξ is a Lagrangian submanifold of $\mathcal{O}_{\min}$

$\Longrightarrow G \overset{?}{\curvearrowright} L^2(\Xi)$

L^2-model of minimal rep.

V: simple Jordan algebra

G = (a finite covering of) the conformal group of V

L^2-model of minimal rep.

V: simple Jordan algebra

G = (a finite covering of) the conformal group of V

Ex 1 $V = \mathrm{Symm}(n, \mathbb{R})$
$G = Mp(n, \mathbb{R})$, a double cover of $Sp(n, \mathbb{R})$

Ex 2 $V = \mathbb{R}^{p,q+1}$
$G = O(p+1, q+1)$

L^2-model of minimal rep.

V: simple Jordan algebra

G = (a finite covering of) the conformal group of V

$\mathcal{O}_{\min}$: minimal nilpotent coadjoint orbit of G

$\Xi := \mathcal{O}_{\min} \cap V$

<u>Observation</u>

$$\begin{array}{ccc} \mathfrak{g} & \simeq & \mathfrak{g}^* \\ \cup & & \cup \\ V & & \mathcal{O}_{\min} \end{array}$$

L^2-model of minimal rep.

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Assume that a maximal euclidean Jordan subalgebra of V is simple, and $V \not\simeq \mathbb{R}^{p,q+1}$ with $p+q$: odd.

Theorem (with Hilgert, Moellers, arXiv:1106.3621)
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4) The annihilator of the differential rep $d\pi$ is the Joseph ideal in $U(\mathfrak{g})$ if V is split and $\mathfrak{g} \not\simeq A_n$.

L^2-model of minimal rep.

V: simple Jordan algebra
G = (a finite covering of) the conformal group of V

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<u>Ex 1</u> $V = Symm(n, \mathbb{R})$
$G = Mp(n, \mathbb{R})$
$\Longrightarrow$ Schrödinger model of the Weil representation
$G \curvearrowright L^2(\mathbb{R})_{\text{even}} \simeq L^2(\text{Symm}(n, \mathbb{R}))$

<u>Ex 2</u> $V = \mathbb{R}^{p,q+1}$, $p+q$: even
$G = O(p+1, q+1)$
$\Longrightarrow G \curvearrowright L^2(\Xi)$
(Theorem D)

Inversion element

$$G = PGL(2, \mathbb{C}) \underset{\text{Möbius transform}}{\curvearrowright} \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$$

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$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad z \mapsto -\frac{1}{z}$$

(inversion)

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G is generated by P and w.

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$$\doteqdot O(3,1) \qquad\qquad \doteqdot \mathbb{R}^{2,0}$$

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Towards a global formula

$p+q$: even > 2

$$G = O(p+1, q+1) \curvearrowright L^2(\Xi) \qquad \text{minimal rep.}$$

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Problem What is $\mathcal{F}_\Xi$?

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P-action $\cdots$ translation and multiplication

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Problem What is $\mathcal{F}_\Xi$?

Cf. Analogous operator for the oscillator rep.

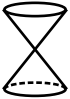
$Mp(n,\mathbb{R}) \curvearrowright L^2(\mathbb{R}^n)$

unitary inversion operator coincides with

Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ (up to scalar)!

New Fourier transform $\mathcal{F}_\Xi$ on Ξ

$$\Xi := \{\xi \in \mathbb{R}^{p+q} : \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2 = 0\}$$

$=$  (figure for $(p,q) = (2,1)$)

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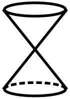
$=$  (figure for $(p, q) = (2, 1)$)

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$	$\mathcal{F}_\Xi$ on $\Xi =$ 

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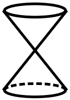
$=$  (figure for $(p, q) = (2, 1)$)

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}_\Xi$ on $\Xi =$ 

Problem
1. Algebraic properties of $\mathcal{F}_\Xi$
2. Analytic formula of $\mathcal{F}_\Xi$.

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

Fourier trans. $\mathcal{F}_{\mathbb{R}^n}$ on $\mathbb{R}^n$

$\mathcal{F}_\Xi$ on $\Xi =$ 

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$$\mathcal{F}^4 = \mathrm{id}$$

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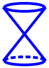
$\mathcal{F}_\Xi$ on $\Xi =$ 

$$\mathcal{F}_\Xi^2 = \mathrm{id}$$

‘Fourier transform’ $\mathcal{F}_\Xi$ on Ξ

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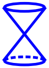
$$Q_j \mapsto -P_j$$
$$P_j \mapsto Q_j$$

$\mathcal{F}_\Xi$ on $\Xi =$ 

$Q_j = x_j$ (multiplication by coordinates function)

$$P_j = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x_j}$$

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$\mathcal{F}_\Xi$ on $\Xi =$ 

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$R_j = {}^{\exists}$second order differential op. on Ξ

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Rediscover Bargmann–Todorov’s operators (1977)

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Notice $\left.\begin{array}{l} Q_1{}^2 + \cdots + Q_p{}^2 - Q_{p+1}{}^2 - \cdots - Q_{p+q}{}^2 = 0 \\ R_1{}^2 + \cdots + R_p{}^2 - R_{p+1}{}^2 - \cdots - R_{p+q}{}^2 = 0 \end{array}\right\}$ on Ξ

Unitary inversion operator $\mathcal{F}_\Xi$

$p+q$: even > 2

$G = O(p+1, q+1) \curvearrowright L^2(\Xi)$ minimal rep.

w-action $\cdots$ $\mathcal{F}_\Xi$ (unitary inversion operator)

Problem Find the unitary operaotr $\mathcal{F}_\Xi$ explicitly.

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Cf. Euclidean case $\varphi(t) = e^{-it}$ (one variable)

$\mathcal{F}_{\mathbb{R}^N} f(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y \rangle) f(y) dy$

Unitary inversion operator $\mathcal{F}_\Xi$

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<u>Thm E</u> (K–Mano, Memoirs AMS, 2011, vol.1000)

$$(\mathcal{F}_\Xi f)(x) = c \int_\Xi \Phi(\langle x, y\rangle) f(y) dy$$

$\mathcal{F}_{\mathbb{R}^N}$ v.s. $\mathcal{F}_{\Xi}$

On $\mathbb{R}^N$

$$(\mathcal{F}_{\mathbb{R}^N} f)(x) = c \int_{\mathbb{R}^N} \varphi(\langle x, y \rangle) f(y) dy$$

$\varphi(t) = e^{-it}$ satisfies

$$\left(\frac{d}{dt} + i\right) \varphi(t) = 0$$

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On Ξ $(\subset \mathbb{R}^{p,q})$

$$(\mathcal{F}_{\Xi} f)(x) = c \int_{\Xi} \Phi(\langle x, y \rangle) f(y) dy$$

$\Phi(t)$ satisfies

$$\left(\left(t\frac{d}{dt}\right)^2 + \frac{1}{2}(p+q-4)t\frac{d}{dt} + 2t\right)\Phi(t) = 0$$

Bessel functions

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{z}{2}\right)^{2j}}{j!\,\Gamma(j+\nu+1)}$$

$$I_\nu(z) := e^{-\frac{\sqrt{-1}\nu\pi}{2}} J_\nu\left(e^{\frac{\sqrt{-1}\pi}{2}} z\right)$$

$$Y_\nu(z) := \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi} \qquad \text{(second kind)}$$

$$K_\nu(z) := \frac{\pi}{2\sin\nu\pi}\left(I_{-\nu}(z) - I_\nu(z)\right) \qquad \text{(third kind)}$$

Bessel distribution

Prop. ([4]) $\Phi_m^{\varepsilon}(t)$ solves the differential equation

$$(\theta^2 + m\theta + 2t)u = 0$$

where $\theta = t\frac{d}{dt}$.

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Explicit forms

$$\Phi_m^0(t) = 2\pi i(2t)_+^{-\frac{m}{2}} J_m(2\sqrt{2t_+})$$

$$\Phi_m^1(t) = \Phi_m^0(t) - \pi i \sum_{l=0}^{m-1} \frac{(-1)^l}{2^l(m-l-1)!}\delta^{(l)}(t)$$

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$$\Phi_m^2(t) = 2\pi i(2t)_+^{-\frac{m}{2}} Y_m(2\sqrt{2t_+}) + 4(-1)^{m+1} i\,(2t)_-^{-\frac{m}{2}} K_m(2\sqrt{2t_-})$$

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$$(\mathcal{F}_\Xi f)(x) = c \int_\Xi \Phi^{\varepsilon(p,q)}_{\frac{1}{2}(p+q-4)}(\langle x, y \rangle) f(y) dy$$

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$$(\mathcal{F}_\Xi f)(x) = c\int_\Xi \Phi^{\varepsilon(p,q)}_{\frac{1}{2}(p+q-4)}(\langle x, y\rangle) f(y)dy$$

Here, $\varepsilon(p,q) = \begin{cases} 0 & \text{if } \min(p,q) = 1, \\ 1 & \text{if } p, q > 1 \text{ are both odd,} \\ 2 & \text{if } p, q > 1 \text{ are both even.} \end{cases}$

Two constructions of minimal reps.

	Group action	Hilbert structure
1. Conformal construction Theorems A, B	Clear	conservative quantity
v.s.		
2. L^2 construction (Schrödinger model) Theorem D	'Fourier transform' $\mathcal{F}_\Xi$	Clear

Clear $\cdots$ advantage of the model

3. Deformation of Fourier transforms (Theorems F, G, H)

Two constructions of minimal reps.

	Group action	Hilbert structure
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Theorems A, B	Clear	Theorem C
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Theorem D		

Clear ··· advantage of the model

3. Deformation of Fourier transforms (Theorems F, G, H)

§1 What are minimal representations?

§2 Conformal model of minimal representations

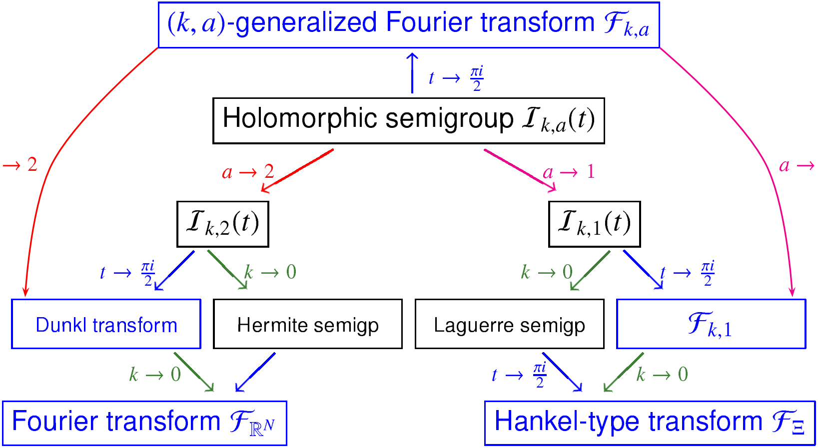
§3 L^2 model of minimal representations

§4 Deformation of Fourier transform

Deformation theory of Fourier transform

- Generalized Fourier transform $\mathcal{F}_{k,a}$ C.R.A.S. Paris (2009)
- Laguerre semigroup and Dunkl operators 74 pp. arXiv:0907.3749 with Ben Saïd and Bent Ørsted
- Inversion and holomorphic extension R. Howe 60th birthday volume, 65 pp. with Mano

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



$\Leftarrow$ 'unitary inversion operator' $\Rightarrow$

the Weil representation of
the metaplectic group $Mp(N, \mathbb{R})$

the minimal representation of
the conformal group $O(N+1, 2)$

Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

$\mathcal{F}_{\Xi}$	$\cdots$	'Fourier transform' on $\Xi \subset \mathbb{R}^{p,q}$
$\mathcal{F}_{\mathbb{R}^N}$	$\cdots$	Fourier transform on $\mathbb{R}^N$

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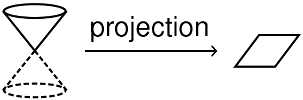
Assume $q = 1$. Set $p = N$.

$\mathbb{R}^{N,1} \supset \Xi =$



$= \mathbb{R}^N$

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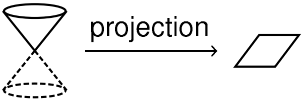
$\mathbb{R}^{N,1} \supset \Xi =$  $= \mathbb{R}^N$

$\mathcal{F}_{\Xi}$	$\mathcal{F}_{\mathbb{R}^N}$
$O(N+1,2)$	$Mp(N,\mathbb{R})$

Interpolation of Fourier transform $\mathcal{F}_{\mathbb{R}^N}$

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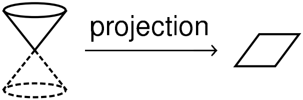
$\mathbb{R}^{N,1} \supset \Xi =$  $= \mathbb{R}^N$

$\mathcal{F}_{\Xi}$ interpolate $\mathcal{F}_{\mathbb{R}^N}$

$a = 1$ $\qquad a = 2$

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Fourier transform

$$\mathcal{F}_{\mathbb{R}^N} = \quad c \quad \exp\left(\frac{\pi i}{4}(\quad \Delta \quad - |x|^2)\right)$$

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Fourier transform

self-adjoint op. on $L^2(\mathbb{R}^N)$

$$\mathcal{F}_{\mathbb{R}^N} = c \exp\left(\frac{\pi i}{4}(\Delta - |x|^2)\right)$$

phase factor $= e^{\frac{\pi i N}{4}}$ Laplacian

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Hermite semigroup

$$I(t) := \exp \frac{t}{2}(\Delta - |x|^2)$$

Mehler kernel using $\exp(-x^2)$

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

(k,a)-generalized Fourier transform

self-adjoint op. on $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$

$$\mathcal{F}_{k,a} = c \exp\left(\frac{\pi i}{2a}\left(|x|^{2-a}\Delta_k - |x|^a\right)\right)$$

phase factor $= e^{i\frac{\pi(N+2\langle k\rangle+a-2)}{2a}}$

Dunkl Laplacian

(k,a)-deformation of Hermite semigroup

$$\mathcal{I}_{k,a}(t) := \exp \frac{t}{a}\left(|x|^{2-a}\Delta_k - |x|^a\right)$$

Mehler kernel using $\exp(-x^2)$

k: multiplicity on root system $\mathcal{R}$, $a > 0$

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

Hankel-type transform on Ξ

self-adjoint op. on $L^2(\mathbb{R}^N, \frac{dx}{|x|})$

$$\mathcal{F}_\Xi = c \exp\left(\frac{\pi i}{2}\left(\overbrace{|x|\Delta} - |x|\right)\right)$$

phase factor $= e^{\frac{\pi i(N-1)}{2}}$ Laplacian

"Laguerre semigroup" ([K–Mano], 2007)

$$\mathcal{I}(t) := \exp t(|x|\Delta - |x|)$$

$\operatorname{Re} t > 0$

closed formula using Bessel function

(k,a)-deformation of $\exp \frac{t}{2}(\Delta - |x|^2)$

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Dunkl Laplacian

(k,a)-deformation of Hermite semigroup ([with Ben Saïd, Ørsted])

$$\mathcal{I}_{k,a}(t) := \exp \frac{t}{a}(|x|^{2-a}\Delta_k - |x|^a) \qquad \operatorname{Re} t > 0$$

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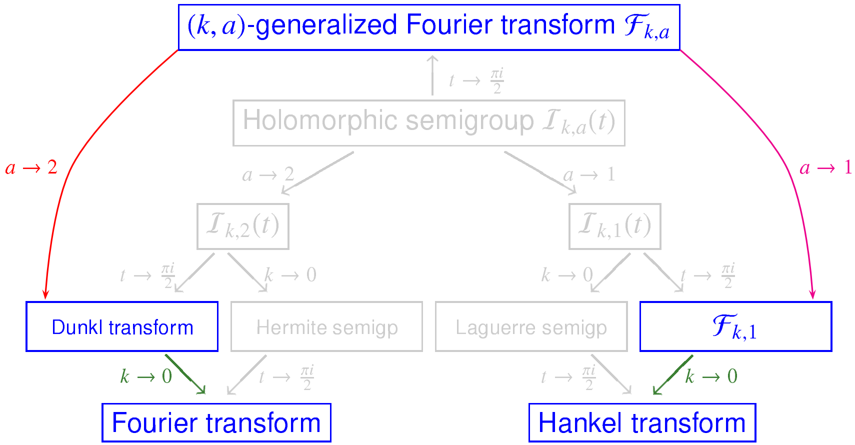
Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$



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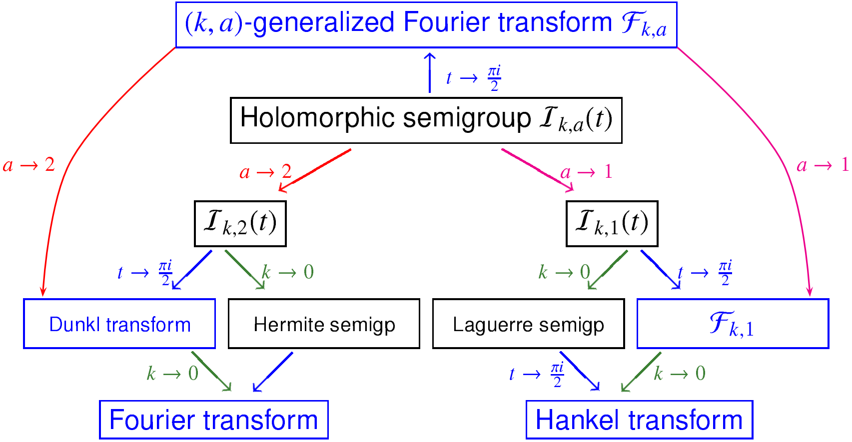

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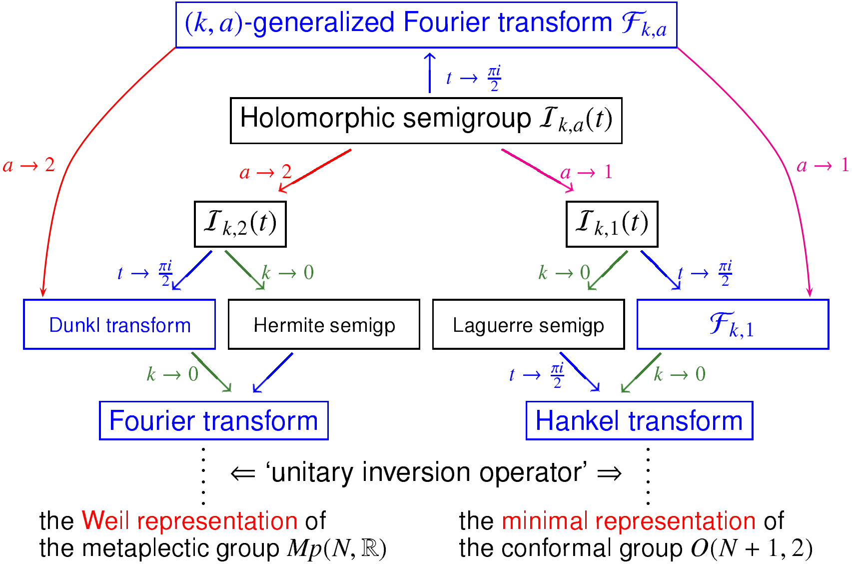

Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c\,\mathcal{I}_{k,a}\left(\frac{\pi i}{2}\right)$$

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$$\mathcal{F}_{k,a} = c\,\mathcal{I}_{k,a}(\frac{\pi i}{2}) = c \exp\left(\frac{\pi i}{2a}(|x|^{2-a}\Delta_k - |x|^a)\right)$$

Thm G ([arXiv:0907.3749])

1) $\mathcal{F}_{k,a}$ is a unitary operator

Generalized Fourier transform $\mathcal{F}_{k,a}$

$$\mathcal{F}_{k,a} = c\,\mathcal{I}_{k,a}\left(\frac{\pi i}{2}\right) = c\exp\left(\frac{\pi i}{2a}\left(|x|^{2-a}\Delta_k - |x|^a\right)\right)$$

Thm G ([arXiv:0907.3749])

1) $\mathcal{F}_{k,a}$ is a unitary operator
2) $\mathcal{F}_{0,2}$ = Fourier transform on $\mathbb{R}^N$
   $F_{k,a}$ = Dunkl transform on $\mathbb{R}^N$
   $\mathcal{F}_{0,1}$ = Hankel-type transform on $L^2($$)$
3) $\mathcal{F}_{k,a}$ is of finite order $\iff a \in \mathbb{Q}$
4) $\mathcal{F}_{k,a}$ intertwines $|x|^a$ and $-|x|^{2-a}\Delta_k$

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   $\mathcal{F}_{0,1}$ = Hankel-type transform on $L^2($  $)$
3) $\mathcal{F}_{k,a}$ is of finite order $\Longleftrightarrow a \in \mathbb{Q}$
4) $\mathcal{F}_{k,a}$ intertwines $|x|^a$ and $-|x|^{2-a}\Delta_k$

$\Longrightarrow$ generalization of classical identities such as Hecke identity, Bochner identity, Parseval–Plancherel formulas, Weber's second exponential integral, etc.

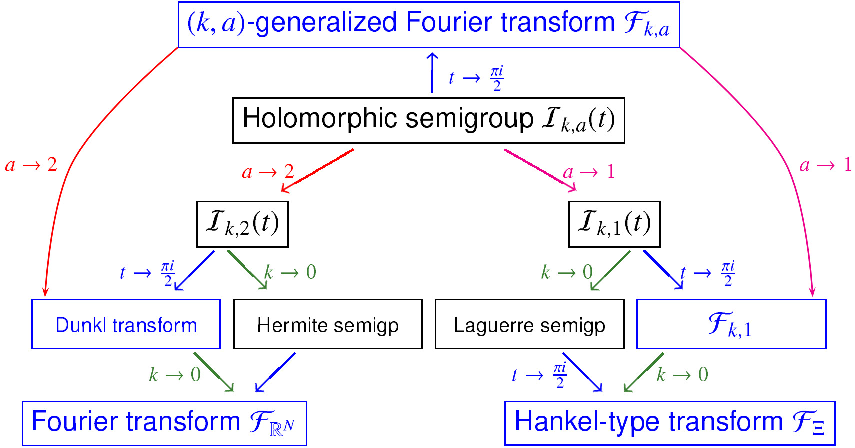
Heisenberg-type inequality

Thm H ([6]) (Heisenberg inequality)

$$\| |x|^{\frac{a}{2}} f(x) \|_k \; \| |\xi|^{\frac{a}{2}} (\mathcal{F}_{k,a} f)(\xi) \|_k \geq \frac{2\langle k \rangle + N + a - 2}{2} \| f(x) \|_k^2$$

$k \equiv 0, a = 2$	$\cdots$ Weyl–Pauli–Heisenberg inequality for Fourier transform $\mathcal{F}_{\mathbb{R}^N}$
k: general, $a = 2$	$\cdots$ Heisenberg inequality for Dunkl transform $\mathcal{D}_k$ (Rösler, Shimeno)
$k \equiv 0, a = 1, N = 1$	$\cdots$ Heisenberg inequality for Hankel transform

Special values of holomorphic semigroup $\mathcal{I}_{k,a}(t)$

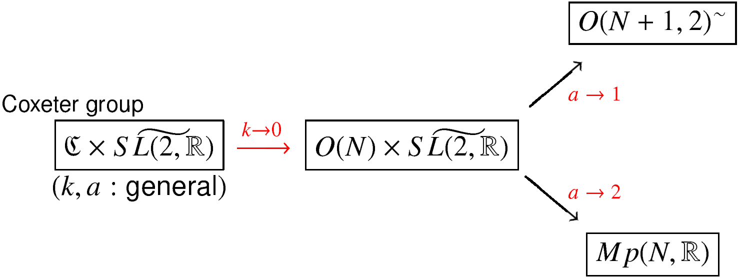


$\Leftarrow$ 'unitary inversion operator' $\Rightarrow$

the Weil representation of the metaplectic group $Mp(N,\mathbb{R})$

the minimal representation of the conformal group $O(N+1,2)$

Hidden symmetries in $L^2(\mathbb{R}^N, \vartheta_{k,a}(x)dx)$



Minimal ⇔ Maximal

(Ambitious) Project:

Use minimal reps to get an inspiration in finding new interactions with other fields of mathematics.

Viewpoint:

Minimal representation ($\Leftarrow$ group)

$\approx$ Maximal symmetries ($\Leftarrow$ rep. space)

Geometric analysis on minimal reps

[1] Schrödinger model of minimal representations of $O(p,q)\cdots$
Memoirs of Amer. Math. Soc. (2011), no.1000, 132 pp.

[2] Algebraic analysis on minimal representations $\cdots$
Publ. RIMS (2011), 28 pp.

[3] Geometric analysis of small unitary reps of $GL(n,\mathbb{R})\cdots$
J. Funct. Anal. (2011)

[4] Special functions associated to a fourth order differential equation $\cdots$
Ramanujan J. Math (2011)

[5] Minimal representations via Bessel operators $\cdots$ 66 pp. arXiv:1106.3621

[6] Laguerre semigroup and Dunkl operators $\cdots$ 74 pp. arXiv:0907.3749

[7] Analysis on minimal representations $\cdots$
Adv. Math. (2003) I, II, III, 110 pp.

[8] Generalized Fourier transforms $\mathcal{F}_{k,a}\cdots$ C.R.A.S. Paris 2009

[9] Inversion and holomorphic extension $\cdots$
R. Howe 60th birthday volume (2007), 65 pp.

Collaborated with S. Ben Saïd, J. Hilgert, G. Mano, J. Möllers and B. Ørsted